Lecture Notes on a Graduate Course on Ordinary Differential Equations

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GODE 01: Introduction to ODEs

INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

I Definitions

- An ordinary differential equation (ODE) is an equation that contains one or more derivatives of the unknown function. A function that satisfies the equation is called a solution of the ODE.
- The most general form of an ODE is:

$$\boxed{F(x, y(x), y'(x), y''(x), \dots, y^{(n)}(x)) = 0} \qquad (1)$$

with $F : \mathbb{R} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$.
If we define $Y(x) = (y(x), y'(x), y''(x), \dots, y^{(n)}(x))$, then the equation above can be rewritten as:

$$\boxed{F(x, Y(x)) = 0} \qquad (2)$$

- The natural number n is the order of the ODE.

II Linear vs. nonlinear ODEs

Let V be the set of all continuous functions $Y : \mathbb{R} \to \mathbb{R}^n$.
We say that the ODE $F(x, Y(x)) = 0$ is linear if and only

if F satisfies

$$\forall x,\lambda,\mu \in \mathbb{R} : \forall Y,Z \in V : F(x,\lambda Y+\mu Z) = \lambda F(x,Y) + \mu F(x,Z)$$

otherwise we say that the ODE is nonlinear.

- It can be shown that the most general form of a linear ODE is:

$$p_n(x)y^{(n)}(x) + \dots + p_2(x)y''(x) + p_1(x)y'(x) + p_0(x)y(x) = q(x)$$

- Types of ODE problems

We distinguish between the following types of ODE problems:

(1) → Initial Value Problem

These are problems of the form:

$$\boxed{\begin{cases} F(x, y(x), y'(x), \dots, y^{(n-1)}(x), y^{(n)}(x)) = 0 \\ y(x_0) = a_0 \wedge y'(x_0) = a_1 \wedge \dots \wedge y^{(n-1)}(x_0) = a_{n-1} \end{cases}}$$

where $y, y', y'', \dots, y^{(n-1)}$ are all fixed at the same point $x_0 \in \mathbb{R}$. These additional equations are called initial conditions.

2 → Boundary Value Problem

These are problems of the form

$$\boxed{\begin{array}{l} F(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0 \\ y^{(k_1)}(x_1) = a_1 \wedge y^{(k_2)}(x_2) = a_2 \wedge \cdots \wedge y^{(k_n)}(x_n) = a_n \end{array}}$$

where $y^{(k_1)}, y^{(k_2)}, \ldots, y^{(k_n)}$ are specified on more than just a unique point. These additional equations are called <u>boundary conditions.</u>

● Techniques for solving ODEs

Solution techniques are classified under the following categories.

a) <u>Exact analytic methods:</u> We obtain an exact solution in closed form.

b) <u>Approximate methods:</u> We obtain an approximate solution in closed form.

   i) <u>Local methods:</u> We obtain an approximate solution which is good in a neighborhood of some special point.

   ii) <u>Global methods:</u> Obtain an approximate solution which is good on the entire domain of the ODE.

c) Numerical methods: We obtain an approximate discretized solution with the use of a computer.

d) Existence/Uniqueness: We prove rigorously that a given ODE problem has a unique solution, without actually being able to find the solution exactly or approximately.

Systems of ODEs

- A system of m ODEs is any problem of the form

$$\boxed{\begin{cases} F_1(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0 \\ F_2(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0 \\ \vdots \\ F_m(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0 \end{cases}}$$

where we require the logical conjunction of all equations.

- Every n^{th}-order ODE of the form

$$y^{(n+1)} = F(x, y(x), y'(x), \dots, y^{(n)}(x))$$

can be rewritten as a system of 1st-order equations.

$$\begin{cases} y_0' = y_1 \\ y_1' = y_2 \\ \vdots \\ y_{n-1}' = y_n \\ y_n' = F(x, y_0, y_1, y_2, \dots, y_n) \end{cases}$$

GODE 02: First-order ODEs

FIRST-ORDER ODEs

• A <u>1st-order ordinary differential equation</u> (ODE) is an equation of the form $y' = f(x,y)$ satisfied by a function $y(x)$ of x. A corresponding <u>1st-order initial value problem</u> is a problem of the form

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

with $x_0, y_0 \in \mathbb{R}$ given.

• An <u>implicit solution</u> to the initial value problem above is a solution of the form $F(x,y)=0$ where we have shown that

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \Longleftrightarrow F(x,y) = 0$$

• An <u>explicit solution</u> to the initial value problem above is a solution of the form $y = g(x)$ such that

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \Longleftrightarrow y = g(x).$$

• There is no general solution method that can give an implicit or explicit solution to a 1st-order ODE. However, solution methods exist for some special cases, including the following:

(1) → Separable ODEs

These are problems of the form $\boxed{\begin{cases} y' = g(x)h(y) \\ y(x_0) = y_0 \end{cases}}$ (1)

Note that we say that

y_0 is a fixed point of Eq.(1) $\Leftrightarrow h(y_0) = 0$

If we initialize the system at a fixed point, then $y' = 0$, and we expect $y(x)$ to remain at the fixed point for all $x \in \mathbb{R}$. Furthermore, if we initialize at y_0 with $h(y_0) \neq 0$ then the solution cannot cross over any fixed point. We can therefore expect that $h(y(x)) \neq 0$ for all $x \in \mathbb{R}$ for which $y(x)$ can be obtained.

Methodology: Based on the above remarks we begin by assuming that $h(y) \neq 0$, and therefore:

$$y' = g(x)h(y) \Leftrightarrow \frac{y'}{h(y)} = g(x) \Leftrightarrow \int \frac{dy}{h(y)} = \int g(x)dx \Leftrightarrow$$

$$\Leftrightarrow H(y) = G(x) + c_1$$

To determine c we use the initial condition $y(x_0) = y_0$:

$H(y_0) = G(x_0) + c_1 \Leftrightarrow c_1 = H(y_0) - G(x_0)$.

Note that in the above argument we assume that the system has not been initialized at a fixed point. If the goal is to find a general solution, then it is necessary to explore whether the general solution continuous to hold when y_0 is a fixed point.

EXAMPLES

a) Solve the initial value problem

$$\begin{cases} y'(x) = (1+y^2(x))\cos x \\ y(0) = 1 \end{cases}$$

Solution

Since $1+y^2 > 0$, then the system has no fixed points.

We note that

$$y' = (1+y^2)\cos x \Leftrightarrow \frac{y'}{1+y^2} = \cos x \Leftrightarrow \int \frac{dy}{1+y^2} = \int \cos x \, dx \quad (1)$$

with $\int \cos x \, dx = \sin x + c_1$ and $\int \frac{dy}{1+y^2} = \text{Arctan}(y) + c_2$

thus

$$(1) \Leftrightarrow \text{Arctan}(y) = \sin x + c \Leftrightarrow y = \tan(\sin x + c)$$

From the initial condition:

$$y(0) = 1 \Leftrightarrow \text{Arctan}(1) = \sin 0 + c \Leftrightarrow$$
$$\Leftrightarrow c = \text{Arctan}(1) = \pi/4$$

and therefore: $y(x) = \tan(\sin x + \pi/4)$.

We note that with increasing x, this solution becomes singular

when:

$$\sin x + \pi/4 = \pi/2 \Leftrightarrow \sin x = \pi/4 - \pi/2 \Leftrightarrow \sin x = \pi/4 \in [-1,1]$$
$$\Leftrightarrow x = \text{Arcsin}(\pi/4).$$

▶ We say that the solution has a <u>finite-time singularity</u> at $x = \text{Arcsin}(\pi/4)$.

6) Solve the initial value problem

$$\begin{cases} y' = y^2 \\ y(0) = y_0 \end{cases}$$

Solution

We note that $y=0$ is a fixed point. We assume that initially $y_0 \neq 0$. Then $y \neq 0$, and it follows that

$$y' = y^2 \Leftrightarrow \frac{y'}{y^2} = 1 \Leftrightarrow \int \frac{dy}{y^2} = \int dx \Leftrightarrow \frac{y^{-1}}{-1} = x + c_1$$

$$\Leftrightarrow y^{-1} = -x - c \Leftrightarrow y = \frac{1}{-x-c} = \frac{-1}{x+c_1}$$

Since $y(0) = y_0 \Leftrightarrow y_0^{-1} = -0 - c \Leftrightarrow c = -y_0^{-1} = \frac{-1}{y_0}$
it follows that

$$y = \frac{-1}{x+c_1} = \frac{-1}{x - y_0^{-1}} = \frac{-y_0}{y_0(x - y_0^{-1})} = \frac{-y_0}{y_0 x - 1}, \text{ with } y_0 \neq 0$$

For the fixed point initialization $y_0 = 0$, the above equation correctly gives $y = \frac{-0}{0x - 1} = 0$, therefore it is valid for all $y_0 \in \mathbb{R}$.

The solution has a finite time singularity when

$y_0 x - 1 = 0 \Leftrightarrow y_0 x = 1 \Leftrightarrow x = 1/y_0$.

c) Solve the initial value problem

$$\begin{cases} y' = 2x(y-1) \\ y(1) = y_0 \end{cases}$$

Solution

We note that $y-1=0 \Leftrightarrow y=1$, so $y=1$ is the fixed point. We assume initialization $y_0 \neq 1$, thus $y \neq 1$. Then,

$$y' = 2x(y-1) \Leftrightarrow \frac{y'}{y-1} = 2x \Leftrightarrow \int \frac{dy}{y-1} = \int 2x dx$$

$$\Leftrightarrow \ln|y-1| = x^2 + c \qquad (1)$$

From the initial condition

$$y(1) = y_0 \Leftrightarrow \ln|y_0 - 1| = 1^2 + c \Leftrightarrow c = \ln|y_0-1| - 1$$

and therefore

$$\ln|y-1| = x^2 + \ln|y_0-1| - 1 \Leftrightarrow$$

$$\Leftrightarrow |y-1| = \exp(x^2 + \ln|y_0-1| - 1) = \exp(x^2-1)\exp(\ln|y_0-1|)$$

$$= |y_0-1|\exp(x^2-1) \Leftrightarrow$$

$$\Leftrightarrow y-1 = \pm|y_0-1|\exp(x^2-1) \qquad (2)$$

Since $y=1$ is a fixed point, for $y_0-1>0$ we will have $y-1>0$ and for $y_0-1<0$ we will have $y-1<0$. It follows that

$$(2) \Leftrightarrow y-1 = (y_0-1)\exp(x^2-1) \Leftrightarrow$$

$$\Leftrightarrow y = 1 + (y_0-1)\exp(x^2-1) \quad \text{for } y_0 \neq 1.$$

For $y_0 = 1$, the above solution gives $y=1$, so the general solution also works for $y_0 = 1$.

(2) → Homogeneous ODEs

<u>Def</u>: A homogeneous ODE is an equation of the form
$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

<u>Solution method</u>: Let $y(x) = xu(x)$. It follows that:
$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \Leftrightarrow x\frac{du}{dx} + u = f(u) \Leftrightarrow x\frac{du}{dx} = f(u) - u \Leftrightarrow$$
$$\Leftrightarrow \frac{1}{f(u)-u}\frac{du}{dx} = \frac{1}{x} \Leftrightarrow \int \frac{d\tilde{u}}{f(\tilde{u})-\tilde{u}} = \int \frac{dx}{x} \Leftrightarrow \text{etc...}$$

EXAMPLE

Solve $\dfrac{dy}{dx} = \dfrac{2xy+y^2}{x^2}$ with $y_0 = -1/2$ for $x_0 = 1$.

<u>Solution</u>

We note that
$$\frac{dy}{dx} = \frac{2xy+y^2}{x^2} = \frac{2xy}{x^2} + \frac{y^2}{x^2} = 2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \quad (1)$$

Let $y = xu \Rightarrow u = y/x$. It follows that
$$(1) \Leftrightarrow x\frac{du}{dx} + u = 2u + u^2 \Leftrightarrow x\frac{du}{dx} = u^2 + 2u - u \Leftrightarrow$$
$$\Leftrightarrow x\frac{du}{dx} = u(u+1) \Leftrightarrow \frac{1}{u(u+1)}\frac{du}{dx} = \frac{1}{x} \Leftrightarrow$$
$$\Leftrightarrow \int \frac{du}{u(u+1)} = \int \frac{dx}{x} \quad (2)$$

Since $\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$ with

$A = \frac{1}{u+1}\bigg|_{u=0} = \frac{1}{0+1} = 1$, and

$B = \frac{1}{u}\bigg|_{u=-1} = \frac{1}{-1} = -1$

it follows that

$$\int \frac{du}{u(u+1)} = \int \left(\frac{1}{u} - \frac{1}{u+1}\right) du = \ln|u| - \ln|u+1| + C_1$$

$$= \ln\left|\frac{u}{u+1}\right| + C_1$$

and

$$\int \frac{dx}{x} = \ln|x| + c_2$$

and therefore

$$(2) \Leftrightarrow \ln\left|\frac{u}{u+1}\right| = \ln|x| + C \qquad (3)$$

Apply the initial condition:

$$y(1) = -1/2 \Leftrightarrow u(1) = y(1)/1 = -1/2 \Leftrightarrow$$

$$\Leftrightarrow \ln\left|\frac{-1/2}{-1/2+1}\right| = \ln|1| + C \Leftrightarrow$$

$$\Leftrightarrow c = \ln\left|\frac{-1/2}{-1/2+1}\right| = \ln\left|\frac{-1}{-1+2}\right| = \ln|-1| = 0$$

and therefore:

$$(3) \Leftrightarrow \ln\left|\frac{u}{u+1}\right| = \ln|x| \Leftrightarrow \left|\frac{u}{u+1}\right| = |x| \Leftrightarrow$$

$$\Leftrightarrow \frac{u}{u+1} = x \ \vee \ \frac{u}{u+1} = -x \qquad (4)$$

From the initial condition $u(1) = -1/2$ we note that $\frac{u}{u+1} < 0$ and $x > 0$, and therefore we reject the first equation on (4) and have:

$$(4) \Leftrightarrow \frac{u}{u+1} = -x \Leftrightarrow u = -x(u+1) \Leftrightarrow u = -xu - x \Leftrightarrow$$

$$\Leftrightarrow (1+x)u = -x \Leftrightarrow u = \frac{-x}{1+x} \Leftrightarrow \frac{y}{x} = \frac{-x}{x+1}$$

$$\Leftrightarrow y = \frac{-x^2}{x+1}$$

(3) → Integrating Factors Method

This method can be applied to ODEs of the form:

$$\boxed{y' + f(x)y = g(x)}$$

with f, g continuous on $\mathbb{R}$.

Solution method

Define $h(x) = \exp\left(\int f(x)\,dx\right)$ and note that $h'(x) = f(x)h(x)$.

Then we multiply both sides of the ODE with $h(x)$:

$$y' + f(x)y = g(x) \Leftrightarrow y'h(x) + h(x)f(x)\,y = g(x)h(x) \Leftrightarrow$$

$$\Leftrightarrow y'h(x) + h'(x)\;y = g(x)h(x) \Leftrightarrow$$

$$\Leftrightarrow \frac{d}{dx}\left[y h(x)\right] = h(x)g(x) \Leftrightarrow$$

$$\Leftrightarrow h(x)y = \int h(x)g(x)\,dx + C_1$$

$$\Leftrightarrow y = \frac{1}{h(x)}\int h(x)g(x)\,dx + \frac{C_1}{h(x)} \qquad (1)$$

↳ Note that for $g(x) = 0$, the above solution simplifies to

$$y = \frac{C_1}{h(x)} = C_1 \exp\left(-\int f(x)\,dx\right)$$

This is called the <u>homogeneous term</u> to Eq. (1).

The integral term is called the <u>particular term.</u>

EXAMPLE

a) Solve the ODE $y' + xy = x^2$ with $y(0) = y_0$.

Solution

Use the integrating factor

$$h(x) = \exp\left(\int x\,dx\right) = \exp(x^2/2) \Rightarrow h'(x) = x h(x)$$

and therefore:

$$y' + xy = x^2 \Leftrightarrow y' h(x) + x h(x) y = x^2 h(x) \Leftrightarrow y' h(x) + h'(x) y = x^2 h(x) \Leftrightarrow$$

$$\Leftrightarrow [y h(x)]' = x^2 h(x) \Leftrightarrow y h(x) = c + \int_0^x t^2 h(t)\,dt \qquad (1)$$

For $x = 0$: $y_0 h(0) = c + 0 \Leftrightarrow c = y_0 h(0) = y_0 \exp(0) = y_0$

and therefore,

$$(1) \Leftrightarrow y h(x) = y_0 + \int_0^x t^2 h(t)\,dt \Leftrightarrow$$

$$\Leftrightarrow y = \frac{y_0}{h(x)} + \frac{1}{h(x)} \int_0^x t^2 h(t)\,dt =$$

$$= \frac{y_0}{\exp(x^2/2)} + \frac{1}{\exp(x^2/2)} \int_0^x t^2 \exp(t^2/2)\,dt =$$

$$= y_0 \exp(-x^2/2) + \exp(-x^2/2) \int_0^x t^2 \exp(t^2/2)\,dt$$

→ The integrating factor method can be applied to the more general problem of the form

$$\boxed{f(x)y' + g(x)y = h(x)}$$

However, if $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then x_0 is a singular point of the ODE and the ODE will only yield a unique solution if x is restricted to an interval between neighboring singular points.

EXAMPLE

Solve the ODE $(x^2-1)y' + xy = 0$ with $y(x_0) = y_0$.

Solution

We have

$$(x^2-1)y' + xy = 0 \Leftrightarrow y' + \frac{x}{x^2-1}y = 0 \qquad (1)$$

We will use the integrating factor

$$h(x) = \exp\left(\int \frac{x}{x^2-1}\,dx\right) = \exp\left(\frac{1}{2}\int \frac{(x^2-1)'}{x^2-1}\,dx\right) =$$

$$= \exp\left(\frac{1}{2}\ln|x^2-1|\right) = \exp\left(\ln\sqrt{|x^2-1|}\right) =$$

$$= \sqrt{|x^2-1|}$$

$\Rightarrow h'(x) = h(x)\dfrac{x}{x^2-1}$. It follows that

$$(1) \Leftrightarrow y' h(x) + \frac{x}{x^2-1} h(x) y = 0 \Leftrightarrow y' h(x) + y h'(x) = 0$$

$$\Leftrightarrow (d/dx)\left[y h(x)\right] = 0 \Leftrightarrow (d/dx)\left[y\sqrt{|x^2-1|}\,\right] = 0$$

$$\Leftrightarrow y\sqrt{|x^2-1|} = c_1 \Leftrightarrow y = \frac{c_1}{\sqrt{|x^2-1|}}$$

We note that the ODE has singular points on $x=1$ and $x=-1$. From the initial condition:

$$y(x_0) = y_0 \Leftrightarrow \frac{c_1}{\sqrt{|x_0^2-1|}} = y_0 \Leftrightarrow c = y_0\sqrt{|x_0^2-1|}$$

and therefore:

$$y = \frac{y_0\sqrt{|x_0^2-1|}}{\sqrt{|x^2-1|}}$$

We distinguish between the following cases:

Case 1 : If $x_0 \in (-\infty,-1)$, then $|x_0^2-1| = x_0^2-1$ and

$$y = \frac{y_0\sqrt{x_0^2-1}}{\sqrt{x^2-1}}, \quad \forall x \in (-\infty,-1)$$

Case 2 : If $x_0 \in (-1,1)$, then $|x_0^2-1| = 1-x_0^2$ and

$$y = \frac{y_0\sqrt{1-x_0^2}}{\sqrt{1-x^2}}, \quad \forall x \in (-1,1)$$

Case 3 : If $x_0 \in (1,+\infty)$, then $|x_0^2-1| = x_0^2-1$ and

$$y = \frac{y_0\sqrt{x_0^2-1}}{\sqrt{x^2-1}}, \quad \forall x \in (1,+\infty).$$

Homework: First-order ODEs

Homework 01: First-order ODEs

1. The logistic population model is intended to model population growth under finite resources. If $y(t)$ is the population at time t, λ is the population growth rate, and N is the carrying capacity, then according to the logistic model, $y(t)$ is governed by

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \lambda y(N - y)$$

   (a) Using the initial condition $y(0) = y_0$, show that

$$y(t) = \frac{N y_0}{y_0 + (N - y_0)\exp(-\lambda N t)}$$

   Show the validity of this result regardless of whether or not y_0 is a fixed point.

   (b) Show that $y(t)$ has an inflection point at $y = N/2$, using directly the differential equation instead of the explicit solution.

   (c) Assuming initialization at $y_0 \in (0, N/2)$, find the time t at which the solution reaches the inflection point

2. Consider an ordinary differential equation of the form

$$M(x, y) + N(x, y)y' = 0$$

   such that

$$\forall \lambda \in (0, +\infty) : \begin{cases} M(\lambda x, \lambda y) = \lambda^a M(x, y) \\ N(\lambda x, \lambda y) = \lambda^a N(x, y) \end{cases}$$

   with $a \in \mathbb{R}$. Show that the substitution $u = y/x$ reduces this differential equation to the separable form

$$\frac{1}{x} + \frac{N(1, u)}{M(1, u) + uN(1, u)} \frac{\mathrm{d}u}{\mathrm{d}x} = 0$$

3. Consider the initial value problem

$$\begin{cases} y' - 2xy = 1 \\ y(0) = y_0 \end{cases}$$

   Show that its unique solution is given by

$$y(x) = \exp(x^2) \left[\frac{\pi}{2} \operatorname{erf}(x) + y_0 \right]$$

   with $\operatorname{erf}(x)$ the error function, defined as

$$\operatorname{erf}(x) = \frac{2}{\pi} \int_0^x \exp(-t^2)\, \mathrm{d}t$$

4. A Bernoulli ordinary differential equation is an equation of the form

$$y' + p(x)y = q(x)y^n$$

   with $n \in \mathbb{N}$.

   (a) Show that the substitution $u = y^{1-n}$ reduces the Bernoulli equation to a linear ordinary differential equation of the form

$$u' + (1 - n)p(x)u = (1 - n)q(x)$$

   (b) Use this substitution to solve the following Bernoulli initial value problem:

$$\begin{cases} y' + xy = xy^2 \\ y(0) = y_0 \end{cases}$$

GODE 03: Linear Differential Equations

LINEAR DIFFERENTIAL EQUATIONS

▼ Basic Definitions - Terminology

- A linear differential equation is any equation of the form
$$p_n(x)y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = f(x). \quad (1)$$
- The functions $p_0, p_1, \ldots, p_n$ are called the coefficients of the linear differential equation and it is usually assumed that they are continuous functions.
- $n \in \mathbb{N}^*$ is the order of the linear differential equation.
- Given the linear differential equation of Eq.(1), we say that for a point $x_0 \in \mathbb{R}$:
$$x_0 \text{ is regular} \Leftrightarrow p_n(x_0) \neq 0$$
$$x_0 \text{ is singular} \Leftrightarrow p_n(x_0) = 0$$
- A linear differential equation of the form of Eq.(1) is homogeneous on a set $A \subseteq \mathbb{R}$ if and only if
$$\forall x \in A : f(x) = 0.$$
otherwise, we say that it is inhomogeneous.
- If an linear differential equation is regular for every point in some interval $A \subseteq \mathbb{R}$ (i.e. if $\forall x \in A: p_n(x) \neq 0$) then we can rewrite it as:
$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x) \quad (2)$$
with
$$\forall k \in [n-1] \cup \{0\} : a_k(x) = \frac{p_k(x)}{p_n(x)} \quad \text{and} \quad g(x) = \frac{f(x)}{p_n(x)}.$$

Function operators and linear operators

- Let $A \subseteq \mathbb{R}$ be an interval. We define the following function spaces via belonging conditions as follows:

  a) Space of continuous functions $C^0(A)$:
  $$y \in C^0(A) \Leftrightarrow \begin{cases} y : A \to \mathbb{R} \\ y \text{ continuous on } A. \end{cases}$$

  b) Space of n-times continuously differentiable functions $C^n(A)$.
  $$y \in C^n(A) \Leftrightarrow \begin{cases} y : A \to \mathbb{R} \\ y \text{ n-times differentiable on } A \\ y^{(n)} \text{ continuous on } A \end{cases}$$

  c) Space of infinitely differentiable functions $C^\infty(A)$.
  $$y \in C^\infty(A) \Leftrightarrow \forall n \in \mathbb{N} : y \in C^n(A)$$

- Given the linear differential equation from Eq.(2) we define the mapping $L : C^n(A) \to C^0(A)$ such that
  $$\forall y \in C^n(A) : L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y \quad (3)$$
  Then, the linear differential equation
  $$y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \dots + a_1(x) y'(x) + a_0(x) y(x) = g(x)$$
  can be rewritten as:
  $$L(y) = g. \quad \text{or also: } Ly = g.$$
  Note that by analogy the <u>operator L</u> is to a function $y \in C^n(A)$ what a matrix A is to some vector $x \in \mathbb{R}^n$.

- The operator L defined by Eq. (3) satisfies the following definition of a <u>linear operator</u>

<u>Def</u>: Consider an operator $L : C^n(A) \to C^0(A)$. We say that L is a linear operator if and only if it satisfies the following conditions:

a) $\forall y_1, y_2 \in C^n(A) : L(y_1 + y_2) = Ly_1 + Ly_2$

b) $\forall \lambda \in \mathbb{R} : \forall y \in C^n(A) : L(\lambda y) = \lambda L(y)$.

<u>Prop</u>: Let $L : C^n(A) \to C^0(A)$ be a linear operator. Then:

$$\forall \lambda, \mu \in \mathbb{R} : \forall y_1, y_2 \in C^n(A) : L(\lambda y_1 + \mu y_2) = \lambda L(y_1) + \mu L(y_2).$$

<u>Proof</u>

Let $\lambda, \mu \in \mathbb{R}$ and $y_1, y_2 \in C^n(A)$ be given. Then:

$$L(\lambda y_1 + \mu y_2) = L(\lambda y_1) + L(\mu y_2)$$
$$= \lambda L(y_1) + \mu L(y_2).$$

It follows that

$$\forall \lambda, \mu \in \mathbb{R} : \forall y_1, y_2 \in C^n(A) : L(\lambda y_1 + \mu y_2) = \lambda L(y_1) + \mu L(y_2).$$

→ Note that the definition

$$Ly = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$$

is given in terms of function algebra, i.e function addition and function multiplication. In terms of regular algebra, we write:

$$\forall x \in A : (Ly)(x) = y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x).$$

▼ Homogeneous linear differential equations

We begin by presenting the theory needed for solving homogeneous linear differential equations of the form $Ly = 0$ given a linear operator $L : C^n(A) \to C^0(A)$.

● Solution set of the homogeneous ODE

We begin by stating some needed definitions. Then we state the main result without proof.

Def: Let $y_1, y_2, \ldots, y_n \in C^0(A)$ be functions. We say that $y_1, y_2, \ldots, y_n$ linearly independent $\Leftrightarrow$
$$\Leftrightarrow \forall \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R} : (\lambda_1 y_1 + \cdots + \lambda_n y_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0)$$

↳ We note that this definition is analogous to the linear independence of vectors on $\mathbb{R}^n$. However, the statement
$\lambda_1 y_1 + \cdots + \lambda_n y_n = 0$
is equivalent to the algebraic statement
$\forall x \in A : \lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x) = 0.$

Def: Let $y_1, y_2, \ldots, y_n \in C^0(A)$. We define the space spanned by the functions $y_1, \ldots, y_n$ as
$$\operatorname{span}\{y_1, y_2, \ldots, y_n\} = \{\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n \mid \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}\}$$

↳ The corresponding belonging condition reads:

$y \in \text{span}\{y_1, y_2, \dots, y_n\} \iff$

$\iff \exists \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}: y = \lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_n y_n.$

<u>Def</u>: Let $L: C^n(A) \to C^0(A)$ be an operator. We define the null space of L as:

$$\text{null}(L) = \{y \in C^n(A) \mid Ly = \mathbf{0}\}.$$

↳ Thus, the problem of solving the homogeneous linear differential equation $Ly = \mathbf{0}$ is equivalent to the problem of finding the null space $\text{null}(L)$ or the operator L.

<u>Thm</u>: Let $a_0, a_1, \dots, a_{n-1} \in C^0(A)$ for some interval $A \subseteq \mathbb{R}$ and define the operator $L: C^n(A) \to C^0(A)$ such that

$\forall y \in C^n(A): Ly = y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y$

Then there exist $y_1, y_2, \dots, y_n \in C^n(A)$ such that they satisfy the following conditions:

(a) $y_1, y_2, \dots, y_n$ are linearly independent

(b) $\text{null}(L) = \text{span}\{y_1, y_2, \dots, y_n\}$

↳ It follows from this theorem that the general solution to the linear differential equation $Ly = \mathbf{0}$ takes the form

$\forall x \in A: y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \dots + \lambda_n y_n(x)$

where $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ are constant coefficients and $y_1, y_2, \dots, y_n$ are linearly independent functions.

The initial value problem

In an initial value problem we consider the homogeneous linear differential equation $Ly = 0$ where we introduce the restrictions

$$y(x_0) = a_0 \wedge y'(x_0) = a_1 \wedge y''(x_0) = a_2 \wedge \cdots \wedge y^{(n-1)}(x_0) = a_{n-1}$$

Given the general solution

$$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x)$$

the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ can be uniquely solved by the following system of equations:

$$\begin{cases} \lambda_1 y_1(x_0) + \lambda_2 y_2(x_0) + \cdots + \lambda_n y_n(x_0) = a_0 \\ \lambda_1 y_1'(x_0) + \lambda_2 y_2'(x_0) + \cdots + \lambda_n y_n'(x_0) = a_1 \\ \vdots \\ \lambda_1 y_1^{(n-1)}(x_0) + \lambda_2 y_2^{(n-1)}(x_0) + \cdots + \lambda_n y_n^{(n-1)}(x_0) = a_{n-1} \end{cases}$$

which can be rewritten in terms of matrices as follows:

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

The determinant of the matrix is called the Wronskian and we will prove later that it is non-zero. It follows that solving with respect to the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ will give a unique solution.

The Wronskian and its properties

Def: Let $y_1, y_2, \dots, y_n \in C^{n-1}(A)$, for some interval $A \subseteq \mathbb{R}$. We define:

a) The matrix $W[y_1, \dots, y_n](x)$ as:

$$\forall x \in A : W[y_1, \dots, y_n](x) = \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & & & \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix}$$

b) The Wronskian $w[y_1, \dots, y_n](x)$ as:

$$\forall x \in A : w[y_1, \dots, y_n](x) = \det W[y_1, \dots, y_n](x)$$

We now show that the Wronskian satisfies the following properties:

1 → Nonzero Wronskian implies linear independence

Thm: Let $y_1, y_2, \dots, y_n \in C^{n-1}(A)$ with $A \subseteq \mathbb{R}$ an interval. Then:

$(\exists x \in A : w[y_1, \dots, y_n](x) \neq 0) \Rightarrow$

$\Rightarrow y_1, y_2, \dots, y_n$ linearly independent

Proof

Assume that $\exists x \in A : w[y_1, \dots, y_n](x) \neq 0$. Choose an $x_0 \in A$ such that $w[y_1, \dots, y_n](x_0) \neq 0$. It is sufficient to show that $\forall \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R} : (\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0)$

Let $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ be given and assume that

$\lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_n y_n = \mathbf{0} \Rightarrow$

$\Rightarrow \forall x \in A: \lambda_1 y_1(x) + \lambda_2 y_2(x) + \dots + \lambda_n y_n(x) = 0$

Differentiating with respect to x gives the equations:

$\forall x \in A: \lambda_1 y_1'(x) + \lambda_2 y_2'(x) + \dots + \lambda_n y_n'(x) = 0$

$\forall x \in A: \lambda_1 y_1''(x) + \lambda_2 y_2''(x) + \dots + \lambda_n y_n''(x) = 0$

$\vdots$

$\forall x \in A: \lambda_1 y_1^{(n-1)}(x) + \lambda_2 y_2^{(n-1)}(x) + \dots + \lambda_n y_n^{(n-1)}(x) = 0$

These equations are equivalent to the matrix equation

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ y_1''(x) & y_2''(x) & \cdots & y_n''(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \forall x \in A$$

We define $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and the matrix equation is

$W[y_1, y_2, \dots, y_n](x)\Lambda = \mathbf{0}$, $\forall x \in A$.

For $x = x_0$, we have:

$$\begin{cases} W[y_1, \dots, y_n](x_0)\Lambda = \mathbf{0} \\ \det W[y_1, \dots, y_n](x_0) = w[y_1, \dots, y_n](x_0) \neq 0 \end{cases} \Rightarrow$$

$\Rightarrow \Lambda = \mathbf{0} \Rightarrow (\lambda_1, \lambda_2, \dots, \lambda_n) = (0, 0, \dots, 0)$

$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$

We have thus shown that

$\forall \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}: (\lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_n y_n = \mathbf{0} \Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0)$

$\Rightarrow y_1, y_2, \dots, y_n$ linearly independent $\square$

(2) $\to$ Linearly independent solutions of a linear differential equation give a non-zero Wronskian

The previous property can be used to prove that a set of functions are linearly independent, if the corresponding Wronskian is nonzero for at least one point. The converse statement is not always true. However we will now show that if some functions $y_1, \dots, y_n$ solve the SAME linear differential equation and are linearly independent, then they will give a nonzero Wronskian for all points.

Thm: Define the operator $L : C^n(A) \to C^0(A)$, for some interval $A \subseteq \mathbb{R}$, such that:

$$\forall x \in A : Ly(x) = y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \dots + a_1(x) y_1'(x) + a_0(x) y(x)$$

We assume that

a) $y_1, y_2, \dots, y_n \in C^n(A)$ are linearly independent

b) $\forall k \in [n] : Ly_k = 0$

Then, it follows that

a) $\forall x \in A : w'[y_1, \dots, y_n](x) + a_{n-1}(x) w[y_1, \dots, y_n](x) = 0$

b) For some $c \in A$:

$$\forall c, x \in A : w[y_1, \dots, y_n](x) = w[y_1, \dots, y_n](c) \exp\left(-\int_c^x a_{n-1}(t)\,dt\right)$$

c) $\forall x \in A : w[y_1, \dots, y_n](x) \neq 0$

Proof

a) Define the vector-valued function $y: A \to \mathbb{R}^n$ with $y = (y_1, y_2, \dots, y_n)$. Since

$$(\forall k \in [n] : L y_k = 0) \Rightarrow \left(\forall k \in [n] : y_k^{(n)} = -\sum_{p=0}^{n-1} a_p y_k^{(p)}\right) \Rightarrow$$

$$\Rightarrow y^{(n)} = -\sum_{p=0}^{n-1} a_p y^{(p)} \qquad (1)$$

Note that $y^{(p)}$ is a vector-valued function whereas a_p is a scalar function. It follows that

$$(d/dx)\, w[y_1, \dots, y_n](x) = (d/dx) \det(y, y', y'', \dots, y^{(n-1)}) =$$
$$= \det(y, y', \dots, y^{(n-2)}, y^{(n)}) =$$
$$= \det\left(y, y', \dots, y^{(n-2)}, -\sum_{p=0}^{n-1} a_p y^{(p)}\right) =$$
$$= \sum_{p=0}^{n-1} \det(y, y', \dots, y^{(n-2)}, -a_p y^{(p)}) =$$
$$= -\sum_{p=0}^{n-1} a_p \det(y, y', \dots, y^{(n-2)}, y^{(p)}) =$$
$$= -\sum_{p=0}^{n-2} a_p \det(y, \dots, y^{(p)}, \dots, y^{(n-2)}, y^{(p)}) +$$
$$a_{n-1} \det(y, \dots, y^{(n-2)}, y^{(n-1)})$$

$$= 0 - a_{n-1} w[y] = -a_{n-1} w[y] \Rightarrow$$
$$\Rightarrow \forall x \in A : \; w'[y](x) + a_{n-1}(x)\, w[y](x) = 0$$

b) Define the integrating factor

$$\forall x \in A: h(x) = \exp\left(\int_c^x a_{n-1}(t)\,dt\right)$$

and note that

$$\forall x \in A: h'(x) = (d/dx)\exp\left(\int_c^x a_{n-1}(t)\,dt\right) =$$

$$= \exp\left(\int_c^x a_{n-1}(t)\,dt\right)\frac{d}{dx}\int_c^x a_{n-1}(t)\,dt =$$

$$= h(x)\,a_{n-1}(x).$$

We may now solve the differential equation satisfied by the Wronskian as follows:

$$w'[y](x) + a_{n-1}(x)\,w[y](x) = 0 \Leftrightarrow$$

$$\Leftrightarrow w'[y](x)\,h(x) + h(x)\,a_{n-1}(x)\,w[y](x) = 0 \Leftrightarrow$$

$$\Leftrightarrow w'[y](x)\,h(x) + w[y](x)\,h'(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow (d/dx)\,[w[y](x)\,h(x)] = 0 \Leftrightarrow w[y](x)\,h(x) = c_0$$

$$\Leftrightarrow w[y](x) = \frac{c_0}{h(x)} = c_0 \exp\left(-\int_c^x a_{n-1}(t)\,dt\right)$$

For $x = c$: $w[y](c) = c_0 \cdot 1 = c_0$, and therefore

$$\forall x \in A: w[y](x) = w[y](c)\exp\left(-\int_c^x a_{n-1}(t)\,dt\right)$$

c) From (b) we see that it is sufficient to show that $\exists c \in A: w[y](c) \neq 0$. To show a contradiction, we assume the opposite statement: $\forall c \in A: w[y](c) = 0$. Choose some $c \in A$ and consider the linear system of equations $W[y](c)\Lambda = \mathbf{0}$ with $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$. It follows that

$w[y](c)=0 \Rightarrow \det W[y](c)=0 \Rightarrow \exists \Lambda \in \mathbb{R}^n-\{0\}: W[y]\Lambda = \mathbf{0}$

Choose some $\Lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbb{R}^n-\{\mathbf{0}\}$ such that $W[y](c)\Lambda=\mathbf{0}$ and define the function $f: A\to\mathbb{R}$ with

$\forall x\in A: f(x)=\lambda_1 y_1(x)+\lambda_2 y_2(x)+\dots+\lambda_n y_n(x)$

It follows that

$$Lf = L\left(\sum_{k=1}^{n}\lambda_k y_k\right)=\sum_{k=1}^{n}L(\lambda_k y_k)=\sum_{k=1}^{n}\lambda_k L y_k =$$

$$=\sum_{k=1}^{n}\lambda_k\cdot\mathbf{0}=\mathbf{0}\Rightarrow f\in \operatorname{null}(L).$$

We also know that

$$\forall p\in\{0\}\cup[n-1]: f^{(p)}(c)=\sum_{k=1}^{n}\lambda_k y_k^{(p)}(c)=\sum_{k=1}^{n}[W[y](c)]_{pk}\Lambda_k =$$

$$=[W[y](c)\Lambda]_p = 0$$

We will now claim that given the initial condition

$f(c)=f'(c)=f''(c)=\dots=f^{(n-1)}(c)=0$

the function f will satisfy $\forall x\in A: f(x)=0$. To show this, we rewrite the equation as a system of first-order ODEs by defining

$\forall k\in[n]:\forall x\in A: g_k(x)=f^{(k-1)}(x)$.

The ODE $Lf=\mathbf{0}$ can be rewritten as

$$\begin{cases} g_1'(x)=g_2(x)\\ g_2'(x)=g_3(x)\\ \vdots\\ g_{n-1}'(x)=g_n(x)\\ g_n'(x)=-\displaystyle\sum_{k=1}^{n}a_{k-1}(x)\,g_k(x)\end{cases}$$

and the corresponding initial condition is

$g_1(x) = g_2(x) = \cdots = g_n(x) = 0$

It is easy to see that all derivatives $g_1'(x), g_2'(x), \ldots, g_n'(x)$ are then zero, and therefore all functions $g_1, \ldots, g_n$ will remain constant and be equal to zero for all $x \in A$. This proves the claim. From the claim we have:

$(\forall x \in A : f(x) = 0) \Rightarrow f = \mathbf{0} \Rightarrow \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n = \mathbf{0}$ (1)

By hypothesis, we also know that

$y_1, y_2, \ldots, y_n$ linearly independent (2)

From Eq.(1) and Eq.(2):

$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0 \Rightarrow \lambda = \mathbf{0}$

This is a contradiction, since by construction λ satisfies $\lambda \in \mathbb{R}^n - \{\mathbf{0}\}$. It follows that

$\exists c \in A : w[y](c) \neq 0$

Fix a $c \in A$ such that $w[y](c) \neq 0$. Then, from (6), it follows that

$$\forall x \in A : w[y](x) = w[y](c) \exp\left(-\int_c^x a_{n-1}(t)\,dt\right) \neq 0$$

because $\forall x \in \mathbb{R} : \exp(x) > 0$. This concludes the proof. □

• Solving homogeneous linear differential equations

To solve a homogeneous linear differential equation
$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0$
we need to find the linearly independent solutions
$y_1(x), y_2(x), \ldots, y_n(x)$ that form the general solution
$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x)$
There is no general method for finding the functions $y_1(x), \ldots,$ $y_n(x)$. However, an exact solution is possible for the following cases.

(1) → Constant coefficient case

Consider the linear ODE
$y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = 0$
with $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{R}$ given constants. Let L be the corresponding operator.

Solution method

- 1 Find the characteristic polynomial $P(b)$:
$$L(e^{bx}) = (b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0)e^{bx} = P(b)e^{bx}$$
- 2 Let $\rho_1, \rho_2, \ldots, \rho_n \in \mathbb{C}$ be the zeroes of the characteristic polynomial P. Then:
  a) Each single zero ρ_k contributes a solution
  $y_k(x) = \exp(\rho_k x)$

b) Each zero ρ_k with multiplicity m (i.e. $P(\beta)$ has a factor $(x-\rho_k)^m$) contributes the following linearly independent solutions:

$y_k(x) = \exp(\rho_k x)$

$y_{k+1}(x) = x\exp(\rho_k x)$

$y_{k+2}(x) = x^2\exp(\rho_k x)$

$\vdots$

$y_{k+m-1}(x) = x^{m-1}\exp(\rho_k x)$.

• 3 We write the general solution and apply the initial conditions if given.

↳ Remark : Complex zeroes appear as complex conjugate pairs $\rho_k = \gamma + i\omega$ and $\rho_{k+1} = \gamma - i\omega$, because the coefficients of the characteristic polynomial are real numbers. We use the De Moivre identity:

$\forall\vartheta\in\mathbb{R}:\ e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$

and note that the corresponding solutions satisfy:

$$y_k(x) = \exp(\rho_k x) = \exp((\gamma+i\omega)x) = \exp(\gamma x + i\omega x) =$$
$$= \exp(\gamma x)\exp(i\omega x) = e^{\gamma x}(\cos(\omega x) + i\sin(\omega x))$$
$$y_{k+1}(x) = \exp(\rho_{k+1}x) = \exp((\gamma - i\omega)x) = \exp(\gamma x - i\omega x) =$$
$$= \exp(\gamma x)\exp(-i\omega x) = e^{\gamma x}(\cos(-\omega x) + i\sin(-\omega x)) =$$
$$= e^{\gamma x}(\cos(\omega x) - i\sin(\omega x))$$

It follows that any linear combination of $y_k(x)$ and $y_{k+1}(x)$ can be rewritten as:

$$\lambda_k y_k(x) + \lambda_{k+1} y_{k+1}(x) =$$
$$= \lambda_k e^{\gamma x}(\cos(\omega x) + i\sin(\omega x)) + \lambda_{k+1} e^{\gamma x}(\cos(\omega x) - i\sin(\omega x)) =$$
$$= e^{\gamma x}\left[(\lambda_k + \lambda_{k+1})\cos(\omega x) + i(\lambda_k - \lambda_{k+1})\sin(\omega x)\right] =$$
$$= (\lambda_k + \lambda_{k+1})\left[e^{\gamma x}\cos(\omega x)\right] + i(\lambda_k - \lambda_{k+1})\left[e^{\gamma x}\sin(\omega x)\right]$$
$$= \mu_k e^{\gamma x}\cos(\omega x) + \mu_{k+1} e^{\gamma x}\sin(\omega x)$$

with

$$\begin{cases} \mu_k = \lambda_k + \lambda_{k+1} \\ \mu_{k+1} = i(\lambda_k - \lambda_{k+1}) \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} \lambda_k \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} \mu_k \\ \mu_{k+1} \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \lambda_k \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1}\begin{bmatrix} \mu_k \\ \mu_{k+1} \end{bmatrix} =$$
$$= \frac{1}{-i-i}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}\begin{bmatrix} \mu_k \\ \mu_{k+1} \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \lambda_k = \frac{-i\mu_k - \mu_{k+1}}{-2i} = \frac{\mu_{k+1} + i\mu_k}{2i} \quad \wedge$$
$$\wedge \; \lambda_{k+1} = \frac{-i\mu_k + \mu_{k+1}}{-2i} = \frac{-\mu_{k+1} + i\mu_k}{2i}$$

It follows that an equivalent set of solutions are

$$z_k(x) = e^{\gamma x}\cos(\omega x)$$
$$z_{k+1}(x) = e^{\gamma x}\sin(\omega x)$$

In general: given complex conjugate zeroes $\gamma + i\omega$ and $\gamma - i\omega$ with multiplicity m, it is best practice to use the following set of linearly independent solutions:

$$y_k(x) = e^{\gamma x}\cos(\omega x), \quad y_{k+2} = x e^{\gamma x}\cos(\omega x), \ldots,$$
$$y_{k+1}(x) = e^{\gamma x}\sin(\omega x) \quad y_{k+3} = x e^{\gamma x}\sin(\omega x)$$

$$y_{k+2m-2}(x) = x^{m-1} e^{\gamma x} \cos(\omega x)$$
$$y_{k+2m-1}(x) = x^{m-1} e^{\gamma x} \sin(\omega x).$$

EXAMPLE

a) Write the general solution to $y'''(x) - 2y'(x) = 0$.

Solution

Define $Ly(x) = y'''(x) - 2y'(x)$ and note that

$$L(e^{bx}) = (e^{bx})''' - 2(e^{bx})' = b^3 e^{bx} - 2b\, e^{bx} =$$
$$= (b^3 - 2b) e^{bx} = b(b^2-2) e^{bx} = b(b-\sqrt{2})(b+\sqrt{2}) e^{bx}$$

The characteristic polynomial $P(b) = b(b-\sqrt{2})(b+\sqrt{2})$ has zeroes: $0, \sqrt{2}, -\sqrt{2}$ and therefore

$$y(x) = \lambda_1 e^{0x} + \lambda_2 e^{\sqrt{2}x} + \lambda_3 e^{-\sqrt{2}x} =$$
$$= \lambda_1 + \lambda_2 e^{x\sqrt{2}} + \lambda_3 e^{-x\sqrt{2}}$$

b) Solve the initial value problem

$$\begin{cases} y''(x) - 8y'(x) + 16y(x) = 0 \\ y(0) = 1 \wedge y'(0) = 3 \end{cases}$$

Solution

Define $Ly(x) = y''(x) - 8y'(x) + 16y(x)$ and note that

$$L(e^{bx}) = (e^{bx})'' - 8(e^{bx})' + 16e^{bx} =$$
$$= b^2 e^{bx} - 8b e^{bx} + 16 e^{bx} = (b^2 - 8b + 16) e^{bx}$$
$$= (b-4)^2 e^{bx}$$

The characteristic polynomial $P(b) = (b-4)^2$ has zeroes: $4, 4$ and therefore:

$$y(x) = \lambda_1 e^{4x} + \lambda_2 x e^{4x}$$

To apply the initial condition, we note that

$$y'(x) = \lambda_1 (e^{4x})' + \lambda_2 (xe^{4x})' = 4\lambda_1 e^{4x} + \lambda_2 (e^{4x} + 4xe^{4x})$$
$$= (4\lambda_1 + \lambda_2) e^{4x} + 4\lambda_2 x e^{4x}$$

and therefore

$$\begin{cases} y(0)=1 \\ y'(0)=3 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 e^0 + \lambda_2 e^0 0 = 1 \\ (4\lambda_1+\lambda_2)e^0 + 4\lambda_2 \cdot 0 e^0 = 3 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 + 0\lambda_2 = 1 \\ 4\lambda_1 + \lambda_2 = 3 \end{cases}$$
$$\Leftrightarrow \begin{cases} \lambda_1 = 1 \\ 4\cdot 1 + \lambda_2 = 3 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 3-4 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}$$

It follows that the solution is

$$y(x) = e^{4x} - xe^{4x} = (1-x)e^{4x}.$$

c) Linear Oscillator problem:

Solve the initial value problem

$$\begin{cases} y''(x) + w^2 y(x) = 0 \\ y(0) = y_0 \wedge y'(0) = y_1 \end{cases}$$

Solution

Define $Ly(x) = y''(x) + w^2 y(x)$ and note that

$$L(e^{bx}) = (e^{bx})'' + w^2 e^{bx} = b^2 e^{bx} + w^2 e^{bx} = (b^2 + w^2) e^{bx}$$
$$= (b+iw)(b-iw)e^{bx}$$

The characteristic polynomial $P(b) = (b+iw)(b-iw)$ has zeroes $+iw, -iw$. It follows that

$$y(x) = \lambda_1 e^{0x} \cos(wx) + \lambda_2 e^{0x} \sin(wx) =$$
$$= \lambda_1 \cos(wx) + \lambda_2 \sin(wx)$$

To apply the initial conditions, we calculate:

$$y'(x) = \lambda_1(-\omega\sin(\omega x)) + \lambda_2(\omega\cos(\omega x)) =$$
$$= -\omega\lambda_1\sin(\omega x) + \omega\lambda_2\cos(\omega x)$$

and therefore

$$\begin{cases} y(0) = y_0 \\ y'(0) = y_1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1\cos 0 + \lambda_2\sin 0 = y_0 \\ -\omega\lambda_1\sin 0 + \omega\lambda_2\cos 0 = y_1 \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} \lambda_1 + 0\lambda_2 = y_0 \\ -0\lambda_1 + \omega\lambda_2 = y_1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = y_0 \\ \omega\lambda_2 = y_1 \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} \lambda_1 = y_0 \\ \lambda_2 = y_1/\omega \end{cases}$$

It follows that the solution is:

$$y(x) = y_0\cos(\omega x) + (y_1/\omega)\sin(\omega x).$$

2 → Equidimensional case (Euler-Cauchy equation)

Consider the linear ODE:

$$x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y(x) = 0$$

with $a_0, a_1, a_2, \ldots, a_{n-1} \in \mathbb{R}$ given constants. Let L be the corresponding operator.

