Homework 03: Series solution of linear differential equations

1. Derive the complete series expansion for the following functions around the indicated points and find the corresponding convergence radius

   (a) $f(x) = e^x \sin x$, around $x = x_0$

   (b) $f(x) = e^x \ln(1+x)$, around $x = x_0$

2. The binomial series is given by

   $$\forall x \in (-1,1) : (1+x)^a = \sum_0^{+\infty} \binom{a}{n} x^n$$

   with

   $$\binom{a}{0} = 1 \text{ and } \forall n \in \mathbb{N}^* : \binom{a}{n} = \prod_{k=1}^{n} \frac{a+1-k}{k}$$

   (a) Show that:

   $$\forall a \in (1,+\infty) : \forall n \in \mathbb{N}^* : \binom{1/a}{n} = (-1)^n \frac{\Gamma(n-1/a)}{n\Gamma(n)\Gamma(-1/a)}$$

   (b) For the special case $a = -2$, show that

   $$\forall n \in \mathbb{N}^* : \binom{-1/2}{n} = (-1)^n \frac{(2n-1)!!}{(2n)!!}$$

   with the double factorial $n!!$ defined via:

   $$0!! = 1 \ \wedge \ 1!! = 1$$

   $$\forall n \in \mathbb{N}^* : (2n)!! = \prod_{k=1}^{n} 2k \ \wedge \ (2n+1)!! = \prod_{k=1}^{n} (2k+1)$$

3. Find all terms of the unique power series solution to the following initial value problem:

   $$\begin{cases} y''(x) - 2xy'(x) + 2y(x) = 0 \\ y(0) = 1 \wedge y'(0) = 0 \end{cases}$$

4. Use the Frobenius method to show that the general homogeneous solution for the equation

   $$4xy''(x) + 2y'(x) + y(x) = 0$$

   is given by

   $$\forall x \in (0,+\infty) : y(x) = \lambda_1 \cos(\sqrt{x}) + \lambda_2 \sin(\sqrt{x})$$

5. Use the Frobenius method to show that the general homogeneous solution for the equation

   $$x(1-x)y''(x) + (1-5x)y'(x) - 4y(x) = 0$$

   is given by

   $$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$$

   with

   $$y_1(x) = \sum_{n=0}^{+\infty} (1+n)^2 x^n$$

   $$y_2(x) = y_1(x) \ln|x| - 2 \sum_{n=1}^{+\infty} n(n+1) x^n$$