Solution method

- 1 We evaluate the characteristic polynomial P from:

$$L(x^b) = P(b) x^b$$

- 2 Let $p_1, p_2, \ldots, p_n \in \mathbb{C}$ be the zeroes of $P(b)$. Then
  (a) If p_k is a single zero, it contributes a solution

$$y_k(x) = x^{p_k}.$$

  (b) If p_k is a zero with multiplicity m, it contributes the following linearly independent solutions.

$$y_k(x) = x^{p_k}$$
$$y_{k+1}(x) = x^{p_k} \ln x$$
$$y_{k+2}(x) = x^{p_k} [\ln x]^2$$
$$\vdots$$
$$y_{k+m-1}(x) = x^{p_k} [\ln x]^{m-1}$$

  (c) Given a complex conjugate pair $p_k = \gamma + i\omega$ and $p_{k+1} = \gamma - i\omega$, from (a) we obtain (see remark below) the following linearly independent solutions:

$$y_k(x) = x^\gamma \cos(\omega \ln x)$$
$$y_{k+1}(x) = x^\gamma \sin(\omega \ln x)$$

(d) Given a complex conjugate pair $\rho_k = \gamma + i\omega$ and $\rho_k = \gamma - i\omega$ of multiplicity m, from (B), we obtain the following linearly independent solutions:

$$y_k(x) = x^\gamma \cos(\omega \ln x)$$
$$y_{k+1}(x) = x^\gamma \sin(\omega \ln x)$$
$$y_{k+2}(x) = x^\gamma \cos(\omega \ln x) \ln x$$
$$y_{k+3}(x) = x^\gamma \cos(\omega \ln x) \ln x$$
$$\vdots$$
$$y_{k+2m-2}(x) = x^\gamma \cos(\omega \ln x)[\ln x]^{m-1}$$
$$y_{k+2m-1}(x) = x^\gamma \sin(\omega \ln x)[\ln x]^{m-1}$$

• 3 We write the general solution and apply the initial conditions, if given.

→ <u>Remark</u>: For the case of a single pair of complex conjugate zeroes $\rho_k = \gamma + i\omega$ and $\rho_{k+1} = \gamma - i\omega$, we have the following contributed solutions:

$$y_k(x) = x^{\rho_k} = x^{\gamma + i\omega} = \exp((\gamma + i\omega)\ln x) = \exp(\gamma \ln x)\exp(i\omega \ln x) =$$
$$= x^\gamma [\cos(\omega \ln x) + i \sin(\omega \ln x)]$$

and similarly:

$$y_{k+1}(x) = x^{\rho_{k+1}} = x^{\gamma - i\omega} = x^\gamma [\cos(\omega \ln x) - i \sin(\omega \ln x)]$$

Via an argument similar to that of case 1, we obtain the following alternate linearly independent solutions:

$$z_k(x) = x^\gamma \cos(\omega \ln x)$$
$$z_{k+1}(x) = x^\gamma \sin(\omega \ln x)$$

EXAMPLES

a) Solve the initial value problem

$$\begin{cases} x^2y''(x)+xy'(x)+4y(x)=0 \\ y(2)=p \land y'(2)=q \end{cases}$$

Solution

Define $Ly(x)=x^2y''(x)+xy'(x)+4y(x)$. It follows that

$$\begin{aligned} L(x^b) &= x^2(x^b)''+x(x^b)'+4x^b = \\ &= x^2 b(b-1)x^{b-2}+x\,bx^{b-1}+4x^b = \\ &= b(b-1)x^b+bx^b+4x^b = [b(b-1)+b+4]x^b = \\ &= (b^2-b+b+4)x^b = (b^2+4)x^b = (b+2i)(b-2i)x^b \end{aligned}$$

which gives the characteristic polynomial

$$P(b)=(b+2i)(b-2i)$$

with zeroes $\rho_1=2i$ and $\rho_2=-2i$. It follows that the general solution reads

$$y(x)=\lambda_1\cos(2\ln x)+\lambda_2\sin(2\ln x)$$

To apply the initial condition we note that

$$y(2)=\lambda_1\cos(2\ln 2)+\lambda_2\sin(2\ln 2)$$

and

$$\begin{aligned} y'(x) &= \lambda_1[\cos(2\ln x)]'+\lambda_2[\sin(2\ln x)]' = \\ &= \lambda_1[-\sin(2\ln x)](2\ln x)'+\lambda_2[\cos(2\ln x)](2\ln x)' = \\ &= (2/x)[-\lambda_1\sin(2\ln x)+\lambda_2\cos(2\ln x)] \Longrightarrow \end{aligned}$$

$$\begin{aligned} \Rightarrow y'(2) &= (2/2)[-\lambda_1\sin(2\ln 2)+\lambda_2\cos(2\ln 2)] = \\ &= -\lambda_1\sin(2\ln 2)+\lambda_2\cos(2\ln 2) \end{aligned}$$

and it follows that:

$$\begin{cases} y(2) = p \\ y'(2) = q \end{cases} \Leftrightarrow \begin{cases} \lambda_1 \cos(2\ln 2) + \lambda_2 \sin(2\ln 2) = p \\ -\lambda_1 \sin(2\ln 2) + \lambda_2 \cos(2\ln 2) = q \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \cos(2\ln 2) & \sin(2\ln 2) \\ -\sin(2\ln 2) & \cos(2\ln 2) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \cos(2\ln 2) & \sin(2\ln 2) \\ -\sin(2\ln 2) & \cos(2\ln 2) \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \end{bmatrix} =$$

$$= \frac{1}{\cos^2(2\ln 2) + \sin^2(2\ln 2)} \begin{bmatrix} \cos(2\ln 2) & -\sin(2\ln 2) \\ \sin(2\ln 2) & \cos(2\ln 2) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

$$= \frac{1}{1} \begin{bmatrix} p\cos(2\ln 2) - q\sin(2\ln 2) \\ p\sin(2\ln 2) + q\cos(2\ln 2) \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda_1 = p\cos(2\ln 2) - q\sin(2\ln 2) \\ \lambda_2 = p\sin(2\ln 2) + q\cos(2\ln 2) \end{cases}$$

Thus, the solution reads

$$y(x) = [p\cos(2\ln 2) - q\sin(2\ln 2)]\cos(2\ln x) + [p\sin(2\ln 2) + q\cos(2\ln 2)]\sin(2\ln x)$$

$$= p\,[\cos(2\ln 2)\cos(2\ln x) + \sin(2\ln 2)\sin(2\ln x)] +$$

$$+ q\,[-\sin(2\ln 2)\cos(2\ln x) + \sin(2\ln x)\cos(2\ln 2)] =$$

$$= p\cos(2\ln x - 2\ln 2) + q\sin(2\ln x - 2\ln 2)$$

b) Solve the initial value problem

$$\begin{cases} 4x^2y''(x)+8xy'(x)+y(x)=0 \\ y(3)=p \wedge y'(3)=q \end{cases}$$

Solution

Define $Ly(x)=4x^2y''(x)+8xy'(x)+y(x)$. It follows that

$$\begin{aligned} L(x^b) &= 4x^2(x^b)''+8x(x^b)'+x^b= \\ &= 4x^2b(b-1)x^{b-2}+8x\,bx^{b-1}+x^b= \\ &= 4b(b-1)x^b+8bx^b+x^b=[4b(b-1)+8b+1]x^b= \\ &= (4b^2-4b+8b+1)x^b=(4b^2+4b+1)x^b=(2b+1)^2x^b \end{aligned}$$

which gives the characteristic polynomial $P(b)=(2b+1)^2$ with a double zero $p=-1/2$. Thus, the general solution reads:

$$y(x)=\lambda_1x^{-1/2}+\lambda_2x^{-1/2}\ln x=\frac{\lambda_1+\lambda_2\ln x}{\sqrt{x}}$$

To apply the initial condition, we note that

$$y(3)=\frac{\lambda_1+\lambda_2\ln 3}{\sqrt{3}}$$

and

$$\begin{aligned} y'(x) &= \frac{(\lambda_1+\lambda_2\ln x)'\sqrt{x}-(\lambda_1+\lambda_2\ln x)(\sqrt{x})'}{(\sqrt{x})^2}= \\ &= \frac{1}{x}\left[\lambda_2\frac{1}{x}\sqrt{x}-\frac{\lambda_1+\lambda_2\ln x}{2\sqrt{x}}\right]= \\ &= \frac{1}{x}\left[\lambda_2\frac{1}{\sqrt{x}}-\frac{\lambda_1+\lambda_2\ln x}{2\sqrt{x}}\right]= \\ &= \frac{1}{2x\sqrt{x}}\left[2\lambda_2-(\lambda_1+\lambda_2\ln x)\right]=\frac{(2\lambda_2-\lambda_1)-\lambda_2\ln x}{2x\sqrt{x}} \end{aligned}$$

$$\Rightarrow y'(3) = \frac{(2\lambda_2 - \lambda_1) - \lambda_2 \ln 3}{2 \cdot 3\sqrt{3}} = \frac{-\lambda_1 + (2 - \ln 3)\lambda_2}{6\sqrt{3}}$$

and therefore

$$\begin{cases} y(3) = p \\ y'(3) = q \end{cases} \Leftrightarrow \begin{cases} \lambda_1 + \lambda_2 \ln 3 = p\sqrt{3} \\ -\lambda_1 + (2 - \ln 3)\lambda_2 = 6q\sqrt{3} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} 1 & \ln 3 \\ -1 & 2 - \ln 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} p\sqrt{3} \\ 6q\sqrt{3} \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & \ln 3 \\ -1 & 2 - \ln 3 \end{bmatrix}^{-1} \begin{bmatrix} p\sqrt{3} \\ 6q\sqrt{3} \end{bmatrix} =$$

$$= \frac{1}{(2 - \ln 3) + \ln 3} \begin{bmatrix} 2 - \ln 3 & -\ln 3 \\ +1 & 1 \end{bmatrix} \begin{bmatrix} p\sqrt{3} \\ 6q\sqrt{3} \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} (2 - \ln 3) p\sqrt{3} - 6q\sqrt{3} \ln 3 \\ p\sqrt{3} + 6q\sqrt{3} \end{bmatrix}$$

It follows that the solution to the initial value problem is:

$$y(x) = \frac{\lambda_1 + \lambda_2 \ln x}{\sqrt{x}} =$$

$$= \frac{1}{2\sqrt{x}} \left[(2 - \ln 3) p\sqrt{3} - 6q\sqrt{3} \ln 3 + (p\sqrt{3} + 6q\sqrt{3}) \ln x \right] =$$

$$= \frac{1}{2\sqrt{x}} \left[p\sqrt{3} (2 - \ln 3 + \ln x) + 6q\sqrt{3} (\ln x - \ln 3) \right]$$

c) Solve the initial value problem

$$\begin{cases} x^3y'''(x) - xy'(x) - 3y(x) = 0 \\ y(1) = 0 \wedge y'(1) = 0 \wedge y''(1) = p \end{cases}$$

<u>Solution</u>

Define $Ly(x) = x^3y'''(x) - xy'(x) - 3y(x)$. It follows that

$$L(x^b) = x^3(x^b)''' - x(x^b)' - 3x^b =$$
$$= x^3 b(b-1)(b-2)x^{b-3} - x(bx^{b-1}) - 3x^b =$$
$$= b(b-1)(b-2)x^b - bx^b - 3x^b = [b(b-1)(b-2) - b - 3]x^b =$$
$$= [b(b^2-3b+2) - b - 3]x^b = (b^3-3b^2+2b-b-3)x^b$$
$$= (b^3-3b^2+b-3)x^b$$

and therefore the characteristic polynomial is:

$$P(b) = b^3 - 3b^2 + b - 3 = b^2(b-3) + (b-3) = (b-3)(b^2+1)$$

with zeroes $\rho_1 = 3$; $\rho_2 = i$, and $\rho_3 = -i$. Thus, the general solution is given by:

$$y(x) = \lambda_1 x^3 + \lambda_2 \cos(\ln x) + \lambda_3 \sin(\ln x).$$

To apply the initial condition, we note that

$$y(1) = \lambda_1 \cdot 1^3 + \lambda_2 \cos(\ln 1) + \lambda_3 \sin(\ln 1) =$$
$$= \lambda_1 + \lambda_2 \cos 0 + \lambda_3 \sin 0 = \lambda_1 + \lambda_2$$

and

$$y'(x) = 3\lambda_1 x^2 + \lambda_2[\cos(\ln x)]' + \lambda_3[\sin(\ln x)]' =$$
$$= 3\lambda_1 x^2 + \lambda_2[-\sin(\ln x)](\ln x)' + \lambda_3[\cos(\ln x)](\ln x)'$$
$$= 3\lambda_1 x^2 + \frac{-\lambda_2 \sin(\ln x) + \lambda_3 \cos(\ln x)}{x} \Rightarrow$$

$$\Rightarrow y'(1) = 3\lambda_1 \cdot 1^2 + \frac{-\lambda_2 \sin(\ln 1) + \lambda_3 \cos(\ln 1)}{1} =$$

$$= 3\lambda_1 - \lambda_2 \sin 0 + \lambda_3 \cos 0 = 3\lambda_1 - 0\lambda_2 + \lambda_3.$$

and

$$y''(x) = 6\lambda_1 x + \frac{d}{dx}\left[\frac{-\sin(\ln x)}{x}\right]\lambda_2 + \frac{d}{dx}\left[\frac{\cos(\ln x)}{x}\right]\lambda_3 =$$

$$= (6x)\lambda_1 + \frac{-[(\sin(\ln x))' x - \sin(\ln x)(x)']}{x^2}\lambda_2$$

$$+ \frac{[\cos(\ln x)]' x - \cos(\ln x)(x)'}{x^2}\lambda_3 =$$

$$= (6x)\lambda_1 + \frac{-[\cos(\ln x)(\ln x)' x - \sin(\ln x)]}{x^2}\lambda_2$$

$$+ \frac{-\sin(\ln x)(\ln x)' x - \cos(\ln x)}{x^2}\lambda_3 =$$

$$= (6x)\lambda_1 + \frac{\sin(\ln x) - \cos(\ln x)}{x^2}\lambda_2 + \frac{-\sin(\ln x) - \cos(\ln x)}{x^2}\lambda_3$$

$$\Rightarrow y''(1) = 6\lambda_1 + [\sin(\ln 1) - \cos(\ln 1)]\lambda_2 - [\sin(\ln 1) + \cos(\ln 1)]\lambda_3 =$$

$$= 6\lambda_1 + (\sin 0 - \cos 0)\lambda_2 - (\sin 0 + \cos 0)\lambda_3 =$$

$$= 6\lambda_1 - \lambda_2 - \lambda_3$$

and therefore:

$$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ 3\lambda_1 + \lambda_3 = 0 \\ 6\lambda_1 - \lambda_2 - \lambda_3 = p \end{cases} \iff \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}$$

Apply Cramer rule:

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 6 & -1 & -1 \end{vmatrix} \begin{matrix} \\ \leftarrow \\ \cdot 1 \end{matrix} = \begin{vmatrix} 1 & 1 & 0 \\ 9 & -1 & 0 \\ 6 & -1 & -1 \end{vmatrix} = +(-1)\begin{vmatrix} 1 & 1 \\ 9 & -1 \end{vmatrix} = -(-1-9) = 10$$

$$D_1 = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & -1 & -1 \end{vmatrix} = p \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = p$$

$$D_2 = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 6 & p & -1 \end{vmatrix} = (-1)p \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = -p(1\cdot 1 - 0\cdot 3) = -p$$

$$D_3 = \begin{vmatrix} 1 & 1 & 0 \\ 3 & 0 & 0 \\ 6 & -1 & p \end{vmatrix} = (+1)p \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} = p(1\cdot 0 - 1\cdot 3) = -3p$$

thus

$$\lambda_1 = \frac{D_1}{D} = \frac{p}{10}$$

$$\lambda_2 = \frac{D_2}{D} = \frac{-p}{10}$$

$$\lambda_3 = \frac{D_3}{D} = \frac{-3p}{10}$$

and the solution reads

$$y(x) = \frac{px^3}{10} - \frac{p\cos(\ln x)}{10} - \frac{3p\sin(\ln x)}{10}$$

$$= (p/10)\left[x^3 - \cos(\ln x) - 3\sin(\ln x)\right]$$

Solving inhomogeneous linear differential equations

We will now consider the general problem of the linear inhomogeneous linear differential equation of the form

$\forall x \in A: y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x) = f(x) \quad (1)$

with $a_0, a_1, a_2, \ldots, a_{n-1}, f \in C^0(A)$. The general method is as follows:

1) Given the solutions $y_1, \ldots, y_n$ of the homogeneous equation and at least one solution y_p of the inhomogeneous equation we show that the general solution of Eq. (1) is:

$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x) + y_p(x)$

2) Given $y_1, y_2, \ldots, y_n$ there is a general result that gives the solution y_p.

Terminology: The terms $\lambda_1 y_1 + \cdots + \lambda_n y_n$ are the homogeneous solution and y_p are the particular solution to the problem.

We now give the details of the theory:

Thm: Consider the linear operator $L: C^n(A) \to C^0(A)$ for some interval $A \subseteq \mathbb{R}$ such that $Ly = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$ with $a_0, a_1, \ldots, a_n \in C^0(A)$. Let $f \in C^0(A)$, and assume that

(a) $\operatorname{null}(L) = \operatorname{span}\{y_1, y_2, \ldots, y_n\}$ with $y_1, y_2, \ldots, y_n \in C^n(A)$.

(b) $L y_p = f$

Then: $Ly = f \Leftrightarrow \exists \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}: y = y_p + \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n$

Proof

($\Rightarrow$): Assume that $Ly=f$. Then it follows that

$L(y-y_p) = Ly - Ly_p = f - f = 0 \Rightarrow (y-y_p) \in null(L) \Rightarrow$

$\Rightarrow y - y_p \in span\{y_1, y_2, \dots, y_n\} \Rightarrow$

$\Rightarrow \exists \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}: y - y_p = \lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_n y_n$

$\Rightarrow \exists \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}: y = y_p + \lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_n y_n$

($\Leftarrow$): Assume that: $\exists \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}: y = y_p + \lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_n y_n$

Then, it follows that:

$$Ly = L\left(y_p + \sum_{k=1}^{n} \lambda_k y_k\right) = Ly_p + L\left(\sum_{k=1}^{n} \lambda_k y_k\right) = f + \sum_{k=1}^{n} L(\lambda_k y_k) =$$

$$= f + \sum_{k=1}^{n} \lambda_k (Ly_k) = f + \sum_{k=1}^{n} \lambda_k 0 = f + 0 = f. \quad \square$$

Thm: Let $L: C^n(A) \to C^0(A)$, with $A \subseteq \mathbb{R}$ an interval, be a linear operator defined as:

$\forall y \in C^n(A): Ly = y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y$

with $a_0, a_1, \dots, a_{n-1} \in C^0(A)$, and let $f \in C^0(A)$. Assume that $null(L) = span\{y_1, y_2, \dots, y_n\}$. Then, the inhomogeneous ODE $Ly = f$ has a particular solution $y_p \in C^n(A)$ such that $Ly_p = f$ given by:

$$\forall x \in A: \quad y_p(x) = \int_A G(x,t) f(t) dt$$

with $\forall x, t \in A:$
$$G(x,t) = \begin{cases} \sum_{k=1}^{n} B_k(t) y_k(x) &, \text{ if } x \geq t \\ 0 &, \text{ if } x < t \end{cases}$$

where $B_1(t), B_2(t), \dots, B_n(t)$ is the unique solution of the system

$$W[y_1, y_2, \dots, y_n](t) (B_1(t), B_2(t), \dots, B_n(t)) = (0, 0, \dots, 0, 1)$$

Remarks :

a) The proof of this theorem is based on generalized functions and will be given later.

b) An alternative proof is to substitute the solution $y_p \in C^n(\mathbb{R})$ to the equation $Ly_p = f$ and confirm that the solution satisfies the equation. This method is known as "variation of parameters".

c) The function $G(x,t)$ is called the Green's function. It captures the effect of the value of the forcing function f at t to the solution y_p at x. The Green's function is not unique, but can be made unique if we introduce the assumption that $G(x,t) = 0$ for $x < t$. This is known as the causality assumption that "the future value $f(t)$ should not have an effect on the past solution $y_p(x)$".

Special case: 2nd-order linear ODE on $A = [c,d]$

Consider the 2nd-order linear ODE of the form

$$y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x). \quad \text{with } a_0, a_1, f \in C^0(A)$$

Given two linearly independent solutions $y_1, y_2 \in C^2(A)$ such that

$$\begin{cases} y_1''(x) + \alpha_1(x)y_1'(x) + a_0(x)y_1(x) = 0 \\ y_2''(x) + \alpha_1(x)y_2'(x) + a_0(x)y_2(x) = 0 \end{cases}$$

a corresponding particular solution $y_p \in C^2(A)$ is given by

$$y_p(x) = -y_1(x)\int_c^x \frac{f(t)y_2(t)}{W(t)}\,dt + y_2(x)\int_c^x \frac{f(t)y_1(t)}{W(t)}\,dt$$

with $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

Proof

The Green's function is given by

$$G(x,t)=\begin{cases} B_1(t)y_1(x)+B_2(t)y_2(x) & , \text{ if } x\geq t \\ 0 & , \text{ if } x<t \end{cases}$$

with $B_1(t), B_2(t)$ given by:

$$W[y_1,y_2](t)\,(B_1(t),B_2(t)) = (0,1) \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}\begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \frac{1}{y_1(t)y_2'(t)-y_1'(t)y_2(t)}\begin{bmatrix} y_2'(t) & -y_2(t) \\ -y_1'(t) & y_1(t) \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \frac{1}{W(t)}\begin{bmatrix} -y_2(t) \\ y_1(t) \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow B_1(t) = \frac{-y_2(t)}{W(t)} \wedge B_2(t) = \frac{y_1(t)}{W(t)}.$$

and therefore, a particular solution is:

$$y_p(x) = \int_c^d G(x,t)f(t)\,dt = \int_c^x [B_1(t)y_1(x)+B_2(t)y_2(x)]\,f(t)\,dt =$$

$$= y_1(x)\int_c^x B_1(t)f(t)\,dt + y_2(x)\int_c^x B_2(t)f(t)\,dt =$$

$$= -y_1(x)\int_c^x \frac{f(t)y_2(t)}{W(t)}\,dt + y_2(x)\int_c^x \frac{f(t)y_1(t)}{W(t)}$$

Note that the lower limit $-\infty$ can be replaced with any constant c. Then the $(-\infty, c)$ integrals gives a contribution that can be moved to the homogeneous solution.

EXAMPLES

a) Solve the initial-value problem:

$$\begin{cases} y''(x) - 2y'(x) + y(x) = (3x+2)e^x \\ y(0) = y_0 \wedge y'(0) = y_1 \end{cases}$$

Solution

Define $\forall y \in C^2(\mathbb{R}) : Ly = y'' - 2y' + y$, and note that

$$Le^{bx} = (e^{bx})'' - 2(e^{bx})' + e^{bx} = b^2 e^{bx} - 2b e^{bx} + e^{bx} =$$
$$= (b^2 - 2b + 1)e^{bx} = (b-1)^2 e^{bx}.$$

The characteristic polynomial $P(b) = (b-1)^2$ has a double zero $b = 1$, therefore $null(L) = span\{y_1, y_2\}$ with $\forall x \in \mathbb{R} : (y_1(x) = e^x \wedge y_2(x) = xe^x)$.

The corresponding Wronskian is:

$$w(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t) =$$
$$= e^x(xe^x)' - (e^x)'(xe^x) = e^x(e^x + xe^x) - e^x xe^x =$$
$$= e^{2x} + xe^{2x} - xe^{2x} = e^{2x}$$

and a particular solution is:

$$y_p(x) = -y_1(x)\int_0^x \frac{f(t)y_2(t)}{w(t)}\,dt + y_2(x)\int_0^x \frac{f(t)y_1(t)}{w(t)}\,dt$$

with $f(t) = (3t+2)e^t$. It follows that

$$y_p(x) = -e^x\int_0^x \frac{(3t+2)e^t \cdot te^t}{e^{2t}}\,dt + xe^x\int_0^x \frac{(3t+2)e^t \cdot e^t}{e^{2t}}\,dt$$
$$= -e^x\int_0^x (3t^2+2t)\,dt + xe^x\int_0^x (3t+2)\,dt =$$

$$= -e^x \left[\frac{3t^3}{3} + \frac{2t^2}{2}\right]_0^x + xe^x \left[\frac{3t^2}{2} + 2t\right]_0^x =$$

$$= -e^x\left(x^3+x^2\right) + xe^x\left(\frac{3x^2}{2} + 2x\right) =$$

$$= -x^2e^x(x+1) + x^2e^x(3x/2+2) =$$

$$= \frac{+x^2e^x}{2}\left(-2(x+1)+3x+4\right) = (1/2)x^2e^x(-2x-2+3x+4)$$

$$= (1/2)x^2e^x(x+2).$$

and therefore the general solution is

$$y(x) = \lambda_1 e^x + \lambda_2 xe^x + (1/2)x^2e^x(x+2)$$

To apply the initial condition, we note that

$$y(0) = \lambda_1 e^0 + \lambda_2 \cdot 0e^0 + (1/2)0^2e^0(0+2) = \lambda_1$$

and

$$y'(x) = \lambda_1(e^x)' + \lambda_2(xe^x)' + (1/2)\left[x^2e^x(x+2)\right]' =$$

$$= \lambda_1 e^x + \lambda_2(e^x + xe^x) + (1/2)\left[e^x(x^3+2x^2)\right]' =$$

$$= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)\left[(e^x)'(x^3+2x^2) + e^x(x^3+2x^2)'\right] =$$

$$= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)\left[e^x(x^3+2x^2+3x^2+4x)\right]$$

$$= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)e^x(x^3+5x^2+4x)$$

$$= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)e^x x(x^2+5x+4)$$

$$= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)xe^x(x+4)(x+1) \Rightarrow$$

$$\Rightarrow y'(0) = (\lambda_1+\lambda_2)e^0 + \lambda_2 \cdot 0 \cdot e^0 + (1/2)\cdot 0e^0(0+4)(0+1) =$$

$$= \lambda_1 + \lambda_2$$

and therefore:

$$\begin{cases} y(0) = y_0 \\ y'(0) = y_1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = y_0 \\ \lambda_1+\lambda_2 = y_1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = y_0 \\ y_0+\lambda_2 = y_1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = y_0 \\ \lambda_2 = y_1 - y_0 \end{cases}$$

thus $y(x) = y_0 e^x + (y_1-y_0)xe^x + (1/2)x^2e^x(x+2).$

b) Solve the ODE value problem

$$x^3y'''(x)+x^2y''(x)-2xy'(x)+2y(x)=f(x)\ ,\ \forall x\in[1,+\infty)$$

Solution

Define $Ly(x)=x^3y'''(x)+x^2y''(x)-2xy'(x)+2y(x)$. Then, since

$$Lx^b = x^3(x^b)'''+x^2(x^b)''-2x(x^b)'+2x^b =$$
$$= x^3b(b-1)(b-2)x^{b-3}+x^2b(b-1)x^{b-2}-2xbx^{b-1}+2x^b=$$
$$=[b(b-1)(b-2)+b(b-1)-2b+2]x^b$$

the characteristic polynomial is given by

$$P(b)=b(b-1)(b-2)+b(b-1)-2b+2=b(b^2-3b+2)+b^2-b-2b+2$$
$$=b^3-3b^2+2b+b^2-b-2b+2=$$
$$=b^3+(-3+1)b^2+(2-1-2)b+2$$
$$=b^3-2b^2-b+2=b^2(b-2)-(b-2)=(b^2-1)(b-2)$$
$$=(b-1)(b+1)(b-2)$$

and has single zeroes $b_1=-1\wedge b_2=1\wedge b_3=2$.

Thus the general solution is:

$$y(x)=\lambda_1x^{-1}+\lambda_2x+\lambda_3x^2+y_p(x)$$

Define: $y_1(x)=x^{-1}\wedge y_2(x)=x\wedge y_3(x)=x^2\ ,\ \forall x\in[1,+\infty)$

The particular solution is given by

$$y_p(x)=\int_1^{+\infty}G(x,t)f(t)\,dt\ ,\ \forall x\in[1,+\infty)$$

with $$G(x,t)=\begin{cases}B_1(t)x^{-1}+B_2(t)x+B_3(t)x^2, & \text{if } x\geq t\\ 0 & \text{, if } x<t\end{cases}$$

with $B_1(t), B_2(t), B_3(t)$ the solution of

$W[y_1,y_2,y_3](B_1(t),B_2(t),B_3(t))=(0,0,1)$.

and therefore

$$y_p(x) = \int_1^{+\infty} G(x,t) f(t)\,dt = \int_1^x [B_1(t)x^{-1} + B_2(t)x + B_3(t)x^2] f(t)\,dt$$

$$= x^{-1}\int_1^x B_1(t) f(t)\,dt + x\int_1^x B_2(t) f(t)\,dt + x^2\int_1^x B_3(t) f(t)\,dt$$

Since

$$W[y_1, y_2, y_3](t) = \begin{bmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \\ y_1''(t) & y_2''(t) & y_3''(t) \end{bmatrix} = \begin{bmatrix} t^{-1} & t & t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{bmatrix}$$

it follows that

$$\begin{bmatrix} t^{-1} & t & t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{bmatrix} \begin{bmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We apply Cramer's rule:

$$D = \begin{vmatrix} t^{-1} & t & t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{vmatrix} (-t) = \begin{vmatrix} t^{-1}+t^{-1} & t-t & t^2-2t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{vmatrix} =$$

$$= \begin{vmatrix} 2t^{-1} & 0 & -t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2t^{-1} & -t^2 \\ 2t^{-3} & 2 \end{vmatrix} = (2t^{-1})2 - (-t^2)(2t^{-3})$$

$$= 4t^{-1} + 2t^{-1} = 6t^{-1}$$

and

$$D_1 = \begin{vmatrix} 0 & t & t^2 \\ 0 & 1 & 2t \\ 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t(2t) - t^2 = 2t^2 - t^2 = t^2.$$

$$D_2 = \begin{vmatrix} t^{-1} & 0 & t^2 \\ -t^{-2} & 0 & 2t \\ 2t^{-3} & 1 & 2 \end{vmatrix} = - \begin{vmatrix} t^{-1} & t^2 \\ -t^{-2} & 2t \end{vmatrix} = -\left[t^{-1}(2t) - t^2(-t^{-2})\right] =$$

$$= -(2+1) = -3$$

and

$$D_3 = \begin{vmatrix} t^{-1} & t & 0 \\ -t^{-2} & 1 & 0 \\ 2t^{-3} & 0 & 1 \end{vmatrix} = \begin{vmatrix} t^{-1} & t \\ -t^{-2} & 1 \end{vmatrix} = t^{-1} - t(-t^{-2}) = t^{-1} + t^{-1} = 2t^{-1}$$

and therefore:

$$B_1(t) = \frac{D_1(t)}{D(t)} = \frac{t^2}{6t^{-1}} = \frac{t^3}{6}$$

$$B_2(t) = \frac{D_2(t)}{D(t)} = \frac{-3}{6t^{-1}} = \frac{-t}{2}$$

$$B_3(t) = \frac{D_3(t)}{D(t)} = \frac{2t^{-1}}{6t^{-1}} = \frac{1}{3}$$

The particular solution is:

$$y_p(x) = x^{-1}\int_1^x \frac{t^3}{6} f(t)\,dt + x\int_1^x \frac{-t}{2} f(t)\,dt + x^2\int_1^x \frac{1}{3} f(t)\,dt =$$

$$= \frac{1}{6x}\int_1^x t^3 f(t)\,dt - \frac{x}{2}\int_1^x t f(t)\,dt + \frac{x^2}{3}\int_1^x f(t)\,dt.$$

It follows that the general solution is given by

$$y(x) = \left[\lambda_1 + \int_1^x \frac{t^3 f(t)}{6}\,dt\right] x^{-1} + \left[\lambda_2 - \int_1^x \frac{t f(t)}{2}\,dt\right] x + \left[\lambda_3 + \int_1^x \frac{f(t)}{3}\,dt\right] x^2$$

$\hookrightarrow$ Note that the integrals can start from numbers other than 1. This will result in a constant shift (i.e. independent of x) in the value of the integrals that can be absorbed by $\lambda_1, \lambda_2, \lambda_3$. In general, it is convenient for the integrals to begin at the location where the initial condition is given.

Homework: Linear Differential Equations

Homework 02: Linear Differential Equations

1. Consider a general linear differential equation of the form

$$\forall x \in A : y''(x) + a(x)y'(x) + b(x)y(x) = 0$$

for some interval $A \subseteq \mathbb{R}$ with $a, b \in \mathcal{C}^0(A)$. Assume that $y_1 \in \mathcal{C}^2(A)$ is a solution, and define $y_2 \in \mathcal{C}^2(A)$ as:

$$\forall x \in A : y_2(x) = y_1(x) \int_c^x \frac{Q(t)}{[y_1(t)]^2} \, \mathrm{d}t$$

with $c \in A$ and with $Q(t)$ given by

$$\forall t \in A : Q(t) = \exp\left(- \int a(t) \, \mathrm{d}t \right)$$

(a) Show that $y_2(x)$ is also a solution.

(b) Show that y_1, y_2 are linearly independent.

Remark: An immediate consequence of (a) and (b) this is that if we define an operator $L : \mathcal{C}^2(A) \to \mathcal{C}^0(A)$ with $Ly = \mathbf{0}$, then it follows that its null space is given by

$$\text{null}(A) = \text{span}\{y_1, y_2\}$$

The corresponding general solution of the equation $Ly = \mathbf{0}$ is given by

$$\forall x \in A : y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$$

Remark: This exercise shows that if we can guess one solution of the second order linear ODE $Ly = \mathbf{0}$, we have an equation that can be used to find a second linearly independent solution. Then, given the aforementioned theorems, we have the null space and the general solution.

2. Find all solutions of the form $\forall x \in \mathbb{R} : y_1(x) = e^{bx}$ for the linear ODE

$$\forall x \in \mathbb{R} : y''(x) + 2ay'(x) + a^2 y(x) = 0$$

with $a \in \mathbb{R}$. Use the previous exercise to find the second linearly independent solution and write the corresponding general solution.

3. Show that the initial value problem

$$\begin{cases} y'(x) - 2(p+a)y'(x) + p^2 y(x) = 0 \\ y(0) = 0 \ \wedge \ y'(0) = 1 \end{cases}$$

with $a, p \in (0, +\infty)$ has solution

$$y(x|a,p) = \frac{\exp(A(p,a)x) - \exp(B(p,a)x)}{2\sqrt{a(2p+a)}}$$

with

$$A(p,a) = p + a + \sqrt{a(2p+a)}$$
$$B(p,a) = p + a - \sqrt{a(2p+a)}$$

without substituting the solution to the ODE. Then, show that:

$$\lim_{a \to 0^+} y(x|a,p) = xe^{px}$$

Remark: This result shows that when considering a second order linear differential equation, in which the two distinct zeroes of the corresponding characteristic polynomial approach each other, the solution obtained using the initial condition $y(0) = 0 \ \wedge \ y'(0) = 1$ converges continuously to the "screwball" $y(x) = xe^{pt}$ solution that we find when the two zeros of the characteristic polynomial are exactly equal to each other. Note that this argument does not establish a solution for the case where the zeros coincide; it only shows that the transition into that case does not exhibit any discontinuities.

4. Show that the linear differential equation

$$ax^3y'''(x) + (b + 3a)x^2y''(x) + (a + b + c)xy'(x) + dy(x) = 0$$

with $a, b, c, d \in \mathbb{R}$ has characteristic polynomial

$$p(x) = ax^3 + bx^2 + cx + d.$$

Remark: This solves the inverse problem of constructing an equidimensional linear differential equation that has a desired characteristic polynomial.

5. Solve the general damped oscillator problem, which is defined as the following initial value problem:

$$\begin{cases} y''(x) + \beta y'(x) + \omega^2 y(x) = f(x) \\ y(0) = y_0 \ \wedge \ y'(x)(0) = y_1 \end{cases}$$

with $\beta, \omega \in (0, +\infty)$ and $y_0, y_1 \in \mathbb{R}$. Distinguish between the following cases:

(a) *Case 1:* $\beta < 2\omega$ (underdamped oscillator)

(b) *Case 2:* $\beta = 2\omega$ (critically damped oscillator)

(c) *Case 3:* $\beta > 2\omega$ (overdamped oscillator)

Remark: It is easier to solve the combined case $\beta \neq 2\omega$, allowing the use of exponentials of complex numbers for the underdamped subcase. This gives a common solution form for both cases $\beta < 2\omega$ and $\beta > 2\omega$, but for the underdamped case, additional work is then needed to convert the exponentials involving complex numbers into trigonometric functions. This approach will be more economical than attempting to handle the underdamped case from scratch.

GODE 04: Series Solution of Linear Differential Equations

SERIES SOLUTION OF ODES

We begin by reviewing, and in some cases, extending, results from Calculus II needed for solving linear ODEs via convergent series methods.

▼ The Gamma function

We recall from my Calculus 2 lecture notes the definition of the factorial and the double factorial:

▸ Factorial:

$$\boxed{\begin{array}{l} 0! = 1 \\ \forall n \in \mathbb{N}^*: \; n! = \prod_{k=1}^{n} k = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \end{array}}$$

▸ Double Factorial:

$$\boxed{\begin{array}{l} 0!! = 1 \;\text{ and }\; 1!! = 1 \\ \forall n \in \mathbb{N}^*: \; (2n)!! = \prod_{k=1}^{n} (2k) = 2^n n! \\ \forall n \in \mathbb{N}^*: \; (2n+1)!! = \prod_{k=1}^{n} (2k+1) = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1) \end{array}}$$

The Gamma function $\Gamma(n)$ generalizes the factorial and is defined, first on $(0, +\infty)$ and then on a wider set as follows.

Def: (Gamma function on $(0, +\infty)$)

$$\boxed{\forall n \in (0, +\infty): \; \Gamma(n) = \int_{0^+}^{+\infty} x^{n-1} e^{-x} \, dx}$$

Then, we show that:

Prop:

a) $\forall n \in (0,+\infty)$: The $\Gamma(n)$ integral converges

b) $\Gamma(1) = 1$

c) $\forall n \in (0,+\infty): \Gamma(n+1) = n\Gamma(n)$

It immediately follows that

$$\boxed{\forall n \in \mathbb{N}^* : \Gamma(n) = (n-1)!}$$

But n is a continuous variable and $\Gamma(n)$ has been defined on $n \in (0,+\infty)$. So, $\Gamma(n)$ generalizes the factorial on a continuous set. We can now use the equation $\Gamma(n) = \Gamma(n+1)/(n+1)$ to extend the definition of the Gamma function for negative n as follows:

$$\forall n \in (-1,0): \Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\forall n \in (-2,-1): \Gamma(n) = \frac{\Gamma(n+1)}{n} = \frac{\Gamma(n+2)}{n(n+1)}$$

$$\forall n \in (-3,-2): \Gamma(n) = \frac{\Gamma(n+2)}{n(n+1)} = \frac{\Gamma(n+3)}{n(n+1)(n+2)}$$

and so on. The general definition of the Gamma function for negative numbers is:

Def : (Gamma function for negative numbers)

$$\forall k \in \mathbb{N}^* : \forall n \in (-k, -k+1) : \Gamma(n) = \frac{\Gamma(n+k)}{\prod_{a=0}^{k-1} (n-a)} = \frac{\Gamma(n+k)}{n(n+1)\cdots(n+k-1)}$$

● **Proof of proposition**

The proof requires the following lemma.

Lemma: $\boxed{\forall a \in \mathbb{R} : \lim_{x \to +\infty} x^a e^{-x} = 0}$

Proof

Let $a \in \mathbb{R}$ be given. We distinguish between the following cases.

Case 1 : For $a \in (-\infty, 0)$, we have

$$\left(\lim_{x \to +\infty} x^a = 0 \wedge \lim_{x \to +\infty} e^{-x} = 0\right) \Rightarrow \lim_{x \to +\infty} x^a e^{-x} = 0.$$

Case 2 : For $a = 0$, we have

$$\lim_{x \to +\infty} x^a e^{-x} = \lim_{x \to +\infty} x^0 e^{-x} = \lim_{x \to +\infty} e^{-x} = 0$$

Case 3 : For $a \in (0, +\infty)$, we define $n = \max\{k \in \mathbb{N} \mid a - k > 0\}$.

We evaluate the limit by applying De L'Hospital $n+1$ times:

$$\lim_{x \to +\infty} x^a e^{-x} = \lim_{x \to +\infty} \frac{x^a}{e^x} = \lim_{x \to +\infty} \frac{a(a-1)\cdots(a-n)\, x^{a-(n+1)}}{e^x} =$$

$$= a(a-1)\cdots(a-n) \lim_{x \to +\infty} x^{a-(n+1)} e^{-x} = 0$$

because, by definition of n, $a-(n+1) < 0$. □

For the convergence proof we use the following theorems from Calculus II:

1) Comparison test

$$\left.\begin{array}{l}\forall x \in S: 0 \leq f(x) \leq g(x) \\ \int_S g(x)\,dx \text{ converges}\end{array}\right\} \Rightarrow \int_S f(x)\,dx \text{ converges.}$$

2) Ratio test

$$\left.\begin{array}{l}\forall x \in S: (f(x) \geq 0 \wedge g(x) \geq 0) \\ \lim_{x \to \sigma} \dfrac{f(x)}{g(x)} = 0\end{array}\right\} \Rightarrow \left(\int_S g(x)\,dx \text{ converges} \Rightarrow \int_S f(x)\,dx \text{ converges}\right)$$

The proofs are as follows:

Proof of (a): Let $n \in (0, +\infty)$ be given.

We write

$$\Gamma(n) = \int_{0^+}^{+\infty} x^{n-1} e^{-x}\,dx = \int_{0^+}^{1} x^{n-1} e^{-x}\,dx + \int_{1}^{+\infty} x^{n-1} e^{-x}\,dx$$

For the $(1, +\infty)$ integral, we define

$$\begin{cases}\forall x \in (1, +\infty): f(x) = x^{n-1} e^{-x} > 0 \\ \forall x \in (1, +\infty): g(x) = 1/x^2 > 0\end{cases}$$

and therefore:

$$\forall x \in (1, +\infty): \frac{f(x)}{g(x)} = \frac{x^{n-1} e^{-x}}{1/x^2} = x^2 x^{n-1} e^{-x} = x^{n+1} e^{-x}$$

$$\Rightarrow \lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} (x^{n+1} e^{-x}) = 0 \qquad (1)$$

From Eq.(1) and the ratio test it follows that:

$$\int_1^{+\infty} \frac{dx}{x^2} \text{ converges} \Rightarrow \int_1^{+\infty} x^{n-1} e^{-x}\, dx \text{ converges.} \quad (2)$$

For the $(0,1)$ integral, let $x \in (0,1)$ be given. Then:
$x \in (0,1) \Rightarrow 0 < x < 1 \Rightarrow -1 < -x < 0 \Rightarrow 0 < e^{-x} < e^0 \Rightarrow$
$\Rightarrow 0 < e^{-x} < 1 \Rightarrow 0 < x^{n-1} e^{-x} < x^{n-1}$ (since $x^{n-1} > 0$).
It follows that
$\forall x \in (0,1) : 0 < x^{n-1} e^{-x} < x^{n-1}$ (3)
From Eq. (3) and via the comparison test, we argue that

$$n > 0 \Rightarrow n-1 > -1 \Rightarrow \int_{0^+}^1 x^{n-1}\, dx \text{ converges} \Rightarrow \int_{0^+}^1 x^{n-1} e^{-x}\, dx \text{ converges.} \quad (4)$$

From Eq. (2) and Eq. (4): $\Gamma(n) = \int_{0^+}^{+\infty} x^{n-1} e^{-x}\, dx$ converges. $\square$

<u>Proof of (b)</u> : Claim $\Gamma(1) = 1$

$$\Gamma(1) = \int_{0^+}^{+\infty} x^{1-1} e^{-x}\, dx = \int_{0^+}^{+\infty} x^0 e^{-x}\, dx = \int_{0^+}^{+\infty} e^{-x}\, dx = \Big[-e^{-x} \Big]_{0^+}^{+\infty} =$$

$$= \lim_{x \to +\infty} (-e^{-x}) - \lim_{x \to 0^+} (-e^{-x}) = (-0) - (-e^0) = 1$$

<u>Proof of (c)</u> : Claim $\forall n \in (0, +\infty) : \Gamma(n+1) = n\Gamma(n)$
Let $n \in (0, +\infty)$ be given. Then:

$$\Gamma(n+1) = \int_{0^+}^{+\infty} x^{(n+1)-1} e^{-x}\, dx = \int_{0^+}^{+\infty} x^n e^{-x}\, dx = \int_{0^+}^{+\infty} x^n (-e^{-x})'\, dx$$

$$= \Big[-x^n e^{-x} \Big]_{0^+}^{+\infty} - \int_{0^+}^{+\infty} (x^n)' (-e^{-x})\, dx =$$

$$= \lim_{x \to +\infty} (-x^n e^{-x}) - (-0^n e^0) - \int_{0^+}^{+\infty} n x^{n-1} (-e^{-x})\, dx =$$

$$= 0 - 0 + n\int_{0^+}^{+\infty} x^{n-1} e^{-x}\,dx = n\Gamma(n)$$

and therefore $\forall n \in (0, +\infty): \Gamma(n+1) = n\Gamma(n)$. □

● Value of $\Gamma(1/2)$: $\boxed{\Gamma(1/2) = \sqrt{\pi}}$

To show that $\Gamma(1/2) = \sqrt{\pi}$ we use the following result from Calculus 3:

$$\int_0^{+\infty} dx \int_0^{+\infty} dy\, f(x,y) = \int_0^{+\infty} r\,dr \int_0^{\pi/2} d\theta\, f(r\cos\theta, r\sin\theta)$$

Proof

We define $u = \sqrt{x} \Rightarrow du = \dfrac{dx}{2\sqrt{x}} \Rightarrow x^{-1/2}dx = 2du$

and note that $x = 0 \Leftrightarrow u = 0$ and $x \to +\infty \Leftrightarrow u \to +\infty$.

It follows that

$$\Gamma(1/2) = \int_{0^+}^{+\infty} x^{1/2-1} e^{-x}\,dx = \int_{0^+}^{+\infty} x^{-1/2} e^{-x}\,dx = \int_{0^+}^{+\infty} e^{-u^2}\,2du$$

$$= 2\int_0^{+\infty} \exp(-u^2)\,du \Rightarrow$$

$$\Rightarrow [\Gamma(1/2)]^2 = \left[2\int_0^{+\infty}\exp(-u^2)\,du\right]\left[2\int_0^{+\infty}\exp(-v^2)\,dv\right] =$$

$$= 4\int_0^{+\infty} du \int_0^{+\infty} dv\, \exp(-u^2 - v^2)$$

$$= 4\int_0^{+\infty} r\,dr \int_0^{\pi/2} d\theta\, \exp(-r^2\cos^2\theta - r^2\sin^2\theta)$$

$$= 4\int_0^{+\infty} r\,dr \int_0^{\pi/2} d\vartheta\ \exp\left(-r^2(\sin^2\vartheta + \cos^2\vartheta)\right)$$

$$= 4\int_0^{+\infty} r\,dr \int_0^{\pi/2} d\vartheta\ \exp(-r^2) = 4\int_0^{+\infty} r\exp(-r^2)\left[\int_0^{\pi/2} d\vartheta\right] dr$$

$$= 4\int_0^{+\infty} r\exp(-r^2)\,(\pi/2)\,dr = \pi\int_0^{+\infty} 2r\exp(-r^2)\,dr =$$

$$= \pi\int_0^{+\infty} \left[-\exp(-r^2)\right]' dr = \pi\left[-\exp(-r^2)\right]_0^{+\infty} =$$

$$= \pi\left[\lim_{x\to+\infty}\left(-\exp(-r^2)\right) - \left(-\exp(-0)\right)\right] = \pi\left[0-(-1)\right] = \pi$$

$$\Rightarrow \Gamma(1/2) = \sqrt{\pi}\ \vee\ \Gamma(1/2) = -\sqrt{\pi}. \qquad (1)$$

Since $(\forall u \in (0, +\infty) : \exp(-u^2) \geq 0) \Rightarrow$

$$\Rightarrow \Gamma(1/2) = 2\int_0^{+\infty} \exp(-u^2)\,du \geq 0 \qquad (2)$$

From Eq. (1) and Eq. (2) it follows that $\Gamma(1/2) = \sqrt{\pi}$.

EXAMPLE

Use proof by induction to show that given an $a \in \mathbb{R} - (-1)\mathbb{N}^*$ with $(-1)\mathbb{N}^* = \{-x \mid x \in \mathbb{N}^*\} = \{-1, -2, -3, \ldots\}$, we have:

$$\boxed{\forall n \in \mathbb{N}^*: \prod_{k=1}^{n} (k+a) = \frac{\Gamma(n+1+a)}{\Gamma(a+1)}}$$

↳ This result is VERY useful for rewritting products in terms of Gamma functions.

Solution

For $n = 1$, we have:

$$\prod_{k=1}^{n} (k+a) = (1+a) = \frac{(a+1)\Gamma(a+1)}{\Gamma(a+1)} = \frac{\Gamma(a+2)}{\Gamma(a+1)} = \frac{\Gamma(1+1+a)}{\Gamma(a+1)} =$$

$$= \frac{\Gamma(n+1+a)}{\Gamma(a+1)}$$

For $n = m$, we assume that $\prod_{k=1}^{m} (k+a) = \dfrac{\Gamma(m+1+a)}{\Gamma(a+1)}$

For $n = m+1$, we will show that $\prod_{k=1}^{m+1} (k+a) = \dfrac{\Gamma((m+1)+1+a)}{\Gamma(a+1)}$ as follows:

$$\prod_{k=1}^{m+1} (k+a) = (m+1+a)\prod_{k=1}^{m} (k+a) = (m+1+a) \cdot \frac{\Gamma(m+1+a)}{\Gamma(a+1)} =$$

$$= \frac{(m+1+a)\Gamma(m+1+a)}{\Gamma(a+1)} = \frac{\Gamma(m+1+a+1)}{\Gamma(a+1)} = \frac{\Gamma((m+1)+1+a)}{\Gamma(a+1)}$$

▼ Review of power series

We review basic results from Calculus II concerning power series expansion of functions.

● Definitions

- A power series is a series of the form

$$\forall x \in A : f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n$$

with $a \in \mathrm{Seq}(\mathbb{R})$ and $x_0 \in \mathbb{R}$.

- The domain A is chosen to be the widest possible subset of $\mathbb{R}$ for which the series converges. If $A = (x_0 - \mu, x_0 + \mu)$ then we say that $\mu > 0$ is the radius of convergence.

Def: Let $f : A \to \mathbb{R}$ be a function with $x_0 \in A$. We say that f analytic at $x = x_0 \iff$

$$\iff \exists a \in \mathrm{Seq}(\mathbb{R}) : \exists \mu \in (0, +\infty) : \forall x \in (x_0 - \mu, x_0 + \mu) : f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n$$

f analytic on $S \subseteq A \iff \forall x_0 \in S : f$ analytic on $x = x_0$

- The space of all functions analytic on S is denoted as $C^\omega(S)$. Note that $C^\omega(S) \subseteq C^\infty(S)$ which means

that in general
$$f \in C^{\omega}(S) \Rightarrow f \in C^{\infty}(S).$$
However, the converse statement is not always true.

General properties of power series

Let f, g be two functions that are analytic at $x = x_0$ such that
$$\forall x \in (x_0 - \mu, x_0 + \mu) : \left(f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n \wedge g(x) = \sum_{n=0}^{+\infty} b_n (x - x_0)^n \right)$$

Then, we can show that:

a) $(\forall x \in (x_0 - \mu, x_0 + \mu) : f(x) = g(x)) \Leftrightarrow (\forall n \in \mathbb{N} : a_n = b_n)$

b) $\forall x \in (x_0 - \mu, x_0 + \mu) : \quad f(x) + g(x) = \sum_{n=0}^{+\infty} (a_n + b_n)(x - x_0)^n$

c) $\forall x \in (x_0 - \mu, x_0 + \mu) : \quad f(x) g(x) = \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} a_k b_{n-k} \right] (x - x_0)^n$

d) $\forall x \in (x_0 - \mu, x_0 + \mu) : \quad f'(x) = \sum_{n=1}^{+\infty} n a_n (x - x_0)^{n-1}$

e) $\forall k \in \mathbb{N}^* : \forall x \in (x_0 - \mu, x_0 + \mu) : f^{(k)}(x) = \sum_{n=k}^{+\infty} \left[\prod_{l=0}^{k-1} (n - l) \right] a_n (x - x_0)^{n-k}$
$$= \sum_{n=k}^{+\infty} \frac{n!}{(n-k)!} a_n (x - x_0)^{n-k}$$

e) $\forall x_1, x_2 \in (x_0 - \mu, x_0 + \mu) : \quad \int_{x_1}^{x_2} f(t)\, dt = \sum_{n=0}^{+\infty} \left[a_n \int_{x_1}^{x_2} (t - x_0)^n \, dt \right]$
$$= \sum_{n=0}^{+\infty} \left[\frac{a_n \left[(x_2 - x_0)^{n+1} - (x_1 - x_0)^{n+1} \right]}{n+1} \right]$$

Some important power series

$$\forall x \in (-1,1): \quad \frac{1}{1-x} = \sum_{k=0}^{+\infty} x^k = 1+x+x^2+\cdots$$

$$\forall x \in (-1,1): \quad (1+x)^p = \sum_{n=0}^{+\infty} \binom{p}{n} x^n = 1+px+\frac{p(p-1)}{2\cdot 1}x^2+\cdots$$

$$\forall x \in \mathbb{R}: \quad e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$\forall x \in \mathbb{R}: \quad \sin x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$$

$$\forall x \in \mathbb{R}: \quad \cos x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$$

$$\forall x \in (-1,1]: \quad \ln(1+x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1} = x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$$

$$\forall x \in [-1,1]: \quad \operatorname{Arctan} x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots$$

The detailed theory on the above series is given in my Calculus 2 notes.

Convergence tests

The proofs of the relevant theorems for series solution of linear ODEs depend on the comparison test and the absolute ratio test. Applied on power series these tests reduce to the following statements:

(1) → Comparison test

Given $a, b \in Seq(\mathbb{R})$ and $x_0 \in \mathbb{R}$, then

$$\forall x \in \mathbb{R}: \left(\begin{cases} \forall n \in \mathbb{N}: |a_n| \leq b_n \\ \sum_{n=0}^{+\infty} b_n (x-x_0)^n \text{ converges} \end{cases} \Rightarrow \sum_{n=0}^{+\infty} a_n (x-x_0)^n \text{ converges} \right)$$

(2) → Absolute Ratio test

Given $a \in Seq(\mathbb{R})$ and $x_0 \in \mathbb{R}$, then:

$$\left(\lim_{n \in \mathbb{N}} \left| \frac{a_{n+1}(x-x_0)}{a_n} \right| < 1 \Rightarrow \sum_{n=0}^{+\infty} a_n (x-x_0)^n \text{ converges} \right), \forall x \in \mathbb{R}$$

$$\left(\lim_{n \in \mathbb{N}} \left| \frac{a_{n+1}(x-x_0)}{a_n} \right| > 1 \Rightarrow \sum_{n=0}^{+\infty} a_n (x-x_0)^n \text{ diverges} \right), \forall x \in \mathbb{R}$$

In practice we get convergence for free via the relevant theorems as we solve the linear ODE. Therefore the above convergence tests are only required in the proofs of the necessary theorems.

• Merten' theorem

Thm: Let (a_n) and (b_n) be two sequences with $n \in \mathbb{N}$. Then, we have:

$$\begin{cases} \sum_{n=0}^{+\infty} |a_n| \text{ converges} \\ \sum_{n=0}^{+\infty} b_n \text{ converges} \end{cases} \Rightarrow \left[\sum_{n=0}^{+\infty} a_n\right]\left[\sum_{n=0}^{+\infty} b_n\right] = \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n} a_k b_{n-k}\right]$$

Merten's theorem can be used safely to multiply power series because when they converge, they converge absolutely. A useful shortcut is to note that if

$$\boxed{\forall x \in A : \left(f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n \wedge g(x) = \sum_{n=0}^{+\infty} b_n (x-x_0)^n \right)}$$

then, it follows that

$$\boxed{\forall x \in A : f(x) g(x) = \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n} a_k b_{n-k}\right](x-x_0)^n}$$

For more details, see my Calculus 2 lecture notes.

EXAMPLES

a) Write the series expansion around $x_0 = 0$ of the function

$$f(x) = \frac{e^x}{2x+1}$$

and find the radius of convergence.

Solution

We have:

$$f(x) = \frac{e^x}{2x+1} = e^x \cdot \frac{1}{1-(-2x)} = \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-2x)^n\right]$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n 2^n x^n\right] =$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^k 2^k}{(n-k)!}\right] x^n$$

The series expansion of e^x converges on $\mathbb{R}$. The series expansion of $1/(1-(-2x))$ requires $|-2x| < 1$. Since:

$$|-2x| < 1 \Leftrightarrow |2x| < 1 \Leftrightarrow 2|x| < 1 \Leftrightarrow |x| < 1/2 \Leftrightarrow$$

$$\Leftrightarrow -1/2 < x < 1/2 \Leftrightarrow x \in (-1/2, 1/2).$$

b) Write the series expansion of the function $f(x) = e^x \cos x$ and find the radius of convergence

Solution

Since:

$$f(x) = e^x \cos x = \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right] =$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right]$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right] + \left[\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right]$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{2k}}{(2k)!} (-1)^{n-k} \frac{x^{2n-2k}}{(2n-2k)!}\right] +$$

$$+ \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{2k+1}}{(2k+1)!} (-1)^{n-k} \frac{x^{2n-2k}}{(2n-2k)!}\right] =$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(2k)!\,(2n-2k)!}\right] x^{2n} +$$

$$+ \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(2k+1)!\,(2n-2k)!}\right] x^{2n+1}$$

c) Write a series expansion of $f(x) = \sin(2x)$ around $x = \pi/8$ and find the radius of convergence.

<u>Solution</u>

$$f(x) = \sin(2x) = \sin(2x - \pi/4 + \pi/4) = \sin(2(x-\pi/8) + \pi/4) =$$

$$= \sin(2(x-\pi/8))\cos(\pi/4) + \cos(2(x-\pi/8))\sin(\pi/4) =$$

$$= (\sqrt{2}/2)\left[\cos(2(x-\pi/8)) + \sin(2(x-\pi/8))\right]$$

$$= \frac{\sqrt{2}}{2}\left[\sum_{n=0}^{+\infty} (-1)^n \frac{[2(x-\pi/8)]^{2n}}{(2n)!} + \sum_{n=0}^{+\infty} (-1)^n \frac{[2(x-\pi/8)]^{2n+1}}{(2n+1)!}\right]$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{\sqrt{2}\, 2^{2n}}{2(2n)!} (x-\pi/8)^{2n} + \sum_{n=0}^{+\infty} (-1)^n \frac{\sqrt{2}\, 2^{2n+1}}{2(2n+1)!} (x-\pi/8)^{2n+1}$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{2^{2n-1}\sqrt{2}}{(2n)!} (x-\pi/8)^{2n} + \sum_{n=0}^{+\infty} (-1)^n \frac{2^{2n}\sqrt{2}}{(2n+1)!} (x-\pi/8)^{2n+1}$$

The convergence set for all series expansions here is $\mathbb{R}$

Series solution of 2nd-order linear ODES

We consider a 2nd-order linear ordinary differential equation of the form

$$y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0$$

and we seek the general solution approximated as a power series around the point $x = x_0$.

We distinguish between the following 3 cases:

1) $\left.\begin{array}{l} x = x_0 \text{ is a} \\ \underline{\text{regular point}} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} p(x) \text{ analytic at } x = x_0 \\ q(x) \text{ analytic at } x = x_0 \end{array}\right.$

2) $\left.\begin{array}{l} x = x_0 \text{ is a} \\ \underline{\text{regular singular}} \\ \underline{\text{point}} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} x = x_0 \text{ is NOT a regular point} \\ (x-x_0)\,p(x) \text{ analytic at } x = x_0 \\ (x-x_0)^2 q(x) \text{ analytic at } x = x_0 \end{array}\right.$

3) $\left.\begin{array}{l} x = x_0 \text{ is an} \\ \underline{\text{irregular singular}} \\ \underline{\text{point}} \end{array}\right\} \Longleftrightarrow \begin{array}{l} \big((x-x_0)\,p(x) \text{ NOT analytic at } x = x_0\big) \vee \\ \vee \big((x-x_0)^2 q(x) \text{ NOT analytic at } x = x_0\big). \end{array}$

- The first two cases can be solved with convergent power series methods. The third case can be only investigated with asymptotic techniques or may be current research.

1 → Regular linear ODEs

Thm: Consider an initial value problem of the form

$$\begin{cases} y''(x) + p(x)y'(x) + q(x)y(x) = 0 \\ y(x_0) = a_0 \wedge y'(x_0) = a_1 \end{cases}$$

with $p, q \in C^{\omega}((x_0 - \mu, x_0 + \mu))$ (i.e. p, q analytic at $x = x_0$) such that

$$\forall x \in (x_0 - \mu, x_0 + \mu) : \left(p(x) = \sum_{n=0}^{+\infty} p_n (x - x_0)^n \wedge q(x) = \sum_{n=0}^{+\infty} q_n (x - x_0)^n \right)$$

The unique solution to this initial value problem is given by

$$\forall x \in (x_0 - \mu, x_0 + \mu) : \quad y(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n$$

with $a \in \text{Seq}(\mathbb{R})$ a sequence defined recursively by

$$\forall n \in \mathbb{N} : \quad a_{n+2} = \frac{-1}{(n+1)(n+2)} \sum_{k=0}^{n} \left[(k+1) a_{k+1} p_{n-k} + a_k q_{n-k} \right]$$

with $a_0, a_1 \in \mathbb{R}$ given via the above initial conditions.

Remarks

1) The unique sequence defined by the above recursion combined with initial values $a_0, a_1 \in \mathbb{R}$ will be denoted for convenience as: $a_n = A_n(a_0, a_1 | p, q)$.

2) The convergence of the power series for $y(x)$ is provided for by the theorem and has the same radius of convergence as the functions p, q. It is therefore not necessary to establish convergence when solving problems.

3) To find the two linearly independent solutions y_1, y_2 we solve, by convention, the following initial value problems:

$$\begin{cases} y(x_0)=1 \\ y'(x_0)=0 \end{cases} \longleftrightarrow y_1(x)=\sum_{n=0}^{+\infty} b_n(x-x_0)^n$$

$$\begin{cases} y(x_0)=0 \\ y'(x_0)=1 \end{cases} \longleftrightarrow y_2(x)=\sum_{n=0}^{+\infty} c_n(x-x_0)^n$$

with $\forall n\in\mathbb{N}: \begin{cases} b_n = A_n(1,0\,|\,p,q) \\ c_n = A_n(0,1\,|\,p,q) \end{cases}$

To show that y_1, y_2 are indeed linearly independent we note that

$$\begin{cases} y_1(x_0)=1 \\ y_1'(x_0)=0 \end{cases} \wedge \begin{cases} y_2(x_0)=0 \\ y_2'(x_0)=1 \end{cases}$$

and therefore:

$$w[y_1,y_2](x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1\cdot 1 - 0\cdot 0 = 1$$

$\Rightarrow y_1, y_2$ linearly independent.

4) In practice it is customary to derive the recursion formula for the power series on a case by case basis. However, given the theorem, it is not necessary to prove convergence.

EXAMPLES

a) Find the general solution to the $\boxed{\text{Airy equation}}$ initial value problem

$$\begin{cases} y''(x) - xy(x) = 0 \\ y(0) = a_0 \wedge y'(0) = a_1 \end{cases}$$

Solution

Consider a solution of the form

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

and note that

$$y'(x) = \frac{d}{dx} \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n$$

and

$$y''(x) = \frac{d}{dx} \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{+\infty} n(n+1) a_{n+1} x^{n-1} =$$

$$= \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n .$$

Then, we have:

$$y''(x) - xy(x) = 0 \Leftrightarrow \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n - x \sum_{n=0}^{+\infty} a_n x^n = 0$$

$$\Leftrightarrow \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{+\infty} a_n x^{n+1} = 0$$

$$\Leftrightarrow \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{+\infty} a_{n-1} x^n = 0$$

$$\Leftrightarrow (0+1)(0+2)a_2 + \sum_{n=1}^{+\infty} \left[(n+1)(n+2)a_{n+2} - a_{n-1} \right] x^n = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a_2 = 0 \\ \forall n \in \mathbb{N}^* : (n+1)(n+2)a_{n+2} - a_{n-1} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a_2 = 0 \\ \forall n \in \mathbb{N}^* : a_{n+2} = \dfrac{a_{n-1}}{(n+1)(n+2)} \end{cases}$$

$$\Leftrightarrow \begin{cases} a_2 = 0 \\ \forall n \in \mathbb{N} - \{0,1,2\} : a_n = \dfrac{a_{n-3}}{n(n-1)} \end{cases}$$

▸ Now we can derive direct results for the sequence a_n. Starting from a_0, we get $a_3, a_6, \dots, a_{3k}, \dots$ we have:

$$\forall n \in \mathbb{N}^* : a_{3n} = a_0 \prod_{\lambda=1}^{n} \frac{1}{3\lambda(3\lambda-1)} = a_0 \prod_{\lambda=1}^{n} \frac{3\lambda-2}{3\lambda(3\lambda-1)(3\lambda-2)}$$

$$= a_0 \frac{1}{(3n)!} \prod_{\lambda=1}^{n} (3\lambda-2)$$

$$= a_0 \frac{3^n}{(3n)!} \prod_{\lambda=1}^{n} (\lambda - 2/3)$$

$$= a_0 \frac{3^n \Gamma(n-2/3+1)}{(3n)! \Gamma(-2/3+1)}$$

$$= a_0 \frac{3^n \Gamma(n+1/3)}{(3n)! \Gamma(1/3)}$$

and we note that this equation is also satisfied for $n=0$.

Starting from a_1, we get $a_4, a_7, \dots, a_{3n+1}, \dots$ and therefore

$$\forall n \in \mathbb{N}^* : a_{3n+1} = a_1 \prod_{\lambda=1}^{n} \frac{1}{(3\lambda+1)((3\lambda+1)-1)} =$$

$$= a_1 \prod_{\lambda=1}^{n} \frac{1}{3\lambda(3\lambda+1)} = a_1 \prod_{\lambda=1}^{n} \frac{3\lambda-1}{(3\lambda-1)(3\lambda)(3\lambda+1)}$$

$$= a_1 \frac{1}{(3n+1)!} \prod_{\lambda=1}^{n} (3\lambda-1) = a_1 \frac{3^n}{(3n+1)!} \prod_{\lambda=1}^{n} (\lambda - 1/3)$$

$$= a_1 \frac{3^n \Gamma(n-1/3+1)}{(3n+1)!\, \Gamma(-1/3+1)} = a_1 \frac{3^n \Gamma(n+2/3)}{(3n+1)!\, \Gamma(2/3)}$$

and we note that this equation is also satisfied for $n=0$.

Since $a_2 = 0$, it follows that $\forall n \in \mathbb{N} : a_{3n+2} = 0$

It follows that the general solution is:

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} a_{3n} x^{3n} + \sum_{n=0}^{+\infty} a_{3n+1} x^{3n+1}$$

$$= \sum_{n=0}^{+\infty} a_0 \frac{3^n \Gamma(n+1/3)}{(3n)!\, \Gamma(1/3)} x^{3n} + \sum_{n=0}^{+\infty} a_1 \frac{3^n \Gamma(n+2/3)}{(3n+1)!\, \Gamma(2/3)} x^{3n+1}$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

with

$$y_1(x) = \sum_{n=0}^{+\infty} \frac{3^n \Gamma(n+1/3)}{(3n)!\, \Gamma(1/3)} x^{3n} \wedge y_2(x) = \sum_{n=0}^{+\infty} \frac{3^n \Gamma(n+2/3)}{(3n+1)!\, \Gamma(2/3)} x^{3n+1}$$

These series will converge uniformly on $\mathbb{R}$ and define the two linearly independent homogeneous solutions that span the null-space of the Airy equation.

↳ In the above argument, we have used the following identity

$$\boxed{\prod_{k=1}^{n} (k+a) = \frac{\Gamma(n+a+1)}{\Gamma(a+1)}}$$

to eliminate the products in the formula for $y_1(x)$ and $y_2(x)$ and extend their validity to from $n \in \mathbb{N}^*$ to $n \in \mathbb{N}$.

EXAMPLE

Solve the linear ODE: $y''(x) + \cos(x) y(x) = 0$.
with a series around $x = 0$.

Solution

We consider a solution of the form

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

and note that

$$y'(x) = \frac{d}{dx} \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n \Rightarrow$$

$$\Rightarrow y''(x) = \frac{d}{dx} \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{+\infty} n(n+1) a_{n+1} x^{n-1} =$$

$$= \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n$$

and

$$(\cos x) y(x) = \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}\right]\left[\sum_{n=0}^{+\infty} a_n x^n\right] =$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}\right]\left[\sum_{n=0}^{+\infty} a_{2n} x^{2n} + \sum_{n=0}^{+\infty} a_{2n+1} x^{2n+1}\right] =$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}\right]\left[\sum_{n=0}^{+\infty} a_{2n} x^{2n}\right] + \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}\right]\left[\sum_{n=0}^{+\infty} a_{2n+1} x^{2n+1}\right]$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{(2n-2k)}}{(2n-2k)!} a_{2k} x^{2k}\right] + \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{2n-2k}}{(2n-2k)!} a_{2k+1} x^{2k+1}\right]$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{a_{2k}}{(2n-2k)!}\right] x^{2n} + \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{a_{2k+1}}{(2n-2k)!}\right] x^{2n+1}$$

It follows that

$y''(x) - \cos(x) y(x) = 0 \Leftrightarrow$

$$\Leftrightarrow \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{a_{2k}}{(2n-2k)!} \right] x^{2n} -$$

$$- \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{a_{2k+1}}{(2n-2k)!} \right] x^{2n+1} = 0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{n=0}^{+\infty} \left[(2n+1)(2n+2) a_{2n+2} - \sum_{k=0}^{n} \frac{a_{2k}}{(2n-2k)!} \right] x^{2n} +$$

$$+ \sum_{n=0}^{+\infty} \left[((2n+1)+1)((2n+1)+2) a_{(2n+1)+2} - \sum_{k=0}^{n} \frac{a_{2k+1}}{(2n-2k)!} \right] x^{2n+1} = 0$$

$$\Leftrightarrow \sum_{n=0}^{+\infty} \left[(2n+1)(2n+2) a_{2n+2} - \sum_{k=0}^{n} \frac{a_{2k}}{(2n-2k)!} \right] x^{2n} +$$

$$+ \sum_{n=0}^{+\infty} \left[(2n+2)(2n+3) a_{2n+3} - \sum_{k=0}^{n} \frac{a_{2k+1}}{(2n-2k)!} \right] x^{2n+1} = 0$$

$$\Leftrightarrow \forall n \in \mathbb{N} : \begin{cases} (2n+1)(2n+2) a_{2n+2} - \displaystyle\sum_{k=0}^{n} \frac{a_{2k}}{(2n-2k)!} = 0 \\ (2n+2)(2n+3) a_{2n+3} - \displaystyle\sum_{k=0}^{n} \frac{a_{2k+1}}{(2n-2k)!} = 0 \end{cases}$$

$$\Leftrightarrow \boxed{\forall n \in \mathbb{N} : \begin{cases} a_{2n+2} = \dfrac{1}{(2n+1)(2n+2)} \displaystyle\sum_{k=0}^{n} \frac{a_{2k}}{(2n-2k)!} \\ a_{2n+3} = \dfrac{1}{(2n+2)(2n+3)} \displaystyle\sum_{k=0}^{n} \frac{a_{2k+1}}{(2n-2k)!} \end{cases} \quad (1)}$$

Initializing the power series requires a_0 and a_1.

↳ Note that it is not possible to express the series in closed form. We can only use Eq. (1) to generate as many series terms as needed. To obtain two linearly independent solutions $y_1(x)$ and $y_2(x)$ we initialize Eq. (1) using $(a_0, a_1) = (1, 0)$ and $(a_0, a_1) = (0, 1)$ respectively, ~~to~~ This will yield the power series for $y_1(x)$ and $y_2(x)$.

↳ In order to multiply power series expansions to calculate $\cos(x) y(x)$ we used Merten's theorem from my Calculus 2 lecture notes:

$$\boxed{\begin{cases} \displaystyle\sum_{n=0}^{+\infty} |a_n| \text{ converges} \\ \displaystyle\sum_{n=0}^{+\infty} b_n \text{ converges} \end{cases} \Rightarrow \left[\sum_{n=0}^{+\infty} a_n\right]\left[\sum_{n=0}^{+\infty} b_n\right] = \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n} a_k b_{n-k}\right]}$$

The required assumptions are always satisfied by power series within their convergence interval.

(2) → Regular singular linear ODEs (Frobenius method)

We consider a linear ODE of the form

$$y''(x) + \frac{p(x)}{x-x_0}\, y'(x) + \frac{q(x)}{(x-x_0)^2}\, y(x) = 0 \qquad (1)$$

or equivalently

$$(x-x_0)^2 y''(x) + (x-x_0)p(x)y'(x) + q(x)y(x) = 0$$

with p, q analytic at $x = x_0$ with power-series expansions

$$\forall x \in (x_0-\mu, x_0+\mu): \left(p(x) = \sum_{n=0}^{+\infty} p_n (x-x_0)^n \wedge q(x) = \sum_{n=0}^{+\infty} q_n (x-x_0)^n \right)$$

Since $x = x_0$ is not a regular point, the ODE does not admit linearly independent solutions $y_1(x), y_2(x)$ that can be expressed as a power series. Nonetheless, a general solution method for Eq. (1), where $x = x_0$ is a regular singular point, has been developed by Frobenius as follows.

Prop: Consider a function y defined as

$$y(x) = |x-x_0|^\lambda \sum_{n=0}^{+\infty} a_n (x-x_0)^n$$

If $y(x)$ solves Eq. (1), then:

(a) $F(\lambda | p_0, q_0) \equiv \lambda(\lambda-1) + p_0\lambda + q_0 = 0$

(b) $\forall n \in \mathbb{N}^*: F(\lambda+n | p_0, q_0)\, a_n = -\sum_{k=0}^{n-1} [(k+\lambda)p_{n-k} + q_{n-k}]\, a_k$

Remarks

(a) The polynomial $F(\lambda|p_0,q_0)$ is the indicial polynomial and the equation

$$\lambda(\lambda-1)+p_0\lambda+q_0=0$$

is the indicial equation associated with the linear ODE Eq.(1).

(b) Using the recurrence for the sequence a_n given by the above proposition with a given initial value a_0, we can show that $a_1, a_2, \ldots, a_n, \ldots$ are proportional to a_0 and the resulting sequence will be denoted as

$$\forall n\in\mathbb{N}^*: \ a_n = a_0\Phi_n(\lambda|p,q)$$

with $p,q\in \mathrm{Seq}(\mathbb{R})$ representing the sequences $p_1,p_2,\ldots,p_n,\ldots$ and $q_1,q_2,\ldots,q_n,\ldots$ Note that Φ_n is independent of x_0.

(c) We may now define the general function

$$y(x,\lambda|p,q)=|x-x_0|^{\lambda}\sum_{n=0}^{+\infty}\Phi_n(\lambda|p,q)(x-x_0)^n$$

For most values of λ this function does not solve Eq.(1). From the following propositions we see that $y(x,\lambda|p,q)$ solves Eq.(1) when λ is one of the zeroes of the indicial equation.

Prop: If p,q converge on $(x_0-\mu, x_0+\mu)$ then the series expansion for $y(x,\lambda|p,q)$ also converges both uniformly and absolutely on $(x_0-\mu,x_0+\mu)$.

Prop: Let $A = (x_0 - \rho, x_0 + \rho)$ and let $L : C^2(A) \to C^0(A)$ be the linear operator associated with the linear ODE Eq. (1) such that

$$\forall y \in C^2(A) : (Ly)(x) = y''(x) + \frac{p(x)}{x - x_0} y'(x) + \frac{q(x)}{(x - x_0)^2} y(x).$$

It follows that

$$\begin{aligned} Ly(x, \lambda | p, q) &= |x - x_0|^{\lambda - 2} F(\lambda | p_0, q_0) \\ &= |x - x_0|^{\lambda - 2} [\lambda(\lambda - 1) + p_0 \lambda + q_0] \end{aligned}$$

Using the above results and notations, and some additional considerations needed for the proofs, we establish the main result:

Thm: Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the zeroes of the indicial polynomial $F(\lambda | p_0, q_0)$ and with no loss of generality we assume that $Re(\lambda_1) \geq Re(\lambda_2)$. We then distinguish between the following cases:

Case 1: If $\lambda_1 \neq \lambda_2 \wedge \lambda_1 - \lambda_2 \notin \mathbb{N}^*$ then Eq. (1) has two linearly independent solutions given by:

$$\forall x \in (x_0 - \rho, x_0 + \rho) : \begin{cases} y_1(x) = y(x, \lambda_1 | p, q) = |x - x_0|^{\lambda_1} \sum_{n=0}^{+\infty} \phi_n(\lambda_1 | p, q)(x - x_0)^n \\ y_2(x) = y(x, \lambda_2 | p, q) = |x - x_0|^{\lambda_2} \sum_{n=0}^{+\infty} \phi_n(\lambda_2 | p, q)(x - x_0)^n \end{cases}$$

Case 2: If $\lambda_1 = \lambda_2$, then the two linearly independent solutions are

$$y_1(x) = y(x, \lambda_1 | p, q) = |x - x_0|^{\lambda_1} \sum_{n=0}^{+\infty} \phi_n(\lambda_1 | p, q)(x - x_0)^n$$

$$y_2(x) = \frac{\partial}{\partial \lambda} y(x, \lambda | p, q) \Big|_{\lambda = \lambda_1} =$$

$$= y_1(x) \ln|x - x_0| + |x - x_0|^{\lambda_1} \sum_{n=0}^{+\infty} b_n (x - x_0)^n$$

with $\forall n \in \mathbb{N} : b_n = \frac{\partial}{\partial \lambda} \phi_n(\lambda | p, q) \Big|_{\lambda = \lambda_1}$

Case 3: If $\lambda_1 - \lambda_2 = N \in \mathbb{N}^*$, then the two linearly independent solutions are

$$y_1(x) = y(x, \lambda_1 | p, q) = |x - x_0|^{\lambda_1} \sum_{n=0}^{+\infty} \phi_n(\lambda_1 | p, q)(x - x_0)^n$$

$$y_2(x) = \frac{\partial}{\partial \lambda} \left[(\lambda - \lambda_2) y(x, \lambda | p, q) \right] \Big|_{\lambda = \lambda_2} =$$

$$= C_1 y_1(x) \ln|x - x_0| + |x - x_0|^{\lambda_2} \sum_{n=0}^{+\infty} c_n (x - x_0)^n$$

with $C_1 = \lim_{\lambda \to \lambda_2} \left[(\lambda - \lambda_2) \phi_N(\lambda | p, q) \right]$

$$\forall n \in \mathbb{N} : \quad c_n = \frac{\partial}{\partial \lambda} \left[(\lambda - \lambda_2) \phi_n(\lambda | p, q) \right] \Big|_{\lambda = \lambda_2}$$

↳ Given the solutions $y_1(x)$ and $y_2(x)$, the general solution is:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

with $c_1, c_2 \in \mathbb{R}$.

Methodology / Remarks

(a) It is recommended that you use the above theorems and propositions to determine the indicial polynomial and the recurrence relationship defining the sequence $a_n = a_0 \Phi_n(\lambda|p,q)$. Although both can be obtained from substituting the solution forms to the original ODE, that tends to be cumbersome.

(b) An explicit expression for a_n as a function of λ is needed for cases 2,3 in order to differentiate them with respect to λ. For case 1 it is not needed, and it is sufficient to have explicit equations for a_n only for $\lambda=\lambda_1$ and $\lambda=\lambda_2$

(c) For the calculation of $y_2(x)$ in cases 2,3 it is often necessary to calculate the derivatives (with respect to λ) of a function defined as a product or ratio of a large number of factors. A technique known as logarithmic differentiation can be used to evaluate such products as follows:

$$\boxed{\frac{d}{dx}\prod_{a=1}^{n}[f_a(x)]^{c_a} = \prod_{a=1}^{n}[f_a(x)]^{c_a}\left[\sum_{a=1}^{n} c_a \frac{f_a'(x)}{f_a(x)}\right]}$$

as long as $\forall a \in [n]: f_a(x) \neq 0$.

(d) Gamma functions are used to simplify linear products:

$$\boxed{\prod_{k=1}^{n}(ak+b) = a^n \prod_{k=1}^{n}(k+b/a) = \frac{a^n\Gamma(n+1+b/a)}{\Gamma(1+b/a)}}$$

EXAMPLES

a) Solve the linear ODE

$$x^2 y''(x) + x(x-1/2)y'(x) + (1/2)y(x) = 0$$

with a series around $x=0$.

Solution

We rewrite the ODE as:

$$y''(x) + \frac{1}{x}(x-1/2)y'(x) + \frac{1}{x^2}\frac{1}{2}y(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow y''(x) + \frac{p(x)}{x}y'(x) + \frac{q(x)}{x^2}y(x) = 0$$

with $p(x) = x - 1/2 \to p_0 = -1/2 \wedge p_1 = 1 \wedge p_2 = p_3 = \cdots = 0$
and $q(x) = 1/2 \Rightarrow q_0 = 1/2 \wedge q_1 = q_2 = \cdots = 0$

Consider a solution

$$y(x) = |x|^\lambda \sum_{n=0}^{+\infty} a_n x^n$$

Substituting to the ODE gives the indicial polynomial

$$F(\lambda) = \lambda(\lambda-1) + p_0\lambda + q_0 = \lambda(\lambda-1) - (1/2)\lambda + 1/2 =$$
$$= \lambda(\lambda-1) - (1/2)(\lambda-1) = (\lambda - 1/2)(\lambda-1)$$

and the recurrence

$$\forall n \in \mathbb{N}^*: F(\lambda+n)a_n = -\sum_{k=0}^{n-1}[(k+\lambda)p_{n-k} + q_{n-k}]a_k =$$

$$= -[(n-1+\lambda)p_1 + q_1]a_{n-1} - \sum_{k=0}^{n-2}[(k+\lambda)p_{n-k} + q_{n-k}]a_k$$

$$= -[(n+\lambda-1)p_1 + 0]a_{n-1} = -(n+\lambda-1)a_{n-1} \Leftrightarrow$$

$$\Leftrightarrow (\lambda+n-1/2)(\lambda+n-1)a_n = -(\lambda+n-1)a_{n-1} \Leftrightarrow$$
$$\Leftrightarrow (\lambda+n-1/2)a_n = -a_{n-1} \Leftrightarrow a_n = \frac{-1}{\lambda+n-1/2} a_{n-1}.$$

It follows that
$$\forall n\in\mathbb{N}^*: a_n = a_0 \prod_{k=1}^{n}\left(\frac{-1}{\lambda+k-1/2}\right) = a_0(-1)^n \prod_{k=1}^{n}\frac{1}{\lambda+k-1/2}.$$

Solving the indicial equation:
$$F(\lambda)=0 \Leftrightarrow (\lambda-1/2)(\lambda-1)=0 \Leftrightarrow \lambda-1/2=0 \lor \lambda-1=0 \Leftrightarrow$$
$$\Leftrightarrow \lambda=1/2 \lor \lambda=1.$$

For $\lambda=1/2$:
$$\forall n\in\mathbb{N}^*: a_n = a_0\prod_{k=1}^{n}\frac{-1}{1/2+k-1/2} = a_0(-1)^n\prod_{k=1}^{n}\frac{1}{k} =$$
$$= a_0\frac{(-1)^n}{n!}$$

and therefore the first homogeneous solution is:
$$y_1(x) = |x|^{1/2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}x^n = |x|^{1/2}\sum_{n=0}^{+\infty}\frac{(-x)^n}{n!} = |x|^{1/2}e^{-x}$$

For $\lambda=1$:
$$\forall n\in\mathbb{N}^*: a_n = a_0\prod_{k=1}^{n}\frac{-1}{1+k-1/2} = a_0(-1)^n\prod_{k=1}^{n}\frac{1}{k+1/2} =$$
$$= a_0(-1)^n\left[\prod_{k=1}^{n}(k+1/2)\right]^{-1} = a_0(-1)^n\left[\frac{\Gamma(n+1+1/2)}{\Gamma(1+1/2)}\right]^{-1}$$
$$= a_0\frac{(-1)^n\Gamma(3/2)}{\Gamma(n+3/2)}$$

and therefore the second homogeneous solution is:

$$y_2(x) = |x| \sum_{n=0}^{+\infty} \frac{(-1)^n \Gamma(3/2)}{\Gamma(n+3/2)} x^n$$

The general solution is:

$$y(x) = \lambda_1 |x|^{1/2} e^{-x} + \lambda_2 |x| \sum_{n=0}^{+\infty} \frac{(-1)^n \Gamma(3/2)}{\Gamma(n+3/2)} x^n.$$

Since p, q converge on $\mathbb{R}$, the general solution $y(x)$ converges on $\mathbb{R}$.

b) Solve the linear ODE

$$x(1-x)y''(x) + (1-x)y'(x) - y(x) = 0$$

around $x=0$.

Solution

We note that

$$x(1-x)y''(x) + (1-x)y'(x) - y(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow y''(x) + \frac{1-x}{x(1-x)}y'(x) - \frac{1}{x(x-1)}y(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow y''(x) + \frac{1}{x}y'(x) + \frac{1}{x^2}\frac{-x}{x-1}y(x) = 0$$

$$\Leftrightarrow y''(x) + \frac{1}{x}p(x)y'(x) + \frac{1}{x^2}q(x)y(x) = 0$$

with $p(x) = 1 = \sum_{n=0}^{+\infty} p_o x^n \Rightarrow p_0 = 1 \wedge p_1 = p_2 = \cdots = 0$

and $q(x) = \frac{-x}{1-x} = (-x)\frac{1}{1-x} = (-x)\sum_{n=0}^{+\infty} x^n =$

$$= \sum_{n=0}^{+\infty}(-x^{n+1}) = \sum_{n=1}^{+\infty}(-1)x^n = \sum_{n=0}^{+\infty} q_n x^n \Rightarrow$$

$$\Rightarrow q_0 = 0 \wedge q_1 = q_2 = \cdots = -1$$

Note that the convergence interval for $q(x)$ is $(-1,1)$.

Using a candidate solution

$$y(x) = |x|^\lambda \sum_{n=0}^{+\infty} a_n x^n$$

we find that the indicial polynomial is:

$$F(\lambda) = \lambda(\lambda-1) + p_0\lambda + q_0 = \lambda(\lambda-1) + \lambda + 0 = \lambda(\lambda-1+1) = \lambda^2$$

and the sequence a_n must satisfy

$$\forall n \in \mathbb{N}^*: \ F(\lambda+n)a_n = -\sum_{k=0}^{n-1} [(k+\lambda)p_{n-k} + q_{n-k}]a_k =$$

$$= -\sum_{k=0}^{n-1} (k+\lambda)p_{n-k}a_k - \sum_{k=0}^{n-1} q_{n-k}a_k =$$

$$= -0 - \sum_{k=0}^{n-1} (-1)a_k = a_0 + a_1 + \cdots + a_{n-1} \Leftrightarrow$$

$$\Leftrightarrow (\lambda+n)^2 a_n = a_0 + a_1 + \cdots + a_{n-1}, \ \forall n \in \mathbb{N}^* \quad (1)$$

For $n=1$: $(\lambda+1)^2 a_1 = a_0 \Leftrightarrow a_1 = \dfrac{a_0}{(\lambda+1)^2}$ (2)

For $n \geq 2$ we note that $(\lambda+n-1)^2 a_{n-1} = a_0 + a_1 + \cdots + a_{n-2}$ and therefore from Eq. (1)

$$(1) \Leftrightarrow (\lambda+n)^2 a_n = (a_0 + a_1 + \cdots + a_{n-2}) + a_{n-1} =$$

$$= (\lambda+n-1)^2 a_{n-1} + a_{n-1} =$$

$$= [(\lambda+n-1)^2 + 1]a_{n-1} \Leftrightarrow$$

$$\Leftrightarrow a_n = \frac{(\lambda+n-1)^2+1}{(\lambda+n)^2} a_{n-1} \quad (3)$$

Note that for $n=1$, $(\lambda+n-1)^2+1 = (\lambda+1-1)^2+1 = \lambda^2+1 \neq 0$ so equation (3) does not reduce to equation (2) for $n=1$. To mitigate that, we choose $a_0 = \lambda^2+1$. Then:

$$a_1 = \frac{\lambda^2+1}{(\lambda+1)^2} = \frac{(\lambda+1-1)^2+1}{(\lambda+1)^2}$$

and it follows that

$$\forall n \in \mathbb{N}^*: \ a_n = \prod_{k=1}^{n} \frac{(\lambda+k-1)^2+1}{(\lambda+k)^2}$$

Solving the indicial equation gives:

$F(\lambda) = 0 \Leftrightarrow \lambda^2 = 0 \Leftrightarrow \lambda = 0 \longleftarrow$ double zero.

For $\lambda = 0$:

$$\forall n \in \mathbb{N}^*: \ a_n = \prod_{k=1}^{n} \frac{(0+k-1)^2+1}{(0+k)^2} = \frac{1}{(n!)^2} \prod_{k=1}^{n} [(k-1)^2+1]$$

$$= \frac{1}{(n!)^2} \prod_{k=0}^{n-1} (k^2+1)$$

and $a_0 = 0^2+1 = 1$, therefore the first homogeneous solution is given by

$$y_1(x) = 1 + \sum_{n=1}^{+\infty} \left[\frac{1}{(n!)^2} \prod_{k=0}^{n-1} (k^2+1) \right] x^n.$$

and the second linearly independent solution is given by:

$$y_2(x) = y_1(x) \ln|x| + \sum_{n=0}^{+\infty} b_n x^n \quad \text{with } b_n = \left. \frac{\partial a_n}{\partial \lambda} \right|_{\lambda=0}$$

To calculate b_n, we note that

$$\frac{\partial a_0}{\partial \lambda} = \frac{\partial}{\partial \lambda} (\lambda^2+1) = 2\lambda \Rightarrow b_0 = \left. \frac{\partial a_0}{\partial \lambda} \right|_{\lambda=0} = \left. 2\lambda \right|_{\lambda=0} = 0$$

and

$$\forall n \in \mathbb{N}^*: \frac{\partial a_n}{\partial \lambda} = \frac{\partial}{\partial \lambda} \prod_{k=1}^{n} \frac{(\lambda+k-1)^2+1}{(\lambda+k)^2} =$$

$$= \prod_{k=1}^{n} \left(\frac{(\lambda+k-1)^2+1}{(\lambda+k)^2} \right) \left[\sum_{k=1}^{n} \frac{(\partial/\partial\lambda)[(\lambda+k-1)^2+1]}{(\lambda+k-1)^2+1} - \right.$$

$$\left. -2 \sum_{k=1}^{n} \frac{(\partial/\partial\lambda)(\lambda+k)}{\lambda+k} \right] =$$

$$= \left[\prod_{k=1}^{n} \frac{(\lambda+k-1)^2+1}{(\lambda+k)^2} \right] \left[\sum_{k=1}^{n} \frac{2(\lambda+k-1)}{(\lambda+k-1)^2+1} - 2\sum_{k=1}^{n} \frac{1}{\lambda+k} \right] \Rightarrow$$

$$\Rightarrow b_n = \left.\frac{\partial a_n}{\partial \lambda}\right|_{\lambda=0} = a_n \sum_{k=1}^{n} \left[\frac{2(k-1)}{(k-1)^2+1} - \frac{2}{k} \right] =$$

$$= a_n \sum_{k=1}^{n} \left[\frac{2(k-1)k - 2[(k-1)^2+1]}{k[(k-1)^2+1]} \right] =$$

$$= a_n \sum_{k=1}^{n} \left[\frac{2k^2 - 2k - 2(k^2-2k+1+1)}{k(k^2-2k+1+1)} \right] =$$

$$= a_n \sum_{k=1}^{n} \frac{2k^2 - 2k - 2k^2 + 4k - 4}{k(k^2-2k-2)} =$$

$$= a_n \sum_{k=1}^{n} \frac{2k-4}{k(k^2-2k-2)} =$$

$$= \frac{1}{(n!)^2} \prod_{k=0}^{n-1} (k^2+1) \left[\sum_{k=1}^{n} \frac{2(k-2)}{k(k^2-2k-2)} \right]$$

It follows that the second solution is given by

$$y_2(x) = y_1(x)\ln|x| + \sum_{n=1}^{+\infty} \left[\frac{1}{(n!)^2} \left(\prod_{k=0}^{n-1} (k^2+1) \right) \left(\sum_{k=1}^{n} \frac{2(k-2)}{k(k^2-2k-2)} \right) \right] x^n$$

and the general solution is $y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$.
The solution will converge on $(-1,1)$ since p converges on $\mathbb{R}$ and q converges on $(-1,1)$.

c) Solve the linear ODE $xy''(x)+2y'(x)-y(x)=0$ with a series around $x=0$

Solution

We note that

$$xy''(x)+2y'(x)-y(x)=0 \Leftrightarrow y''(x)+(2/x)y'(x)-(1/x)y(x)=0$$
$$\Leftrightarrow y''(x)+(1/x)2y'(x)+(1/x^2)(-x)y(x)=0$$
$$\Leftrightarrow y''(x)+(1/x)p(x)y'(x)+(1/x^2)q(x)y(x)=0$$

with

$$p(x)=2=\sum_{n=0}^{+\infty} p_n x^n \Rightarrow p_0=2 \wedge p_1=p_2=\cdots=0$$

and

$$q(x)=-x=\sum_{n=0}^{+\infty} q_n x^n \Rightarrow q_0=0 \wedge q_1=-1 \wedge q_2=q_3=\cdots=0.$$

Using a candidate solution $y(x)=|x|^\lambda \sum_{n=0}^{+\infty} a_n x^n$ the corresponding indicial polynomial is

$$F(\lambda)=\lambda(\lambda-1)+p_0\lambda+q_0=\lambda(\lambda-1)+2\lambda=\lambda(\lambda-1+2)=\lambda(\lambda+1)$$

and a_n satisfies:

$$\forall n\in\mathbb{N}^*: F(\lambda+n)a_n=-\sum_{k=0}^{n-1}[(k+\lambda)p_{n-k}+q_{n-k}]a_k=$$
$$=-\sum_{k=0}^{n-1}(k+\lambda)p_{n-k}a_k-\sum_{k=0}^{n-1}q_{n-k}a_k=$$
$$=-0-q_1a_{n-1}=-(-1)a_{n-1}=a_{n-1}\Leftrightarrow$$
$$\Leftrightarrow (\lambda+n)(\lambda+n+1)a_n=a_{n-1}\Leftrightarrow a_n=\frac{1}{(\lambda+n)(\lambda+n+1)}a_{n-1}$$

and therefore $\forall n\in\mathbb{N}^*: a_n=a_0\prod_{k=1}^{n}\frac{1}{(\lambda+k+1)(\lambda+k)}$

Solving the indicial equation gives:

$F(\lambda)=0 \Leftrightarrow \lambda(\lambda+1)=0 \Leftrightarrow \lambda=0 \vee \lambda+1=0 \Leftrightarrow \lambda=0 \vee \lambda=-1.$

For $\lambda=0$, we have

$$a_n = a_0 \prod_{k=1}^{n} \frac{1}{(0+k+1)(0+k)} = a_0 \prod_{k=1}^{n} \frac{1}{k(k+1)} =$$

$$= a_0 \left[\prod_{k=1}^{n} \frac{1}{k}\right]\left[\prod_{k=1}^{n} \frac{1}{k+1}\right] =$$

$$= \frac{a_0}{n!} \prod_{k=2}^{n+1} \frac{1}{k} = \frac{a_0}{n!} \prod_{k=1}^{n+1} \frac{1}{k} = \frac{a_0}{n!\,(n+1)!} =$$

$$= \frac{a_0}{(n!)^2(n+1)}, \quad \forall n \in \mathbb{N}^*$$

and the corresponding solution is:

$$y_1(x) = \sum_{n=0}^{+\infty} \frac{x^n}{(n!)^2(n+1)}$$

Since $0-(-1)=1$, the second solution is

$$y_2(x) = C_1 y_1(x) \ln|x| + |x|^{-1} \sum_{n=0}^{+\infty} c_n x^n$$

Using $a_0(\lambda)=a_0$, we have:

$$C_1 = \lim_{\lambda\to -1} \left[(\lambda-(-1))\, a_1(\lambda)\right] = \lim_{\lambda \to -1} \left[(\lambda+1) \frac{a_0}{(\lambda+1)(\lambda+2)}\right] =$$

$$= \lim_{\lambda\to -1} \frac{a_0}{\lambda+2} = \frac{a_0}{-1+2} = a_0$$

and

$$c_n = \frac{\partial}{\partial\lambda}\Big[(\lambda-(-1))a_n(\lambda)\Big]\Big|_{\lambda=-1} = \frac{\partial}{\partial\lambda}\Big[(\lambda+1)\,a_n(\lambda)\Big]\Big|_{\lambda=-1}$$

$$= \frac{\partial}{\partial\lambda}\Big[(\lambda+1)\,a_0\prod_{k=1}^{n}\frac{1}{(\lambda+k+1)(\lambda+k)}\Big]\Big|_{\lambda=-1}, \quad \forall n\in\mathbb{N}^*$$

We distinguish between the following cases.

For $n=0$:

$$c_0 = \frac{\partial}{\partial\lambda}\Big[(\lambda+1)a_0\Big]\Big|_{\lambda=-1} = a_0\Big|_{\lambda=-1} = a_0$$

For $n=1$:

$$c_1 = \frac{\partial}{\partial\lambda}\Big[(\lambda+1)a_0\,\frac{1}{(\lambda+1+1)(\lambda+1)}\Big]\Big|_{\lambda=-1} = \frac{\partial}{\partial\lambda}\Big[\frac{a_0}{\lambda+2}\Big]\Big|_{\lambda=-1}$$

$$= \Big[\frac{-a_0(\partial/\partial\lambda)(\lambda+2)}{(\lambda+2)^2}\Big]\Big|_{\lambda=-1} = \Big[\frac{-a_0}{(\lambda+2)^2}\Big]_{\lambda=-1} =$$

$$= \frac{-a_0}{(-1+2)^2} = \frac{-a_0}{1^2} = -a_0 .$$

For $n>1$:

$$c_n = \frac{\partial}{\partial\lambda}\Big[(\lambda+1)a_n(\lambda)\Big]\Big|_{\lambda=-1} =$$

$$= \frac{\partial}{\partial\lambda}\Big[(\lambda+1)a_0\prod_{k=1}^{n}\Big(\frac{1}{(\lambda+k+1)(\lambda+k)}\Big)\Big]\Big|_{\lambda=-1} =$$

$$= \frac{\partial}{\partial\lambda}\Big[a_0\prod_{k=1}^{n}\Big(\frac{1}{\lambda+k+1}\Big)\prod_{k=2}^{n}\Big(\frac{1}{\lambda+k}\Big)\Big]\Big|_{\lambda=-1} =$$

$$= a_0\frac{\partial}{\partial\lambda}\Big[\frac{1}{\lambda+n+1}\Big(\prod_{k=2}^{n}\frac{1}{\lambda+k}\Big)^2\Big]\Big|_{\lambda=-1} =$$

$$= a_0 \frac{1}{\lambda+n+1}\left(\prod_{k=2}^{n}\frac{1}{\lambda+k}\right)^2\left[\frac{-(\partial/\partial\lambda)(\lambda+n+1)}{\lambda+n+1}+\sum_{k=2}^{n}\frac{-2(\partial/\partial\lambda)(\lambda+k)}{\lambda+k}\right]\Bigg|_{\lambda=-1}$$

$$= -a_0 \frac{1}{\lambda+n+1}\left(\prod_{k=2}^{n}\frac{1}{\lambda+k}\right)^2\left[\frac{1}{\lambda+n+1}+\sum_{k=2}^{n}\frac{2}{\lambda+k}\right]\Bigg|_{\lambda=-1}$$

$$= -a_0 \frac{1}{-1+n+1}\left(\prod_{k=2}^{n}\frac{1}{-1+k}\right)^2\left[\frac{1}{-1+n+1}+\sum_{k=2}^{n}\frac{2}{-1+k}\right]$$

$$= -a_0 \frac{1}{n}\left(\prod_{k=1}^{n-1}\frac{1}{k}\right)^2\left[\frac{1}{n}+\sum_{k=1}^{n-1}\frac{2}{k}\right] =$$

$$= -a_0 \frac{1}{n[(n-1)!]^2}\left[\frac{-1}{n}+\sum_{k=1}^{n}\frac{2}{k}\right]$$

$$= \frac{-a_0}{n!(n-1)!}\left[\frac{-1}{n}+2\sum_{k=1}^{n}\frac{1}{k}\right]$$

Note that this result, for $n=1$, agrees with our previous result for $n=1$. It follows that the second solution is

$$y_2(x) = y_1(x)\ln|x| + |x|^{-1}\left[1-\sum_{n=1}^{+\infty}\frac{1}{n!(n-1)!}\left[\frac{-1}{n}+2\sum_{k=1}^{n}\frac{1}{k}\right]x^n\right]$$

The general solution is $y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$.

Homework: Series solution of linear differential equations

Homework 03: Series solution of linear differential equations

1. Derive the complete series expansion for the following functions around the indicated points and find the corresponding convergence radius
   (a) $f(x) = e^x \sin x$, around $x = x_0$
   (b) $f(x) = e^x \ln(1+x)$, around $x = x_0$

2. The binomial series is given by
$$\forall x \in (-1,1) : (1+x)^a = \sum_0^{+\infty} \binom{a}{n} x^n$$
   with
$$\binom{a}{0} = 1 \text{ and } \forall n \in \mathbb{N}^* : \binom{a}{n} = \prod_{k=1}^{n} \frac{a+1-k}{k}$$
   (a) Show that:
$$\forall a \in (1,+\infty) : \forall n \in \mathbb{N}^* : \binom{1/a}{n} = (-1)^n \frac{\Gamma(n-1/a)}{n\Gamma(n)\Gamma(-1/a)}$$
   (b) For the special case $a = -2$, show that
$$\forall n \in \mathbb{N}^* : \binom{-1/2}{n} = (-1)^n \frac{(2n-1)!!}{(2n)!!}$$
   with the double factorial $n!!$ defined via:
$$0!! = 1 \ \wedge\ 1!! = 1$$
$$\forall n \in \mathbb{N}^* : (2n)!! = \prod_{k=1}^{n} 2k \ \wedge\ (2n+1)!! = \prod_{k=1}^{n} (2k+1)$$

3. Find all terms of the unique power series solution to the following initial value problem:
$$\begin{cases} y''(x) - 2xy'(x) + 2y(x) = 0 \\ y(0) = 1 \wedge y'(0) = 0 \end{cases}$$

4. Use the Frobenius method to show that the general homogeneous solution for the equation
$$4xy''(x) + 2y'(x) + y(x) = 0$$
   is given by
$$\forall x \in (0,+\infty) : y(x) = \lambda_1 \cos(\sqrt{x}) + \lambda_2 \sin(\sqrt{x})$$

5. Use the Frobenius method to show that the general homogeneous solution for the equation
$$x(1-x)y''(x) + (1-5x)y'(x) - 4y(x) = 0$$
   is given by
$$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$$
   with
$$y_1(x) = \sum_{n=0}^{+\infty} (1+n)^2 x^n$$
$$y_2(x) = y_1(x) \ln|x| - 2 \sum_{n=1}^{+\infty} n(n+1) x^n$$

GODE 05: Asymptotic Methods

Asymptotic methods for ODEs

We distinguish between the following methods:

1. **Local analysis:** provide an approximate solution which is accurate only in a local region
2. **Global methods:** provide an approximate solution valid of the entire domain.
   (a) Boundary layer theory
   (b) Multiple scale analysis

1 Asymptotic relations

Asymptotic methods are centered around the concepts of asymptotic equations and asymptotic inequalities, given by the following definition:

Definition 1.1. Let f, g be two functions and ler σ be an accumulation point for both functions. We say that:

$$f(x) \ll g(x), \quad \text{as } x \to \sigma \Longleftrightarrow \lim_{x\to\sigma} \frac{f(x)}{g(x)} = 0$$
$$f(x) \gg g(x), \quad \text{as } x \to \sigma \Longleftrightarrow g(x) \ll f(x), \quad \text{as } x \to \sigma$$
$$f(x) \sim g(x), \quad \text{as } x \to \sigma \Longleftrightarrow \lim_{x\to\sigma} \frac{f(x)}{g(x)} = 1$$

• The statement $f(x) \ll g(x)$ reads: "$f(x)$ is much smaller than $g(x)$ as $x \to \sigma$"
• The statement $f(x) \sim g(x)$ reads: "$f(x)$ is asymptotically equal to $g(x)$"
• A function cannot be asymptotically equal to zero.
• An immediate consequence of the definitions is that:

$$f(x) \ll g(x), \quad \text{as } x \to \sigma \Longrightarrow f(x) + g(x) \sim g(x), \quad \text{as } x \to \sigma$$

• Both relations satisfy the transitive property, which allows intuitive multi-step calculations:

$$\begin{cases} f(x) \sim g(x), & \text{as } x \to \sigma \\ g(x) \sim h(x), & \text{as } x \to \sigma \end{cases} \Longrightarrow f(x) \sim h(x), \quad \text{as } x \to \sigma$$
$$\begin{cases} f(x) \ll g(x), & \text{as } x \to \sigma \\ g(x) \ll h(x), & \text{as } x \to \sigma \end{cases} \Longrightarrow f(x) \ll h(x), \quad \text{as } x \to \sigma$$

• The following results are immediate consequences of the definition and useful in calculations:

$$\forall a, b \in \mathbb{R} : (a < b \Longrightarrow x^a \ll x^b, \quad \text{as } x \to +\infty$$

$$\forall a, b \in \mathbb{R} : (a < b \Longrightarrow x^a \gg x^b, \text{ as } x \to 0^+$$
$$\forall a \in (0, +\infty) : \ln x \ll x^a, \text{ as } x \to +\infty$$
$$\forall a \in (0, +\infty) : \ln x \ll x^{-a}, \text{ as } x \to 0^+$$
$$\forall a \in (0, +\infty) : x^a \ll \exp(x), \text{ as } x \to +\infty$$

2 Asymptotic power series

Linear ordinary differential equations at an irregular singular point could be solved by using asymptotic power series, although doing so is very cumbersome.

Definition 2.1. Let y be a function. We say that y has an asymptotic power series

$$y(x) \sim \sum_{n=0}^{+\infty} a_n (x - x_0)^{\gamma n}, \text{ as } x \to x_0$$

if and only if

$$\forall N \in \mathbb{N} - \{0\} : y(x) - \sum_{n=0}^{N} a_n (x - x_0)^{\gamma n} \ll (x - x_0)^{\gamma N}, \text{ as } x \to x_0$$

• An asymptotic series is not necessarily convergent, however, when truncated, it provides approximations to the function $y(x)$ that are asymptotically valid in the limit $x \to x_0$.
• An asymptotic series will be convergent if and only if

$$\lim_{N \in \mathbb{N}} \sum_{n=N}^{+\infty} a_n (x - x_0)^{\gamma n} = 0$$

• If a function $y(x)$ has an asymptotic series of the form

$$y(x) \sim \sum_{n=0}^{+\infty} a_n (x - x_0)^{\gamma n}, \text{ as } x \to x_0$$

then the coefficients a_n are uniquely determined by the following equations:

$$a_0 = \lim_{x \to x_0} y(x)$$
$$a_1 = \lim_{x \to x_0} \frac{y(x) - a_0}{(x - x_0)^{\gamma}}$$
$$a_n = \lim_{x \to x_0} \frac{y(x) - \sum_{k=0}^{n-1} a_k (x - x_0)^{\gamma k}}{(x - x_0)^{\gamma n}}, \forall n \in \mathbb{N}^*$$

This ensures that, given γ, the asymptotic power series expansion of a function $y(x)$, if it exists, is unique. However, not every function has an asymptotic power series expansion. In order for a function to have an asymptotic power series expansion, it is necessary that all of the above limit calculations converge.

• Although every function $y(x)$ can only have a unique asymptotic expansion, if one exists, each asymptotic power series expansion is asymptotic to many functions.

3 Properties of asymptotic expansions

• The following theorem shows that we can add, multiply, and divide functions with asymptotic power series expansions and obtain new functions that also have asymptotic power series expansions.

Theorem 3.1. *Let f, g be two functions with*

$$f(x) \sim \sum_{n\in\mathbb{N}} a_n(x-x_0)^{\gamma n}, \quad \text{as } x \to x_0$$
$$g(x) \sim \sum_{n\in\mathbb{N}} b_n(x-x_0)^{\gamma n}, \quad \text{as } x \to x_0$$

Then, we have:

1. *Every linear combination of f and g has an asymptotic series*

$$\forall \lambda, \mu \in \mathbb{R} : \lambda f(x) + \mu g(x) \sim \sum_{n\in\mathbb{N}} (\lambda a_n + \mu b_n)(x-x_0)^{\gamma n}, \quad \text{as } x \to x_0$$

2. *The product $f(x)g(x)$ has an asymptotic series*

$$f(x)g(x) \sim \sum_{n\in\mathbb{N}} c_n(x-x_0)^{\gamma n}, \quad \text{as } x \to x_0,$$

with c_n given by

$$\forall n \in \mathbb{N} : c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

3. *The ratio $f(x)/g(x)$ has an asymptotic series*

$$\frac{f(x)}{g(x)} \sim \sum_{n\in\mathbb{N}} d_n(x-x_0)^{\gamma n}, \quad \text{as } x \to x_0,$$

with d_n given by

$$\begin{cases} \forall n \in \mathbb{N}^* : d_n = (1/b_0)\left[a_n - \sum_{k=0}^{n-1} d_k b_{n-k}\right] \\ d_0 = a_0/b_0 \end{cases}$$

• The following theorem shows that if a function has an asymptotic power series expansion, then the integral of that function also has an asymptotic power series expansion:

Theorem 3.2. *Let f be a function. Then, we have:*

$$f(x) \sim \sum_{n\in\mathbb{N}} a_n (x-x_0)^{\gamma n}, \quad \textit{as } x \to x_0 \Longrightarrow$$
$$\Longrightarrow \int_{x_0}^{x} f(t)\,\mathrm{d}t \sim \sum_{n\in\mathbb{N}} \frac{a_n (x-x_0)^{\gamma n+1}}{\gamma n + 1}, \quad \textit{as } x \to x_0$$

• Differentiation of asymptotic series expansions does not always work. There is a complicated collection of theorems called *Tauberian theorems* that can be used to justify differentiation. For the local analysis of ODEs, the following result can be used:

Theorem 3.3. *If $y(x)$ is a solution to a linear ODE of the form*

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_1(x) y' + p_0(x) y = 0$$

given by

$$y(x) \sim \sum_{n\in\mathbb{N}} a_n (x-x_0)^{\gamma n}, \quad \textit{as } x \to x_0$$

and the functions $p_k(x)$ are asymptotic to power series of the form

$$\forall k \in \{0, 1, \dots, n\} : p_k(x) \sim \sum_{n\in\mathbb{N}} b_{nk} (x-x_0)^{\gamma n}, \quad \textit{as } x \to x_0$$

then $y(x)$ can be differentiated term-by-term n times, with

$$\forall k \in \{1, 2, \dots, n\} : y^{(k)}(x) \sim \sum_{n\in\mathbb{N}} a_n (\mathrm{d}/\mathrm{d}x)^k (x-x_0)^{\gamma n}, \quad \textit{as } x \to x_0$$

4 Method of dominant balance

• There is no general theory for solving linear ODEs near an irregular singular point. However, there is an ad hoc asymptotic method known as the method of dominant balance.

• Suppose that the linear ODE $y''(x)+p(x)y'(x)+q(x)y(x)=0$ has an irregular singular point x_0. We are expecting to find general solutions of the form:

$$y(x) \sim \ell(x) \sum_{n\in\mathbb{N}} a_n (x-x_0)^{\gamma n}, \quad \text{as } x \to x_0$$

where $\ell(x)$ is the leading-order factor of the solution, which is expected to have an essential singularity at $x=x_0$, that involves $\ell(x)$ having a factor

$$\ell_0(x) = \exp[a(x-x_0)^{-b} \text{ with } b>0$$

By contrast, for regular singular points $x=x_0$, we have seen that the leading-order factor of the solution typically takes the form $\ell(x)=|x-x_0|^{\lambda}$, which does not have an essential singularity.
• To find the leading-order factor $\ell(x)$, we work as follows:

1. Define $S(x)$ such that $y(x)=\exp(S(x))$. Then, we have:

$$\begin{aligned} y'(x) &= S'(x)\exp(S(x)) = S'(x)y(x) \\ y''(x) &= S''(x)y(x) + S'(x)y'(x) = S''(x)y(x) + S'(x)[S'(x)y(x)] \\ &= [S''(x) + [S'(x)]^2]y(x) \end{aligned}$$

   and the linear ODE is equivalently

$$\begin{aligned} & y''(x)+p(x)y'(x)+q(x)y(x)=0 \\ & \quad \Longleftrightarrow [S''(x)+[S'(x)]^2+p(x)S'(x)+q(x)]y(x)=0 \\ & \quad \Longleftrightarrow S''(x)+[S'(x)]^2+p(x)S'(x)+q(x)=0 \end{aligned}$$

2. If x_0 is an irregular singular point of the equation, then we can guess that perhaps $S''(x) \ll [S'(x)]^2, \;$ as $x\to x_0$. If we assume so, then we obtain the following asymptotic differential equation:

$$[S'(x)]^2 + p(x)S'(x) \sim -q(x), \quad \text{as } x \to x_0$$

3. To solve this equation, we assume that there is dominant balance between two out of three terms, meaning that the third term is subdominant, and use that to solve for $S(x)$. Then, we check whether the solution satisfies the subdominance assumption, and if it doesn't then it is inconsistent and another combination should be attempted. For example:

   (a) We can assume that $[S'(x)]^2 \ll p(x)S'(x), \;$ as $x\to x_0$ and solve the equation $p(x)S'(x) \sim -q(x), \;$ as $x\to x_0$.

(b) We can assume that $[S'(x)]^2 \gg p(x)S'(x)$, as $x \to x_0$ and solve the equation $[S'(x)]^2 \sim -q(x)$, as $x \to x_0$.

4. When a self-consistent solution is found, we confirm that it also satisfies the assumption $S''(x) \ll [S'(x)]^2$, as $x \to x_0$
5. Let $S_0(x)$ be the resulting leading contribution to $S(x)$. To find the next-order contribution, write $S(x) = S_0(x) + C(x)$ and substitute to the exact governing equation for $S(x)$, to obtain an exact governing equation for $C(x)$.
6. Assume that $C(x) \ll S_0(x)$, as $x \to x_0$, and try to solve for $C(x)$ by using some self-consistent dominant balance between two terms of the resulting asymptotic equation. Some simplifications could be introduced as a consequence of the assumption $C(x) \ll S_0(x)$, as $x \to x_0$.
7. This process is repeated recursively to obtain $S(x) \sim S_0(x) + S_1(x) + \cdots$, as $x \to x_0$ until we encounter a logarithmic term $a \ln |x - x_0|$, which is the weakest possible singularity, for a leading factor that contains an essential singularity.

• The assumption $S''(x) \ll [S'(x)]^2$, as $x \to x_0$, is a consequence of the expectation that the solution of a linear ODE near an irregular singular point is likely to have an essential singularity. To show that, assume that

$$y(x) = \exp[a(x - x_0)^{-b} \text{ with } b > 0$$

Then, we have:

$$\begin{aligned}
S(x) = \ln y(x) = a(x - x_0)^{-b} &\Longrightarrow S'(x) = -ab(x - x_0)^{-(b+1)} \\
&\Longrightarrow \begin{cases} S''(x) = ab(b+1)(x - x_0)^{-(b+2)} \\ (S'(x))^2 = (ab)^2(x - x_0)^{-(2b+2)} \end{cases} \\
&\Longrightarrow \begin{cases} S''(x) = ab(b+1)(x - x_0)^{-(b+2)} \\ (S'(x))^2 = (ab)^2(x - x_0)^{-(2b+2)} \end{cases} \\
&\Longrightarrow \lim_{x \to x_0} \frac{S''(x)}{[S'(x)]^2} = \lim_{x \to x_0} \frac{ab(b+1)(x - x_0)^{-(b+2)}}{(ab)^2(x - x_0)^{-(2b+2)}} = \\
&\qquad = \frac{b+1}{ab} \lim_{x \to x_0} (x - x_0)^b = 0 \\
&\Longrightarrow S''(x) \ll [S'(x)]^2, \text{ as } x \to x_0
\end{aligned}$$

therefore the assumption is satisfied.

• Furthermore, we can argue that if the point $x = x_0$ is regular singular, then the same assumption is not satisfied. To show that, assume that

$$y(x) = |x - x_0|^\lambda$$

Then, we have:

$$\begin{aligned}
&S(x) = \ln y(x) = \lambda \ln|x - x_0| \Longrightarrow S'(x) = \frac{\lambda}{x - x_0} \\
&\Longrightarrow \begin{cases} S''(x) = \dfrac{-\lambda}{(x - x_0)^2} \\ (S'(x))^2 = \dfrac{\lambda^2}{(x - x_0)^2} \end{cases} \\
&\Longrightarrow [S'(x)]^2 \sim -\lambda S''(x), \quad \text{as } x \to x_0
\end{aligned}$$

therefore the assumption is not satisfied.

Example 4.1. Find the leading order solutions to the equation

$$y''(x) + (2/x)y'(x) - (1/x^4)y = 0, \quad \text{as } x \to 0$$

Solution. Let $y(x) = \exp(S(x))$. Then, we have $y'(x) = S'(x)y(x)$ and $y''(x) = [S''(x) + (S'(x))^2]y(x)$ and it follows that

$$y''(x) + 2x^{-1}y'(x) - x^{-4}y = 0 \Longleftrightarrow S''(x) + (S'(x))^2 + 2x^{-1}S'(x) - x^{-4} = 0$$

Assume that $S''(x) \ll (S'(x))^2$, as $x \to 0$ and consider the asymptotic equation

$$(S'(x))^2 + 2x^{-1}S'(x) \sim x^{-4}, \quad \text{as } x \to 0$$

• We investigate the assumption that $(S'(x))^2 \ll 2x^{-1}S'(x)$, as $x \to 0$. Then, we have

$$\begin{aligned}
2x^{-1}S'(x) \sim x^{-4}, \quad \text{as } x \to 0 &\Longleftrightarrow S'(x) \sim (1/2)x^{-3}, \quad \text{as } x \to 0 \\
&\Longleftrightarrow S(x) \sim (-1/4)x^{-2}, \quad \text{as } x \to 0
\end{aligned}$$

To check for consistency, we note that

$$(S'(x))^2 \sim (1/4)x^{-6} \sim (2x^{-1})(1/8)x^{-5} \gg (2x^{-1})(1/2)x^{-3} \sim 2x^{-1}S'(x), \quad \text{as } x \to 0$$

which contradicts with the assumption $(S'(x))^2 \ll 2x^{-1}S'(x)$, as $x \to 0$. Therefore, this is not the dominant balance.

• We investigate the assumption that $(2/x)S'(x) \ll (S'(x))^2$. Then, we have

$$(S'(x))^2 \sim x^{-4}, \quad \text{as } x \to 0 \Longleftrightarrow S'(x) \sim \pm x^{-2}, \quad \text{as } x \to 0$$

$$\Longleftrightarrow S(x) \sim \pm \int x^{-2} \mathrm{d}x \sim \mp x^{-1}, \text{ as } x \to 0$$

To check for consistency, we note that

$$S'(x) \sim \pm x^{-2}, \text{ as } x \to 0 \Longrightarrow \begin{cases} (S'(x))^2 \sim x^{-4} \\ 2x^{-1}S'(x) \sim \pm 2x^{-3} \end{cases}, \text{ as } x \to 0$$
$$\Longrightarrow 2x^{-1}S'(x) \ll (S'(x))^2, \text{ as } x \to 0$$

which confirms the consistency.

• To confirm the main underlying assumption, we note that

$$\begin{cases} S''(x) \sim \mp 2x^{-3} \\ (S'(x))^2 \sim x^{-4} \end{cases}, \text{ as } x \to 0 \Longrightarrow S''(x) \ll (S'(x))^2, \text{ as } x \to 0$$

We have thus shown that the dominant balance is consistent and the leading order term is $S(x) \sim \pm x^{-1}$.

• *Subleading contribution:* Define $S(x) = \pm x^{-1} + C(x)$ and assume that $C(x) \ll \pm x^{-1}$, as $x \to 0$. Then, we have:

$$\begin{aligned} S'(x) &= \mp x^{-2} + C'(x) \\ S''(x) &= \pm 2x^{-3} + C''(x) \\ [S'(x)]^2 &= (\mp x^{-2} + C'(x))^2 = [C'(x)]^2 \mp 2x^{-2}C'(x) + x^{-4} \end{aligned}$$

and it follows that

$$\begin{aligned} &S''(x) + [S'(x)]^2 + 2x^{-1}S(x) - x^{-4} = 0 \\ &\Longleftrightarrow [\pm 2x^{-3} + C''(x)] + [[C'(x)]^2 \mp 2x^{-2}C'(x) + x^{-4}] + 2x^{-1}(\pm x^{-1} + C(x)) - x^{-4} = 0 \\ &\Longleftrightarrow \pm 2x^{-3} + C''(x) + [C'(x)]^2 \mp 2x^{-2}C'(x) \pm 2x^{-2} + 2x^{-1}C(x) = 0 \\ &\Longleftrightarrow C''(x) + [C'(x)]^2 \mp 2x^{-2}C'(x) + 2x^{-1}C(x) = \mp 2x^{-3} \mp 2x^{-2} \end{aligned}$$

Since $x^{-2} \ll x^{-3}$, as $x \to 0$, it follows that $C(x)$ satisfies the following asymptotic equation:

$$C''(x) + [C'(x)]^2 \mp 2x^{-2}C'(x) + 2x^{-1}C(x) \sim \mp 2x^{-3}, \text{ as } x \to 0$$

This equation can be further simplified because

$$C(x) \ll \pm x^{-1}, \text{ as } x \to 0 \Longrightarrow C'(x) \ll \mp x^{-2}, \text{ as } x \to 0$$
$$\Longrightarrow [C'(x)]^2 = C'(x)C'(x) \ll x^{-2}C'(x), \text{ as } x \to 0$$

resulting in the simplification:

$$C''(x) \mp 2x^{-2}C'(x) + 2x^{-1}C(x) \sim \mp 2x^{-3}, \ \text{ as } x \to 0$$

If $C(x)$ follows a power law or logarithmic dependence on x, then we may expect both $C'(x)$ and $C''(x)$ to follow power law dependence on x, from which we would expect that $C''(x) \sim x^{-1}C'(x), \ \text{ as } x \to 0$. Consequently, we assume that $C''(x) \ll 2x^{-2}C'(x), \ \text{ as } x \to 0$. This simplifies the asymptotic equation to:

$$\mp 2x^{-2}C'(x) + 2x^{-1}C(x) \sim \mp 2x^{-3}, \ \text{ as } x \to 0 \Longleftrightarrow \mp C'(x) + xC(x) \sim \mp x^{-1}, \ \text{ as } x \to 0$$

To find the appropriate dominant balance, we distinguish between the following cases:
• *Case 1:* Assume that: $C'(x) \ll xC(x), \ \text{ as } x \to 0$. Then, we have:

$$xC(x) \sim \mp x^{-1}, \ \text{ as } x \to 0 \Longleftrightarrow C(x) \sim \mp x^{-2}, \ \text{ as } x \to 0$$

which is inconsistent because

$$\begin{cases} C'(x) \sim \pm - 2x^{-3} \\ xC(x) \sim \mp x^{-1} \end{cases}, \ \text{ as } x \to 0 \Longrightarrow xC(x) \ll C'(x), \ \text{ as } x \to 0$$

We conclude that this case is inconsistent.
• *Case 2:* Assume that: $xC(x) \ll C'(x), \ \text{ as } x \to 0$. Then, we have:

$$\mp C'(x) \sim \mp x^{-1}, \ \text{ as } x \to 0 \Longleftrightarrow C'(x) \sim x^{-1}, \ \text{ as } x \to 0 \Longleftrightarrow C(x) \sim \ln|x|, \ \text{ as } x \to 0$$

To check for consistency, we note that:

$$\lim_{x\to 0} \frac{xC(x)}{C'(x)} = \lim_{x\to 0} \frac{x\ln|x|}{x^{-1}} = \lim_{x\to 0} \frac{\ln|x|}{x^{-2}} = \lim_{x\to 0} \frac{x^{-1}}{-2x^{-3}} = \lim_{x\to 0} \frac{-x^2}{2} = 0$$
$$\Longrightarrow xC(x) \ll C'(x), \ \text{ as } x \to 0$$

therefore this case is consistent.

Furthermore, we confirm the remaining dominant balance assumptions by noting that

$$C(x) \sim \ln|x| \ll \pm x^{-1}, \ \text{ as } x \to 0$$

and

$$C(x) \sim \ln|x|, \ \text{ as } x \to 0 \Longrightarrow \begin{cases} C'(x) \sim x^{-1} \\ C''(x) \sim -x^{-2} \end{cases}, \ \text{ as } x \to 0$$
$$\Longrightarrow \begin{cases} x^{-2}C'(x) \sim x^{-3} \\ C''(x) \sim -x^{-2} \end{cases}, \ \text{ as } x \to 0$$

$$\implies C''(x) \ll x^{-2}C'(x), \quad \text{as } x \to 0$$

We have thus confirmed all assumptions needed for consistency and conclude that

$$S(x) \sim \pm 1/x + \ln|x|, \quad \text{as } x \to 0 \implies y(x) \sim \exp(1/x + \ln|x|) \sim |x|e^{\pm 1/x}, \quad \text{as } x \to 0$$

Thus, the leading order term of the solution is

$$y(x) \sim |x|e^{\pm 1/x}, \quad \text{as } x \to 0$$

□

Example 4.2. Find the leading order solutions to the equation

$$x^3 y''(x) = y(x)$$

Solution. Let $y(x) = \exp(S(x))$. Then, we have $y'(x) = S'(x)y(x)$ and $y''(x) = [S''(x) + (S'(x))^2]y(x)$ and it follows that

$$x^3 y''(x) = y \iff x^3[S''(x) + (S'(x))^2]y(x) = y(x) \iff S''(x) + (S'(x))^2 = x^{-3}$$

• Assume that $S''(x) \ll (S'(x))^2$, as $x \to 0$.Then, we have:

$$(S'(x))^2 \sim x^{-3}, \quad \text{as } x \to 0 \iff S''(x) \sim \pm x^{-3/2}, \quad \text{as } x \to 0$$
$$\iff S(x) \sim \pm \frac{x^{-1/2}}{-1/2} \sim \mp 2x^{-1/2}, \quad \text{as } x \to 0$$

• To check for consistency we note that

$$S''(x) = (S'(x))' \sim (\pm x^{-3/2})' = \mp(3/2)x^{-5/2} \ll x^{-3} \sim (S'(x))^2, \quad \text{as } x \to 0$$
$$\implies S''(x) \ll (S'(x))^2, \quad \text{as } x \to 0$$

We conclude that the dominant balance is consistent.
• *Subleading solution:* Now consider the subleading contribution

$$S(x) = \mp 2x^{-1/2} + C(x) \text{ with } C(x) \ll x^{-1/2}, \quad \text{as } x \to 0$$

Then, we have:

$$S'(x) = \mp 2(-1/2)x^{-3/2} + C'(x) = \pm x^{-3/2} + C'(x)$$
$$S''(x) = \mp(3/2)x^{-5/2} + C''(x)$$
$$(S'(x))^2 = [C'(x) \pm x^{-3/2}]^2 = (C'(x))^2 \pm 2x^{-3/2}C'(x) + x^{-3}$$

and it follows that

$$\begin{aligned}&S''(x)+(S'(x))^2=x^{-3}\iff\\&\qquad\iff\mp(3/2)x^{-5/2}+C''(x)+(C'(x))^2\pm 2x^{-3/2}C'(x)+x^{-3}=x^{-3}\\&\qquad\iff C''(x)+(C'(x))^2\pm 2x^{-3/2}C'(x)=\pm(3/2)x^{-5/2}\end{aligned}$$

• To construct a dominant balance we note that:

$$\begin{aligned}&C(x)\ll x^{-1/2},\ \text{ as } x\to 0\implies C'(x)\ll x^{-3/2},\ \text{ as } x\to 0\\&\quad\implies (C'(x))^2\ll x^{-3/2}C'(x),\ \text{ as } x\to 0\end{aligned}$$

and we assume that $C''(x)\ll 2x^{-3/2}C'(x),\ \text{ as } x\to 0$. Then, we have the following dominant balance

$$\begin{aligned}&\pm 2x^{-3/2}C'(x)\sim\pm(3/2)x^{-5/2},\ \text{ as } x\to 0\iff C'(x)\sim(3/4)x^{-1},\ \text{ as } x\to 0\\&\qquad\iff C(x)\sim(3/4)\ln|x|,\ \text{ as } x\to 0\end{aligned}$$

• To check for consistency we note that

$$\begin{aligned}&2x^{-3/2}C'(x)\sim 2x^{-3/2}[(3/4)x^{-1}\sim(3/2)x^{-5/2}\gg-(3/4)x^{-2}\sim C''(x),\ \text{ as } x\to 0\\&\qquad\implies C''(x)\ll 2x^{-3/2}C'(x),\ \text{ as } x\to 0\end{aligned}$$

We conclude that the dominant balance is consistent and therefore:

$$S(x)\sim\pm 2x^{-1/2}+(3/4)\ln|x|,\ \text{ as } x\to 0\implies y(x)\sim|x|^{3/4}\exp(\mp 2/\sqrt{x}),\ \text{ as } x\to 0$$

□

5 General n^{th}-order Schrodinger equation

The previous example is a special case of the more general n^{th}-order Schrodinger equation problem, for which a very general solution can be established for an asymptotic solution around an irregular singular point.

Theorem 5.1. *Consider the equation $y^{(n)}(x)=Q(x)y(x)$ and assume that*

$$Q'(x)\ll Q(x)^{1/n+1},\ \text{ as } x\to\sigma$$

with $\sigma=0^{\pm}$ or $\sigma=\pm\infty$. Then, the leading solutions of the equation are:

$$\begin{cases}y(x)\sim|Q(x)|^{\mu}\exp\left(\omega\int Q^{1/n}(x)\,\mathrm{d}x\right),\ \text{ as } x\to\sigma\\ \mu=\dfrac{1-n}{n^2}\end{cases}$$

with ω one of the n^{th} the roots of unity, such that $\omega^n=1$.

Remark 1. The condition $Q'(x) \ll Q(x)^{1/n+1}$, as $x \to \sigma$ follows from assuming that there is an irregular singular point at $x \to \sigma$. For example, consider the case $Q(x) = x^a$ and define Δ such that $Q'(x)/Q^{1/n+1}(x) \sim ax^{\Delta}$, as $x \to \sigma$. Then, we have:

$$\Delta = (a-1) - a(1/n+1) = a - 1 - a/n - a = -1 - a/n = -(a+n)/n$$

and therefore

$$Q'(x) \ll Q^{1/n+1}(x), \text{ as } x \to 0^+ \Longleftrightarrow \Delta > 0 \Longleftrightarrow a + n < 0 \Longleftrightarrow a < -n$$

We have thus shown that this condition is indeed equivalent to the necessary and sufficient condition for the equation $y^{(n)}(x) - x^a y(x) = 0$ having an irregular singular point at $x = 0$.

Proof. Define $S(x)$ such that $y(x) = \exp(S(x))$. To find the leading term, we assume that $S''(x) \ll (S'(x))^2$, as $x \to \sigma$. We shall use proof by induction to show that it follows that:

$$\forall n \in \mathbb{N} - \{0, 1\} : y^{(n)}(x) \sim [S'(x)]^n y(x)$$

• For $n = 2$, we have:

$$y'(x) = (\exp(S(x)))' = S'(x)\exp(S(x)) = S'(x)y(x)$$

and therefore,

$$\begin{aligned}
y''(x) &= (S'(x)y(x))' = S''(x)y(x) + S'(x)y'(x) \\
&= S''(x)y(x) + S'(x)[S'(x)y(x)] \\
&= [S''(x) + (S'(x))^2]y(x) \\
&\sim (S'(x))^2 y(x), \text{ as } x \to \sigma && [\text{ via } S''(x) \ll (S'(x))^3]
\end{aligned}$$

• For $n = k$, we assume that $y^{(k)}(x) \sim (S'(x))^k y(x)$, as $x \to \sigma$.
• For $n = k + 1$, we will show that $y^{(k+1}(x) \sim (S'(x))^{k+1} y(x)$, as $x \to \sigma$. from the induction hypothesis, we have:

$$\begin{aligned}
y^{(k+1)}(x) &\sim [(S'(x))^k y(x)]' \\
&\sim [(S'(x))^k]' y(x) + (S'(x))^k y'(x) \\
&\sim k(S'(x))^{k-1} S''(x)y(x) + (S'(x))^k [S'(x)y(x)] \\
&\sim [k(S'(x))^{k-1} S''(x) + (S'(x))^{k+1}] \\
&\sim (S'(x))^{k+1} y(x)[k(S''(x)/(S'(x))^2) + 1] \\
&\sim (S'(x))^{k+1} y(x), \text{ as } x \to \sigma && [\text{ via } S''(x) \ll (S'(x))^2]
\end{aligned}$$

We have thus shown, by induction, that $y^{(n)}(x) \sim (S'(x))^n y(x), \ \text{ as } x \to \sigma$, for all $n \in \mathbb{N}$ with $n \geq 2$. From the resulting dominant balance on the governing equation $y^{(n)}(x) = Q(x)y(x)$, we obtain:

$$\begin{aligned}
(S'(x))^n y(x) \sim Q(x)y(x), \ \text{ as } x \to \sigma &\Longleftrightarrow (S'(x))^n \sim Q(x), \ \text{ as } x \to \sigma \\
&\Longleftrightarrow S'(x) \sim \omega Q^{1/n}(x), \ \text{ as } x \to \sigma \\
&\Longleftrightarrow S(x) \sim \omega \int Q^{1/n}(x)\, \mathrm{d}x, \ \text{ as } x \to \sigma
\end{aligned}$$

To confirm the dominant balance assumption $S''(x) \ll (S'(x))^2, \ \text{ as } x \to \sigma$, we argue as follows:

$$\begin{aligned}
S''(x) &\sim \omega(Q^{1/n}(x))' \sim \omega Q^{1/n-1}(x)Q'(x) \\
&\ll \omega^2 Q^{1/n-1}(x)Q^{1/n+1}(x) && [\text{ via } Q'(x) \ll Q^{1/n+1}(x)] \\
&\sim \omega^2 Q^{2/n}(x) \sim [\omega Q^{1/n}(x)]^2 \sim (S'(x))^2, \ \text{ as } x \to \sigma \\
&\Longrightarrow S''(x) \ll (S'(x))^2, \ \text{ as } x \to \sigma
\end{aligned}$$

• To obtain the subleading contribution, we write

$$S(x) = C(x) + \omega \int Q^{1/n}(x)\, \mathrm{d}x$$

assuming that

$$C(x) \ll \omega \int Q^{1/n}(x)\, \mathrm{d}x, \ \text{ as } x \to \sigma$$

and we derive a dominant balance asymptotic equation for $C(x)$. Instead of working with the exact governing equation for $C(x)$, which is very cumbersome to write explicitly, we can begin with an asymptotic equation for $C(x)$, as long as enough terms are included for canceling out the leading contributions and capturing the leading behavior of $C(x)$. We use as a starting point

$$y^{(n-1)}(x) \sim (S'(x))^{n-1} y(x), \ \text{ as } x \to \sigma$$

and write

$$\begin{aligned}
y^{(n)}(x) &\sim [(S'(x))^{n-1}]' y(x) + (S'(x))^{n-1} S'(x) y(x) \\
&\sim [(n-1)(S'(x))^{n-2} S''(x) + (S'(x))^n] y(x), \ \text{ as } x \to \sigma
\end{aligned}$$

From the governing equation, we have:

$$y^{(n)}(x) = Q(x)y(x) \Longleftrightarrow (n-1)(S'(x))^{n-2} S''(x) \sim Q(x) - (S'(x))^n, \ \text{ as } x \to \sigma$$

For the LHS, we have:

$$\begin{aligned}
S(x) &= C(x) + \omega \int Q^{1/n}(x)\,\mathrm{d}x \sim \omega \int Q^{1/n}(x)\,\mathrm{d}x, \quad \text{as } x \to \sigma \\
&\implies S'(x) \sim \omega Q^{1/n}(x), \quad \text{as } x \to \sigma \\
&\implies S''(x) \sim \omega(1/n) Q^{1/n-1}(x) Q'(x) \sim (1/n)[\omega Q^{1/n}(x)][Q'(x)/Q(x)], \quad \text{as } x \to \sigma \\
&\implies (n-1)(S'(x))^{n-2} S''(x) \sim (n-1)[\omega Q^{1/n}(x)]^{n-2}(1/n)[\omega Q^{1/n}(x)][Q'(x)/Q(x)] \\
&\qquad\qquad\qquad\qquad \sim \frac{n-1}{n}[\omega Q^{1/n}(x)]^{n-1}[\ln|Q(x)|]', \quad \text{as } x \to \sigma
\end{aligned}$$

Likewise, for the RHS, we have:

$$\begin{aligned}
Q(x) - [S'(x)]^n &= Q(x) - [C(x) + \omega Q^{1/n}(x)]^n \\
&= Q(x) - [Q(x) + nC'(x)(\omega Q^{1/n}(x))^{n-1} + O((C'(x))^2)] \\
&\sim -nC'(x)[\omega Q^{1/n}(x)]^{n-1}, \quad \text{as } x \to \sigma
\end{aligned}$$

The dominant balance between the LHS and RHS gives:

$$\begin{aligned}
&(n-1)(S'(x))^{n-2} S''(x) \sim Q(x) - [S'(x)]^n, \quad \text{as } x \to \sigma \\
&\quad\iff \frac{n-1}{n}[\omega Q^{1/n}(x)]^{n-1}[\ln|Q(x)|]' \sim -nC'(x)[\omega Q^{1/n}(x)]^{n-1}, \quad \text{as } x \to \sigma \\
&\quad\iff C'(x) \sim \frac{1-n}{n^2}[\ln|Q(x)|]', \quad \text{as } x \to \sigma \\
&\quad\iff C(x) \sim \frac{1-n}{n^2}\ln|Q(x)|, \quad \text{as } x \to \sigma
\end{aligned}$$

To confirm that $C(x)$ is a subleading contribution, we use the hypothesis $Q'(x) \ll Q^{1/n+1}(x)$ to argue that:

$$\begin{aligned}
C'(x) &\sim \frac{1-n}{n^2}[\ln|Q(x)|]' \sim \frac{1-n}{n^2}\frac{Q'(x)}{Q(x)} \ll \frac{Q^{1/n+1}(x)}{Q(x)} \sim Q^{1/n}(x), \quad \text{as } x \to \sigma \\
&\implies C'(x) \ll \omega Q^{1/n}(x), \quad \text{as } x \to \sigma \\
&\implies C(x) \ll \omega \int Q^{1/n}(x)\,\mathrm{d}x, \quad \text{as } x \to \sigma
\end{aligned}$$

We conclude that

$$\begin{aligned}
y(x) = \exp(S(x)) &\sim \exp\left(\frac{1-n}{n^2}\ln|Q(x)| + \omega \int Q^{1/n}(x)\,\mathrm{d}x\right) \\
&\sim |Q(x)|^{(1-n)/n^2} \exp\left(\omega \int Q^{1/n}(x)\,\mathrm{d}x\right), \quad \text{as } x \to \sigma
\end{aligned}$$

This concludes the argument. □

Example 5.2. Find the leading solution to the equation $y''(x) = y/x^5$ as $x \to 0^+$.

Solution. For $n = 2$ and $Q(x) = x^5$, we confirm that

$$\frac{Q'(x)}{Q^{1/n+1}(x)} = \frac{(x^{-5})'}{(x^{-5})^{1/2+1}} = \frac{-5x^{-6}}{(x^{-5})^{3/2}} = \frac{-5x^{-12/2}}{x^{-15/2}} = -5x^{3/2}$$
$$\implies \lim_{x\to 0^+} \frac{Q'(x)}{Q^{1/n+1}(x)} = \lim_{x\to 0^+}(-5x^{3/2}) = 0 \implies Q'(x) \ll Q(x)^{1/n+1}, \quad \text{as } x \to 0^+$$

so the required assumption is satisfied. Since,

$$\int Q^{1/n}(x)\,\mathrm{d}x = \int (x^{-5})^{1/2}\,\mathrm{d}x = \int x^{-5/2}\,\mathrm{d}x = \frac{x^{-3/2}}{-3/2} = \frac{-2}{3x\sqrt{x}}$$

and

$$\frac{1-n}{n^2} = \frac{1-2}{2^2} = \frac{-1}{4}$$

it follows that the leading term of the asymptotic solution is given by

$$y(x) \sim c|Q(x)|^{(1-n)/n^2} \exp\left(\omega \int Q^{1/n}(x)\,\mathrm{d}x\right)$$
$$\sim c|x^{-5}|^{-1/4} \exp\left(\frac{-2\omega}{3x\sqrt{x}}\right) \sim cx^{5/4} \exp\left(\frac{-2\omega}{3x\sqrt{x}}\right), \quad \text{as } x \to 0^+$$

□

Example 5.3. Find the leading order solution of the Airy equation $y''(x) = xy(x)$ as $x \to +\infty$.

Solution. This is a special case of $y^{(n)}(x) = Q(x)y(x)$ with $n = 2$ and $Q(x) = x$. We confirm that

$$\frac{Q'(x)}{Q^{1/n+1}(x)} = \frac{(x)'}{(x)^{1/2+1}} = \frac{1}{x^{3/2}} \implies \lim_{x\to+\infty} \frac{Q'(x)}{Q^{1/n+1}(x)} = \lim_{x\to+\infty} \frac{1}{x^{3/2}} = 0$$
$$\implies Q'(x) \ll Q^{1/n+1}(x), \quad \text{as } x \to +\infty$$

It follows that since

$$\int Q^{1/n}(x)\,\mathrm{d}x = \int x^{1/2}\,\mathrm{d}x \sim \frac{x^{3/2}}{3/2} \sim (2/3)x^{3/2}, \quad \text{as } x \to +\infty$$
$$\frac{1-n}{n^2} = \frac{(1-2)}{2^2} = \frac{-1}{4}$$

It follows that the leading order term of the solution is given by

$$\begin{aligned} y(x) &\sim CQ^{(1-n)/n^2}(x)\exp\left(\omega\int Q^{1/n}(x)\,\mathrm{d}x\right) \\ &\sim Cx^{-1/4}\exp(\omega(2/3)x^{3/2}) \\ &\sim Cx^{-1/4}\exp\left(\pm\frac{2x^{3/2}}{3}\right), \quad \text{as } x\to+\infty \end{aligned}$$

□

GODE 06: Introduction to autonomous dynamical systems

INTRODUCTION

Autonomous dynamical systems

- An autonomous dynamical system is a system of n differential equations of the form:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3, \dots, x_n) \\ \dot{x}_2 = f_2(x_1, x_2, x_3, \dots, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_1, x_2, x_3, \dots, x_n) \end{cases}$$

▶ notation: $\dot{x}_K = dx_K/dt = x_K'(t)$.

- The system can be also rewritten as:

$$\dot{x} = f(x)$$

with $x : \mathbb{R} \longrightarrow \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

- We assume that an initial value condition is given at $t=0$: $x(0) = x_0$, with $x_0 \in \mathbb{R}^n$.

- Classification of autonomous systems

a) Linear Autonomous systems: These are systems where $f(x) = Ax$ with $A \in GL(n, \mathbb{R})$. Note that

$GL(n,\mathbb{R})$ is the set of all nonsingular $n\times n$ matrices.

b) Nonlinear autonomous systems: These are systems where $f(x)$ is nonlinear.

- Jacobian matrix

The Jacobian matrix of the autonomous system $\dot{x} = f(x)$ is defined as

$$\boxed{[Df]_{ab} = \frac{\partial f_a}{\partial x_b}}$$

Note that for a linear system with $f(x) = Ax$ we have $Df = A$.

- Systems reducible to autonomous

a) High-order ODEs $\longrightarrow$ $\boxed{y^{(n)} = F(y, y', y'', \dots, y^{(n-1)})}$

We let: $x_1 = y,\ x_2 = y',\ x_3 = y'', \dots, x_n = y^{(n-1)}$.

EXAMPLE

$\ddot{x} - b\dot{x} + kx = 0$ (linear oscillator)

Let $x_1 = x$ and $x_2 = \dot{x} \longrightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = bx_2 - kx_1 \end{cases}$

b) Time-dependent system

A time-dependent system of the form

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, x_2, \ldots, x_n) \\ \dot{x}_2 = f_2(t, x_1, x_2, \ldots, x_n) \\ \vdots \\ \dot{x}_n = f_n(t, x_1, x_2, \ldots, x_n) \end{cases}$$

can be rewritten as an autonomous system by setting $x_0 = t$. Then:

$$\begin{cases} \dot{x}_0 = 1 \\ \dot{x}_1 = f_1(x_0, x_1, x_2, \ldots, x_n) \\ \dot{x}_2 = f_2(x_0, x_1, x_2, \ldots, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_0, x_1, x_2, \ldots, x_n) \end{cases}$$

Existence and Uniqueness

Consider the problem

$$\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases}$$

with $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, and $f: \mathbb{R}^n \to \mathbb{R}^n$

We also define

$$\|x\| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$$

with $x = (x_1, x_2, \dots, x_n)$ a vector in $\mathbb{R}^n$.

- <u>Definition</u>:

f Lipschitz continuous in $A \Leftrightarrow$

$\Leftrightarrow \exists L > 0 : \forall x, y \in A : \|f(x) - f(y)\| \leq L\|x - y\|$

with L = Lipschitz constant of f.

If $L < 1 \Rightarrow f$ is a <u>contraction</u>

- <u>Proposition</u>: Let $f: \mathbb{R}^n \to \mathbb{R}^m$. Then

a) $\left.\begin{array}{l} f \text{ differentiable in } A \\ \nabla f \text{ bounded in } A \end{array}\right\} \Rightarrow f$ Lipschitz continuous in A.

b) $\left.\begin{array}{l} f \text{ differentiable in } A \\ f \text{ Lipschitz continuous in } A \end{array}\right\} \Rightarrow \nabla f$ bounded in A

c) $\left.\begin{array}{l} f \text{ differentiable in } A \\ \nabla f \text{ not bounded in } A \end{array}\right\} \Rightarrow f$ not Lipschitz continuous in A.

↳ Note that (c) is the contrapositive of (b).

- <u>Theorem</u> : Assume that

  a) f Lipschitz continuous in $B(x_0,\delta)$ with Lipschitz constant L, where $\delta > 0$ and $B(x_0,\delta) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < \delta\}$

  b) $\forall x \in B(x_0,\delta) : \|f(x)\| \leq M$

  Then $\dot{x} = f(x)$ with $x(t_0) = x_0$ has a unique solution for $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ as long as $0 < \varepsilon < \min(1/L, \delta/M)$

- <u>Theorem</u> : Assume that

  a) f Lipschitz continuous in $B(x_0,\delta)$ with Lipschitz constant L

  b) y solution for $x(t_0) = y_0$ and z solution for $x(t_0) = z_0$ defined for $t \in [t_0, t_1]$

  Then :

$$\boxed{\forall t \in [t_0, t_1] : \|y(t) - z(t)\| \leq \|y_0 - z_0\| e^{L(t - t_0)}}$$

↳ The divergence between two solutions with nearby initial conditions do not grow apart at a faster than exponential rate.

↳ Note that the existence of the unique solution is not guaranteed for infinite time.

EXAMPLES

a) Show that $f(x) = 2x+3$, $\forall x \in \mathbb{R}$ is Lipschitz continuous in $\mathbb{R}$.

Solution

1st method: By definition.

Let $x, y \in \mathbb{R}$ be given. Then

$$|f(x)-f(y)| = |(2x+3)-(2y+3)| = |2x+3-2y-3| =$$
$$= |2x-2y| = |2(x-y)| = |2||x-y|$$
$$= 2|x-y| \Rightarrow |f(x)-f(y)| \leq 2|x-y|.$$

It follows that

$\forall x, y \in \mathbb{R}: |f(x)-f(y)| \leq 2|x-y| \Rightarrow$

$\Rightarrow f$ Lipschitz continuous in $\mathbb{R}$.

2nd method: By proposition

f differentiable in $\mathbb{R}$ with $f'(x) = (2x+3)' = 2$, $\forall x \in \mathbb{R}$ (1)

Since:

$\forall x \in \mathbb{R} : (|f'(x)| = |2| = 2) \Rightarrow \forall x \in \mathbb{R} : (|f'(x)| \leq 2) \Rightarrow$

$\Rightarrow f'$ bounded in $\mathbb{R}$ (2)

From (1) and (2) it follows that f is Lipschitz continuous in $\mathbb{R}$.

b) Show that $f(x) = x^{2/3}$, $\forall x \in (0, +\infty)$ is not Lipschitz continuous on $(0, +\infty)$

Solution

f differentiable in $(0, +\infty)$ with

$$f'(x) = (x^{2/3})' = (2/3)x^{2/3-1} = (2/3)x^{-1/3} =$$

$$= \frac{2}{3\sqrt[3]{x}}, \quad \forall x \in (0, +\infty) \qquad (1)$$

However, since:

$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \frac{2}{3\sqrt[3]{x}} = +\infty \Rightarrow$$

$\Rightarrow f'$ not bounded in $(0, +\infty)$ (2)

From (1) and (2), it follows that f not Lipschitz continuous in $(0, +\infty)$.

↳ Examples on existence and uniqueness

c) $\dot{x} = 1 + x^2$

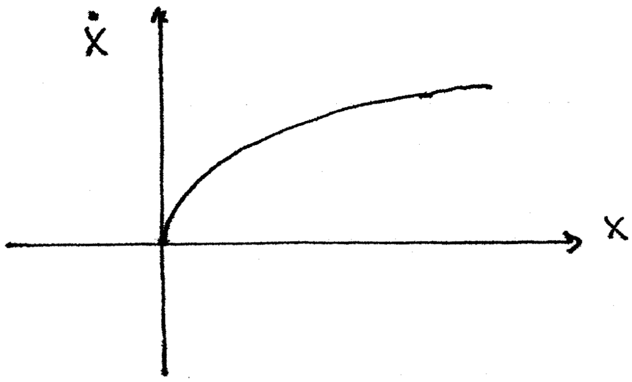
▶ We can show, using standard ODE techniques, that

$$\dot{x} = 1 + x^2 \Leftrightarrow x(t) = \tan(t + c)$$

with c dependant on the initial condition. Obviously, a solution <u>exists</u>. Furthermore, the "if and only if" ($\Leftrightarrow$) ensures the <u>uniqueness</u> of this solution. However, we note that the solution has a singularity occuring at finite time when $t + c = k\pi + \pi/2$, consequently the existence and uniqueness holds for finite time only.

d) $\begin{cases} \dot{x} = 3x^{2/3} \\ x(0) = 0 \end{cases}$

has two solutions: $x(t) = 0$ and $x(t) = t^3$ (!!)
thus we have existence but not uniqueness.
Note that $f(x) = 3x^{2/3}$ is continuous but not Lipschitz continuous.



The solution $x(t) = 0$ is VERY unstable because the slope of the function $f(x) = 3x^{2/3}$ is infinite at $x = 0$.
This instability manifests in the existence of a second solution, such as $x(t) = t^3$ (there is in fact an infinite set of such solutions). From a physical standpoint, $x(t) = 0$ is the solution one would expect if we initialize with $x(0) = 0$.
The lack of uniqueness indicates that the system could spontaneously break into the second solution $x(t) = t^3$ at time $t = 0$ if there is even an infinitesimal deviation in the initial condition, thus giving a 3rd solution:

$$x(t) = \begin{cases} t^3 & , \text{ if } t \geq 0 \\ 0 & , \text{ if } t < 0 \end{cases}$$

that combines the previous two solutions. Note that this spontaneous break can just as well occur at any other time t_0. (see homework).

Fixed points and stability

- Consider the autonomous system $\dot{x} = f(x)$, with $f : \mathbb{R}^n \to \mathbb{R}^n$. We say that
  x_0 is a fixed point $\iff f(x_0) = 0$

- If x_0 is a fixed point, then $\dot{x} = f(x)$ with initial condition $x(t_0) = x_0$ has solution $x(t) = x_0$. Thus if we start at a fixed point, we will stay at the fixed point. The question of stability concerns what happens when we start with an initial condition near a fixed point.

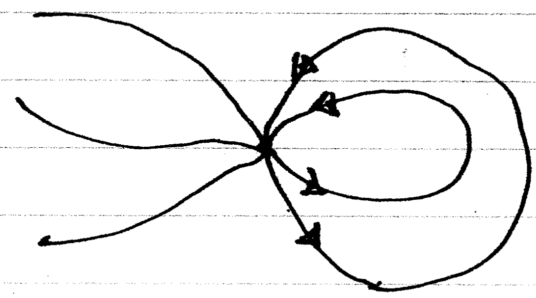
Stability of fixed points

Let $x_0 \in \mathbb{R}^n$ be a fixed point of $\dot{x} = f(x)$.

1. x_0 Lyapunov stable $\iff$
$$\forall \varepsilon > 0 : \exists \delta > 0 : (\|x(t_0) - x_0\| < \delta \Rightarrow$$
$$\Rightarrow (\forall t > t_0 : \|x(t) - x_0\| < \varepsilon))$$
2. x_0 attracting $\iff$
$$\exists \delta > 0 : (\|x(t_0) - x_0\| < \delta \Rightarrow \lim_{t \to +\infty} x(t) = x_0)$$

In a Lyapunov stable fixed point, solutions that start near the fixed point will stay near the fixed point. In an attracting fixed point, solutions that start near the fixed point will eventually converge into the fixed point.

Note that it is possible for a fixed point to be attractive without being Lyapunov stable, as in the following example:



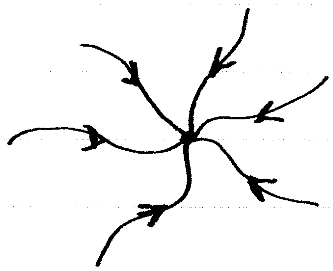
This occurs when there are trajectories that start near the fixed point, then wander far away from the fixed point before returning back to the fixed point for a final approach. This remark motivates the following additional definitions:

(3) x_0 asymptotically stable $\Leftrightarrow \begin{cases} x_0 \text{ Lyapunov stable} \\ x_0 \text{ attracting.} \end{cases}$

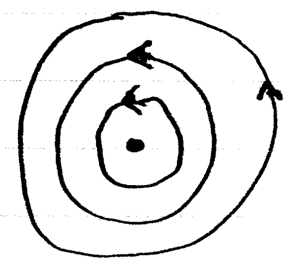
(4) x_0 neutrally stable $\Leftrightarrow \begin{cases} x_0 \text{ Lyapunov stable} \\ x_0 \text{ not attracting} \end{cases}$

(5) x_0 unstable $\Leftrightarrow \begin{cases} x_0 \text{ not Lyapunov stable} \\ x_0 \text{ not attracting.} \end{cases}$

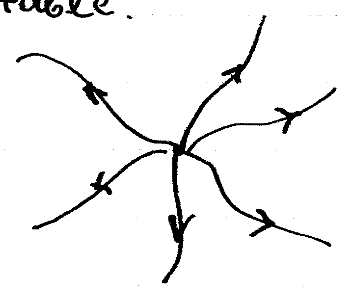
Examples of definitions



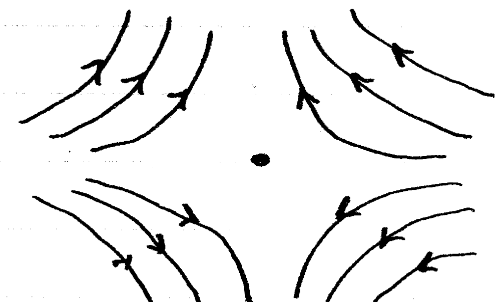
asymptotically stable.



neutrally stable.



unstable (source)



unstable (saddle point)

The distinction between sources and saddle points will be explained later.

(6) x_0 is <u>exponentially stable</u> if and only if

a) x_0 is asymptotically stable AND

b) $\exists\, a, b, \delta \in (0, +\infty) : \big(\|x(t_0) - x_0\| < \delta \Rightarrow$

$\Rightarrow \big(\forall t > t_0 : \|x(t) - x_0\| \leqslant a e^{-bt} \|x(t_0) - x_0\| \big)\big)$

▼ Lyapunov functions

Let $\dot{x} = f(x)$ be an autonomous system with $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and let x_0 be a fixed point such that $f(x_0) = 0$.

Def: We say that a function $V: A \longrightarrow \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ an open set is a Lyapunov function if it satisfies the following conditions:

a) $V(x_0) = 0$

b) $V(x) > 0, \ \forall x \in A - \{x_0\}$

c) V continuous in A

d) $x(t_0) \in A \Rightarrow \forall t > t_0 : V(x(t)) \leq V(x(t_0))$

- The domain A of the Lyapunov function is called a trapping region of the autonomous system.

Thm: (1st Lyapunov Theorem)

If

a) $f(x_0) = 0$

b) There is a Lyapunov function $V: A \to \mathbb{R}$ with $V(x_0) = 0$

Then $x = x_0$ is Lyapunov stable.

Thm: (2nd Lyapunov Theorem)

If:

a) $f(x_0) = 0$ with $x_0 \in A$.

b) There is a Lyapunov function $V : A \to \mathbb{R}$ with $V(x_0) = 0$

* c) $x(0) \in A \Rightarrow \forall t > t_0 : V(x(t)) < V(x(t_0))$ (!)

Then $x = x_0$ is asymptotically stable.

GODE 07: 1D autonomous dynamical systems

1D AUTONOMOUS SYSTEMS

▼ Stability analysis for 1d systems

• We recall from analysis strong differentiability:

<u>Def</u>: Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$. We say that f is strongly differentiable at $x_0 \in A$ if and only if there is a function $g: A \to \mathbb{R}$ such that

$$\begin{cases} \forall x \in A: f(x) = f(x_0) + (x - x_0) f'(x_0) + |x| g(x) \\ \lim\limits_{x \to x_0} g(x) = 0 \end{cases}$$

<u>Prop</u>: Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and let $x_0 \in A$.

$$\left.\begin{array}{l} f \text{ differentiable at } x_0 \\ f' \text{ continuous at } x_0 \end{array}\right\} \Rightarrow f \text{ strongly differentiable at } x_0$$

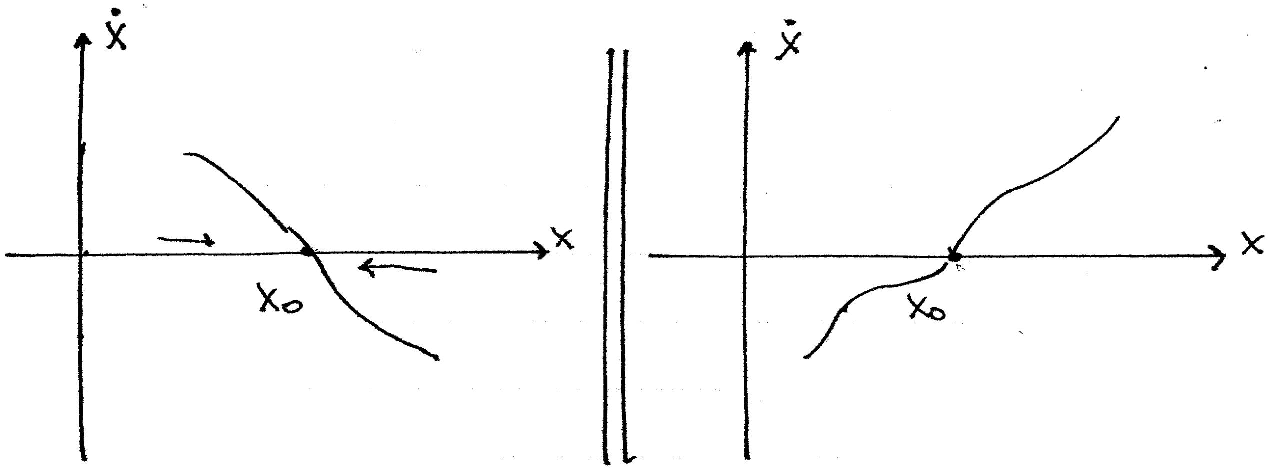
• The stability of 1d autonomous dynamical systems is determined via the following theorem.

<u>Thm</u>: Consider the system $\dot{x} = f(x)$ with $f: \mathbb{R} \to \mathbb{R}$ a function which is strongly differentiable on $\mathbb{R}$. Let $x_0 \in \mathbb{R}$ be a fixed point with $f(x_0) = 0$. Then:

a) $f'(x_0) < 0 \Rightarrow x_0$ asymptotically stable

b) $f'(x_0) > 0 \Rightarrow x_0$ unstable.

↳ The theorem is interpreted according to the following phase portraits:



x_0 asymptotically stable $\qquad\qquad$ x_0 unstable

<u>Proof</u>

Define $y(t) = x(t) - x_0 \Rightarrow x(t) = y(t) + x_0 \Rightarrow$

$$\Rightarrow \dot{y} = \dot{x} = f(x) = f(x_0 + y) = f(x_0) + yf'(x_0) + |y|g(y) =$$
$$= yf'(x_0) + |y|g(y)$$

with $\lim_{y \to 0} g(y) = 0$, since f is strongly differentiable in x_0. It follows that

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in (-\delta, 0) \cup (0, \delta) : |g(y)| < \varepsilon$$

Let $\varepsilon = (1/2)|f'(x_0)|$ and let $\delta > 0$ be the corresponding δ such that $\forall y \in (-\delta, 0) \cup (0, \delta) : |g(y)| < \varepsilon$. We see that:

$$|\dot{y} - yf'(x_0)| = ||y|g(y)| = |y||g(y)| < |y|\varepsilon = |y|(1/2)|f'(x_0)|$$
$$= (1/2)|yf'(x_0)| \Rightarrow$$
$$\Rightarrow |\dot{y} - yf'(x_0)| < (1/2)|yf'(x_0)| \Rightarrow$$
$$\Rightarrow -(1/2)|yf'(x_0)| < \dot{y} - yf'(x_0) < (1/2)|yf'(x_0)| \Rightarrow$$
$$\Rightarrow yf'(x_0) - (1/2)|yf'(x_0)| < \dot{y} < yf'(x_0) + (1/2)|yf'(x_0)|$$

First, we note that:

a) If $yf'(x_0) > 0 \Rightarrow \dot{y} > yf'(x_0) - (1/2)\,|yf'(x_0)| =$
$= yf'(x_0) - (1/2)\,yf'(x_0) =$
$= (1/2)\,yf'(x_0) \Rightarrow \dot{y} > (1/2)\,yf'(x_0)$

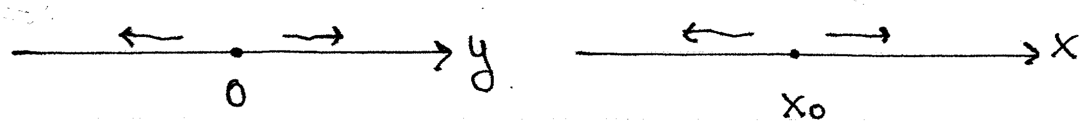
b) If $yf'(x_0) < 0 \Rightarrow$
$\Rightarrow \dot{y} < yf'(x_0) + (1/2)\,|yf'(x_0)| = yf'(x_0) - (1/2)\,yf'(x_0)$
$= (1/2)\,yf'(x_0) \Rightarrow \dot{y} < (1/2)\,yf'(x_0)$.

We have thus shown that for $y \in (-\delta, 0) \cup (0, \delta)$:
$yf'(x_0) > 0 \Rightarrow \dot{y} > (1/2)\,yf'(x_0)$
$yf'(x_0) < 0 \Rightarrow \dot{y} < (1/2)\,yf'(x_0)$

We now distinguish between the following cases:

Case 1: Assume that $f'(x_0) > 0$. Then for
$y \in (0, \delta) \Rightarrow yf'(x_0) > 0 \Rightarrow \dot{y} > (1/2)\,yf'(x_0) > 0 \Rightarrow$
$\Rightarrow y(t)$ increasing.
$y \in (-\delta, 0) \Rightarrow yf'(x_0) < 0 \Rightarrow \dot{y} < (1/2)\,yf'(x_0) < 0 \Rightarrow$
$\Rightarrow y(t)$ decreasing.

It follows that the fixed point x_0 is unstable:



Case 2: Assume that $f'(x_0) < 0$. Then for
$y \in (0, \delta) \Rightarrow yf'(x_0) < 0 \Rightarrow \dot{y} < (1/2)\,yf'(x_0) < 0 \Rightarrow$
$\Rightarrow y(t)$ decreasing $\Rightarrow$ Lyapunov stability.

Since $y = 0$ is a fixed point, it follows that if we initialize at $\cancel{y(0) \in (-\delta, 0)}$ $y(0) \in (0, \delta)$, then $y(t) \geq 0$ and furthermore $y(0)\exp((1/2)f'(x_0)t) \geq y(t) \geq 0 \Rightarrow$
$\Rightarrow \lim_{t \to +\infty} y(t) = 0 \Rightarrow$ fixed point is attracting

Likewise, for

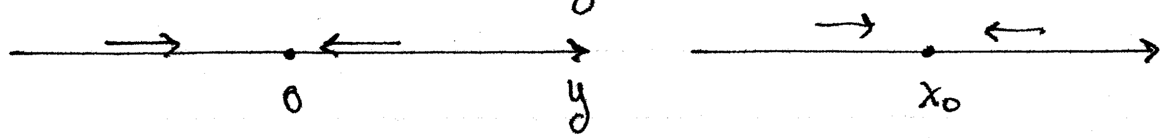
$y \in (-\delta, 0) \Rightarrow y f'(x_0) > 0 \Rightarrow \dot{y} > (1/2) y f'(x_0) > 0 \Rightarrow$

$\Rightarrow y(t)$ increasing $\Rightarrow$ Lyapunov stability

and similarly we can show that

$y(0) \exp\left((1/2) f'(x_0) t \right) \leq y(t) \leq 0 \Rightarrow \lim_{t \to +\infty} y(t) = 0 \Rightarrow$

$\Rightarrow$ fixed point is attracting.



In both cases, initializing at $y(0) \in (-\delta, 0) \cup (0, \delta)$ yields both Lyapunov stability and the attracting property, therefore the fixed point x_0 is asymptotically stable.

EXAMPLES

a) $\begin{cases} \dot{x} = ax \\ x(0) = x_0 \end{cases}$ with $a>0$ (Exponential growth model)

- Exact solution $x(t) = x_0 \exp(at)$
- Fixed points.

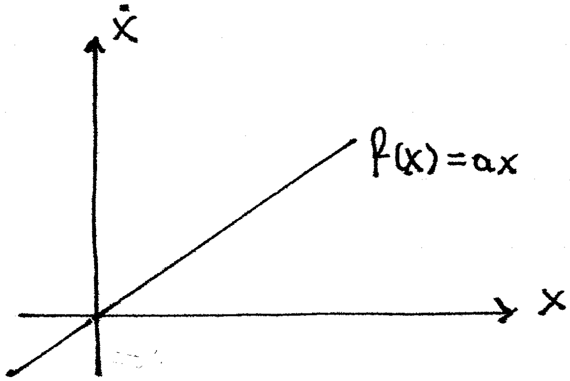
Let $f(x) = ax$. Then

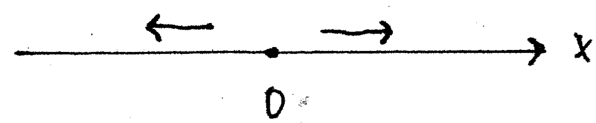
x fixed point $\Leftrightarrow f(x)=0 \Leftrightarrow ax=0 \Leftrightarrow \underline{x=0}$

- Stability.

$f'(x) = (ax)' = a$

At $x=0$: $f'(0) = a>0 \Rightarrow x=0$ is an unstable fixed point.





b) $\begin{cases} \dot{x} = (a/b)x(b-x) \\ x(0) = x_0 \end{cases}$ with $a>0$ and $b>0$ (Logistic Model)

Here a = growth rate
b = carrying capacity.

- Fixed points.

Let $f(x) = (a/b)\,x\,(b-x)$.

x fixed point $\Leftrightarrow f(x)=0 \Leftrightarrow (a/b)x(b-x)=0 \Leftrightarrow$
$\Leftrightarrow x(b-x)=0 \Leftrightarrow x=0 \vee x=b$

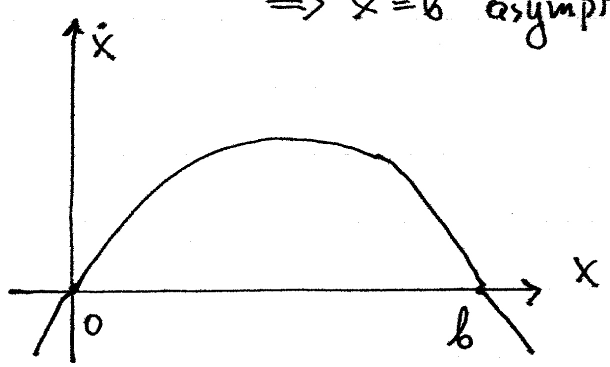
▶ Stability:

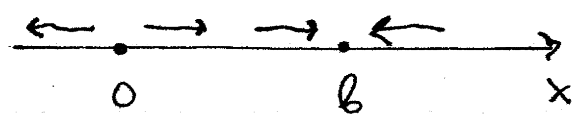
$$f'(x) = \frac{a}{b}\frac{d}{dx}x(b-x) = \frac{a}{b}\frac{d}{dx}(bx-x^2) =$$

$$= \frac{a}{b}(b-2x) = a - \frac{2ax}{b}$$

For $x=0$: $f'(0)=a>0 \Rightarrow x=0$ unstable.
For $x=b$: $f'(b)=a-\frac{2ab}{b}=a-2a=-a<0 \Rightarrow$

$\Rightarrow x=b$ asymptotically stable.





↦ The stability theorem given above may fail if at a fixed point $x=x_0$ we have $f'(x_0)=0$. An alternative method for determining fixed point stability useful in such situations is the construction of a sign table.

c) $\dot{x} = 2x(x-1)^2(x-2)^3$

▶ Fixed points

Let $f(x) = 2x(x-1)^2(x-2)^3$. Then

x fixed point $\Leftrightarrow f(x)=0 \Leftrightarrow 2x(x-1)^2(x-2)^3 = 0 \Leftrightarrow$

$\Leftrightarrow 2x = 0 \vee (x-1)^2 = 0 \vee (x-2)^3 = 0 \Leftrightarrow$

$\Leftrightarrow x = 0 \vee x = 1 \vee x = 2$.

▶ Stability

x		0		1		2	
$2x$	$-$	0	$+$		$+$		$+$
$(x-1)^2$	$+$		$+$	0	$+$		$+$
$(x-2)^3$	$-$		$-$		$-$	0	$+$
$f(x)$	$+$	0	$-$	0	$-$	0	$+$
	$\rightarrow$		$\leftarrow$		$\leftarrow$		$\rightarrow$
		stab.		unst.		unst.	

Thus $x=0$ is asymptotically stable and $x=1$ and $x=2$ are unstable.

▼ Potential and 1d systems

Consider a 1d autonomous system $\dot{x} = f(x)$ with f continuous in $\mathbb{R}$. Then we may define a potential function

$$V(x) = \int_x^c f(t)\,dt \Rightarrow f(x) = -\frac{dV(x)}{dx} = -V'(x).$$

It follows that

$$\frac{dx}{dt} = -V'(x).$$

- Let $x(t)$ be a solution of the autonomous system. We will show that $V(x(t))$ decreases with time, that is the system evolves towards lower potentials. Formally:

$$\boxed{t_1 < t_2 \Rightarrow V(x(t_1)) \geq V(x(t_2))}$$

Proof

$$\frac{d}{dt} V(x(t)) = V'(x(t))\frac{dx(t)}{dt} = V'(x(t)) f(x(t)) =$$

$$= V'(x(t))[-V'(x(t))] = -[V'(x(t))]^2 \Rightarrow$$

$$\Rightarrow V(x(t_2)) - V(x(t_1)) = \int_{t_1}^{t_2} \left[\frac{d}{dt} V(x(t))\right] dt =$$

$$= \int_{t_1}^{t_2} -[V'(x(t))]^2 dt \leq 0 \Rightarrow$$

$$\Rightarrow V(x(t_1)) \geq V(x(t_2)). \qquad \square$$

Remarks

a) Fixed points occur at the min/max points of the potential function $V(x)$.
b) Stable fixed points occur at the min points of $V(x)$.
c) Unstable fixed points occur at the max points of $V(x)$.

$\mapsto$ No periodic solutions

- A 1d autonomous system $\dot{x} = f(x)$ never has any periodic solution that is not constant for all time.

Proof

Let $x(t)$ be a solution of $\dot{x} = f(x)$ such that
$x(t) = x(t+T)$, $\forall t \in \mathbb{R}$. (1)
Let $V(x)$ be the potential function. Then
$(1) \Rightarrow V(x(t)) = V(x(t+T))$, $\forall t \in \mathbb{R} \Rightarrow$

$$\Rightarrow \int_t^{t+T} [V'(x(\tau))]^2 d\tau = 0, \ \forall t \in \mathbb{R} \Rightarrow$$

$$\Rightarrow V'(x(\tau)) = 0 \quad , \forall \tau \in [t, t+T] \, , \forall t \in \mathbb{R} \Rightarrow$$
$$\Rightarrow V'(x(t)) = 0 \quad , \forall t \in \mathbb{R}$$
$$\Rightarrow dx(t)/dt = 0 \quad , \forall t \in \mathbb{R} \Rightarrow x(t) \text{ constant } \square.$$

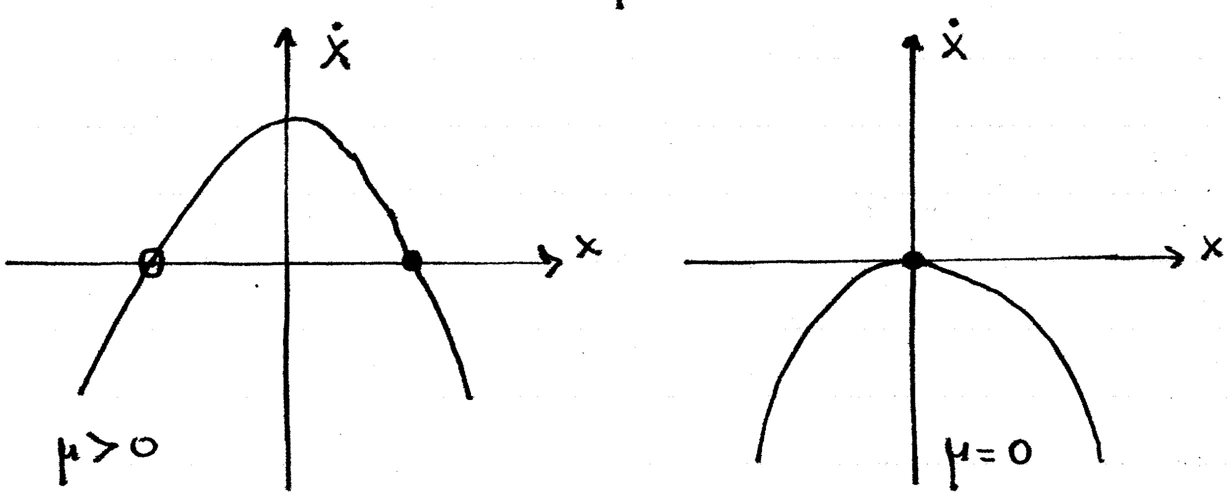
Local Bifurcations with 1d systems

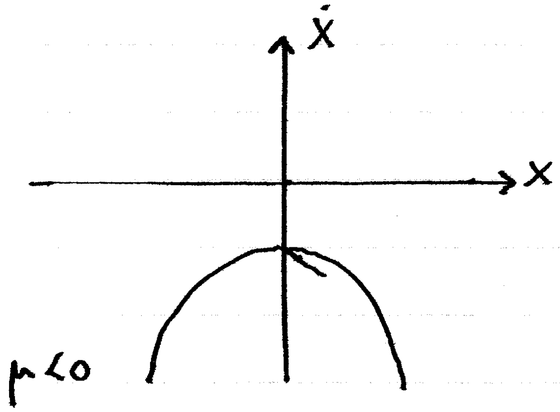
- In general, bifurcations fall under two general categories
  (a) Local Bifurcations
  (b) Global Bifurcations
  However 1d systems admit only local bifurcations.

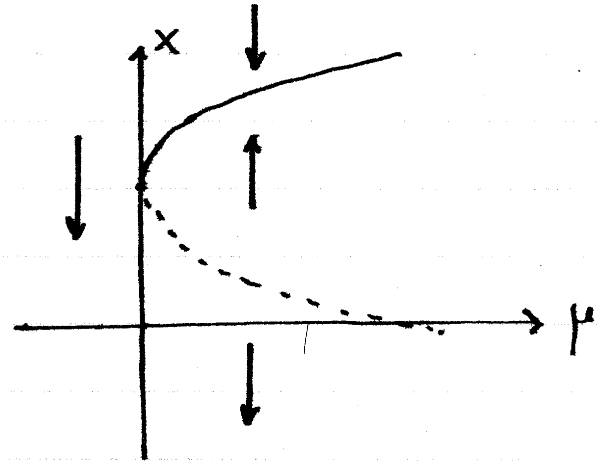
- Consider the 1d autonomous system $\dot{x} = f(x,\mu)$ with $\mu \in \mathbb{R}$ a parameter. A local bifurcation occurs when the number of fixed-points changes as we vary the value of the parameter μ. The three most common types of local bifurcations are:

(1) Saddle-node bifurcation $\longrightarrow$ $\boxed{\dot{x} = \mu - x^2}$

Two fixed-points with opposite stability properties collide into a saddle point which then vanishes:





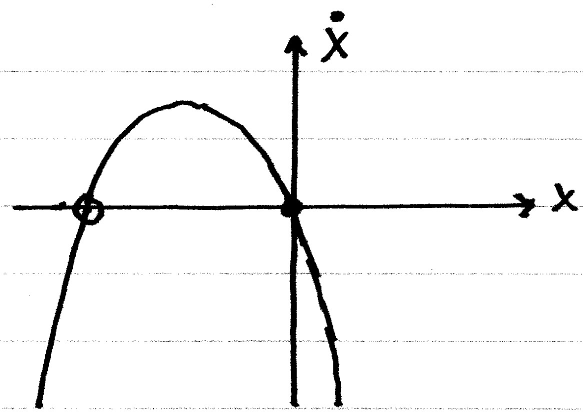


(Bifurcation diagram)

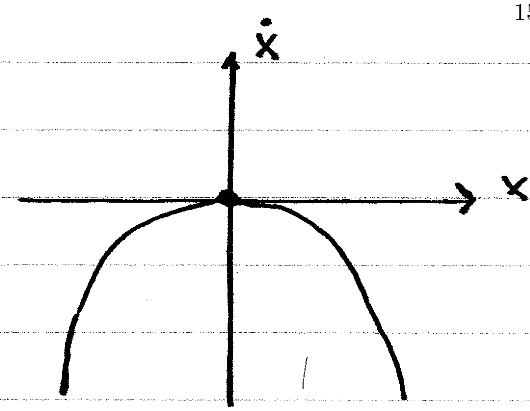
↳ A bifurcation diagram shows the motion of the fixed-points on the x-axis as a function of the parameter μ. We use a solid line to denote the motion of a stable fixed-point and a dotted-line to show the motion of an unstable fixed-point.

(2) → Transcritical Bifurcation → $\boxed{\dot{x} = \mu x - x^2}$

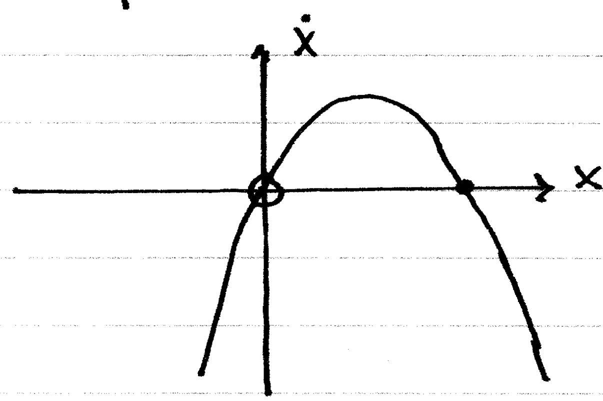
Two fixed points with opposite stability properties collide into a saddle-point which breaks up into two fixed points again with opposite stability properties but also with their stability properties exchanged. One fixed point is independent of μ.



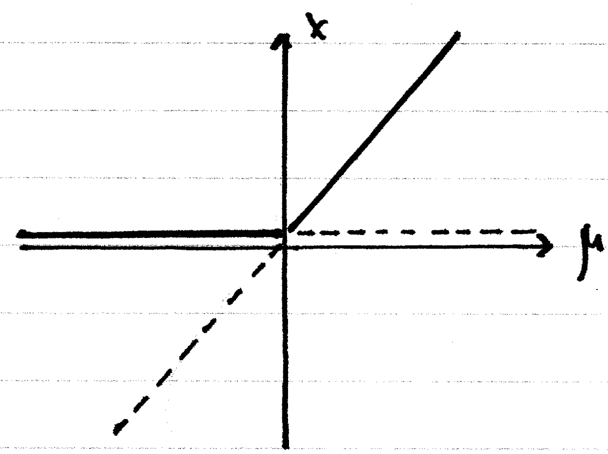
$\mu < 0$



$\mu = 0$



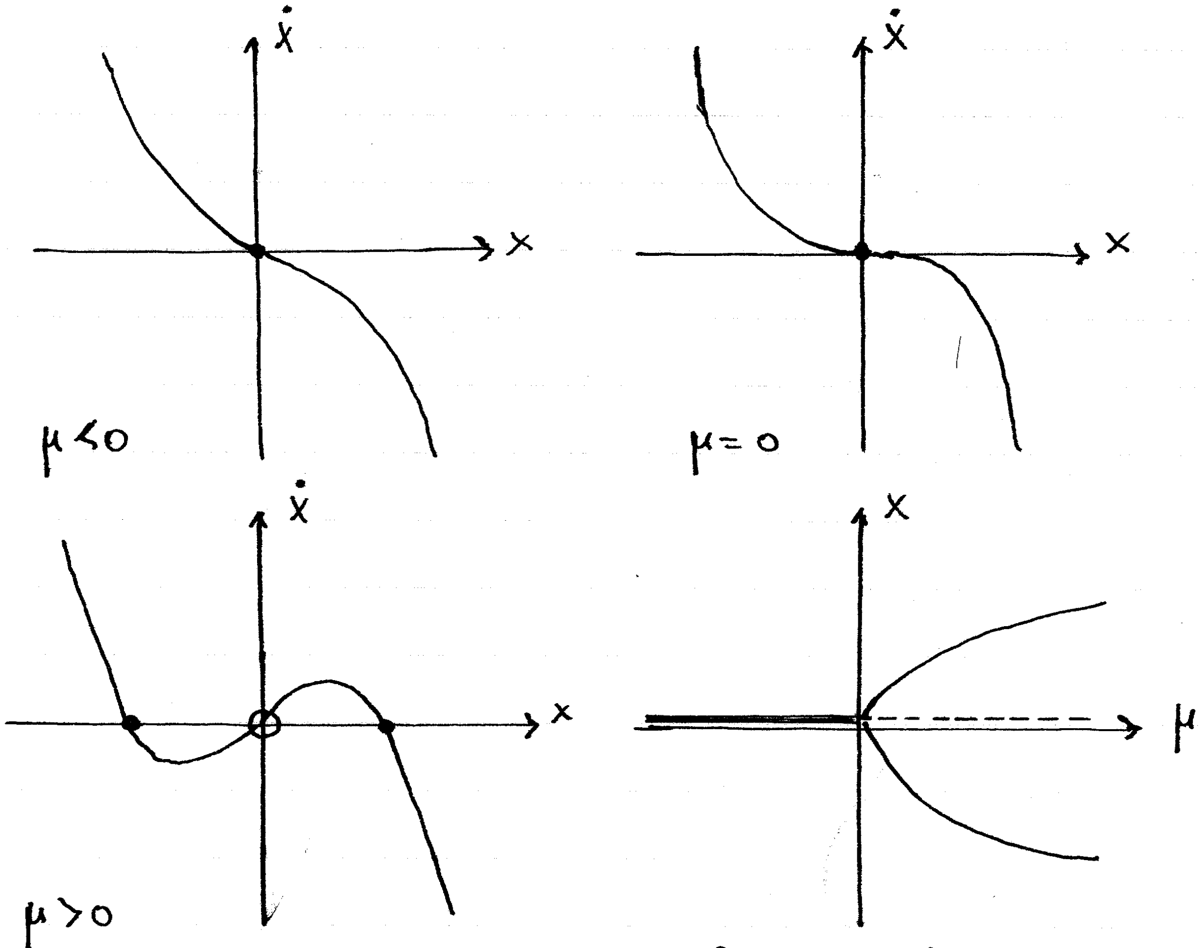
$\mu > 0$



Bifurcation diagram.

③ → <u>Pitchfork Bifurcation</u> → $\boxed{\dot{x} = \mu x - x^3}$

In a pitchfork bifurcation, a fixed-point breaks into 3 fixed-points. The inner fixed-point has opposite stability property with respect to the original fixed-point. The 2 outer fixed-points have the same stability property as the original fixed-point. We call this bifurcation a pitchfork bifurcation because the bifurcation diagram resembles a pitchfork. The inner fixed-point is independant of the parameter μ.



Bifurcation diagram

Tangency condition

From the examples above we see that bifurcations occur always when the graph of $f(x)$ is tangent to the x-axis. It follows that candidate points (x_0, μ_0) for bifurcation events can be located by solving the system of equations:

$$\boxed{\begin{cases} f(x_0, \mu_0) = 0 \\ f_x(x_0, \mu_0) = 0 \end{cases}}$$

Here, the subscripts represent partial derivatives, thus $f_x = \dfrac{\partial f}{\partial x}$.

Sufficient conditions

Once we identify a candidate for a bifurcation event at (x_0, μ_0) it can be classified by confirming the corresponding sufficient conditions. The sufficient conditions for the bifurcations considered above are:

Saddle-node bifurcation	Transcritical bifurcation	Pitchfork bifurcation.
$f(x_0, \mu_0) = 0$	$f(x_0, \mu_0) = 0$	$f(x_0, \mu_0) = 0$
$f_x(x_0, \mu_0) = 0$	$f_x(x_0, \mu_0) = 0$	$f_x(x_0, \mu_0) = 0$
$f_\mu(x_0, \mu_0) \neq 0$	$f_\mu(x_0, \mu_0) = 0$	$f_\mu(x_0, \mu_0) = 0$
$f_{xx}(x_0, \mu_0) \neq 0$	$f_{xx}(x_0, \mu_0) \neq 0$	$f_{xx}(x_0, \mu_0) = 0$
	$f_{x\mu}(x_0, \mu_0) \neq 0$	$f_{x\mu}(x_0, \mu_0) \neq 0$
		$f_{xxx}(x_0, \mu_0) \neq 0$

Procedure

To identify and classify bifurcation events (x_0, μ_0) we work as follows:

- 1 Solve the equations
$$f(x,\mu) = 0$$
$$f_x(x,\mu) = 0$$
to identify candidates (x_0, μ_0).
- 2 Calculate $f_\mu(x_0, \mu_0)$.
  1) If $f_\mu(x_0,\mu_0) \neq 0$, then check that $f_{xx}(x_0,\mu_0) \neq 0$. If so, then $(x_0,\mu_0) \leftarrow$ saddle-node bifurcation
  2) If $f_\mu(x_0,\mu_0) = 0$, then check that $f_{x\mu}(x_0,\mu_0) \neq 0$. Then, if:
     a) $f_{xx}(x_0,\mu_0) \neq 0 \leftarrow$ trancritical bifurcation
     b) $f_{xx}(x_0,\mu_0) = 0 \leftarrow$, then check that $f_{xxx}(x_0,\mu_0) \neq 0 \leftarrow$ pitchfork bifurcation.
- 3 For a saddle-node bifurcation we have 2 fixed points
  a) For $\mu > \mu_0$, if $f_{xx}(x_0,\mu_0) f_\mu(x_0,\mu_0) < 0$
  b) For $\mu < \mu_0$, if $f_{xx}(x_0,\mu_0) f_\mu(x_0,\mu_0) > 0$
  (see exercise 9)
- 4 For a pitchfork bifurcation we have 3 fixed points
  a) For $\mu > \mu_0$, if $f_{xxx}(x_0,\mu_0) f_{x\mu}(x_0,\mu_0) < 0$
  b) For $\mu < \mu_0$, if $f_{xxx}(x_0,\mu_0) f_{x\mu}(x_0,\mu_0) > 0$
  (see exercise 11).

EXAMPLES

a) <u>Saddle-Node Bifurcation:</u> $\dot{x} = \mu - x - e^{-x}$

Let $f(x,\mu) = \mu - x - e^{-x} \Rightarrow f_x(x,\mu) = -1 + e^{-x}$.

$$\begin{cases} f(x,\mu) = 0 \\ f_x(x,\mu) = 0 \end{cases} \Leftrightarrow \begin{cases} \mu - x - e^{-x} = 0 \\ -1 + e^{-x} = 0 \end{cases} \Leftrightarrow \begin{cases} \mu - x - \cancel{e^{-x}} = 0 \\ e^{-x} = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \mu - 0 - e^{-0} = 0 \\ x = 0 \end{cases} \Leftrightarrow \begin{cases} \mu - 1 = 0 \\ x = 0 \end{cases} \Leftrightarrow \begin{cases} \mu = 1 \\ x = 0 \end{cases}$$

thus possible bifurcation at $(x_0,\mu_0) = (0,1)$

$$f_\mu(x,\mu) = 1 \Rightarrow f_\mu(x_0,\mu_0) = 1 \neq 0 \qquad (1)$$

$$f_{xx}(x,\mu) = -e^{-x} \Rightarrow f_{xx}(x_0,\mu_0) = -e^{-0} = -1 \neq 0 \quad (2)$$

From (1) and (2): saddle-node bifurcation at $(x_0,\mu_0) = (0,1)$

Since $f_{xx}(x_0,\mu_0) f_\mu(x_0,\mu_0) = 1 \cdot (-1) = -1 < 0 \Rightarrow$

$\Rightarrow$ two fixed points for $\mu > 1$ and no fixed points for $\mu < 1$.

b) <u>Transcritical Bifurcation:</u> $\dot{x} = \mu \ln x + x - 1$

Let $f(x,\mu) = \mu \ln x + x - 1 \Rightarrow f_x(x,\mu) = \frac{\mu}{x} + 1$

$$\begin{cases} f(x,\mu) = 0 \\ f_x(x,\mu) = 0 \end{cases} \Leftrightarrow \begin{cases} \mu \ln x + x - 1 = 0 \\ \mu/x + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} -x \ln x + x - 1 = 0 \\ \mu = -x \end{cases}$$

Let $g(x) = -x \ln x + x - 1$. Note the obvious solution $x = 1$ since $g(1) = -1 \ln 1 + 1 - 1 = -0 + 0 = 0$. We now show the solution is unique.

$$g'(x) = -(x)'\ln x - x(\ln x)' + 1 = -\ln x - x\frac{1}{x} + 1 =$$
$$= -\ln x - 1 + 1 = -\ln x$$

It follows that $g \nearrow (0,1)$ and $g \searrow (1,+\infty)$ thus $\forall x \in (0,1) \cup (1,+\infty)$: $g(x) < 0$.

We conclude that the solution $x = 1$ is unique and therefore a bifurcation may occur when $(x_0, \mu_0) = (1,-1)$. Note that

$$f_\mu(x,\mu) = \ln x \Rightarrow f_\mu(1,-1) = \ln 1 = 0$$

$$f_{x\mu}(x,\mu) = \frac{1}{x} \Rightarrow f_{x\mu}(1,-1) = \frac{1}{1} = 1 \neq 0$$

$$f_{xx}(x,\mu) = -\frac{\mu}{x^2} \Rightarrow f_{xx}(1,-1) = -\frac{-1}{1^2} = 1 \neq 0$$

It follows that there is a transcritical bifurcation at $(x_0,\mu_0) = (1,-1)$.

c) <u>Pitchfork bifurcation</u> : $\dot{x} = -x + \mu \tanh x$

Let $f(x,\mu) = -x + \mu \tanh x \Rightarrow$
$\Rightarrow f_x(x,\mu) = -1 + \mu(1 - \tanh^2 x)$

$$\begin{cases} f(x,\mu) = 0 \\ f_x(x,\mu) = 0 \end{cases} \Leftrightarrow \begin{cases} -x + \mu \tanh x = 0 \\ -1 + \mu(1 - \tanh^2 x) = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \mu \tanh x = x \\ -1 + \mu - (\mu \tanh x)\tanh x = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \mu \tanh x = x \\ -1 + \mu - x \tanh x = 0 \end{cases} \Leftrightarrow \begin{cases} \mu \tanh x = x \\ \mu = 1 + x\tanh x \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (1+x\tanh x)\tanh x = x & (1) \\ \mu = 1 + x\tanh x \end{cases}$$

Since $\tanh 0 = 0$, $x=0$ is an obvious solution of (1).
We now show that this solution is unique.
Let $g(x) = (1+x\tanh x)\tanh x - x =$

$$= \tanh x + x\tanh^2 x - x \Rightarrow$$

$$\Rightarrow g'(x) = (1-\tanh^2 x) + \tanh^2 x + x(2\tanh x)(1-\tanh^2 x) - 1$$

$$= \underline{1} - \underline{\underline{\tanh^2 x}} + \underline{\underline{\tanh^2 x}} + 2x\tanh x - 2x\tanh^3 x - \underline{1}$$

$$= 2x\tanh x - 2x\tanh^3 x =$$

$$= 2x\tanh x\,(1-\tanh^2 x)$$

Note that $-1 < \tanh x < 1 \Rightarrow 1 - \tanh^2 x > 0$ thus

x		0	
$2x$	$-$	0	$+$
$\tanh x$	$-$	0	$+$
$1-\tanh^2 x$	$+$		$+$
$g'(x)$	$+$	0	$+$
$g(x)$	$\nearrow$		$\nearrow$

Since $g \nearrow (-\infty, 0)$ and $g \nearrow (0, +\infty)$ and $g(0)=0$, it follows that $x=0$ is a unique solution of $g(x)=0$.

For $x=0 \Rightarrow \mu = 1 + 0\tanh 0 = 1$ thus there is a bifurcation at $(x_0, \mu_0) = (0, 1)$.

Now, we note that:

$f_\mu(x,\mu) = \tanh x \Rightarrow f_\mu(0,1) = \tanh 0 = 0$
$f_{x\mu}(x,\mu) = 1-\tanh^2 x \rightarrow f_{x\mu}(0,1) = 1-\tanh^2 0 = 1-0 = 1 \neq 0$
$f_{xx}(x,\mu) = -\mu \frac{\partial}{\partial x} \tanh^2 x =$

$= -2\mu \tanh x \cdot (1-\tanh^2 x) \Rightarrow$
$\Rightarrow f_{xx}(0,1) = -2\cdot 1\cdot 0\cdot(1-0) = 0$, thus we rule out transcritical.

$f_{xxx}(x,\mu) = \frac{\partial}{\partial x}\left[-2\mu\tanh x + 2\mu\tanh^3 x\right] =$

$= -2\mu(1-\tanh^2 x) + 6\mu\tanh^2 x\,(1-\tanh^2 x)$
$= 2\mu(1-\tanh^2 x)\left[-1+3\tanh^2 x\right] \Rightarrow$
$\Rightarrow f_{xxx}(0,1) = 2\cdot 1\cdot(1-0)\left[-1+3\cdot 0\right] =$
$= 2\cdot 1\cdot 1\cdot(-1) = -2 \neq 0.$

It follows that $(x_0,\mu_0) = (0,1)$ are pitchfork bifurcation. Since
$f_{xxx}(0,1)\, f_{x\mu}(0,1) = (-2)\cdot 1 = -2 < 0 \Rightarrow$
$\Rightarrow$ there are 3 fixed points for $\mu > 1$.

More on sufficient conditions for bifurcation events

We will now derive the sufficient conditions for classifying bifurcation events. The proofs are based on the implicit function theorem.

Implicit function theorem

First we define the ball $B((x_0,y_0),\varepsilon)$ as:

$$B((x_0,y_0),\varepsilon)=\{(x,y)\in\mathbb{R}^2 \mid (x-x_0)^2+(y-y_0)^2<\varepsilon^2\}$$

The implicit function theorem states:

<u>Thm</u>: Assume that the function $f: A\to\mathbb{R}$ with $A\subseteq\mathbb{R}^2$ satisfies

a) $f(x_0,y_0)=0$
b) $\forall (x,y)\in B((x_0,y_0),\varepsilon): f_y(x,y)\neq 0$
c) f_x, f_y continuous at $B((x_0,y_0),\varepsilon)$

Then, there is a <u>unique</u> function g such that $\forall (x,y)\in B((x_0,y_0),\varepsilon): f(x,g(x))=0$

Note that condition (b) can be weakened to $f_y(x_0,y_0)\neq 0$. Then, combined with (c) it follows that there is an ε for which both (b) and (c) are satisfied.

(1) → Saddle-node bifurcation conditions

Let us assume that

$f(x_0, \mu_0) = 0 \quad (1)$

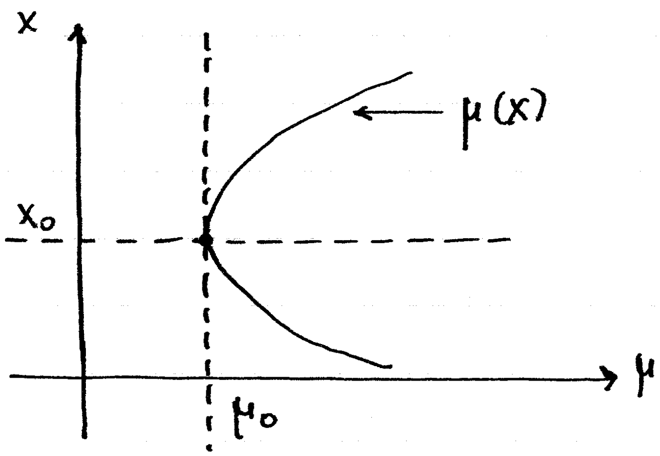
$f_x(x_0, \mu_0) = 0 \quad (2)$

$f_\mu(x_0, \mu_0) \neq 0 \quad (3)$

$f_{xx}(x_0, \mu_0) \neq 0 \quad (4)$

• **Analysis**

The typical bifurcation diagram for a saddle-node bifurcation is shown below:



We see that we have to show that there is a unique function $\mu(x)$ such that

$f(x, \mu(x)) = 0, \forall x \in (x_0 - \varepsilon, x_0 + \varepsilon)$

with

$$\mu'(x_0) = \frac{d}{dx}\mu(x_0) = 0$$

$$\mu''(x_0) = \frac{d^2}{dx^2}\mu(x_0) \neq 0$$

The condition $\mu'(x_0) = 0$ ensures that the bifurcation curve is tangent to $\mu = \mu_0$. The condition $\mu''(x_0) \neq 0$ ensures that x_0 is a minimum or maximum so that the bifurcation curve $\mu(x)$ remains on the same half-plane defined by $\mu = \mu_0$.

• **Construction**: Since $f(x_0, y_0) = 0$ and $f_\mu(x_0, \mu_0) \neq 0$, it follows that the implicit function theorem applies and therefore there is a unique function $\mu(x)$ such that

$$\forall x \in (x_0-\varepsilon, x_0+\varepsilon) : f(x, \mu(x)) = 0 \qquad (5)$$

Thus $\mu(x)$ is hereby constructed.

- Proof : We will now show that $\mu'(x_0) = 0$ and $\mu''(x_0) \neq 0$.

Differentiating (5) with respect to x gives:

$$f_x(x, \mu(x)) + f_\mu(x, \mu(x))\, \mu'(x) = 0 \qquad (6)$$

For $x = x_0$:

$f_x(x_0, \mu(x_0)) = f_x(x_0, \mu_0) = 0$ and

$f_\mu(x_0, \mu(x_0)) = f_\mu(x_0, \mu_0) \neq 0$

thus:

$$\mu'(x_0) = \frac{-f_x(x_0, \mu_0)}{f_\mu(x_0, \mu_0)} = 0 \qquad (7)$$

Differentiating (6) one more time with respect to x gives:

$$f_{xx} + f_{x\mu} \cdot \mu' + (f_{\mu x} + f_{\mu\mu} \cdot \mu')\mu' + f_\mu \cdot \mu'' = 0 \Rightarrow$$

$$\Rightarrow f_{xx} + (2f_{x\mu} + f_{\mu\mu} \cdot \mu')\mu' + f_\mu \cdot \mu'' = 0$$

evaluated at $(x, \mu(x))$. For $x = x_0$, $\mu'(x_0) = 0$, and therefore:

$$f_{xx}(x_0, \mu_0) + f_\mu(x_0, \mu_0) \cdot \mu''(x_0) = 0$$

Since $f_{xx}(x_0, \mu_0) \neq 0$ and $f_\mu(x_0, \mu_0) \neq 0$, it follows that

$$\mu''(x_0) = \frac{-f_{xx}(x_0, \mu_0)}{f_\mu(x_0, \mu_0)} \neq 0.$$

- <u>Stability</u>: We will now show that the two fixed-points that emerge one one of the two half-planes defined by $\mu=\mu_0$ on the bifurcation diagram have opposite stability.

From (6):

$f_x(x,\mu(x)) = -f_\mu(x,\mu(x))\mu'(x)$, $\forall x\in(x_0-\varepsilon, x_0+\varepsilon)$

Since $f_\mu(x_0,\mu_0)\neq 0$, we can choose $\varepsilon>0$ small enough so that

$\forall x\in(x_0-\varepsilon, x_0+\varepsilon): f_\mu(x,\mu(x))\neq 0$

Thus $f_\mu(x,\mu(x))$ maintains its sign in $x\in(x_0-\varepsilon, x_0+\varepsilon)$.

Since $\mu''(x_0)\neq 0$ and $\mu'(x_0)=0$, we expect that $\mu'(x_0)$ changes sign from $x\in(x_0-\varepsilon, x_0)$ to $x\in(x_0, x_0+\varepsilon)$. Thus, so does $f_x(x,\mu(x))$ and it follows that the two fixedpoints, when they exist, have opposite stability.

(2) → Transcritical Bifurcation conditions

Let us assume that

$f(x_0, \mu_0) = 0$ (1)

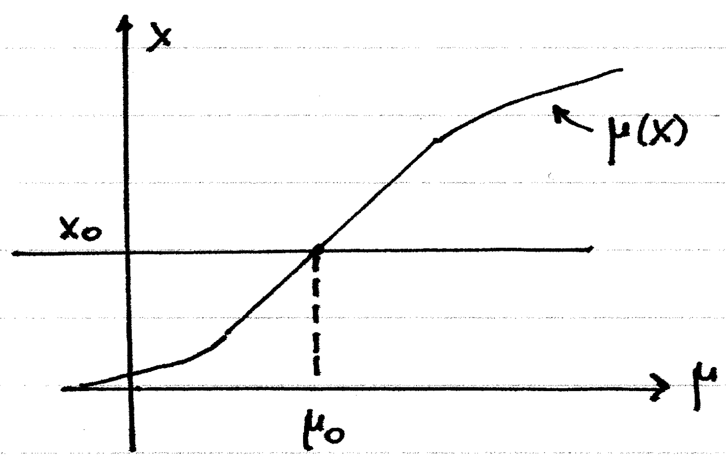
$f_x(x_0, \mu_0) = 0$ (2)

$f_\mu(x_0, \mu_0) = 0$ (3)

$f_{xx}(x_0, \mu_0) \neq 0$ (4)

$f_{x\mu}(x_0, \mu_0) \neq 0$ (5)

The typical bifurcation diagram for a transcritical bifurcation is shown below:



• <u>Analysis</u> : We see that there are two bifurcation curves passing through (x_0, μ_0) :

a) The line (ℓ): $x = x_0$ (independent of the parameter μ)

b) The line (ℓ_2): $\mu = \mu(x)$ passing from one half-plane to the other, separated by $\mu = \mu_0$, with $\mu_0 = \mu(x_0)$.

It follows that :

$f(x_0, \mu) = 0 \quad , \forall \mu \in (\mu_0 - \varepsilon_1, \mu_0 + \varepsilon_1)$
$f(x, \mu(x)) = 0 \,, \ \forall x \in (x_0 - \varepsilon_2, x_0 + \varepsilon_2)$
Note that to get two distinct curves pass through (x_0, μ_0) it is necessary to violate the implicit function theorem. Since $f(x_0, \mu_0) = 0$, to violate the theorem we require that $f_\mu(x_0, \mu_0) = 0$.
Let us now define

$$F(x, \mu) = \begin{cases} f(x, \mu)/(x - x_0) & , x \neq x_0 \\ f_x(x, \mu) & , x = x_0 \end{cases}$$

It follows that $f(x, \mu) = (x - x_0) F(x, \mu)$, thus we assume that $x = x_0$ is a bifurcation curve. We also note that $F(x, \mu)$ retains continuity because

$$\lim_{x \to x_0} F(x, \mu) = \lim_{x \to x_0} \frac{f(x, \mu)}{x - x_0} \overset{\text{L'Hospital}}{=} \lim_{x \to x_0} \frac{f_x(x, \mu)}{1 - 0} =$$
$$= \lim_{x \to x_0} f_x(x, \mu) = f_x(x_0, \mu) = F(x_0, \mu).$$

Note that L'Hospital applies since $\lim_{x \to x_0} f(x, \mu) = f(x_0, \mu) = 0$.

We will now show that $F(x, \mu)$ has a unique curve passing through (x_0, μ_0) and accross $\mu = \mu_0$.

- <u>Construction</u> : We note that
$F(x_0, \mu_0) = f_x(x_0, \mu_0) = 0$ and
$F_\mu(x_0, \mu_0) = f_{x\mu}(x_0, \mu_0) \neq 0$

therefore the implicit function theorem applies.
It follows that there is a unique function $\mu(x)$ such that $F(x,\mu(x))=0$ for all x near x_0.
$x=\mu(x)$ is a bifurcation curve since
$f(x,\mu(x))=(x-x_0)F(x,\mu(x))=(x-x_0)\cdot 0=0$

- Proof : We will now show that the curve $x=\mu(x)$ passes accross $\mu=\mu_0$. To do that, it is sufficient to show that $\mu'(x_0)\neq 0$.
Since $F(x,\mu(x))=0 \Rightarrow$
$\Rightarrow F_x(x,\mu(x))+F_\mu(x,\mu(x))\mu'(x)=0 \Rightarrow$
$$\Rightarrow \mu'(x_0)=\frac{-F_x(x_0,\mu(x_0))}{F_\mu(x_0,\mu(x_0))}=\frac{-f_{xx}(x_0,\mu(x_0))}{f_{x\mu}(x_0,\mu(x_0))}$$
Since $f_{xx}(x_0,\mu_0)\neq 0$ and $f_{x\mu}(x_0,\mu_0)\neq 0$,
$\mu'(x_0)$ is well-defined and $\mu'(x_0)\neq 0$.
It follows that $x=\mu(x)$ does not have a min or max at $x=x_0$, thus it will go accross the line $\mu=\mu_0$.

- Stability: We will now show that both bifurcation lines $(l_1): x=x_0$ and $(l_2): x=\mu(x)$ change stability upon crossing the point (x_0,μ_0).

a) For the line $(l_1): x=x_0$:
$$f_x(x_0,\mu)=f_x(x_0,\mu_0)+\int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm=$$
$$=\int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm$$

Since $f_{x\mu}(x_0, \mu_0) \neq 0 \Rightarrow$

$\Rightarrow \exists \varepsilon > 0 : \forall \mu \in (\mu_0 - \varepsilon, \mu_0 + \varepsilon) : f_{x\mu}(x_0, \mu) \neq 0$

Thus $f_{x\mu}(x_0, \mu)$ maintains its sign in $(\mu_0 - \varepsilon, \mu_0 + \varepsilon)$ therefore $f_x(x_0, \mu)$ changes sign from $\mu > \mu_0$ to $\mu < \mu_0$.

b) For the line $(\ell_2) : x = \mu(x)$

$$f_x(x, \mu(x)) = \frac{\partial}{\partial x}\Big[(x - x_0) F(x, \mu(x))\Big] =$$

$$= F(x, \mu(x)) + (x - x_0) F_x(x, \mu(x))$$
$$= (x - x_0) F_x(x, \mu(x))$$

Here we have used $F(x, \mu(x)) = 0$.

At $x = x_0$: $F_x(x_0, \mu(x_0)) = f_{xx}(x_0, \mu(x_0)) =$
$$= f_{xx}(x_0, \mu_0) \neq 0 \Rightarrow$$

$\Rightarrow \exists \varepsilon > 0 : \forall x \in (x_0 - \varepsilon, x_0 + \varepsilon) : F_x(x, \mu(x)) \neq 0$

Thus $F_x(x, \mu(x))$ does not change sign in $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ but $x - x_0$ does change from negative to positive. It follows that $f_x(x, \mu(x))$ changes sign accross $x = x_0$

From (a) and (b) above we conclude that since for both curves f_x changes sign accross the point (x_0, μ_0), the stability for both curves also changes.

(3) → Pitchfork bifurcation conditions

Let us assume that

$f(x_0, \mu_0) = 0 \qquad (1)$

$f_x(x_0, \mu_0) = 0 \qquad (2)$

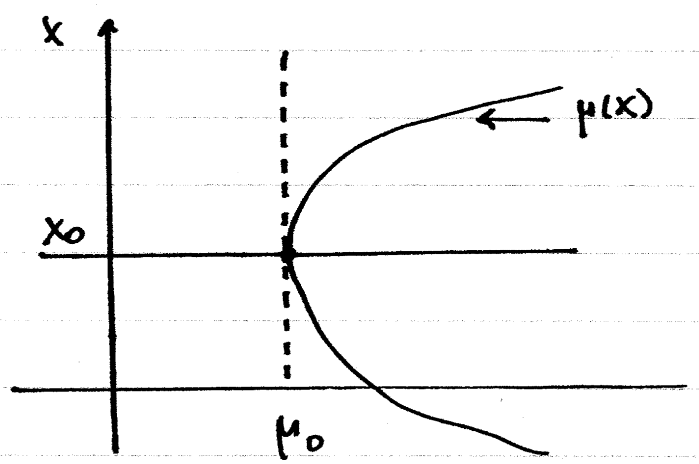
$f_\mu(x_0, \mu_0) = 0 \qquad (3)$

$f_{xx}(x_0, \mu_0) = 0 \qquad (4)$

$f_{x\mu}(x_0, \mu_0) \neq 0 \qquad (5)$

$f_{xxx}(x_0, \mu_0) \neq 0 \qquad (6)$

The typical bifurcation diagram for a pitchfork bifurcation is shown below:



• **Analysis**: The bifurcation diagram has two lines:

(a) The line (l_1): $x = x_0$ which is independent of μ.

(b) The curve (l_2): $\mu = \mu(x)$ which is tangent to the line (l): $\mu = \mu_0$. It follows that μ must satisfy $\mu'(x_0) = 0$ and $\mu''(x_0) \neq 0$.

Both lines intersect at (x_0, μ_0).

Again, in order to have two curves passing through

(x_0, μ_0) it is necessary to violate the implicit function theorem. Since $f(x_0, \mu_0) = 0$, it is thus necessary to have $f_\mu(x_0, \mu_0) = 0$.

Again, let us define

$$F(x, \mu) = \begin{cases} f(x, \mu)/(x - x_0), & \text{if } x \neq x_0 \\ f_x(x_0, \mu), & \text{if } x = x_0 \end{cases}$$

Similarly with our transcritical bifurcation argument, it follows that

$$f(x, \mu) = (x - x_0) F(x, \mu)$$
$$\lim_{x \to x_0} F(x, \mu) = F(x_0, \mu).$$

Thus $(\ell_1) : x = x_0$ is by definition a bifurcation line.

- <u>Construction</u>: We note that

$$F(x_0, \mu_0) = f_x(x_0, \mu_0) = 0$$
$$F_\mu(x_0, \mu_0) = f_{x\mu}(x_0, \mu_0) \neq 0$$

It follows that the implicit function theorem applies and thus there is a unique function $\mu(x)$ such that $F(x, \mu(x)) = 0$. It follows that

$$f(x, \mu(x)) = (x - x_0) F(x, \mu(x)) = (x - x_0) \cdot 0 = 0$$

Thus $\mu(x)$ has been constructed.

- <u>Proof</u>: We will now show that $\mu'(x_0) = 0$ and $\mu''(x_0) \neq 0$. Using a calculation similar to the one we did for the saddle-node proof, it follows that

$$\mu'(x_0) = \frac{-F_x(x_0,\mu_0)}{F_\mu(x_0,\mu_0)} = \frac{-f_{xx}(x_0,\mu_0)}{f_{x\mu}(x_0,\mu_0)} = 0$$

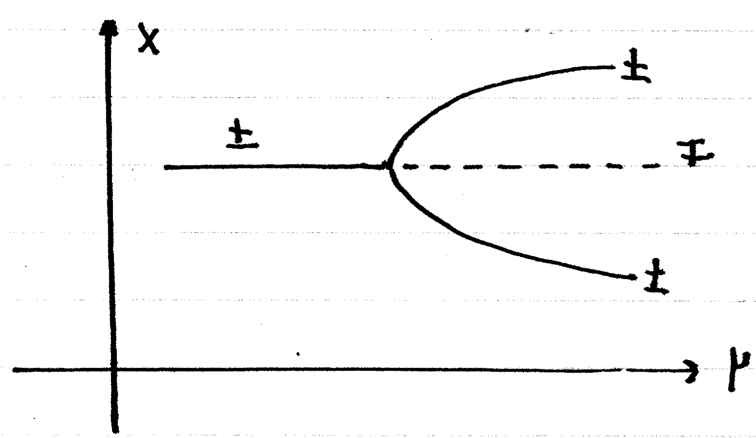
because $f_{xx}(x_0,\mu_0) = 0$

and therefore

$$\mu''(x_0) = \frac{-F_{xx}(x_0,\mu_0)}{F_\mu(x_0,\mu_0)} = \frac{-f_{xxx}(x_0,\mu_0)}{f_{x\mu}(x_0,\mu_0)} \neq 0$$

because $f_{xxx}(x_0,\mu_0) \neq 0$.

- Stability: We will now show that the inner fixed point changes stability accross the point (x_0,μ_0). We will also show that the outer fixed points after, the pitchfork occurs, have the same stability with each other as well as with the inner fixe dpoint BEFORE the fixed point. This is all shown in the diagram below:



a) For the line $x = x_0$:

$$f_x(x_0,\mu) = f_x(x_0,\mu_0) + \int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm =$$
$$= \int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm$$

Since $f_{x\mu}(x_0,\mu_0) \neq 0 \Rightarrow$
$\Rightarrow f_{x\mu}(x_0,m)$ does not change sign accross $m = \mu_0$
$\Rightarrow f_x(x,\mu)$ changes sign accross $\mu = \mu_0$
$\Rightarrow$ The fixed point on the line $x = x_0$ changes stability.

b) For the line $\mu = \mu(x)$:

$$f_x(x,\mu(x)) = \frac{\partial}{\partial x}\Big[(x-x_0)F(x,\mu(x))\Big] =$$
$$= F(x,\mu(x)) + (x-x_0)\,F_x(x,\mu(x))$$
$$= (x-x_0)\,F_x(x,\mu(x)) =$$
$$= (x-x_0)\Big[-F_\mu(x,\mu(x))\,\mu'(x)\Big]$$

Here we used the identity
$$F_x(x,\mu(x)) + F_\mu(x,\mu(x))\,\mu'(x) = 0$$
We note that accross $x = x_0$:
$x - x_0$ changes sign, and
$\mu'(x_0) = 0$ and $\mu''(x_0) \neq 0 \Rightarrow \mu'(x_0)$ changes sign
and $F_\mu(x_0,\mu(x_0)) = F_\mu(x_0,\mu_0) = f_{x\mu}(x_0,\mu_0) \neq 0 \Rightarrow$

$\Rightarrow F_\mu(x,\mu(x))$ does not change sign.
Thus $f_x(x,\mu(x))$ does not change sign. It follows that the two outer fixed points have the same stability.
c) We now compare the stability of the outer fixed points with the inner fixed point. Recall that f_x for these fixed points is:

inner point: $f_x(x_0,\mu) = \int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm$

outer points: $f_x(x,\mu(x)) = -\mu'(x)(x-x_0)F_\mu(x,\mu(x))$

We assume, with no loss of generality, that $\mu''(x_0) > 0$. This implies that $\mu(x)$ has a minimum at $x = x_0$, so the 3 fixed points occur when $\mu > \mu_0$. We may thus assume that $\underline{\mu > \mu_0}$. It also follows that when x is near x_0, $\mu'(x)$ is increasing, and therefore:

$x - x_0 < 0 \Rightarrow \mu'(x) < 0$

$x - x_0 > 0 \Rightarrow \mu'(x) > 0$

Thus: $\mu'(x)(x-x_0) > 0$ when x is near x_0.
It follows that:

$f_x(x,\mu(x))$ opposite sign as $F_\mu(x,\mu(x))$
same sign as $F_\mu(x_0,\mu_0)$ (x near x_0)
same sign as $f_{x\mu}(x_0,\mu_0)$
same sign as $f_{x\mu}(x_0,m)$ (m near μ_0)
same sign as $\int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm = f_x(x_0,\mu)$ (use $\mu > \mu_0$).

Thus $f_x(x,\mu(x))$ has opposite sign from $f_x(x_0,\mu)$, thus outer and inner points have opposite stability.

GODE 08: Linear autonomous systems

LINEAR AUTONOMOUS SYSTEMS

- A linear autonomous system is a system of ordinary differential equations of the form
$$\dot{x} = Ax$$
with $x \in \mathbb{R}^n$ a vector and $A \in M_n(\mathbb{R})$ an $n \times n$ matrix. In detail:

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n \\ \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n \\ \vdots \\ \dot{x}_n = A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nn}x_n \end{cases}$$

▼ Exact solutions

- An exact solution can be written in terms of the matrix exponential.

Def: $$\boxed{\exp(A) = \sum_{n=0}^{+\infty} \frac{A^n}{n!}}$$ (with $A^0 = I$)

→ Properties:
- $AB = BA \Rightarrow \exp(A+B) = \exp(A)\exp(B)$
- $[\exp(A)]^{-1} = \exp(-A)$
- $\frac{d}{dt}\exp(tA) = A\exp(tA) = \exp(tA)A$

- The solution of $\dot{x} = Ax$ with $x(0) = x_0$ is

$$\boxed{x(t) = \exp(tA)\,x(0)}$$

- <u>Eigenvalues and eigenvectors</u>

<u>Def</u>: $\lambda \in \mathbb{C}$ is an eigenvalue of $A \in M_n(\mathbb{R})$ with eigenvector $x \in \mathbb{C}^n$ if and only if $Ax = \lambda x$.

▸ <u>notation</u>: $\lambda(A)$ = the set of all eigenvalues of A.

<u>Thm</u>: $\boxed{\lambda \in \lambda(A) \Longleftrightarrow \det(A - \lambda I) = 0}$

- We note that $p(\lambda) = \det(A - \lambda I)$ is a polynomial called the <u>characteristic polynomial</u> of A.
- Assume that A has n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with corresponding eigenvectors $v_1, v_2, \dots, v_n \in \mathbb{R}^n$. Then,

  a) The eigenvalues $v_1, v_2, \dots, v_n$ are linearly independent. Thus any $x \in \mathbb{R}^n$ can be written as:
  $$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$
  with $c_1, c_2, \dots, c_n$ constant.

  b) For $P = [v_1 v_2 \cdots v_n]$, A can be written as
  $$A = P \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) P^{-1}.$$
  with

$$\mathrm{diag}(\lambda_1,\lambda_2,\dots,\lambda_n)=\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}$$

c) If the initial condition of $\dot{x}=Ax$ satisfies

$$x(0)=c_1v_1+c_2v_2+\dots+c_nv_n$$

then

$$\boxed{x(t)=c_1e^{\lambda_1 t}v_1+c_2e^{\lambda_2 t}v_2+\dots+c_ne^{\lambda_n t}v_n.}$$

<u>Proof</u>

$$x(t)=\exp(tA)x(0)=\exp(tA)[c_1v_1+c_2v_2+\dots+c_nv_n]$$

$$=\sum_{a=1}^{n}c_a\exp(tA)v_a=\sum_{a=1}^{n}c_a\left[\sum_{k=0}^{+\infty}\frac{1}{k!}(tA)^k\right]v_a$$

$$=\sum_{a=1}^{n}c_a\left[\sum_{k=0}^{+\infty}\frac{t^k}{k!}(A^kv_a)\right]=$$

$$=\sum_{a=1}^{n}c_a\left[\sum_{k=0}^{+\infty}\frac{t^k}{k!}\lambda_a^kv_a\right]=$$

$$=\sum_{a=1}^{n}c_a\left[\sum_{k=0}^{+\infty}\frac{(\lambda_at)^k}{k!}\right]v_a=\sum_{a=1}^{n}c_ae^{\lambda_at}v_a \quad \square$$

We see that when the eigenvalues are all distinct, we can find the exact solution without calculating the matrix exponential.

• Matrix Exponential - 2x2 case

Let $A \in M_2(\mathbb{R})$ be a 2×2 matrix with eigenvalues λ_1, λ_2.

a) If $\lambda_1 \neq \lambda_2$, then

$$\boxed{\exp(tA) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} A}$$

b) If $\lambda_1 = \lambda_2 = \lambda$, then

$$\boxed{\exp(tA) = e^{\lambda t}(1 - \lambda t) I + t e^{\lambda t} A}$$

Lyapunov function for $\dot{x} = Ax$

- Consider the linear autonomous system $\dot{x} = Ax$. If $\det A \neq 0$, then $Ax = 0 \Leftrightarrow x = 0$. Thus $x = 0$ is the unique fixed point. It's stability can be investigated by constructing an appropriate Lyapunov function.

Definition of $V(x)$

Let $x, y \in \mathbb{C}^n$ with $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. We define the inner product:

$$\boxed{\langle x | y \rangle = \overline{x_1}\, y_1 + \overline{x_2}\, y_2 + \cdots + \overline{x_n}\, y_n}$$

The bar (e.g. $\bar{x}$) represents the complex conjugate. We note that

$$|\langle x | y \rangle|^2 = \langle x | y \rangle \langle y | x \rangle .$$

For the matrix $A = [A_{ab}]$ we define the Hermitian matrix $A^H = [\overline{A_{ba}}]$. It can then be shown that

$$\langle x | Ay \rangle = \langle A^H x | y \rangle$$
$$\langle Ax | y \rangle = \langle x | A^H y \rangle$$

Let λ_a be the eigenvalues of A with eigenvectors u_a for $a \in \{1, 2, 3, \ldots, n\}$. Also, let $\overline{\lambda_a}$ be the

eigenvalues of A^H with eigenvectors v_a.
We define the Lyapunov function $V(x)$ as:

$$\boxed{V(x) = \sum_a b_a |\langle v_a | x \rangle|^2}$$

Here $b_a > 0$ are arbitrary positive constants.
The sum runs from $a = 1, 2, 3, \ldots, n$.
By definition, it is easy to see that

$V(0) = 0$

$x \neq 0 \Rightarrow V(x) > 0$.

→ Stability theorem

$$\boxed{\begin{array}{l} \mathrm{Re}(\lambda_a) \leqslant 0, \ \forall a \Rightarrow x=0 \text{ is Lyapunov stable} \\ \mathrm{Re}(\lambda_a) < 0, \ \forall a \Rightarrow x=0 \text{ is asymptotically stable} \end{array}}$$

Proof

We note that

$$\langle v_a | Ax \rangle = \langle A^H v_a | x \rangle = \langle \overline{\lambda_a} v_a | x \rangle =$$
$$= \lambda_a \langle v_a | x \rangle$$

and

$$\langle Ax | v_a \rangle = \langle x | A^H v_a \rangle = \langle x | \overline{\lambda_a} v_a \rangle =$$
$$= \overline{\lambda_a} \langle x | v_a \rangle$$

It follows that:

$$\frac{dV}{dt} = \frac{d}{dt}\sum_a b_a |\langle v_a | x\rangle|^2 =$$

$$= \frac{d}{dt}\sum_a b_a \langle v_a | x\rangle\langle x | v_a\rangle =$$

$$= \sum_a \left[b_a\left(\frac{d}{dt}\langle v_a|x\rangle\right)\langle x|v_a\rangle + b_a \langle v_a|x\rangle\left(\frac{d}{dt}\langle x|v_a\rangle\right)\right]$$

$$= \sum_a b_a \left[\langle v_a | Ax\rangle\langle x|v_a\rangle + \langle v_a|x\rangle\langle Ax|v_a\rangle\right] =$$

$$= \sum_a b_a\left[\lambda_a \langle v_a|x\rangle\langle x|v_a\rangle + \langle v_a|x\rangle\left(\overline{\lambda_a}\langle x|v_a\rangle\right)\right]$$

$$= \sum_a b_a (\lambda_a + \overline{\lambda_a})\langle v_a|x\rangle\langle x|v_a\rangle =$$

$$= \sum_a 2 b_a \,\mathrm{Re}(\lambda_a)\,|\langle v_a|x\rangle|^2$$

For $x \neq 0$, $|\langle v_a|x\rangle|^2 > 0$, and by definition $b_a > 0$ for all a. Recall that $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$

a) If $\mathrm{Re}(\lambda_a) \leq 0 \Rightarrow dV/dt \leq 0 \Rightarrow$

$\Rightarrow x = 0$ Lyapunov stable.

b) If $\mathrm{Re}(\lambda_a) < 0 \Rightarrow dV/dt < 0 \Rightarrow$

$\Rightarrow x = 0$ asymptotically stable. □

↳ A matrix A whose eigenvalues satisfy $\mathrm{Re}(\lambda_a) < 0$, $\forall a$ is called <u>negative-definite</u>
Assuming $A \in M_n(\mathbb{R})$, it can be shown that
A negative-definite $\Rightarrow \forall x \in \mathbb{R}^n : \langle x | Ax\rangle < 0$

▼ The 2x2 linear autonomous system

Consider the 2x2 linear autonomous system:

$$\begin{cases} \dot{x}_1 = a x_1 + b x_2 \\ \dot{x}_2 = c x_1 + d x_2 \end{cases} \Leftrightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The eigenvalues λ_1, λ_2 of A are found by solving the equation:

$$\begin{aligned} \det(A - \lambda I) = 0 &\Leftrightarrow (a-\lambda)(d-\lambda) - bc = 0 \Leftrightarrow \\ &\Leftrightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0 \\ &\Leftrightarrow \lambda^2 - \tau\lambda + D = 0 \end{aligned}$$

with: $\tau = \mathrm{tr} A = a + d = \lambda_1 + \lambda_2$

$D = \det A = ad - bc = \lambda_1 \lambda_2$

The solution reads:

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

The general solution of the system reads

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

with v_1, v_2 the eigenvectors corresponding to the eigenvalues λ_1, λ_2.

We note that an eigenvalue λ that satisfies

a) $Re(\lambda) < 0 \longrightarrow$ Gives a contribution that vanishes thus approaching the fixed-point.

b) $Re(\lambda) > 0 \longrightarrow$ Gives a contribution that diverges away from the fixed-point.

c) $Re(\lambda) = 0 \longrightarrow$ Gives a contribution that neither approaches nor diverges from the fixed-point.

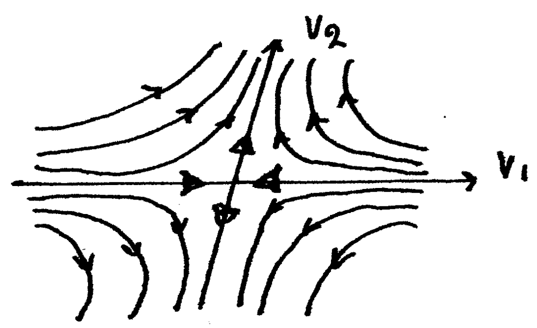
d) $Im(\lambda) \neq 0 \longrightarrow$ Gives a contribution that spirals around the fixed point.

e) $Im(\lambda) = 0 \longrightarrow$ Gives a contribution that does not spiral around the fixed point.

Based on that, we classify the $(0,0)$ fixed point as follows:

Classification of fixed-points in 2d

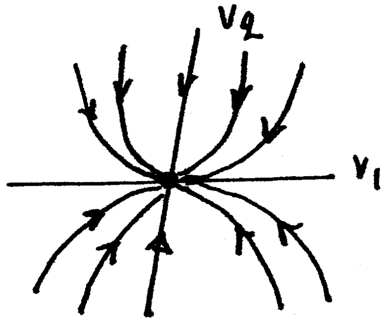
1) Saddle node :



- Eigenvalue condition: $\lambda_1 \lambda_2 < 0$
- τ-D condition: $D < 0$
- Unstable

The shape of the saddle node is controlled by the eigenvectors v_1, v_2.

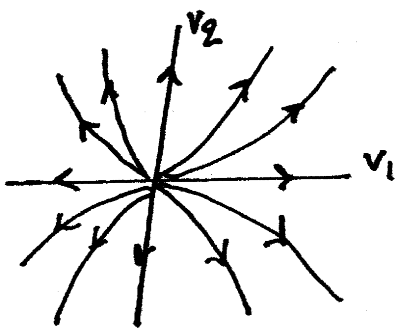
2) Sink :



- Eigenvalue condition
  $\lambda_1, \lambda_2 \in \mathbb{R} \wedge \lambda_1 < 0 \wedge \lambda_2 < 0$
- τ-D condition:
  $D > 0 \wedge \tau^2 - 4D > 0 \wedge \tau < 0$
- Exponentially stable

(v_1 slow, v_2 fast : $|\lambda_1| < |\lambda_2|$)

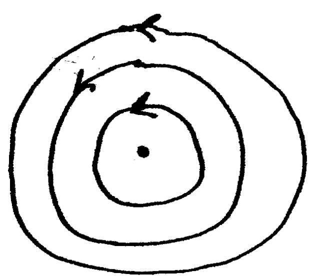
3) Source :



- Eigenvalue condition:
  $\lambda_1, \lambda_2 \in \mathbb{R} \wedge \lambda_1 > 0 \wedge \lambda_2 > 0$
- τ-D condition:
  $D > 0 \wedge \tau^2 - 4D > 0 \wedge \tau > 0$
- Unstable

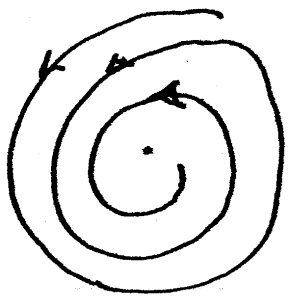
(v_1 fast , v_2 slow : $\lambda_1 > \lambda_2 > 0$)

4) Center :



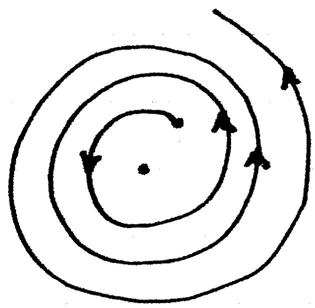
- Eigenvalue condition
  $\begin{cases} Re(\lambda_1) = Re(\lambda_2) = 0 \\ Im(\lambda_1) \neq 0 \wedge Im(\lambda_2) \neq 0 \end{cases}$
- τ-D condition
  $D > 0 \wedge \tau = 0$
- Neutrally stable
  (i.e. Lyapunov stable but not attracting)
- Note that $D > 0 \wedge \tau = 0 \Rightarrow \Delta < 0$

5) Stable spiral :



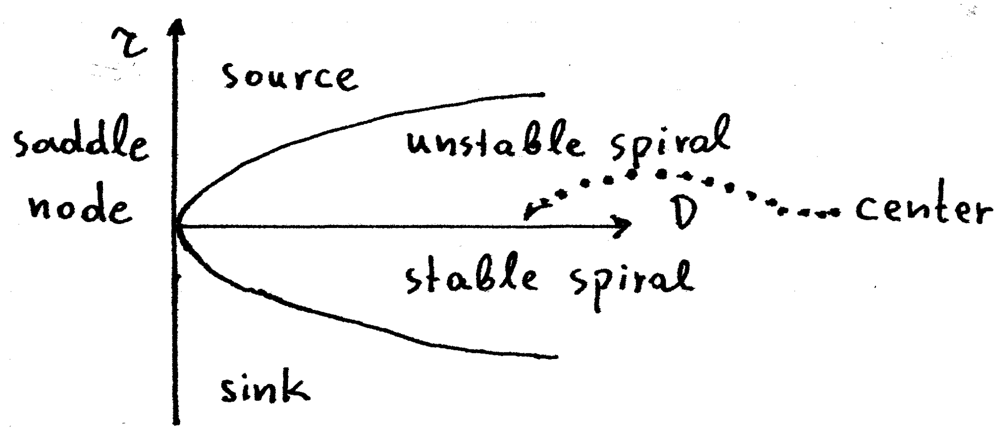
- Eigenvalue condition
$$\begin{cases} Re(\lambda_1) < 0 \wedge Re(\lambda_2) < 0 \\ Im(\lambda_1) \neq 0 \wedge Im(\lambda_2) \neq 0 \end{cases}$$
- τ-D condition
$D > 0 \wedge \tau^2 - 4D < 0 \wedge \tau < 0$
- Exponentially stable

6) Unstable spiral :



- Eigenvalue condition
$$\begin{cases} Re(\lambda_1) > 0 \wedge Re(\lambda_2) > 0 \\ Im(\lambda_1) \neq 0 \wedge Im(\lambda_2) \neq 0 \end{cases}$$
- τ-D condition
$D > 0 \wedge \tau^2 - 4D < 0 \wedge \tau > 0$

Summary of τ-D conditions



$D < 0$: saddle point

$D > 0$: $\tau = 0$: center

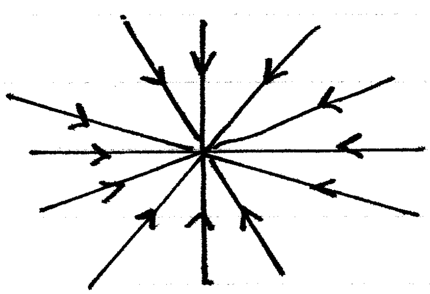
$\tau^2 - 4D > 0$: source $(\tau > 0)$, sink $(\tau < 0)$

$\tau^2 - 4D < 0$: spiral, stable $(\tau < 0)$ or unstable $(\tau > 0)$

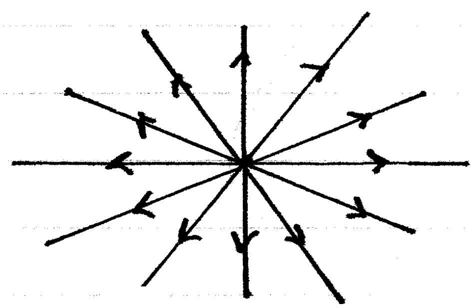
Borderline nodes

Borderline nodes occur when $\lambda_1 = \lambda_2$ which occurs when $\tau^2 - 4D = 0$. Let $E_\lambda = \{ v \in \mathbb{R}^2 \mid Av = \lambda v \}$ be the eigenspace associated with the eigenvalue $\lambda = \lambda_1 = \lambda_2$. We distinguish between two cases: $\dim E_\lambda = 1$ or $\dim E_\lambda = 2$.

7) Stars

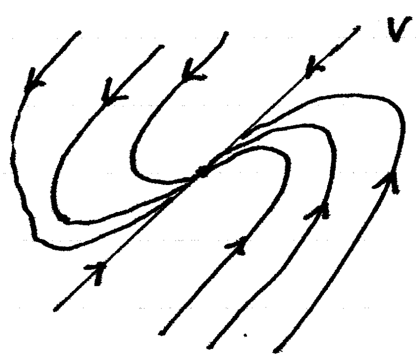


$\lambda_1 = \lambda_2 < 0$
$\dim E_\lambda = 2$

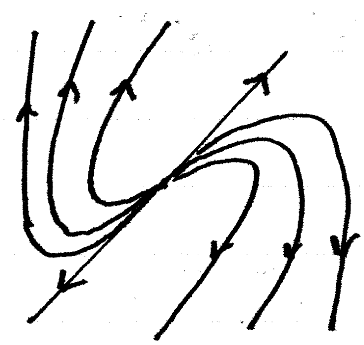


$\lambda_1 = \lambda_2 > 0$
$\dim E_\lambda = 2$

8) Degenerate nodes



$\lambda_1 = \lambda_2 < 0$
$\dim E_\lambda = 1$



$\lambda_1 = \lambda_2 > 0$
$\dim E_\lambda = 1$.

EXAMPLES

a) $\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = 4x_1 - 2x_2 \end{cases} \Leftrightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} =$$

$$= (1-\lambda)(-2-\lambda) - 4 = -2 - \lambda + 2\lambda + \lambda^2 - 4 =$$

$$= \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2) = 0 \Leftrightarrow \underline{\lambda = -3 \vee \lambda = 2}.$$

Since $\begin{cases} \lambda_1, \lambda_2 \in \mathbb{R} \\ \lambda_1 \lambda_2 < 0 \end{cases} \Rightarrow (0,0)$ is a saddle-node.

- To draw a phase portrait we need the eigenvectors.
  In general, for eigenvalue λ
  $Av = \lambda v \Leftrightarrow (A - \lambda I)v = 0 \Leftrightarrow \begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  For $\lambda_1 = 2$:
  $\begin{bmatrix} 1-2 & 1 \\ 4 & -2-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} -x + y = 0 \\ 4x - 4y = 0 \end{cases} \Leftrightarrow y = x$

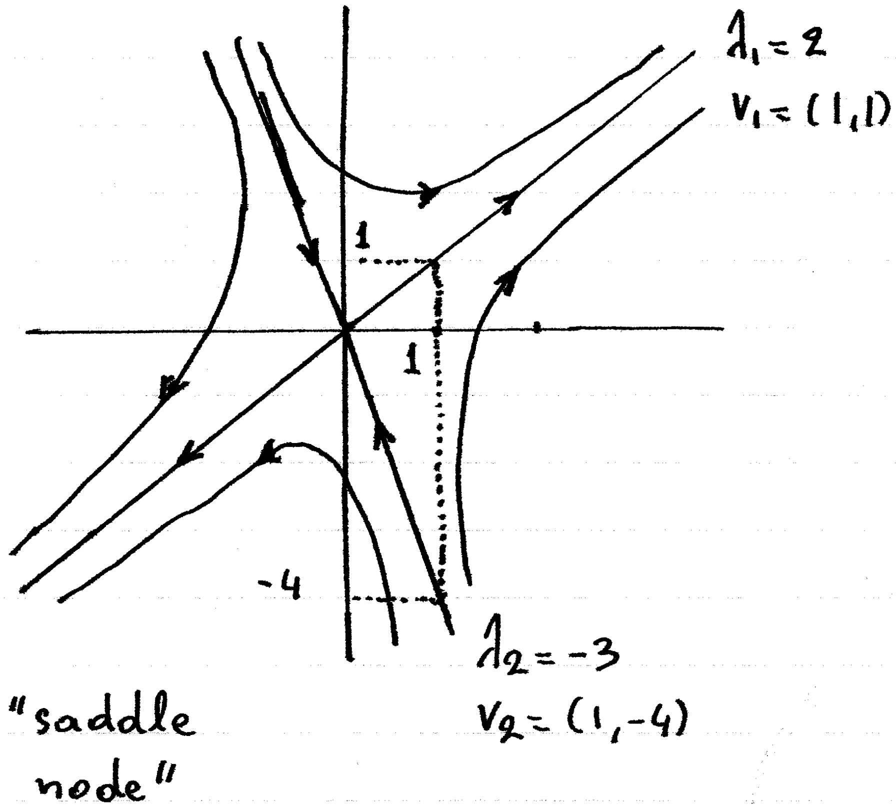
  $\Leftrightarrow (x,y) = (x,x) = x(1,1)$

  thus $v_1 = (1,1)$.

  For $\lambda_2 = -3$:
  $\begin{bmatrix} 1-(-3) & 1 \\ 4 & -2-(-3) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} 4x + y = 0 \\ 4x + y = 0 \end{cases} \Leftrightarrow$

  $\Leftrightarrow y = -4x \Leftrightarrow (x,y) = (x, -4x) = x(1,-4)$

  thus $v_2 = (1,-4)$


$\lambda_1 = 2$
$v_1 = (1,1)$
1
1
-4
$\lambda_2 = -3$
$v_2 = (1,-4)$
"saddle
node"

b) $\begin{cases} \dot{x}_1 = x_1 - 2x_2 \\ \dot{x}_2 = 2x_1 - x_2 \end{cases} \leftarrow A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} =$$

$$= (1-\lambda)(-1-\lambda) + 4 = -1 - \lambda + \lambda + \lambda^2 + 4 =$$

$$= \lambda^2 + 3 = 0 \Leftrightarrow \lambda = \pm i\sqrt{3}$$

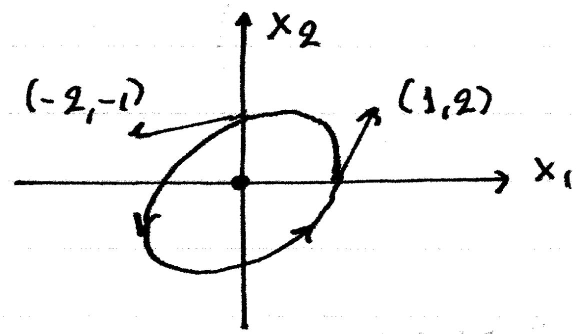
It follows that $(0,0)$ is a center.

- Clockwise or counterclockwise?

The direction of the orbits can be determined by calculating Ax with x a unit vector:

$$\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$



$\rightarrow$ When a linear system has a center, the shape of the orbits can be derived by noting that

$$V(x) = ax_1^2 + bx_1x_2 + cx_2^2$$

with appropriate choice of a, b, c remains constant along the orbits around $(0,0)$.
This $V(x)$ is a Lyapunov function.

For this problem:

$$V(x) = ax_1^2 + bx_1x_2 + cx_2^2 \Rightarrow$$

$$\Rightarrow dV(x)/dt = 2ax_1\dot{x}_1 + b(\dot{x}_1x_2 + x_1\dot{x}_2) + 2cx_2\dot{x}_2 =$$

$$= 2ax_1(x_1 - 2x_2) + b[(x_1 - 2x_2)x_2 + x_1(2x_1 - x_2)] + 2cx_2(2x_1 - x_2)$$

$$= 2ax_1^2 - 4ax_1x_2 + bx_1x_2 - 2bx_2^2 + 2bx_1^2 - bx_1x_2 + 4cx_1x_2 - 2cx_2^2$$

$$= (2a+2b)x_1^2 + (-4a+b-b+4c)x_1x_2 - 2(b+c)x_2^2 =$$

$$= 2(a+b)x_1^2 + 4(c-a)x_1x_2 - 2(b+c)x_2^2$$

Require:

$$\begin{cases} a+b=0 \\ c-a=0 \\ b+c=0 \end{cases} \Leftrightarrow \begin{cases} c-c=0 \\ a=c \\ b=-c \end{cases} \Leftrightarrow \begin{cases} a=c \\ b=-c \end{cases} \Leftrightarrow (a,b,c) = c(1,-1,1)$$

Choose: $(a,b,c) = (1,-1,1)$, thus

$$V(x) = x_1^2 - x_1x_2 + x_2^2.$$

Center orbits have equation:

$$\boxed{(C):\ x_1^2 - x_1x_2 + x_2^2 = C_1}$$

c) $\begin{cases} \dot{x}_1 = -x_1 - x_2 \\ \dot{x}_2 = 3x_1 \end{cases} \longleftarrow \quad A = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ 3 & -\lambda \end{vmatrix} = -\lambda(-1-\lambda) - (-1)\cdot 3$$

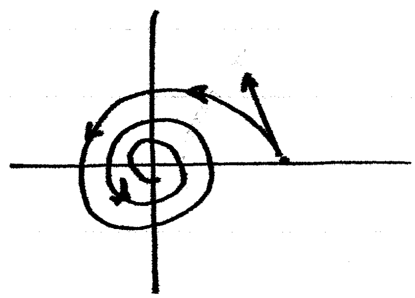
$$\left.\begin{aligned} &= \lambda + \lambda^2 + 3 = \lambda^2 + \lambda + 3 = 0 \\ &\Delta = 1 - 12 = -11 \end{aligned}\right\} \Rightarrow \lambda_{1,2} = \frac{-1 \pm i\sqrt{11}}{2}$$

Since $\lambda_{1,2}$ are complex and $Re(\lambda_1) = Re(\lambda_2) < 0$ it follows that $(0,0)$ is <u>stable spiral.</u>

Since

$$\begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

the direction is counterclockwise.



d) $\begin{cases} \dot{x}_1 = 4x_1 - x_2 \\ \dot{x}_2 = -x_1 + 4x_2 \end{cases} \longleftarrow \quad A = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 1$$

$$\left.\begin{aligned} &= 16 - 8\lambda + \lambda^2 - 1 = \lambda^2 - 8\lambda + 15 \\ &\Delta = 64 - 4\cdot 15 = 64 - 60 = 4 \end{aligned}\right\} \Rightarrow \lambda_{1,2} = \frac{8 \pm 2}{2} = \begin{cases} 5 \\ 3 \end{cases}$$

thus $(0,0)$ is a source.

- Eigenvalues:

For $\lambda_1 = 3$:

$$Ax = 3x \Leftrightarrow \begin{cases} 4x_1 - x_2 = 3x_1 \\ -x_1 + 4x_2 = 3x_2 \end{cases} \Leftrightarrow \begin{cases} x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow x_1 - x_2 = 0 \Leftrightarrow x_1 = x_2 \Leftrightarrow (x_1, x_2) = (1,1)x_1$$

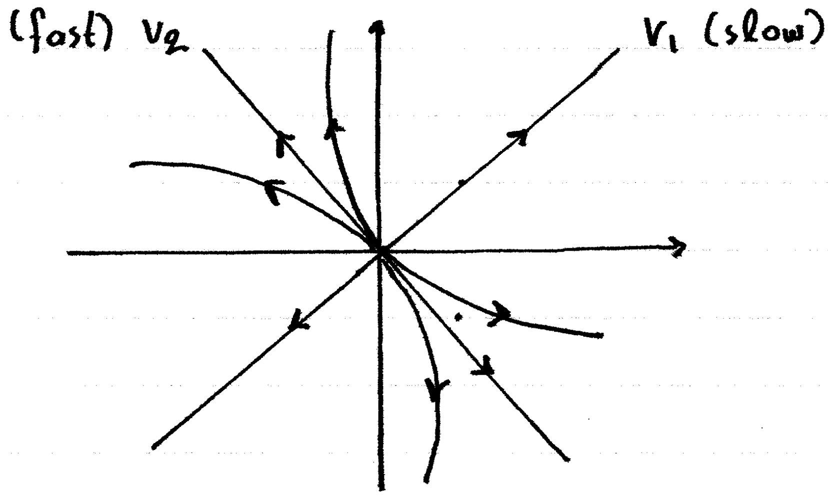
Choose $v_1 = (1,1)$.

For $\lambda_2 = 5$:

$$Ax = 5x \Leftrightarrow \begin{cases} 4x_1 - x_2 = 5x_1 \\ -x_1 + 4x_2 = 5x_2 \end{cases} \Leftrightarrow \begin{cases} -x_1 - x_2 = 0 \\ -x_1 - x_2 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow x_1 = -x_2 \Leftrightarrow (x_1, x_2) = (-1,1)x_2$$

Choose $v_2 = (-1,1)$.



GODE 09: Nonlinear autonomous systems

NONLINEAR AUTONOMOUS SYSTEMS

▼ Local analysis of fixed points

Consider the nonlinear autonomous systems $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$.
Let $x_0 \in \mathbb{R}^n$ be a fixed point with $f(x_0) = 0$.
Let $x_0(t) = x_0$ be a solution with the fixed point as initial condition.

To examine the stability of x_0, we consider the following perturbation around x_0:

$$x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$$

with $0 < \varepsilon \ll 1$. It follows that:

$$\dot{x}(t) = \dot{x}_0 + \varepsilon \dot{x}_1(t) + O(\varepsilon^2) = \varepsilon \dot{x}_1(t) + O(\varepsilon^2)$$

$$f(x) = f(x_0 + \varepsilon x_1) = f(x_0) + (\varepsilon\ \ ) Df(x_0) x_1 + O(\varepsilon^2)$$

$$= \varepsilon Df(x_0) x_1 + O(\varepsilon^2)$$

Equating the ε terms gives the linearization

$$\boxed{\dot{x}_1(t) = Df(x_0) x_1(t)}$$

Here Df is the Jacobian matrix given by

$$\boxed{[Df]_{ab} = \frac{\partial f_a}{\partial x_b}}$$

Def : We say that the fixed point x_0 is a hyperbolic point if and only if

$$\boxed{\forall \lambda \in \lambda(Df(x_0)) : \mathrm{Re}(\lambda) \neq 0}$$

- It can be shown that if x_0 is a hyperbolic fixed-point, then the local behavior of the nonlinear systems is topologically equivalent to the local behaviour of the linearized equation $\dot{x} = Df(x_0)\,x$.
- It follows that hyperbolic fixed-points can be classified according to the eigenvalues of the Jacobian matrix $Df(x_0)$.

EXAMPLE

$$\begin{cases} \dot{x}_1 = x_1(3 - x_1 - x_2) \\ \dot{x}_2 = x_2(x_1 - 1) \end{cases}$$

- Fixed points:

$$\begin{cases} x_1(3-x_1-x_2) = 0 \\ x_2(x_1-1) = 0 \end{cases} \Leftrightarrow \begin{cases} x_1(3-x_1) = 0 \\ x_2 = 0 \end{cases} \vee \begin{cases} 1\cdot(3-1-x_2) = 0 \\ x_1 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \vee \begin{cases} x_1 = 3 \\ x_2 = 0 \end{cases} \vee \begin{cases} x_2 = 2 \\ x_1 = 1 \end{cases}$$

thus set of fixed points: $\{(0,0), (3,0), (1,2)\}$.

- Jacobian

$$\frac{\partial f_1}{\partial x_1} = 1\cdot(3-x_1-x_2)+x_1(-1) = 3-2x_1-x_2$$

$$\frac{\partial f_1}{\partial x_2} = -x_1$$

$$\frac{\partial f_2}{\partial x_1} = x_2$$

$$\frac{\partial f_2}{\partial x_2} = x_1-1$$

thus $Df(x_1,x_2) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} =$

$$= \begin{bmatrix} 3-2x_1-x_2 & -x_1 \\ x_2 & x_1-1 \end{bmatrix}$$

- At $(0,0)$

$$Df(0,0) = \begin{bmatrix} 3-0-0 & 0 \\ 0 & 0-1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Eigenvalues $\lambda_1 = 3$ with $v_1 = (1,0)$
$\lambda_2 = -1$ with $v_2 = (0,1)$

thus $(0,0)$ is a saddle point.

- At $(1,2)$

$$Df(1,2) = \begin{bmatrix} 3-2\cdot 1-2 & -1 \\ 2 & 1-1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}$$

$$p(\lambda) = \det(Df(1,2) - \lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} =$$

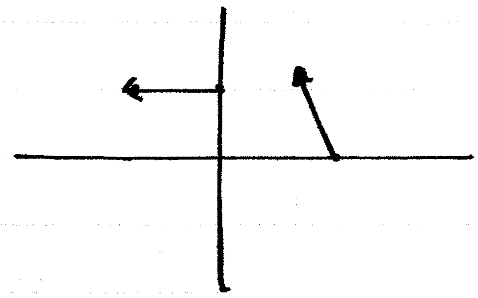
$$= (-1-\lambda)(-\lambda) - (-1)\cdot 2 = \lambda(\lambda+1) + 2 =$$

$$\left.\begin{aligned} &= \lambda^2 + \lambda + 2 \\ &\Delta = 1^2 - 4\cdot 1\cdot 2 = -7 \end{aligned}\right\} \Rightarrow \lambda_{1,2} = \frac{-1 \pm i\sqrt{7}}{2}$$

Note that

$$\begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$



Thus $(1,2)$ is a counterclockwise stable spiral.

• <u>At $(3,0)$</u>

$$Df(3,0) = \begin{bmatrix} 3-2\cdot 3-0 & -3 \\ 0 & 3-1 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 0 & 2 \end{bmatrix}$$

$$p(\lambda) = \det(Df(3,0) - \lambda I) = \begin{vmatrix} -3-\lambda & -3 \\ 0 & 2-\lambda \end{vmatrix} =$$

$$= (-3-\lambda)(2-\lambda) - (-3)\cdot 0 = (\lambda+3)(\lambda-2) = 0 \Leftrightarrow$$

$\Leftrightarrow \lambda_1 = -3$ or $\lambda_2 = 2$. $\leftarrow$ $(3,0)$ is a saddle point.

Eigenvectors:

a) For $\underline{\lambda_1 = -3}$:

$$Ax = -3x \Leftrightarrow \begin{cases} -3x_1 - 3x_2 = -3x_1 \\ 2x_2 = -3x_2 \end{cases} \Leftrightarrow \begin{cases} -3x_2 = 0 \\ 5x_2 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow x_2 = 0 \Leftrightarrow (x_1, x_2) = (1,0)\, x_1 \leftarrow \underline{v_1 = (1,0)}$$

b) For $\underline{\lambda_2 = 2}$:

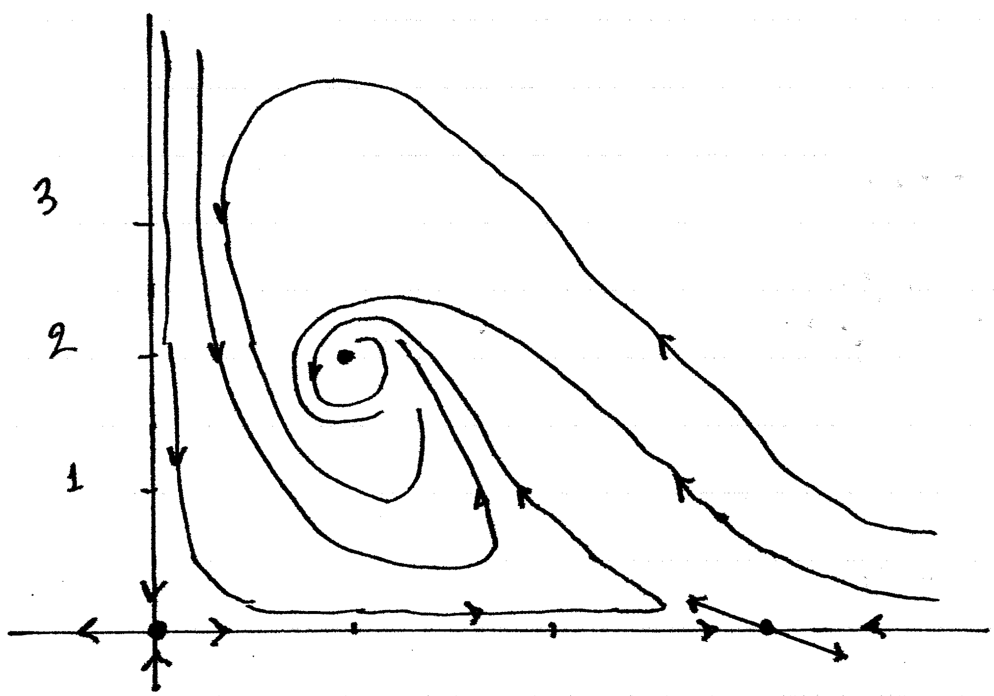
$$Ax = 2x \Leftrightarrow \begin{cases} -3x_1 - 3x_2 = 2x_1 \\ 2x_2 = 2x_2 \end{cases} \Leftrightarrow -5x_1 - 3x_2 = 0 \Leftrightarrow$$

$$\Leftrightarrow x_2 = \frac{-5}{3} x_1 \Leftrightarrow (x_1, x_2) = (1, \frac{-5}{3}) x_1$$

thus choose $v_2 = (3, -5)$.

- Phase Portrait

$(0,0)$ saddle point with $\lambda_1 = 3$, $v_1 = (1,0)$, $\lambda_2 = -1$, $v_2 = (0,1)$
$(1,2)$ counterclockwise stable spiral
$(3,0)$ saddle point with $\lambda_1 = -3$, $v_1 = (1,0)$, $\lambda_2 = 2$, $v_2 = (3,-5)$



Nonlinear Centers

- Fixed points which, according to local linear analysis, appear to be centers are NOT hyperbolic. It follows that the original nonlinear system may or may not be a center. To determine whether a fixed point with

$$\exists\, \lambda \in \lambda(Df(x_0)) : Re(\lambda) = 0$$

is or is not a center, we rely on the following methods:

(1) → Conversion to polar coordinates

A two-dimensional autonomous system of the form

$$\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2) \end{cases}$$

can be rewritten in polar coordinates (r, ϑ) with $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$ using the following identities:

$$\boxed{\dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}} \quad \boxed{\dot{\vartheta} = \frac{x_1\dot{x}_2 - \dot{x}_1 x_2}{r^2}}$$

Proof

$$x_1^2+x_2^2=r^2\cos^2\vartheta+r^2\sin^2\vartheta=r^2(\cos^2\vartheta+\sin^2\vartheta)=r^2 \Rightarrow$$
$$\Rightarrow 2x_1\dot{x}_1+2x_2\dot{x}_2=2r\dot{r} \Rightarrow \dot{r}=\frac{x_1\dot{x}_1+x_2\dot{x}_2}{r}$$

Since $\begin{cases} x_1=r\cos\vartheta \\ x_2=r\sin\vartheta \end{cases} \Rightarrow \begin{cases} \dot{x}_1=\dot{r}\cos\vartheta-r\dot{\vartheta}\sin\vartheta \\ \dot{x}_2=\dot{r}\sin\vartheta+r\dot{\vartheta}\cos\vartheta \end{cases} \Rightarrow$

$$\Rightarrow x_1\dot{x}_2-\dot{x}_1x_2=(r\cos\vartheta)(\dot{r}\sin\vartheta+r\dot{\vartheta}\cos\vartheta)-(\dot{r}\cos\vartheta-r\dot{\vartheta}\sin\vartheta)(r\sin\vartheta)$$
$$=r\dot{r}\cos\vartheta\sin\vartheta+r^2\dot{\vartheta}\cos^2\vartheta-r\dot{r}\cos\vartheta\sin\vartheta+r^2\dot{\vartheta}\sin^2\vartheta=$$
$$=r^2\dot{\vartheta}\cos^2\vartheta+r^2\dot{\vartheta}\sin^2\vartheta=r^2\dot{\vartheta}(\cos^2\vartheta+\sin^2\vartheta)=r^2\dot{\vartheta} \Rightarrow$$
$$\Rightarrow \dot{\vartheta}=\frac{x_1\dot{x}_2-\dot{x}_1x_2}{r^2}$$

EXAMPLE

$$\begin{cases} \dot{x}_1=-x_2+ax_1(x_1^2+x_2^2)=f_1(x_1,x_2) \\ \dot{x}_2=x_1+ax_2(x_1^2+x_2^2)=f_2(x_1,x_2) \end{cases}$$

Solution

Obvious fixed point at $(x_1,x_2)=(0,0)$

Jacobian

$$\left.\begin{aligned} \frac{\partial f_1}{\partial x_1}&=3ax_1^2+ax_2^2 \\ \frac{\partial f_1}{\partial x_2}&=-1+2ax_1x_2 \\ \frac{\partial f_2}{\partial x_1}&=1+2ax_1x_2 \\ \frac{\partial f_2}{\partial x_2}&=ax_1^2+3ax_2^2 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow Df(x_1,x_2) = \begin{bmatrix} 3ax_1^2 + ax_2^2 & -1 + 2ax_1x_2 \\ 1 + 2ax_1x_2 & ax_1^2 + 3ax_2^2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow Df(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(0,0) - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} =$$

$$= (-\lambda)(-\lambda) - (-1)\cdot 1 = \lambda^2 + 1 \Rightarrow$$

$$\Rightarrow \lambda(Df(0,0)) = \{+i, -i\} \leftarrow \text{a center ?}$$

- Convert to polar coordinates:

$$r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2 =$$

$$= x_1[-x_2 + ax_1(x_1^2+x_2^2)] + x_2[x_1 + ax_2(x_1^2+x_2^2)]$$

$$= -x_1x_2 + ax_1^2r^2 + x_1x_2 + ax_2^2r^2 =$$

$$= ax_1^2r^2 + ax_2^2r^2 = ar^2(x_1^2+x_2^2) = ar^4 \Rightarrow$$

$$\Rightarrow \underline{\dot{r} = ar^3}, \text{ and}$$

$$r^2\dot{\vartheta} = x_1\dot{x}_2 - \dot{x}_1x_2 =$$

$$= x_1[x_1 + ax_2(x_1^2+x_2^2)] - [-x_2 + ax_1(x_1^2+x_2^2)]x_2 =$$

$$= x_1^2 + ax_1x_2r^2 + x_2^2 - ax_1x_2r^2 =$$

$$= x_1^2 + x_2^2 = r^2 \Rightarrow \underline{\dot{\vartheta} = 1}$$

Thus $\begin{cases} \dot{r} = ar^3 \\ \dot{\vartheta} = 1 \end{cases}$.

For $a = 0$: $\dot{r} = 0$ and $\dot{\vartheta} = 1 \Rightarrow (0,0)$ is a center.

For $a > 0$: $\dot{r} > 0$ and $\dot{\vartheta} = 1 \Rightarrow$

$\Rightarrow (0,0)$ is unstable counterclockwise spiral.

For $a<0$: $\dot{r}<0$ and $\dot{\theta}=1 \Rightarrow$
$\Rightarrow (0,0)$ is a counterclockwise stable spiral.

② → Conservative systems

Consider a general autonomous system $\dot{x}=f(x)$ with $x\in\mathbb{R}^n$ and $f:\mathbb{R}^n\to\mathbb{R}^n$.

Def: We say that the system is conservative if and only if

a) There is a function $V:\mathbb{R}^n\to\mathbb{R}$ such that $(d/dt)V(x(t))=0$

b) $\forall A\subset\mathbb{R}^n: (A \text{ open set} \Rightarrow V \text{ non-constant in } A)$.

Def: $x_0\in\mathbb{R}^n$ is an isolated fixed-point if and only if

a) $f(x_0)=0$

b) $\exists\varepsilon>0:\forall x\in\mathbb{R}^n: (0<\|x-x_0\|<\varepsilon \Rightarrow f(x)\neq 0)$.

Def: $x(t)$ is a closed orbit if and only if $\forall t>0:\exists\tau>0: x(t+\tau)=x(t)$.

Thm: Assume that

a) $x_0\in\mathbb{R}^n$ is an isolated fixed-point

b) f is continuously differentiable in $\mathbb{R}^n$

c) the system is conservative with $(d/dt)V(x(t))=0$

d) x_0 is a local min or max of $V(x)$.
Then,
$\exists \varepsilon > 0 : (\|x(0) - x_0\| < \varepsilon \Rightarrow x(t)$ is a closed orbit$)$.

<u>Prop</u> : If $\dot{x} = f(x)$ is conservative then it has no attracting fixed points.

<u>Proof</u>

Let $x_0 \in \mathbb{R}^n$ be an attracting fixed point. Let A be the basin of attraction of x_0 such that

$$\forall y \in A : (x(0) = y \Rightarrow \lim_{t \to +\infty} x(t) = x_0).$$

Let $y \in A$ be given and choose $x(0) = y$. Then
$V(y) = V(x(0)) = V(x(t))$, $\forall t > 0 \Rightarrow$

$$\Rightarrow V(y) = \lim_{t \to +\infty} V(x(t)) = V(\lim_{t \to +\infty} x(t)) = V(x_0), \forall y \in A$$

$\Rightarrow V$ constant in $A \leftarrow$ contradiction.
Thus x_0 cannot be an attracting fixed point. □

↳ Thus to show that a system is NOT conservative it is sufficient to show that it has an attracting fixed point.

- Constructing $V(x)$ is easy for systems of the following forms:

Form 1 : $\boxed{\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1) \end{cases}} \longleftarrow \boxed{\ddot{x} = f(x)}$

Consider:

$$x_1\dot{x}_1 + x_2\dot{x}_2 = x_1\dot{x}_1 + x_2 f(x_1) = x_1\dot{x}_1 + \dot{x}_1 f(x_1) =$$
$$= \dot{x}_1(x_1 + f(x_1)) \Rightarrow$$
$$\Rightarrow -f(x_1)\dot{x}_1 + x_2\dot{x}_2 = 0 \longleftarrow \text{easily integrated to yield } V(x).$$

EXAMPLE

$$\begin{cases} \dot{x}_1 = -x_2 - x_2^3 \\ \dot{x}_2 = x_1 \end{cases}$$

- Fixed points

$$\begin{cases} -x_2 - x_2^3 = 0 \\ x_1 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2(1+x_2^2) = 0 \\ x_1 = 0 \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} x_2 = 0 \\ x_1 = 0 \end{cases} \vee \begin{cases} 1 + x_2^2 = 0 \\ x_1 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$
$$\Leftrightarrow (x_1, x_2) = (0,0)$$

- Local linear analysis

$$Df(x_1, x_2) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -1-3x_2^2 \\ 1 & 0 \end{bmatrix} \Rightarrow Df(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(0,0) - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} =$$

$$= (-\lambda)(-\lambda) - (-1)\cdot 1 = \lambda^2 + 1 \Rightarrow$$

$\Rightarrow \lambda(Df(0,0)) = \{+i, -i\} \Rightarrow (0,0)$ is a <u>linear center.</u>

- However we still have to prove that it is a nonlinear center.

- <u>Nonlinear center.</u>

Let:

$$x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_2 - x_2^3) + x_2 x_1 =$$

$$= -x_1x_2 - x_1x_2^3 + x_1x_2 = -x_1x_2^3 =$$

$$= -x_2^3\dot{x}_2 \Rightarrow$$

$$\Rightarrow x_1\dot{x}_1 + (x_2 + x_2^3)\dot{x}_2 = 0 \Rightarrow$$

$$\Rightarrow \frac{d}{dt}\left[\frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{x_2^4}{4}\right] = 0$$

$$\Rightarrow \underline{2x_1^2 + 2x_2^2 + x_2^4 = c} \qquad (1)$$

For $V(x_1,x_2) = 2x_1^2 + 2x_2^2 + x_2^4$ we have $V(0,0) = 0$ and $V(x_1,x_2) > 0$, $\forall (x_1,x_2) \in \mathbb{R}^2 - \{(0,0)\}$, thus $(0,0)$ is a local minimum. It follows that $(0,0)$ is a nonlinear center. The closed trajectories are given by (1).

Form 2 : $\boxed{\begin{aligned}\dot{x}_1 &= f(x_1)\,g_1(x_2)\\ \dot{x}_2 &= f(x_2)\,g_2(x_1)\end{aligned}}$

Consider:

$$\frac{dV}{dt} = \frac{g_2(x_1)}{f(x_1)}\dot{x}_1 - \frac{g_1(x_2)}{f(x_2)}\dot{x}_2 =$$

$$= \frac{g_2(x_1)}{f(x_1)} f(x_1)\,g_1(x_2) - \frac{g_1(x_2)}{f(x_2)} f(x_2)\,g_2(x_1) =$$

$$= g_2(x_1)\,g_1(x_2) - g_1(x_2)\,g_2(x_1) = 0$$

which can then be easily integrated to yield the Lyapunov function.

EXAMPLE

$$\begin{cases}\dot{x}_1 = x_1 - x_1 x_2\\ \dot{x}_2 = -x_2 + x_1 x_2\end{cases}$$

- Fixed points

$$\begin{cases} x_1 - x_1 x_2 = 0\\ -x_2 + x_1 x_2 = 0\end{cases} \Leftrightarrow \begin{cases} x_1(1-x_2) = 0\\ x_2(x_1-1) = 0\end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 = 0\\ x_2(0-1) = 0\end{cases} \vee \begin{cases} x_2 = 1\\ 1\cdot(x_1-1) = 0\end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 = 0\\ x_2 = 0\end{cases} \vee \begin{cases} x_1 = 1\\ x_2 = 1\end{cases} \leftarrow \text{Fixed points: } \{(0,0),(1,1)\}.$$

- Local linear analysis

$$Df(x_1,x_2) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1-x_2 & -x_1 \\ x_2 & x_1-1 \end{bmatrix}$$

- At $(0,0)$:

$$Df(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda(Df(0,0)) = \{+1,-1\} \Rightarrow$$

$\Rightarrow (0,0)$ is a saddle point.

- At $(1,1)$

$$Df(1,1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(1,1) - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} =$$

$$= (-\lambda)(-\lambda) - 1\cdot(-1) = \lambda^2 + 1 \Rightarrow$$

$\Rightarrow \lambda(Df(1,1)) = \{+i, -i\} \Rightarrow (1,1)$ is a linear center.

- Nonlinear center.

We now show that $(1,1)$ is a nonlinear center.

Construct a Lyapunov function:

Note that:

$$\dot{x}_1 = x_1(1-x_2)$$

$$\dot{x}_2 = x_2(x_1-1)$$

so we define:

$$\frac{dV}{dt} = \frac{x_1 - 1}{x_1}\dot{x}_1 - \frac{1-x_2}{x_2}\dot{x}_2 =$$

$$= \frac{x_1-1}{x_1} x_1(1-x_2) - \frac{1-x_2}{x_2} x_2(x_1-1) =$$

$$= (x_1-1)(1-x_2) - (1-x_2)(x_1-1) = 0 \Rightarrow$$

$$\Rightarrow \left(1 - \frac{1}{x_1}\right)\dot{x}_1 + \left(1 - \frac{1}{x_2}\right)\dot{x}_2 = 0 \Rightarrow$$

$$\Rightarrow \frac{d}{dt}\left[x_1 - \ln x_1 + x_2 - \ln x_2\right] = 0 \Rightarrow$$

$$\Rightarrow \underline{x_1 + x_2 - \ln(x_1 x_2) = c} \leftarrow \text{shape of trajectories.}$$

We now show that $(1,1)$ is an extremum by calculating the Hessian:

Let $f(x_1, x_2) = x_1 + x_2 - \ln(x_1 x_2)$. Then

$$\nabla f(x_1, x_2) = (\partial f/\partial x_1, \partial f/\partial x_2) =$$

$$= (1 - 1/x_1, 1 - 1/x_2) \Rightarrow$$

$$\Rightarrow \nabla f(1,1) = (1-1, 1-1) = (0,0).$$

Since:

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(1 - \frac{1}{x_1}\right) = \frac{1}{x_1^2}$$

$$\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(1 - \frac{1}{x_2}\right) = \frac{1}{x_2^2}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(1 - \frac{1}{x_2}\right) = 0$$

the Hessian reads:

$$\Delta(x_1,x_2) = \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 =$$

$$= \frac{1}{x_1^2}\frac{1}{x_2^2} - 0^2 = \left(\frac{1}{x_1 x_2}\right)^2 \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow \Delta(1,1) = 1 > 0 \\ \dfrac{\partial^2 f(1,1)}{\partial x_1^2} = \dfrac{1}{1^2} = 1 > 0 \end{array}\right\} \Rightarrow$$

$\Rightarrow$ $(1,1)$ is a local min of
$f(x_1,x_2) = x_1 + x_2 - \ln(x_1 x_2)$

It follows that $(1,1)$ is a nonlinear center.

$\hookrightarrow$ Recall that for

$$\Delta = \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2$$

we have the following sufficient conditions:

a) $\left.\begin{array}{l} \Delta(x_0) > 0 \\ \partial^2 f(x_0)/\partial x_1^2 > 0 \end{array}\right\} \Rightarrow x_0 \in \mathbb{R}^2$ is a local min

b) $\left.\begin{array}{l} \Delta(x_0) > 0 \\ \partial^2 f(x_0)/\partial x_1^2 < 0 \end{array}\right\} \Rightarrow x_0 \in \mathbb{R}^2$ is a local max.

③ → Reversible systems

- Consider the system $\dot{x} = f(x)$ with $f: \mathbb{R}^n \to \mathbb{R}^n$

Def: We say that a mapping $P: \mathbb{R}^n \to \mathbb{R}^n$ is an involution if and only if
$\forall x \in \mathbb{R}^n : P(P(x)) = x$.

Def: We say that the system $\dot{x} = f(x)$ is reversible if and only if there is an involution P such that

$$\frac{d}{dt} P(x) = -f(P(x))$$

- A reversible system is invariant under the transformation

$$t \to -t$$
$$x \to P(x)$$

- We define the symmetry section of the involution P as:

$$\text{Fix}(P) = \{x \in \mathbb{R}^n \mid P(x) = x\}$$

Thm: Assume that the system $\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2) \end{cases}$ is reversible under the involution P. Then, if

$$\left.\begin{array}{l} x_0 \in \text{Fix}(P) \\ x_0 \text{ linear center} \end{array}\right\} \Rightarrow x_0 \text{ nonlinear center.}$$

We confine our attention to the two-dimensional system

$$\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2) \end{cases} \quad (1)$$

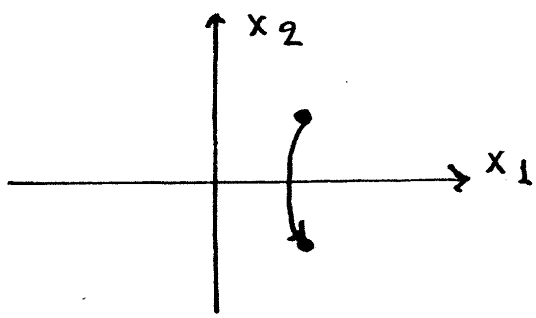
▶ <u>Reflection around x-axis</u>

Assume that

$f(x_1, -x_2) = -f(x_1, +x_2)$, $\forall x_1, x_2 \in \mathbb{R}$

$g(x_1, -x_2) = g(x_1, x_2)$, $\forall x_1, x_2 \in \mathbb{R}$

Then, the system (1) is reversible under the involution $(x_1, -x_2) = P(x_1, x_2)$



We note that the symmetry section is $Fix(P) = \{(x,0) \mid x \in \mathbb{R}\}$.

<u>Proof</u>

Let $x = (x_1, x_2)$ and $F(x) = (f(x_1, x_2), g(x_1, x_2))$.

Then:

$$\frac{d}{dt} P(x) = \frac{d}{dt}(x_1, -x_2) = (\dot{x}_1, -\dot{x}_2) =$$

$$= (f(x_1, x_2), -g(x_1, x_2)) =$$

$$= (-f(x_1, -x_2), -g(x_1, -x_2)) =$$

$$= -(f(x_1, -x_2), g(x_1, -x_2)) = -F(P(x)) \quad \square$$

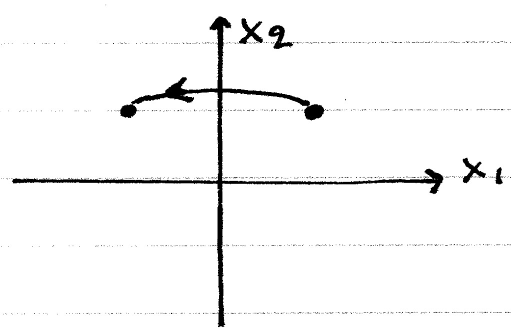
▸ Reflection around y-axis

Assume that

$$f(-x_1,x_2)=f(x_1,x_2) \quad , \forall x_1,x_2\in\mathbb{R}$$
$$g(-x_1,x_2)=-g(x_1,x_2), \forall x_1,x_2\in\mathbb{R}$$

then (1) is reversible under the involution

$$P(x_1,x_2)=(-x_1,x_2)$$



We note that the symmetry section is
$Fix(P)=\{(0,y) \mid y\in\mathbb{R}\}$

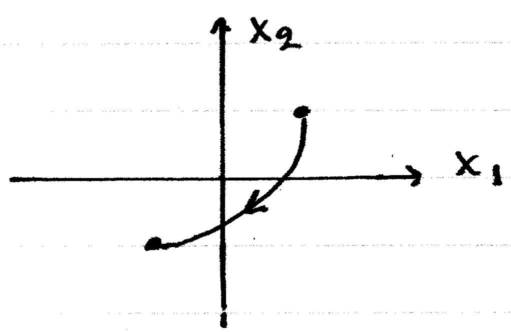
▸ Reflection around x-axis and y-axis

Assume that

$$f(-x_1,-x_2)=f(x_1,x_2) \quad , \forall x_1,x_2\in\mathbb{R}$$
$$g(-x_1,-x_2)=g(x_1,x_2) \quad , \forall x_1,x_2\in\mathbb{R}$$

then (1) is reversible under the involution

$$P(x_1,x_2)=(-x_1,-x_2)$$



We note that the symmetry section is:
$Fix(P)=\{(0,0)\}$.

EXAMPLE

$$\begin{cases} \dot{x}_1 = x_2 - x_2^3 \\ \dot{x}_2 = -x_1 - x_2^2 \end{cases} \longleftarrow \text{Classification of fixed points.}$$

Proof

Let $f(x_1,x_2) = x_2 - x_2^3$ and $g(x_1,x_2) = -x_1 - x_2^2$.

- Fixed points:

$$\begin{cases} f(x_1,x_2) = 0 \\ g(x_1,x_2) = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 - x_2^3 = 0 \\ -x_1 - x_2^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2(1-x_2)(1+x_2) = 0 \\ x_1 = -x_2^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \vee \begin{cases} x_1 = -1 \\ x_2 = 1 \end{cases} \vee \begin{cases} x_1 = -1 \\ x_2 = -1 \end{cases}$$

- Jacobian

$$Df(x_1,x_2) = \begin{bmatrix} 0 & 1-3x_2^2 \\ -1 & -2x_2 \end{bmatrix}$$

- At $(x_1,x_2) = (0,0)$

$$Df(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(0,0) - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - (-1) =$$

$$= \lambda^2 + 1 \Rightarrow \lambda(Df(0,0)) = \{+i, -i\} \Rightarrow$$

$\Rightarrow (0,0)$ is a linear center.

Since $f(x_1,-x_2) = (-x_2) - (-x_2)^3 = -(x_2 - x_2^3) =$

$$= -f(x_1,x_2)$$

and

$g(x_1, -x_2) = -x_1 - (-x_2)^2 = -x_1 - x_2^2 = g(x_1, x_2)$

it follows that the system is reversible. Thus, since

$$\left.\begin{array}{l}(0,0) \text{ is a linear center} \\ (0,0) \in Fix(P) = \{(x,0) \mid x \in \mathbb{R}\}\end{array}\right\} \Rightarrow (0,0) \text{ is a nonlinear center.}$$

• At $(x_1, x_2) = (-1, 1)$

$$Df(-1,1) = \begin{bmatrix} 0 & 1 - 3 \cdot 1^2 \\ -1 & -2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & -2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(-1,1) - \lambda I) = \begin{vmatrix} -\lambda & -2 \\ -1 & -\lambda - 2 \end{vmatrix} =$$

$$= (-\lambda)(-\lambda - 2) - (-1)(-2) = \lambda(\lambda + 2) - 2 = \lambda^2 + 2\lambda - 2.$$

$$\Delta = 2^2 - 4 \cdot 1 \cdot (-2) = 4 + 8 = 12 = 4 \cdot 3 \Rightarrow \lambda_{1,2} = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$$

$\Rightarrow \lambda(Df(-1,1)) = \{-1-\sqrt{3}, -1+\sqrt{3}\} \Rightarrow (-1,1)$ is a saddle point.

• At $(x_1, x_2) = (-1, -1)$

$$Df(-1,-1) = \begin{bmatrix} 0 & 1 - 3(-1)^2 \\ -1 & -2(-1) \end{bmatrix} = \begin{bmatrix} 0 & 1-3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & 2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(-1,-1) - \lambda I) = \begin{vmatrix} -\lambda & -2 \\ -1 & -\lambda + 2 \end{vmatrix} = (-\lambda)(-\lambda + 2) - (-1)(-2)$$

$$= \lambda^2 - 2\lambda - 2$$

$$\Delta = (-2)^2 - 4 \cdot 1 \cdot (-2) = 4 + 8 = 12 = 4 \cdot 3 \Rightarrow \lambda_{1,2} = \frac{-(-2) \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

$\Rightarrow \lambda(Df(-1,-1)) = \{1+\sqrt{3}, 1-\sqrt{3}\} \Rightarrow$

$\Rightarrow (-1,-1)$ is a saddle point.

EXAMPLE

$$\begin{cases} \dot{x}_1 = 2\cos x_1 + \cos x_2 \\ \dot{x}_2 = 2\cos x_2 + \cos x_1 \end{cases} \leftarrow$$ Show that system is reversible but not conservative.

- Reversability.

Let $f(x_1, x_2) = 2\cos x_1 + \cos x_2$ and
$g(x_1, x_2) = 2\cos x_2 + \cos x_1$.

Since:

$$f(-x_1, -x_2) = 2\cos(-x_1) + \cos(-x_2) =$$
$$= 2\cos x_1 + \cos x_2 = f(x_1, x_2)$$

and

$$g(-x_1, -x_2) = 2\cos(-x_2) + \cos(-x_1) =$$
$$= 2\cos x_2 + \cos x_1 = g(x_1, x_2)$$

thus the system is reversible with respect to the involution $P(x_1, x_2) = (-x_1, -x_2)$

- Not conservative

It is sufficient to show that the system has an attracting fixed point.

We first find the fixed points of the system:

$$\begin{cases} 2\cos x_1 + \cos x_2 = 0 \\ 2\cos x_2 + \cos x_1 = 0 \end{cases} \Leftrightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \cos x_1 \\ \cos x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \cos x_1 = 0 \\ \cos x_2 = 0 \end{cases} \Leftrightarrow \exists\, k, \lambda \in \mathbb{Z} : \begin{cases} x_1 = k\pi + \pi/2 \\ x_2 = \lambda\pi + \pi/2 \end{cases}$$

The Jacobian of the system reads:

$$Df(x_1, x_2) = \begin{bmatrix} -2\sin x_1 & -\sin x_2 \\ -\sin x_1 & -2\sin x_2 \end{bmatrix}$$

At $(x_1, x_2) = (\pi/2, \pi/2)$:

$$Df(\pi/2, \pi/2) = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det\left(Df(\pi/2, \pi/2) - \lambda I\right) = \begin{vmatrix} -2-\lambda & -1 \\ -1 & -2-\lambda \end{vmatrix}$$

$$= (-2-\lambda)^2 - (-1)^2 = (\lambda+2)^2 - 1 = 0 \Leftrightarrow$$

$$\Leftrightarrow (\lambda+2)^2 = 1 \Leftrightarrow \lambda + 2 = \pm 1 \Leftrightarrow \lambda = -2 \pm 1 = \begin{cases} -3 \\ -1 \end{cases}$$

thus $\lambda\left(Df(\pi/2, \pi/2)\right) = \{-3, -1\} \Rightarrow$

$\Rightarrow (\pi/2, \pi/2)$ is a sink $\Rightarrow$

$\Rightarrow (\pi/2, \pi/2)$ is asymptotically stable $\Rightarrow$

$\Rightarrow (\pi/2, \pi/2)$ is attracting $\Rightarrow$

$\Rightarrow$ the system is not conservative.

Index theory

Index theory is a global method that provides global information about the phase portrait of a two-dimensional autonomous system.

Definition of the index

Consider the two-dimensional autonomous system

$$\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2) \end{cases}$$

We note that at (x_1, x_2), the angle φ of the vector $(\dot{x}_1, \dot{x}_2)$ is given by

$$\varphi(x_1, x_2) = \operatorname{Arctan}\left(\frac{g(x_1, x_2)}{f(x_1, x_2)}\right)$$

Let C be a simple closed curve. We define the index $I(C)$ of C as:

$$\boxed{I(C) = \oint_C \frac{d\varphi(x_1, x_2)}{2\pi}}$$

Explicit form of the index integral

We note that:

$$d\varphi = d\left(\operatorname{Arctan}(g/f)\right) = \frac{1}{1+(g/f)^2}\, d\left(\frac{g}{f}\right) =$$

$$= \frac{1}{1+(g/f)^2}\,\frac{f\,dg - g\,df}{f^2} =$$

$$= \frac{f\,dg - g\,df}{f^2+g^2} \Rightarrow$$

$$\Rightarrow I(C) = \oint_C \frac{d\varphi}{2\pi} = \oint_C \frac{f\,dg - g\,df}{2\pi(f^2+g^2)}$$

Let $C: \rho(t) \in \mathbb{R}^2$, $t \in [0,1]$ be a parameterization of the curve C. Then, the differentials df and dg are given by:

$$df = [\dot{\rho}(t) \cdot \nabla f(\rho(t))]\,dt$$
$$dg = [\dot{\rho}(t) \cdot \nabla g(\rho(t))]\,dt$$

It follows that:

$$I(C) = \oint_C \frac{f\,dg - g\,df}{f^2+g^2} =$$

$$= \int_0^1 dt\, \frac{f(\rho(t))\nabla g(\rho(t)) \cdot \dot{\rho}(t) - g(\rho(t))\nabla f(\rho(t)) \cdot \dot{\rho}(t)}{2\pi\left[f^2(\rho(t)) + g^2(\rho(t))\right]}$$

$$= \int_0^1 dt\, \dot{\rho}(t) \cdot \left[\frac{f(\rho(t))\nabla g(\rho(t)) - g(\rho(t))\nabla f(\rho(t))}{2\pi\left[f^2(\rho(t)) + g^2(\rho(t))\right]}\right]$$

- Properties of the index

① $\boxed{I(C) \in \mathbb{Z}}$ (i.e. $I(C)$ is an integer).

Proof

Going around the curve C, both initial and final value of φ point in the same direction, therefore the variation $\Delta\varphi$ of the angle must be a multiple of 2π. It follows that

$$\Delta\varphi = \oint_C d\varphi = 2k\pi \ , \text{ with } k \in \mathbb{Z} \Rightarrow$$

$$\Rightarrow I(C) = \frac{1}{2\pi} \oint_C d\varphi = \frac{1}{2\pi} \cdot (2k\pi) = k \in \mathbb{Z} \ \square$$

② Assume that there are no fixed points in the interior of a simple closed curve C. Then $I(C) = 0$.

Proof

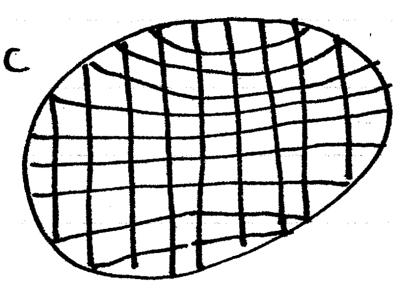


Fig. 1

We divide the interior of the curve C into a mesh of N closed simple curves γ_k with $k \in [N]$. We assume that the loops γ_k are small enough so that the maximum angle variation around γ_K does not exceed $\pi/2$. This is possible only because there are no fixed points in the interior of any γ_K (see fig. 2)

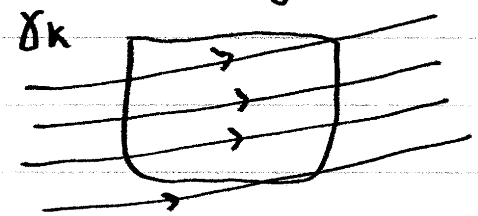


Fig. 2

It follows that

$$\forall K \in [N] : \oint_{\gamma_K} d\varphi = 0$$

and therefore

$$I(C) = \frac{1}{2\pi} \oint_C d\varphi = \frac{1}{2\pi} \left[\sum_{K=1}^{N} \oint_{\gamma_K} d\varphi \right] = 0 \quad \square$$

③ Invariance with contour deformation

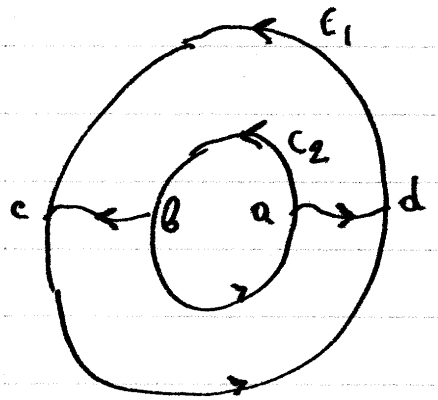
Def : Let C_1, C_2 be two simple closed curves with
$C_1 : \rho_1(t) \in \mathbb{R}^2, \ t \in [0,1]$ and
$C_2 : \rho_2(t) \in \mathbb{R}^2, \ t \in [0,1]$.
We say that $C_1 \sim C_2$ if and only if there is a mapping $\rho : [0,1]^2 \to \mathbb{R}^2$ such that
a) $\forall t \in [0,1] : (\rho(t,0) = \rho_1(t) \wedge \rho(t,1) = \rho_2(t))$
b) ρ continuous at $[0,1]^2$

c) $\forall (t,a) \in [0,1]^2$: $\rho(t,a)$ not a fixed point.

→ $C_1 \sim C_2$ means that C_1 can be continuously deformed into C_2 without crossing any fixed points.

- $\boxed{C_1 \sim C_2 \Rightarrow I(C_1) = I(C_2)}$

Proof



Define:

C_{bc} : path from b to c

C_{ad} : path from a to d

C_{ab} : counterclockwise path from a to b

C_{ba} : counterclockwise path from b to a

C_{cd} : counterclockwise path from c to d.

C_{dc} : counterclockwise path from d to c.

We also let $-C$ represent the path C with its direction reversed. (e.g. $-C_{ab}$ vs. C_{ba}).

Now consider the paths Γ_1 and Γ_2 defined as:

$\Gamma_1 = C_{ad} \cup C_{dc} \cup (-C_{bc}) \cup (-C_{ab})$

$\Gamma_2 = C_{bc} \cup C_{cd} \cup (-C_{ad}) \cup (-C_{ba})$

There are no fixed points in the interiors of Γ_1 and Γ_2, therefore $I(\Gamma_1) = 0$ and $I(\Gamma_2) = 0$.

We note that

$$2\pi I(\Gamma_1) = \int_{C_{ad}} d\varphi + \int_{C_{dc}} d\varphi + \int_{-C_{bc}} d\varphi + \int_{-C_{ab}} d\varphi =$$

$$= \int_{C_{ad}} d\varphi + \int_{C_{dc}} d\varphi - \int_{C_{bc}} d\varphi - \int_{C_{ab}} d\varphi \qquad (1)$$

and

$$2\pi I(\Gamma_2) = \int_{C_{bc}} d\varphi + \int_{C_{cd}} d\varphi + \int_{-C_{ad}} d\varphi + \int_{-C_{ba}} d\varphi =$$

$$= \int_{C_{bc}} d\varphi + \int_{C_{cd}} d\varphi - \int_{C_{ad}} d\varphi - \int_{C_{ba}} d\varphi \qquad (2)$$

Adding (1) and (2) gives: the cancellations: C_{bc}, C_{ad}

$$2\pi[I(\Gamma_1) + I(\Gamma_2)] = \int_{C_{cd}} d\varphi + \int_{C_{dc}} d\varphi - \int_{C_{ab}} d\varphi - \int_{C_{ba}} d\varphi =$$

$$= \oint_{C_1} d\varphi - \oint_{C_2} d\varphi = 2\pi[I(C_1) - I(C_2)]$$

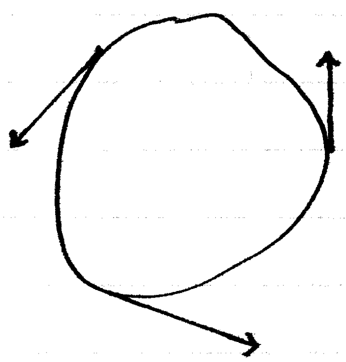
$\Rightarrow I(C_1) - I(C_2) = I(\Gamma_1) + I(\Gamma_2) = 0 + 0 = 0 \Rightarrow$

$\Rightarrow I(C_1) = I(C_2)$. □

④ Index of closed orbits

- If C is a closed orbit of the system then $I(C) = 1$

Proof: If C is a closed orbit of the system, then it is easy to see that the vector $(\dot{x}_1, \dot{x}_2)$ is tangent to C for all points of C. Thus, the total change in the angle φ is $\Delta\varphi = 2\pi$. It follows that



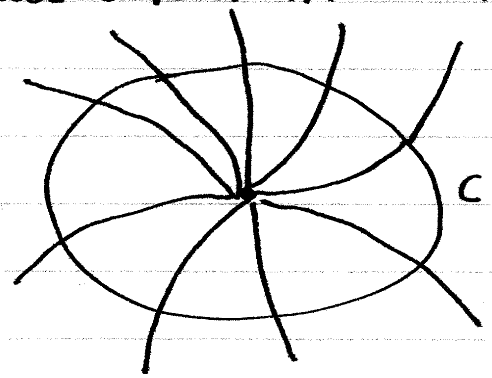
$$I(C) = \frac{1}{2\pi}\oint_C d\varphi = \frac{\Delta\varphi}{2\pi} = \frac{2\pi}{2\pi} = 1 \quad \square$$

- **Index of a fixed point**

Def: Let $x_0 \in \mathbb{R}^2$ be a fixed point. Let C be a counterclockwise curve whose interior contains the fixed point x_0, but no other fixed points. We define the index $I(x_0)$ of the fixed point x_0 as $I(x_0) = I(C)$

↳ We note that from property 3 above, $I(x_0)$ is independent of our choice of C, subject to the stated constraints.

<u>Thm</u> : Let $x_0 \in \mathbb{R}^2$ be a fixed point such that trajectories radiate from or towards x_0 in all directions. Then $I(x_0) = +1$.



<u>Proof</u>

Consider a small enough loop C around x_0 constructed so that it is perpendicular to every trajectory it intersects. Then the total change in the angle around C is $\Delta\varphi = 2\pi$. It follows that:

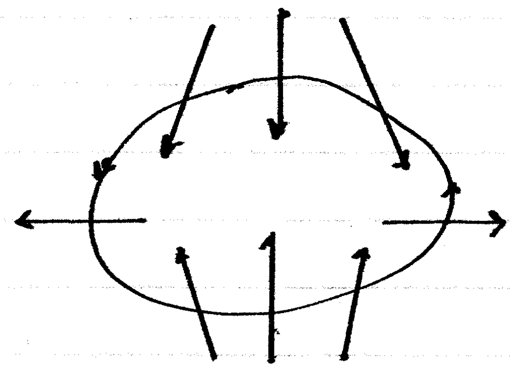
$$I(x_0) = I(C) = \frac{1}{2\pi}\oint_C d\varphi = \frac{\Delta\varphi}{2\pi} = \frac{2\pi}{2\pi} = 1. \quad \square$$

It follows that the following fixed points have $I(x_0) = 1$:

a) sources c) stable spirals e) degenerate nodes

b) sinks d) unstable spirals f) stars.

Thm: $\boxed{\text{Let } x_0 \in \mathbb{R}^2 \text{ be a \underline{saddle node}. Then } I(x_0) = -1.}$



Proof

Let C be a small loop around the saddle node x_0. The angle φ varies clockwise around C with $\Delta\varphi = -2\pi$ (see fig.) It follows that

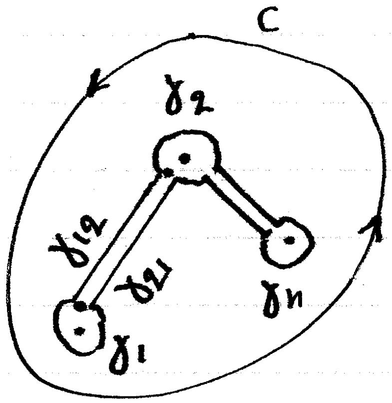
$$I(x_0) = I(C) = \frac{1}{2\pi}\oint_C d\varphi = \frac{\Delta\varphi}{2\pi} = \frac{-2\pi}{2\pi} = -1 \quad \square$$

- **Index of a curve surrounding fixed points.**

Thm: Let C be a simple closed curve containing the fixed points $x_1, x_2, \dots, x_n$. Then:

$$\boxed{I(C) = \sum_{a=1}^{n} I(x_a)}$$

Proof



We deform continuously C into a curve $\Gamma \sim C$ such that Γ consists of small loops γ_a around the fixed points x_a and connecting bridges γ_{ab} connecting x_a to x_b as shown in the figure. We further assume that the gap between γ_{ab} to γ_{ba} tends to zero. That implies that $\gamma_{ab} = -\gamma_{ba}$ and γ_a are closed. It follows that

$$I(C) = I(\Gamma) = \frac{1}{2\pi} \oint_{\Gamma} d\varphi =$$

$$= \frac{1}{2\pi} \left[\sum_{a=1}^{n} \oint_{\gamma_a} d\varphi + \sum_{a=1}^{n-1} \int_{\gamma_{a,a+1}} d\varphi + \sum_{a=1}^{n} \int_{\gamma_{a+1,a}} d\varphi \right]$$

$$= \frac{1}{2\pi} \left[\sum_{a=1}^{n} \oint_{\gamma_a} d\varphi + \sum_{a=1}^{n-1} \int_{\gamma_{a,a+1}} d\varphi - \sum_{a=1}^{n-1} \int_{\gamma_{a,a+1}} d\varphi \right]$$

$$= \frac{1}{2\pi} \left[\sum_{a=1}^{n} \oint_{\gamma_a} d\varphi \right] = \sum_{a=1}^{n} \left[\frac{1}{2\pi} \oint_{\gamma_a} d\varphi \right] =$$

$$= \sum_{a=1}^{n} I(x_a)$$

$\square$

<u>Corollary</u>: Let C be a closed trajectory enclosing the fixed points $x_1, x_2, \ldots, x_n$. Then

$$\sum_{a=1}^{n} I(x_a) = +1$$

<u>Proof</u>

Since C is a closed trajectory, from property 4, we have $I(C) = +1$. Thus, from the theorem:

$$\sum_{a=1}^{n} I(x_a) = I(C) = +1 \qquad \square$$

EXAMPLES

a) Show that the system

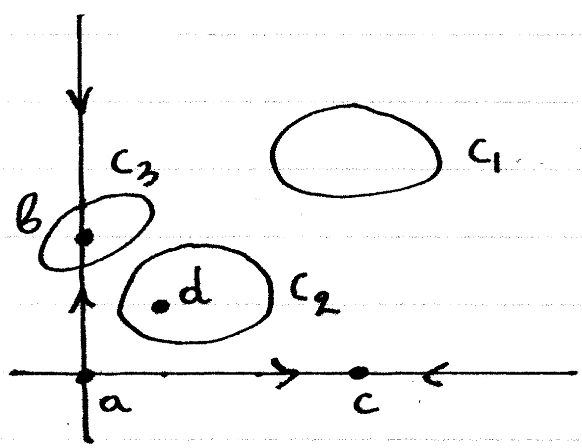
$$\begin{cases} \dot{x}_1 = x_1(3 - x_1 - 2x_2) \\ \dot{x}_2 = x_2(2 - x_1 - x_2) \end{cases}$$

does not have any closed trajectories.

Solution

It can be shown that this system has the following fixed points:

$a = (0,0)$ unstable node $\Rightarrow I(a) = 1$
$b = (0,2)$ stable node $\Rightarrow I(b) = 1$
$c = (3,0)$ stable node $\Rightarrow I(c) = 1$
$d = (1,1)$ saddle node $\Rightarrow I(d) = -1$.



- Let C_1 be a curve enclosing no fixed points. Then
$I(C_1) = 0 \neq 1 \Rightarrow$
$\Rightarrow C_1$ not a trajectory.

- Let C_2 be any curve that encloses only the fixed point $d = (1,1)$. Then
$I(C_2) = I(d) = -1 \neq +1 \Rightarrow C_2$ not a trajectory.

- Let C_3 be any curve enclosing a or b or c or any combination of these three fixed points. Then C_3 will intersect at least the x_1-axis or the x_2-axis (or both). Since both the x_1-axis and the x_2-axis are trajectories, and trajectories cannot intersect, it follows that C_3 is not a trajectory. □

↳ We see that trajectories that cannot be ruled out by index theory, can be eliminated, sometimes, by the constraint that two trajectories cannot intersect.

b) Show that the system

$$\begin{cases} \dot{x}_1 = x_1 e^{-x_1} \\ \dot{x}_2 = 1 + x_1 + x_2^2 \end{cases}$$

does not have any closed trajectories.

Solution

Fixed points:

$$\begin{cases} x_1 e^{-x_1} = 0 \\ 1 + x_1 + x_2^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ 1 + x_1 + x_2^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ 1 + x_2^2 = 0 \end{cases}$$

Since $1 + x_2^2 = 0$ is inconsistent, there are no

fixed points.
Now, let C be any closed curve. Then
$I(C) = 0 \neq 1 \Rightarrow C$ not a trajectory.

GODE 10: Center manifold reduction

Center Manifold Reduction

This technique is based on the following theorem

Theorem : Consider the following system of $n+m$ ordinary differential equations:

$$\begin{cases} \dot{x} = Ax + f(x,y) \\ \dot{y} = By + g(x,y) \end{cases}, \text{ with } \begin{cases} f(\mathbf{0}) = \mathbf{0} \wedge Df(\mathbf{0}) = \mathbf{0} \\ g(\mathbf{0}) = \mathbf{0} \wedge Dg(\mathbf{0}) = \mathbf{0} \end{cases}$$

with $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, $f: \mathbb{R}^{n+m} \to \mathbb{R}^n$, $g: \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$.
Here, A is an $n \times n$ matrix, B is an $m \times m$ matrix, with

$$\begin{cases} \forall \lambda \in \lambda(A) : Re(\lambda) = 0 \\ \forall \lambda \in \lambda(B) : Re(\lambda) < 0 \end{cases}$$

Then there exists a center-manifold W^c given by

$$W^c = \{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = h(x) \}$$

with $h: \mathbb{R}^n \to \mathbb{R}^m$, and $h(\mathbf{0}) = \mathbf{0}$, and $Dh(\mathbf{0}) = \mathbf{0}$ such that the solution of the nonlinear system converges to W^c as $t \to +\infty$ if initialized near enough the fixed point $\mathbf{0}$.

The theorem can be used to analyze non-hyperbolic fixed points where all the eigenvalues of the corresponding Jacobian matrix are either zero or negative. The method cannot be applied if at least one eigenvalue is positive (in the real part $Re(\lambda)$).

Methodology

Let $\dot{x} = f(x)$ with $f: \mathbb{R}^n \to \mathbb{R}^n$ be an autonomous dynamical system. Let $x_0 \in \mathbb{R}^n$ be a fixed point with $f(x_0) = 0$. We assume that x_0 is a non-hyperbolic fixed point such that some of the eigenvalues of $Df(x_0)$ have zero real part and none of the eigenvalues have a strictly positive real part. In other words, we assume that

$$\begin{cases} \exists \lambda \in \lambda(Df(x_0)) : \mathrm{Re}(\lambda) = 0 \\ \forall \lambda \in \lambda(Df(x_0)) : \mathrm{Re}(\lambda) \leq 0 \end{cases}$$

The center manifold reduction technique consists of the following 3 steps:

1) Reduce system to canonical form
2) Apply the center-manifold theorem.
3) Determine series expansion for mapping h.

Reduction to canonical form

- $_1$ We linearize the autonomous system around x_0 and write:

$$\dot{x} = Df(x_0)x + G(x)$$

Here $G(x)$ captures the nonlinear terms of the system.

- $_2$ Assume that $Df(x_0)$ has distinct eigenvalues

$$\lambda(Df(x_0)) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

with corresponding eigenvectors $v_1, v_2, \ldots, v_n$.

We daagonalize $Df(x_0)$ by defining
$$P = [V_1 \; V_2 \; \cdots \; V_n]$$
and writing
$$Df(x_0) = P \, \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) P^{-1}$$
Note that $Df(x_0)$ can be diagonalized even when the eigenvalues are not distinct, into a block-diagonal matrix.

- 3 Define the change of variables $y = P^{-1}x$.
  It follows that $x = Py$, and therefore

$$\dot{y} = P^{-1}\dot{x} = P^{-1}(Df(x_0)x + G(x)) = P^{-1}(Df(x_0)Py + G(Py))$$
$$= [P^{-1}Df(x_0)P]y + G(Py) =$$
$$= [P^{-1}P\,\mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)P^{-1}P]y + G(Py) =$$
$$= \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)y + G(Py)$$
which reduces to the following system of ODEs:
$$\begin{cases} \dot{y}_1 = \lambda_1 y_1 + g_1(y_1, y_2, \dots, y_n) \\ \dot{y}_2 = \lambda_2 y_2 + g_2(y_1, y_2, \dots, y_n) \\ \text{-----} \\ \dot{y}_n = \lambda_n y_n + g_n(y_1, y_2, \dots, y_n) \end{cases}$$

- Determining the center manifold

Let us now assume that $\mathrm{Re}(\lambda_a) = 0, \forall a \in [k]$ and $\mathrm{Re}(\lambda_a) < 0, \forall a \in [n] - [k]$. Then, according to the center manifold theorem, the first k equations for $y_1, y_2, \dots, y_k$

drive the dynamics of the system and the other $n-k$ equations are driven by <u>slaving principles</u> given by

$$\begin{cases} y_{k+1} = h_1(y_1, y_2, \dots, y_k) \\ y_{k+2} = h_2(y_1, y_2, \dots, y_k) \\ \vdots \\ y_n = h_{n-k}(y_1, y_2, \dots, y_k) \end{cases}$$

Let us define $u = (y_1, y_2, \dots, y_k)$ and $v = (y_{k+1}, y_{k+2}, \dots, y_n)$ and rewrite the above system as $v = h(u)$, with $h: \mathbb{R}^k \to \mathbb{R}^{n-k}$

Also, let $(u_0, v_0) = y_0 = P^{-1} x_0$ be the fixed point.

Then according to the center manifold theorem, h has to satisfy:

$$\begin{cases} h(u_0) = 0 \\ Dh(u_0) = 0 \end{cases}$$

Now let us rewrite the original system of ODEs for $y = (u, v)$ as follows:

$$\dot{u} = Au + G_1(u, v)$$
$$\dot{v} = Bv + G_2(u, v)$$

with $A = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_k)$ and $B = \operatorname{diag}(\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_n)$ and G_1, G_2 are given by

$$G_1 = (g_1, g_2, \dots, g_k)$$
$$G_2 = (g_{k+1}, g_{k+2}, \dots, g_n)$$

To write governing PDEs for h, we note that

$$v = h(u) \Rightarrow \dot{v} = Dh(u)\,\dot{u} = Dh(u)\left[Au + G_1(u, v)\right] =$$
$$= Dh(u)\left[Au + G_1(u, h(u))\right]$$

and

$\dot{v} = Bv + G_2(u,v) = Bh(u) + G_2(u,h(u))$

and it follows that

$$Dh(u)[Au + G_1(u,h(u))] = Bh(u) + G_2(u,h(u))$$

Now, let us define the operator

$$\boxed{(Nh)(u) = Dh(u)[Au + G_1(u,h(u))] - [Bh(u) + G_2(u,h(u))]}$$

Then, h is given by the solution of the following initial value problem:

$$\boxed{\begin{cases} (Nh)(u) = 0 \\ h(u_0) = 0 \\ Dh(u_0) = 0 \end{cases}}$$

Note that in terms of components, $(Nh)(u)$ is given by:

$$\boxed{(Nh)_a(u) = \sum_{b=1}^{k} \left[(\lambda_b y_b + g_b(y)) \frac{\partial h_a}{\partial y_b} \right] - (\lambda_{k+a} h_a + g_{k+a}(y))}$$

- <u>The spectral gap theorem</u>

The mapping h can be determined via a power-series technique based on the following spectral gap theorem.

<u>Theorem</u> : Let an arbitrary $\psi : \mathbb{R}^k \to \mathbb{R}^{n-k}$ be given with $\psi(u_0) = \mathbb{0}$ and $D\psi(u_0) = \mathbb{0}$. Then, under the limit $u \to u_0$, we can show that

$$\exists q > 1 : (N\psi)(u) = O(\|u - u_0\|^q) \Rightarrow \|h(u) - \psi(u)\| = O(\|u - u_0\|^q)$$

It follows that power-series techniques can be used to approximate the center manifold to any degree of approximation by solving the equation $(N\psi)(u) = \mathbb{0}$ to the same degree of approximation, as shown in the examples below.

EXAMPLES

a) $\begin{cases} \dot{x} = x^2 y - x^5 \\ \dot{y} = -y + x^2 \end{cases}$ $\leftarrow$ Analyze fixed points.

Solution

• Fixed points and Jacobian.

Let $f(x,y) = x^2 y - x^5$ and $g(x,y) = -y + x^2$.

(x,y) fixed point $\Leftrightarrow \begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 y - x^5 = 0 \\ -y + x^2 = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x^2(y - x^3) = 0 \\ y = x^2 \end{cases} \Leftrightarrow \begin{cases} x^2 = 0 \\ y = x^2 \end{cases} \vee \begin{cases} y = x^3 \\ y = x^2 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x^3 - x^2 = 0 \\ y = x^2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x^2(x-1) = 0 \\ y = x^2 \end{cases}$

$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = 1 \\ y = 1 \end{cases} \Leftrightarrow (x,y) \in \{(0,0), (1,1)\}$.

$$DF(x,y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2xy - 5x^4 & x^2 \\ 2x & -1 \end{bmatrix}$$

• For $(x,y) = (1,1)$:

$$DF(1,1) = \begin{bmatrix} 2-5 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(DF(1,1) - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} =$$

$$= (-3-\lambda)(-1-\lambda) - 2 = (\lambda+3)(\lambda+1) - 2 = \lambda^2 + 4\lambda + 3 - 2$$

$= \lambda^2 + 4\lambda + 1$

$\Delta = b^2 - 4ac = 4^2 - 4\cdot 1\cdot 1 = 16 - 4 = 12 = 4\cdot 3 \Rightarrow$

$\Rightarrow \lambda_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a} = \dfrac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$ (both negative)

thus $\lambda(DF(1,1)) = \{-2-\sqrt{3}, -2+\sqrt{3}\} \Rightarrow (1,1)$ is a stable sink.

• For $(x,y) = (0,0)$:

$DF(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda(DF(0,0)) = \{0, -1\} \Rightarrow$

$\Rightarrow (0,0)$ is a non-hyperbolic fixed point.

▸ Center-Manifold analysis: We note that $\dot{x}$ is the master equation and $\dot{y}$ is the slave equation, the system being already written in canonical form. So, let us consider $y = h(x)$ with $h(0) = 0$ and $h'(0) = 0$. It follows that

$$\left.\begin{aligned} \dot{y} &= h'(x)\dot{x} = h'(x)[x^2 y - x^5] = h'(x)[x^2 h(x) - x^5] \\ \dot{y} &= -y + x^2 = -h(x) + x^2 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow (Nh)(x) = h'(x)[x^2 h(x) - x^5] + h(x) - x^2$

Let $h(x) = ax^2 + bx^3 + O(x^4) \Rightarrow h'(x) = 2ax + 3bx^2 + O(x^3)$

and it follows that

$$\begin{aligned} (Nh)(x) &= [2ax + 3bx^2 + O(x^3)][x^2(ax^2 + bx^3 + O(x^4)) - x^5] \\ &\quad + ax^2 + bx^3 + O(x^4) - x^2 = \\ &= x^2(2ax + 3bx^2)(ax^2 + bx^3) + O(x^9) - x^5[2ax + 3bx^2 + O(x^3)] \\ &\quad + ax^2 + bx^3 + O(x^4) - x^2 = \end{aligned}$$

$= x^2(2a^2x^3 + 2abx^4 + 3abx^4 + 3b^2x^5) - 2ax^6 - 3bx^7 \underline{+ax^2} + \underline{\underline{bx^3}} \underline{-x^2} + O(x^4)$

$= (a-1)x^2 + bx^3 + O(x^4).$

and therefore:

$(Nh)(x) = 0 + O(x^4) \Leftrightarrow (a-1)x^2 + bx^3 + O(x^4) = O(x^4) \Leftrightarrow$

$$\Leftrightarrow \begin{cases} a-1=0 \\ b=0 \end{cases} \Leftrightarrow \begin{cases} a=1 \\ b=0 \end{cases}$$

It follows that $h(x) = x^2 + O(x^4)$.

Then $\dot{x} = x^2y - x^5 = x^2h(x) - x^5 = x^2[x^2 + O(x^4)] - x^5$
$= x^4 + O(x^5)$

and the center-manifold reduction is:

$$\begin{cases} \dot{x} = x^4 + O(x^5) \leftarrow \text{master equation} \\ y = x^2 + O(x^4) \leftarrow \text{slave equation} \end{cases}$$

From the master equation we see that the $(0,0)$ fixed point is unstable.

$\hookrightarrow$ Note that $\lambda(DF(0,0)) = \{0,-1\}$, thus linear stability analysis might suggest that $(0,0)$ is Lyapunov stable, and in the absense of positive eigenvalues there is no hint of instability. On the other hand, because $(0,0)$ is not hyperbolic, so linear stability analysis is not guaranteed to be accurate, and center manifold analysis shows that the $(0,0)$ fixed point is in fact unstable.

$\hookrightarrow$ Note that the center manifold ensures local convergence: if the initial condition is close to the center manifold, it will converge to the center manifold. We can also investigate <u>global convergence</u>, i.e. whether or not ALL initial conditions converge to the center manifold via the following argument:

Since the center manifold is $y = x^2 + O(x^4)$, we define $U(x,y) = (y-x^2)^2$.

It is sufficient to show that $\dot{U}(x,y) < 0$.

$$\dot{U}(x,y) = (d/dt)[(y-x^2)^2] = 2(y-x^2)(\dot{y} - 2x\dot{x}) =$$
$$= 2(y-x^2)(-y+x^2-2x(x^2y-x^5)) =$$
$$= -2(y-x^2)^2 - 4x(y-x^2)(x^2y-x^5)$$
$$= -2(y-x^2)^2 - 4(y-x^2)(x^3y-x^6)$$
$$= -2(y-x^2)^2 - 4(x^3y^2 - x^6y - x^5y + x^8) =$$
$$= -2(y-x^2)^2 - 4x^8 + 4x^3y(-y+x^3+x^2)$$

First two terms are negative, third term is unclear (negative or positive). Let us assume that $y = x^2+\varepsilon$ with ε small. Then, it follows that

$$4x^3y(-y+x^2+x^3) = 4x^3(x^2+\varepsilon)(-x^2-\varepsilon+x^2+x^3) =$$
$$= 4x^3(x^2+\varepsilon)(x^3-\varepsilon)$$
$$= 4x^3(x^5-\varepsilon x^2+\varepsilon x^3-\varepsilon^2)$$
$$= 4x^8 + 4\varepsilon x^3(-x^2+x^3-\varepsilon) \Rightarrow$$

$$\Rightarrow \dot{U}(x,y) = -2(y-x^2)^2 - 4x^8 + 4x^8 + 4\varepsilon x^3(x^3-x^2-\varepsilon) =$$
$$= -2(y-x^2)^2 + 4\varepsilon x^3(x^3-x^2-\varepsilon)$$

$$= -2(x^2+\varepsilon-x^2)^2 + 4\varepsilon x^3(x^3-x^2-\varepsilon)$$
$$= -2\varepsilon^2 + 4\varepsilon x^6 - 4\varepsilon x^5 - 4\varepsilon^2 x^3$$
$$= -2\varepsilon^2 - 4\varepsilon^2 x^3 + O(x^4) = -2\varepsilon^2(1+2x^3) + O(x^4) < 0$$

in the limit $x \to 0$, since for small x, $1+2x^3 > 0$.
It follows that we do not have global convergence towards the center manifold.

b) $\begin{cases} \dot{x} = xy \\ \dot{y} = -y - x^2 \end{cases}$ $\longleftarrow$ Find all fixed points and classify with respect to stability

Solution

▸ Fixed points.

Let $f(x,y) = xy \wedge g(x,y) = -y - x^2$.

(x,y) fixed point $\Leftrightarrow \begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} xy = 0 \\ -y - x^2 = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x(-x^2) = 0 \\ y = -x^2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = -x^2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \Leftrightarrow (x,y) = (0,0).$

▸ Jacobian

Let $F(x,y) = (f(x,y), g(x,y))$.

$DF(x,y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} y & x \\ -2x & -1 \end{bmatrix} \Rightarrow$

$\Rightarrow DF(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda(DF(0,0)) = \{0, -1\} \Rightarrow$

$\Rightarrow (0,0)$ non-hyperbolic fixed point.

▸ Center-Manifold reduction.

System is already in canonical form with $\dot{x} = xy$ the master equation and $\dot{y} = -y - x^2$ the slave equation. Thus, in the limit $x \to 0$, let us define

$y = h(x) = ax^2 + bx^3 + cx^4 + dx^5 + O(x^6)$

Then:

$\dot{y} = h'(x)\dot{x} = h'(x)xy = h'(x)xh(x) = xh(x)h'(x)$

$\dot{y} = -y - x^2 = -h(x) - x^2$

therefore, let us define

$N(x) = xh(x)h'(x) - [-h(x) - x^2] = xh(x)h'(x) + h(x) + x^2 =$

$= x(ax^2 + bx^3 + cx^4 + dx^5)(2ax + 3bx^2 + 4cx^3 + 5dx^4) + O(x^{12})$

$\quad + (ax^2 + bx^3 + cx^4 + dx^5) + O(x^6) + x^2 =$

$= (ax^3 + bx^4 + cx^5 + dx^6)(2ax + 3bx^2 + 4cx^3 + 5dx^4) +$

$\quad + ax^2 + bx^3 + cx^4 + dx^5 + x^2 + O(x^6) =$

$= 2a^2x^4 + 3abx^5 + 2abx^5 + O(x^6) + ax^2 + bx^3 + cx^4 + dx^5 + x^2 + O(x^6)$

$= (a+1)x^2 + bx^3 + (c + 2a^2)x^4 + (5ab + d)x^5 + O(x^6)$

It follows that

$N(x) = O(x^6) \Leftrightarrow (a+1)x^2 + bx^3 + (c+2a^2)x^4 + (5ab+d)x^5 + O(x^6) = O(x^6)$

$$\Leftrightarrow \begin{cases} a+1=0 \\ b=0 \\ c+2a^2=0 \\ 5ab+d=0 \end{cases} \Leftrightarrow \begin{cases} a=-1 \\ b=0 \\ c=-2a^2 \\ d=-5ab \end{cases} \Leftrightarrow \begin{cases} a=-1 \\ b=0 \\ c=-2 \\ d=0 \end{cases}$$

and therefore $h(x) = -x^2 - 2x^4 + O(x^6)$.

Thus, $\dot{x} = xy = xh(x) = x(-x^2 - 2x^4 + O(x^6)) =$

$\quad = -x^3 - 2x^5 + O(x^7)$

$\quad y = -x^2 - 2x^4 + O(x^6)$

and the centermanifold reduction reads:

$$\begin{cases} \dot{x} = -x^3 - 2x^5 + O(x^7) \\ y = -x^2 - 2x^4 + O(x^6) \end{cases}$$

It follows that $(0,0)$ is stable since the fixed point $x=0$ of $\dot{x} = -x^3 - 2x^5 + O(x^7)$ is stable.

Local vs. global convergence

Let us consider the 1st-order approximation

$$y = -x^2 + O(x^4)$$

of the center manifold and therefore define

$$U(x,y) = (y+x^2)^2$$

It follows that

$$\begin{aligned}\dot{U}(x,y) &= 2(y+x^2)(\dot{y}+2x\dot{x}) = 2(y+x^2)(-y-x^2+2x(xy)) = \\ &= 2(y+x^2)(-y-x^2+2x^2y) \\ &= -2(y+x^2)^2 + 4x^2y\,(y+x^2)\end{aligned}$$

Note that the 1st term is negative but the 2nd term can be positive or negative. Choose $y = -x^2+\varepsilon$ with ε small. Then

$$\begin{aligned}\dot{U}(x,y) &= -2(-x^2+\varepsilon+x^2)^2 + 4x^2(-x^2+\varepsilon)(-x^2+\varepsilon+x^2) \\ &= -2\varepsilon^2 + 4\varepsilon x^2(-x^2+\varepsilon) = -2\varepsilon^2 - 4\varepsilon x^{+4} + 4\varepsilon^2 x^2 \\ &= -4\varepsilon x^4 + 2\varepsilon^2(2x^2-1) = 2\varepsilon^2(2x^2-1) + O(x^4)\end{aligned}$$

For small enough ε, $\dot{U}(x,y) < 0$, thus we have local but not global convergence

Inclusion of Linearly Unstable Directions

- The center manifold analysis is still applicable even if in the canonical formulation of the original ODE system some eigenvalues have $Re(\lambda) > 0$.
- Consider the system, in canonical form

$$\begin{cases} \dot{x} = Ax + f(x,y,z) \\ \dot{y} = By + g(x,y,z) \\ \dot{z} = Cz + h(x,y,z) \end{cases}$$

with $(x,y,z) \in \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c$ and

$$\begin{cases} \forall \lambda \in \lambda(A) : Re(\lambda) = 0 \\ \forall \lambda \in \lambda(B) : Re(\lambda) < 0 \\ \forall \lambda \in \lambda(C) : Re(\lambda) > 0 \end{cases}$$

and

$$\begin{cases} f(\mathbf{0}) = \mathbf{0} \wedge g(\mathbf{0}) = \mathbf{0} \wedge h(\mathbf{0}) = \mathbf{0} \\ Df(\mathbf{0}) = \mathbf{0} \wedge Dg(\mathbf{0}) = \mathbf{0} \wedge Dh(\mathbf{0}) = \mathbf{0} \end{cases}$$

(i.e. $\mathbf{0}$ is a fixed point and f, g, h capture only the nonlinear terms).

Then the center-manifold is given by

$$W^c = \{(x,y,z) \in \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c \mid y = h_1(x) \wedge z = h_2(x)\}$$

with $h_1(\mathbf{0}) = \mathbf{0} \wedge h_2(\mathbf{0}) = \mathbf{0} \wedge Dh_1(\mathbf{0}) = \mathbf{0} \wedge Dh_2(\mathbf{0}) = \mathbf{0}$

- To determine h_1 and h_2, we note that

$$\dot{y} = Dh_1(x)\dot{x} = Dh_1(x)[Ax + f(x,y,z)] =$$
$$= Dh_1(x)[Ax + f(x, h_1(x), h_2(x))]$$

$$\dot{y} = By + g(x,y,z) = Bh_1(x) + g(x, h_1(x), h_2(x))$$
$$\dot{z} = Dh_2(x)\dot{x} = Dh_2(x)[Ax + f(x,y,z)] =$$
$$= Dh_2(x)[Ax + f(x, h_1(x), h_2(x))]$$
$$\dot{z} = Cz + h(x,y,z) = Ch_2(x) + h(x, h_1(x), h_2(x))$$

It follows that if we define

$$(N_1(h_1,h_2))(x) = Dh_1(x)[Ax + f(x, h_1(x), h_2(x))] - Bh_1(x) - g(x, h_1(x), h_2(x))$$
$$(N_2(h_1,h_2))(x) = Dh_2(x)[Ax + f(x, h_1(x), h_2(x))] - Ch_2(x) - h(x, h_1(x), h_2(x))$$

then $(N_1(h_1,h_2))(x) = \mathbf{0} \wedge (N_2(h_1,h_2))(x) = \mathbf{0}$.

Consequently, h_1 and h_2 are the solutions of the following initial value problem:

$$\begin{cases} (N_1(h_1,h_2))(x) = \mathbf{0} \\ (N_2(h_1,h_2))(x) = \mathbf{0} \\ h_1(x) = \mathbf{0} \wedge Dh_1(x) = \mathbf{0} \\ h_2(x) = \mathbf{0} \wedge Dh_2(x) = \mathbf{0} \end{cases}$$

which can still be solved via power-series methods.

EXAMPLES

a) Analyze the stability of the fixed point $(x,y,z)=(0,0,0)$ for the system

$$\begin{cases} \dot{x} = xz \\ \dot{y} = -y+x^2 \\ \dot{z} = z-xy \end{cases}$$

using center-manifold reduction.

Solution

Define $f(x,y,z)=xz \wedge g(x,y,z)=-y+x^2 \wedge h(x,y,z)=z-xy$. and $F=(f,g,h)$. It follows that

$$DF(x,y,z)=\begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial z \end{bmatrix} = \begin{bmatrix} z & 0 & x \\ 2x & -1 & 0 \\ -y & -x & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0,0)=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda(DF(0,0,0))=\{0,-1,1\}$$

$\Rightarrow (0,0,0)$ is a non-hyperbolic fixed point.

- Center Manifold analysis.

We note that $\dot{x}=xz$ is the master equation. Thus let $y=h_1(x)$ and $z=h_2(x)$. It follows that

$\dot{y}=h_1'(x)\dot{x}=h_1'(x)xz = xh_1'(x)h_2(x)$

$\dot{y}=-y+x^2=-h_1(x)+x^2$

therefore, we define $N_1(x)=xh_1'(x)h_2(x)-[-h_1(x)+x^2]$

Likewise,

$$\dot{z} = h_2'(x)\dot{x} = h_2'(x)\, xz = x h_2'(x) h_2(x)$$

$$\dot{z} = z - xy = h_2(x) - x h_1(x)$$

therefore, we define

$$N_2(x) = x h_2'(x) h_2(x) - [h_2(x) - x h_1(x)].$$

For $h_1(x) = ax^2 + bx^3 + O(x^4)$

$h_2(x) = cx^2 + dx^3 + O(x^4)$

we have

$$N_1(x) = x h_1'(x) h_2(x) + h_1(x) - x^2 =$$

$$= x(2ax + 3bx^2)(cx^2 + dx^3) + O(x^8) + ax^2 + bx^3 - x^2 + O(x^4)$$

$$= (a-1)x^2 + bx^3 + O(x^4)$$

$$N_2(x) = x h_2'(x) h_2(x) - h_2(x) + x h_1(x)$$

$$= x(2cx + 3dx^2)(cx^2 + dx^3) + O(x^8) - cx^2 - dx^3 + O(x^4)$$

$$+ x(ax^2 + bx^3) + O(x^5)$$

$$= -cx^2 - dx^3 + ax^3 + O(x^4) = -cx^2 + (a-d)x^3 + O(x^4)$$

It follows that

$$\begin{cases} N_1(x) = (a-1)x^2 + bx^3 + O(x^4) = O(x^4) \\ N_2(x) = -cx^2 + (a-d)x^3 + O(x^4) = O(x^4) \end{cases} \iff$$

$$\iff \begin{cases} a - 1 = 0 \\ b = 0 \\ -c = 0 \\ a - d = 0 \end{cases} \iff \begin{cases} a = 1 \\ b = 0 \\ c = 0 \\ d = a \end{cases} \iff \begin{cases} a = 1 \\ b = 0 \\ c = 0 \\ d = 1 \end{cases}$$

and therefore $h_1(x) = x^2 + O(x^4)$ and $h_2(x) = x^3 + O(x^4)$

It follows that the master equation reads

$$\dot{x} = xz = x h_2(x) = x(x^3 + O(x^4)) = x^4 + O(x^5).$$

The center-manifold reduction near the $(0,0,0)$ fixed point is given by:

$$\begin{cases} \dot{x} = x^4 + O(x^5) \\ y = x^2 + O(x^4) \\ z = x^3 + O(x^4) \end{cases}$$

$\longleftarrow$ thus $(0,0,0)$ is unstable source.

EXERCISES

(1) Study the dynamics of the following systems near the origin $(x,y)=(0,0)$ via center-manifold analysis for the following autonomous dynamical systems.

a) $\begin{cases} \dot{x} = -x+y^2 \\ \dot{y} = -\sin x \end{cases}$ b) $\begin{cases} \dot{x} = x-2y \\ \dot{y} = x+y+x^4 \end{cases}$ c) $\begin{cases} \dot{x} = -2x+3y+y^3 \\ \dot{y} = 2x-3y+x^3 \end{cases}$

d) $\begin{cases} \dot{x} = y+x^2 \\ \dot{y} = -y-x^2 \end{cases}$ e) $\begin{cases} \dot{x} = -x+y \\ \dot{y} = -e^x+e^{-x}+2x \end{cases}$

f) $\begin{cases} \dot{x} = -x-y+z^2 \\ \dot{y} = 2x+y-z^2 \\ \dot{z} = x+2y-z \end{cases}$

Application of Center Manifold to Local Bifurcations

The center manifold technique can be used to investigate local bifurcations for multidimensional autonomous dynamical systems without an explicit determination of the local fixed point, as in the following example:

EXAMPLE

a) Investigate the possible bifurcation at $\mu=0$ for the following system, using center-manifold reduction.

$$\begin{cases} \dot{x} = \mu x - x^3 + xy \\ \dot{y} = -y + y^2 - x^2 \end{cases}$$

Solution

- Fixed point: There is an obvious fixed point at $(x,y)=(0,0)$.
- Linearization

Define: $f(x,y) = \mu x - x^3 + xy$ and $g(x,y) = -y + y^2 - x^2$ and $F(x,y) = (f(x,y), g(x,y))$. Then:

$$DF(x,y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} \mu - 3x^2 + y & x \\ -2x & -1+2y \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0) = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \cancel{(0,0)}\ \lambda(DF(0,0)) = \{\mu, -1\} \Rightarrow$$

$\Rightarrow (0,0)$ stable for $\mu<0$, unstable for $\mu>0$.
Note that stability is unknown for $\mu=0$.

► Center Manifold analysis: Note that center manifold reduction cannot be applied to the given dynamical system in the absense of zero-eigenvalues. However, we can "cheat" by turning μ into a variable and rewriting the system as:

$$\begin{cases} \dot{x} = \mu x - x^3 + xy \\ \dot{y} = -y + y^2 - x^2 \\ \dot{\mu} = 0 \end{cases}$$

Define $f(x,y,\mu) = \mu x - x^3 + xy$, $g(x,y,\mu) = -y + y^2 - x^2$, and $h(x,y,\mu) = 0$, and also define

$$F(x,y,\mu) = (f(x,y,\mu), g(x,y,\mu), h(x,y,\mu))$$

It follows that

$$DF(x,y,\mu) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial \mu \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial \mu \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial \mu \end{bmatrix} =$$

$$= \begin{bmatrix} \mu - 3x^2 + y & x & x \\ -2x & -1 + 2y & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \lambda(DF(0,0,0)) = \{0, -1\}$$

$\Rightarrow (0,0,0)$ is a non-hyperbolic fixed point.

Write the linearized equations around $(x,y,\mu) = (0,0,0)$ as follows:

$$\begin{cases} \dot{x} = 0x + (\mu x - x^3 + xy) \\ \dot{y} = -y + (y^2 - x^2) \\ \dot{\mu} = 0\mu \end{cases}$$

Note that $\dot{x}$ and $\dot{\mu}$ are the master equations and $\dot{y}$ is the slave equation. Therefore, let us write $y = H(x,\mu)$. It follows that

$$\dot{y} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial \mu}\dot{\mu} = \frac{\partial H}{\partial x}\dot{x} = (\mu x - x^3 + xy)\frac{\partial H}{\partial x} =$$

$$= (\mu x - x^3 + x H(x,\mu))\frac{\partial H}{\partial x}$$

and

$$\dot{y} = -y + (y^2 - x^2) = -H(x,\mu) + H(x,\mu)^2 - x^2$$

therefore we define

$$N(x,\mu) = (\mu x - x^3 + xH(x,\mu))\frac{\partial H}{\partial x} + H(x,\mu) - H(x,\mu)^2 + x^2 .$$

Under the limit $x \to 0$, consider the expansion

$$H(x,\mu) = a(\mu)x^2 + b(\mu)x^3 + O(x^4) \Rightarrow \frac{\partial H(x,\mu)}{\partial x} = 2a(\mu)x + 3b(\mu)x^2 + O(x^3)$$

It follows that

$$N(x,\mu) = [\mu x - x^3 + x(a(\mu)x^2 + b(\mu)x^3 + O(x^4))][2a(\mu)x + 3b(\mu)x^2 + O(x^3)]$$
$$+ [a(\mu)x^2 + b(\mu)x^3 + O(x^4)] - [a(\mu)x^2 + b(\mu)x^3 + O(x^4)]^2 + x^2 =$$
$$= \underline{2\mu a(\mu)x^2} + \underline{\underline{3\mu b(\mu)x^3}} + O(x^4) + \underline{a(\mu)x^2} + \underline{\underline{b(\mu)x^3}} + O(x^4) - O(x^4) + \underline{x^2} =$$
$$= [2\mu a(\mu) + a(\mu) + 1]x^2 + [3\mu b(\mu) + b(\mu)]x^3 + O(x^4) =$$
$$= [(2\mu + 1)a(\mu) + 1]x^2 + b(\mu)(3\mu + 1)x^3 + O(x^4)$$

and therefore, if we restrict μ to $\mu \in (-1/3, 1/3)$, we have:

$$N(x,\mu) = O(x^4) \Longleftrightarrow \begin{cases} (2\mu+1)\,a(\mu) + 1 = 0 \\ b(\mu)(3\mu+1) = 0 \end{cases} \Longleftrightarrow \begin{cases} a(\mu) = \dfrac{-1}{2\mu+1} \\ b(\mu) = 0 \end{cases}$$

and therefore

$$H(x,\mu) = \frac{-x^2}{2\mu+1} + O(x^4)$$

It follows that the $\dot{x}$ master equation reads:

$$\dot{x} = \mu x - x^3 + xy = \mu x - x^3 + x H(x,\mu) = \mu x - x^3 + x\left[\frac{-x^2}{2\mu+1} + O(x^4)\right]$$

$$= \mu x - x^3 - \frac{1}{2\mu+1}x^3 + O(x^4) = \mu x - \left(1 + \frac{1}{2\mu+1}\right)x^3 + O(x^4)$$

$$= \mu x - \frac{2\mu+2}{2\mu+1}x^3 + O(x^4)$$

and consequently, the center manifold reduction reads:

$$\begin{cases} \left.\begin{aligned} \dot{x} &= \mu x - \frac{2\mu+2}{2\mu+1}x^3 + O(x^4) \\ \dot{\mu} &= 0 \end{aligned}\right\} \text{master equations} \\ \left. y = \dfrac{-x^2}{2\mu+1} + O(x^4). \right\} \text{slave equation} \end{cases}$$

► Local Bifurcation at $\mu = 0$ and $(x,y) = (0,0)$

We can analyze the local bifurcation of $(x,y) = (0,0)$ at $\mu = 0$, by studying the master equation $\dot{x}$ instead of the original two-dimensional system.

Define $G(x,\mu) = \mu x - \dfrac{2\mu+2}{2\mu+1}x^3 + O(x^4)$.

We note that

$G(0,0) = 0$

$$G_x(x,\mu) = \mu - \frac{2\mu+2}{2\mu+1}(3x^2) + O(x^3) \Rightarrow G_x(0,0) = 0$$

$$G_\mu(x,\mu) = x - x^3 \frac{\partial}{\partial\mu}\left(\frac{2\mu+2}{2\mu+1}\right) + O(x^4) \Rightarrow G_\mu(0,0) = 0$$

$$G_{xx}(x,\mu) = 0 - \frac{2\mu+2}{2\mu+1}(6x) + O(x^2) \Rightarrow G_{xx}(0,0) = 0$$

$$G_{x\mu}(x,\mu) = \frac{\partial}{\partial\mu}\left[\mu - \frac{2\mu+2}{2\mu+1}(3x^2) + O(x^3)\right] =$$

$$= 1 - (3x^2)\frac{\partial}{\partial\mu}\left(\frac{2\mu+2}{2\mu+1}\right) + O(x^3) \Rightarrow$$

$$\Rightarrow G_{x\mu}(0,0) = 1 - 0 = 1 \neq 0$$

$$G_{xxx}(x,\mu) = \frac{-(2\mu+2)}{2\mu+1} \cdot 6 + O(x) \Rightarrow$$

$$\Rightarrow G_{xxx}(0,0) = \frac{-(0+2)}{0+1} \cdot 6 + 0 = -12 \neq 0.$$

To summarize:

$$\left\{\begin{array}{l} G(0,0) = G_x(0,0) = G_\mu(0,0) = G_{xx}(0,0) \\ G_{x\mu}(0,0) = 1 \neq 0 \\ G_{xxx}(0,0) = -12 \neq 0 \end{array}\right. \Rightarrow$$

$\Rightarrow$ At $\mu = 0$, the $(x,y) = (0,0)$ fixed point undergoes a pitchfork bifurcation.

b) Analyze the bifurcation at the origin for the Lorenz equations, given below, using the center-manifold reduction method

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = -bz + xy \end{cases} \quad \text{with } b>0, \sigma>0, \text{ and } \rho>0.$$

<u>Solution</u>

We note that $(x,y,z) = (0,0,0)$ is an obvious fixed point.

▶ Direct linearization

Define $f(x,y,z) = \sigma(y-x)$, $g(x,y,z) = \rho x - y - xz$, and $h(x,y,z) = -bz + xy$. Also define

$$F(x,y,z) = (f(x,y,z), g(x,y,z), h(x,y,z))$$

It follows that

$$DF(x,y,z) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial z \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -b \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(DF(0,0,0) - \lambda I) = \begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ \rho & -1-\lambda & 0 \\ 0 & 0 & -b-\lambda \end{vmatrix} =$$

$$= (-b-\lambda)\begin{vmatrix} -\sigma-\lambda & \sigma \\ \rho & -1-\lambda \end{vmatrix} =$$

$= (-b-\lambda)\left[(-\sigma-\lambda)(-1-\lambda)-\rho\sigma\right] =$
$= (-b-\lambda)\left[(\lambda+\sigma)(\lambda+1)-\rho\sigma\right] =$
$= (-b-\lambda)(\lambda^2+(\sigma+1)\lambda+\sigma-\rho\sigma] =$
$= \ -(\lambda+b)(\lambda^2+(\sigma+1)\lambda+\sigma(1-\rho))$

↳ Note that for $\sigma(1-\rho)\neq 0$, the zeroes λ_1, λ_2 of the quadratic factor $\lambda^2+(\sigma+1)\lambda+\sigma(1-\rho)$ will satisfy $\lambda_1\lambda_2\neq 0$ consequently none of the eigenvalues is zero and therefore we cannot apply the center manifold method. On the other hand for $\rho=1$, we have

$p(\lambda) = -(\lambda+b)(\lambda^2+(\sigma+1)\lambda) = -\lambda(\lambda+b)(\lambda+\sigma+1) \Rightarrow$
$\Rightarrow \lambda(DF(0,0,0)) = \{-b, -\sigma-1, 0\} \Rightarrow$
$\Rightarrow (0,0,0)$ non-hyperbolic fixed point.

▸ Center Manifold reduction

To make center manifold reduction applicable, we turn ρ into a variable governed by $\dot{\rho}=0$ with initial condition $\rho=1$ (at $t=0$). To center the 4D fixed point to the origin, we define $\mu=\rho-1$ and rewrite the Lorenz equations as:

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = \mu x + x - y - xz \\ \dot{z} = -bz + xy \\ \dot{\mu} = 0 \end{cases}$$

This extended 4D system has an obvious fixed point at $(x,y,z,\mu) = (0,0,0,0)$.

We define $f(x,y,z,\mu) = \sigma(y-x)$,

$$g(x,y,z,\mu) = \mu x + x - y - xz,$$
$$h(x,y,z,\mu) = -bz + xy$$

and also we define

$$F(x,y,z,\mu) = (f(x,y,z,\mu), g(x,y,z,\mu), h(x,y,z,\mu))$$
$$\mathcal{F}(x,y,z,\mu) = (f(x,y,z,\mu), g(x,y,z,\mu), h(x,y,z,\mu), 0)$$

It follows that

$$DF(x,y,z,\mu) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial z \end{bmatrix} =$$

$$= \begin{bmatrix} -\sigma & \sigma & 0 \\ \mu+1-z & -1 & -x \\ y & x & -b \end{bmatrix} \Longrightarrow$$

$$\Rightarrow DF(0,0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

which is the same as the previous Jacobian matrix with $\rho = 1$, and therefore:

$$\lambda(DF(0,0,0,0)) = \{0, -\sigma-1, -b\}$$

$\hookrightarrow$ Note that it is not necessary to write the full Jacobian for the 4×4 system explicitly since its Jacobian has a block diagonal structure

$$D\mathcal{F}(0,0,0,0) = \begin{bmatrix} DF(0,0,0,0) & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}$$

▸ To diagonalize the system we find the corresponding eigenvectors:

$\lambda_1 = -b$ has eigenvector $v_1 = (0,0,1)$

$\lambda_2 = 0$ has eigenvector $v_2 = (1,1,0)$

$\lambda_3 = -(\sigma+1)$ has eigenvector $v_3 = (-\sigma,1,0)$

Consequently, we define

$$P = [v_1\ v_2\ v_3] = \begin{bmatrix} 0 & 1 & -\sigma \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{\sigma+1}\begin{bmatrix} 0 & 0 & \sigma+1 \\ 1 & \sigma & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

and define

$$(u,v,w) = P^{-1}(x,y,z) \Leftrightarrow (x,y,z) = P(u,v,w)$$

Equivalently, we write:

$$\begin{cases} x = v - \sigma w \\ y = v + w \\ z = u \end{cases} \Leftrightarrow \begin{cases} u = z \\ v = (x+\sigma y)/(\sigma+1) \\ w = (-x+y)/(\sigma+1) \end{cases}$$

Now we rewrite the Lorenz equations in terms of the new variables u,v,w:

$$\dot{u} = \dot{z} = -bz + xy = -bu + (v-\sigma w)(v+w)$$

$$\dot{v} = \frac{\dot{x}+\sigma\dot{y}}{\sigma+1} = \frac{\sigma(y-x)+\sigma(\mu x + x - y - xz)}{\sigma+1} =$$

$$= \frac{\sigma(y-x+\mu x+x-y-xz)}{\sigma+1} = \frac{\sigma(\mu x - xz)}{\sigma+1} = \frac{\sigma x(\mu - z)}{\sigma+1} =$$

$$= \frac{\sigma(v-\sigma w)(\mu-u)}{\sigma+1}$$

$$\dot{w} = \frac{\dot{y}-\dot{x}}{\sigma+1} = \frac{(\mu x + x - y - xz) - \sigma(y-x)}{\sigma+1} =$$

$$= \frac{(\mu+\sigma+1)\,x - (\sigma+1)y - xz}{\sigma+1} = \frac{(\sigma+1)(x-y) + (\mu x - xz)}{\sigma+1} =$$

$$= (x-y) + \frac{x(\mu-z)}{\sigma+1} =$$

$$= (v-\sigma w) - (v+w) + \frac{(v-\sigma w)(\mu-u)}{\sigma+1} =$$

$$= -(\sigma+1)w + \frac{(\mu-u)(v-\sigma w)}{\sigma+1}$$

To summarize; the diagonalized equations read:

$$\begin{cases} \dot{u} = -bu + (v-\sigma w)(v+w) \\ \dot{v} = 0v + \sigma(v-\sigma w)(\mu-u)/(\sigma+1) \\ \dot{w} = -(\sigma+1)w + (\mu-u)(v-\sigma w)/(\sigma+1) \\ \dot{\mu} = 0\mu \end{cases}$$

We see that v, μ are the master variables and u, w are the slave variables. Let us write, therefore:

$$u = f(v) \quad \text{and} \quad w = g(v)$$

We note that

$$\dot{u} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial f}{\partial \mu}\frac{\partial \mu}{\partial t} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} =$$

$$= \frac{\partial f}{\partial v}\,\frac{\sigma(v-\sigma w)(\mu-u)}{\sigma+1} = \frac{\partial f}{\partial v}\,\frac{\sigma(v-\sigma g(v))(\mu-f(v))}{\sigma+1}$$

$$\dot{u} = -bu + (v-\sigma w)(v+w) = -bf(v) + (v-\sigma g(v))(v+g(v))$$

$$\dot{w} = \frac{\partial g}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial g}{\partial \mu}\frac{\partial \mu}{\partial t} = \frac{\partial g}{\partial v}\frac{\partial v}{\partial t} =$$

$$= \frac{\partial g}{\partial v}\frac{\sigma(v-\sigma w)(\mu-u)}{\sigma+1} = \frac{\partial g}{\partial v}\frac{\sigma(v-\sigma g(v))(\mu-f(v))}{\sigma+1}$$

$$\dot{w} = -(\sigma+1)w + \frac{(\mu-u)(v-\sigma w)}{\sigma+1} = -(\sigma+1)g(v) + \frac{(\mu-f(v))(v-\sigma g(v))}{\sigma+1}$$

consequently, we define:

$$N_u(v) = \frac{\partial f}{\partial v}\frac{\sigma(v-\sigma g(v))(\mu-f(v))}{\sigma+1} + bf(v) - (v-\sigma g(v))(v+g(v))$$

$$= bf(v) + (v-\sigma g(v))\left[\frac{\partial f}{\partial v}\frac{\sigma(\mu-f(v))}{\sigma+1} - (v+g(v))\right]$$

$$N_w(v) = \frac{\partial g}{\partial v}\frac{\sigma(v-\sigma g(v))(\mu-f(v))}{\sigma+1} + (\sigma+1)g(v) - \frac{(\mu-f(v))(v-\sigma g(v))}{\sigma+1} =$$

$$= (\sigma+1)g(v) + (v-\sigma g(v))\left[\frac{\partial g}{\partial v}\frac{\sigma(\mu-f(v))}{\sigma+1} - \frac{\mu-f(v)}{\sigma+1}\right]$$

$$= (\sigma+1)g(v) + \frac{(v-\sigma g(v))(\mu-f(v))}{\sigma+1}\left[\sigma\frac{\partial g}{\partial v} - 1\right]$$

- Use the expansions

$$f(v) = a_1(\mu)v^2 + a_2(\mu)v^3 + O(v^4) \Rightarrow \partial f/\partial v = 2a_1(\mu)v + 3a_2(\mu)v^2 + O(v^3)$$

$$g(v) = b_1(\mu)v^2 + b_2(\mu)v^3 + O(v^4) \Rightarrow \partial g/\partial v = 2b_1(\mu)v + 3b_2(\mu)v^2 + O(v^3)$$

and it follows that $N_u(v)$ and $N_w(v)$ are given by

$$N_u(v) = N_u^{(1)}(v) + N_u^{(2)}(v) + N_u^{(3)}(v)$$

with

$$N_u^{(1)}(v) = bf(v) = b(a_1v^2 + a_2v^3 + O(v^4)) =$$
$$= ba_1v^2 + ba_2v^3 + O(v^4)$$
$$N_u^{(2)}(v) = (v - \sigma g(v))\frac{\partial f}{\partial v}\,\frac{\sigma(\mu - f(v))}{\sigma+1} =$$

$$= \frac{\sigma}{\sigma+1}(v - \sigma b_1v^2 - \sigma b_2v^3)(2a_1v + 3a_2v^2)(\mu - a_1v^2 - a_2v^3) + O(v^4)$$

$$= \frac{\sigma}{\sigma+1}(2a_1v^2 + 3a_2v^3 - 2\sigma a_1b_1v^3)(\mu - a_1v^2 - a_2v^3) + O(v^4)$$

$$= \frac{\sigma}{\sigma+1}(2a_1\mu v^2) + O(v^4)$$ [Note: We drop μv^3 which is a 4th order]

$$N_u^{(3)}(v) = -(v - \sigma g(v))(v + g(v)) =$$
$$= -(v - \sigma b_1v^2 - \sigma b_2v^3)(v + b_1v^2 + b_2v^3) + O(v^4)$$
$$= -(v^2 + b_1v^3 - \sigma b_1v^3) + O(v^4) =$$
$$= -v^2 - b_1v^3 + \sigma b_1v^3 + O(v^4)$$

and therefore

$$N_u(v) = ba_1v^2 + ba_2v^3 + \frac{2\sigma a_1}{\sigma+1}\mu v^2 - v^2 - b_1v^3 + \sigma b_1v^3 + O(v^4)$$

$$= \left(ba_1 - 1 + \frac{2\mu\sigma}{\sigma+1}a_1\right)v^2 + (ba_2 - b_1 + \sigma b_1)v^3 + O(v^4)$$

Likewise $N_w(v) = N_w^{(1)}(v) + N_w^{(2)}(v) + N_w^{(3)}(v)$ with

$$N_w^{(1)}(v) = (\sigma+1)g(v) = (\sigma+1)b_1v^2 + (\sigma+1)b_2v^3 + O(v^4)$$
$$(v - \sigma g(v))(\mu - f(v)) = (v - \sigma b_1v^2 - \sigma b_2v^3)(\mu - a_1v^2 - a_2v^3) + O(v^4)$$
$$= \mu v - a_1v^3 - \sigma b_1\mu v^2 + O(v^4)$$
$$N_w^{(2)}(v) = \frac{\sigma}{\sigma+1}(v - \sigma g(v))(\mu - f(v))\frac{\partial g}{\partial v} =$$

$$= \frac{\sigma}{\sigma+1}(\mu v - a_1 v^3 - \sigma b_1 \mu v^2)(2b_1 v + 3b_2 v^2) + O(v^4)$$

$$= \frac{\sigma}{\sigma+1}(2b_1 \mu v^2) + O(v^4)$$ [Drop all μv^3 terms because they are 4th order]

$$N_W^{(3)}(v) = \frac{-1}{\sigma+1}(v - \sigma g(v))(\mu - f(v)) =$$

$$= \frac{-1}{\sigma+1}(\mu v - a_1 v^3 - \sigma b_1 \mu v^2) + O(v^4)$$

and therefore

$$N_W(v) = (\sigma+1) b_1 v^2 + (\sigma+1) b_2 v^3 + \frac{2\sigma b_1}{\sigma+1}\mu v^2 +$$

$$+ \frac{-1}{\sigma+1}(\mu v - a_1 v^3 - \sigma b_1 \mu v^2) + O(v^4)$$

$$= \frac{-\mu}{\sigma+1} v + \left[b_1(\sigma+1) + \frac{2\sigma\mu}{\sigma+1} b_1 + \frac{\sigma\mu}{\sigma+1} b_1\right] v^2 + \left[b_2(\sigma+1) + \frac{a_1}{\sigma+1}\right] v^3 + O(v^4)$$

- We disregard the μv term on $N_W^{(3)}(v)$ since it can be paired up with other μv terms that we are not keeping track of. Now, we set the coefficients of v^2 and v^3 equal to zero:

$$\begin{cases} b a_1 - 1 + \frac{2\mu\sigma}{\sigma+1} a_1 = 0 \\ b a_2 - b_1 + \sigma b_1 = 0 \end{cases} \wedge \begin{cases} b_1(\sigma+1) + \frac{3\sigma\mu}{\sigma+1} = 0 \\ b_2(\sigma+1) + \frac{a_1}{\sigma+1} = 0 \end{cases} \iff$$

and it follows that:

$$a_1 = \frac{1}{b + \frac{2\mu\sigma}{\sigma+1}} = \frac{1}{b} - \frac{1}{b^2}\frac{2\mu\sigma}{\sigma+1} + O(\mu^2)$$

$$b_1 = \frac{3\sigma\mu}{(\sigma+1)^2}$$

$$a_2 = (1-\sigma)b_1 = \frac{3\sigma(1-\sigma)\mu}{b(\sigma+1)^2}$$

$$b_2 = \frac{-a_1}{(\sigma+1)^2} = \frac{-1}{b(\sigma+1)^2} + \frac{1}{b^2(\sigma+1)^2}\frac{2\mu\sigma}{\sigma+1} + O(\mu^2)$$

The master equation is given by:

$$\dot{v} = \frac{\sigma}{\sigma+1}(v - \sigma w)(\mu - u) = \frac{\sigma}{\sigma+1}(v - \sigma g(v))(\mu - f(v)) =$$

$$= \frac{\sigma}{\sigma+1}(\mu v - a_1 v^3 - \sigma b_1 \mu v^2) + O(v^4) =$$

$$= \frac{\sigma}{\sigma+1}\left[\mu v - \left(\frac{1}{b} - \frac{2\mu\sigma}{\sigma+1}\frac{1}{b^2}\right)v^3 - \sigma \underbrace{\frac{3\sigma\mu}{(\sigma+1)^2}\mu v^2}_{\mu^2 v^2 \text{ term}}\right] + O(v^4)$$

$$= \frac{\sigma}{\sigma+1}\left[\mu v - \left(\frac{1}{b} - \underbrace{\frac{2\mu\sigma}{\sigma+1}\frac{1}{b^2}}_{\mu v^3 \text{ term can also be dropped}}\right)v^3\right] + O(v^4)$$

$$= \frac{\sigma}{\sigma+1}\left[\mu v - \frac{v^3}{b}\right]$$

which is the standard form of a $\boxed{\text{pitchfork bifurcation.}}$
For $\mu = 0$: $\dot{v} = -[b\sigma/(\sigma+1)]v^3$ which gives stable fixed point.

EXERCISE

(2) Use center manifold reduction to analyze the local bifurcation near the origin for the following autonomous dynamical systems

a) $\begin{cases} \dot{x} = -x + \mu y + y^2 \\ \dot{y} = -\sin x \end{cases}$

b) $\begin{cases} \dot{x} = -x + y + \mu x^2 \\ \dot{y} = -\sin x \end{cases}$

c) $\begin{cases} \dot{x} = 2x + 2y \\ \dot{y} = x + y + x^4 + \mu y^2 \end{cases}$

d) $\begin{cases} \dot{x} = -2x + 3y + y^3 + \mu x^2 \\ \dot{y} = 2x - 3y + x^3 \end{cases}$

e) $\begin{cases} \dot{x} = -x - y + z^2 \\ \dot{y} = 2x + y + \mu y - z^2 \\ \dot{z} = x + 2y - z \end{cases}$

f) $\begin{cases} \dot{x} = -2x + y + z + \mu x + y^2 z \\ \dot{y} = x - 2y + z + \mu x + x z^2 \\ \dot{z} = x + y - 2z + \mu x + x^2 y \end{cases}$