Homework 02: Linear Differential Equations

1. Consider a general linear differential equation of the form

$$\forall x \in A : y''(x) + a(x)y'(x) + b(x)y(x) = 0$$

for some interval $A \subseteq \mathbb{R}$ with $a, b \in \mathcal{C}^0(A)$. Assume that $y_1 \in \mathcal{C}^2(A)$ is a solution, and define $y_2 \in \mathcal{C}^2(A)$ as:

$$\forall x \in A : y_2(x) = y_1(x) \int_c^x \frac{Q(t)}{[y_1(t)]^2} \, \mathrm{d}t$$

with $c \in A$ and with $Q(t)$ given by

$$\forall t \in A : Q(t) = \exp\left(-\int a(t) \, \mathrm{d}t\right)$$

(a) Show that $y_2(x)$ is also a solution.

(b) Show that y_1, y_2 are linearly independent.

Remark: An immediate consequence of (a) and (b) this is that if we define an operator $L : \mathcal{C}^2(A) \to \mathcal{C}^0(A)$ with $Ly = \mathbf{0}$, then it follows that its null space is given by

$$\text{null}(A) = \text{span}\{y_1, y_2\}$$

The corresponding general solution of the equation $Ly = \mathbf{0}$ is given by

$$\forall x \in A : y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$$

Remark: This exercise shows that if we can guess one solution of the second order linear ODE $Ly = \mathbf{0}$, we have an equation that can be used to find a second linearly independent solution. Then, given the aforementioned theorems, we have the null space and the general solution.

2. Find all solutions of the form $\forall x \in \mathbb{R} : y_1(x) = e^{bx}$ for the linear ODE

$$\forall x \in \mathbb{R} : y''(x) + 2ay'(x) + a^2 y(x) = 0$$

with $a \in \mathbb{R}$. Use the previous exercise to find the second linearly independent solution and write the corresponding general solution.

3. Show that the initial value problem

$$\begin{cases} y'(x) - 2(p+a)y'(x) + p^2 y(x) = 0 \\ y(0) = 0 \ \wedge \ y'(0) = 1 \end{cases}$$

with $a, p \in (0, +\infty)$ has solution

$$y(x|a,p) = \frac{\exp(A(p,a)x) - \exp(B(p,a)x)}{2\sqrt{a(2p+a)}}$$

with

$$A(p,a) = p + a + \sqrt{a(2p+a)}$$
$$B(p,a) = p + a - \sqrt{a(2p+a)}$$

without substituting the solution to the ODE. Then, show that:

$$\lim_{a \to 0^+} y(x|a,p) = xe^{px}$$

Remark: This result shows that when considering a second order linear differential equation, in which the two distinct zeroes of the corresponding characteristic polynomial approach each other, the solution obtained using the initial condition $y(0) = 0 \ \wedge \ y'(0) = 1$ converges continuously to the "screwball" $y(x) = xe^{pt}$ solution that we find when the two zeros of the characteristic polynomial are exactly equal to each other. Note that this argument does not establish a solution for the case where the zeros coincide; it only shows that the transition into that case does not exhibit any discontinuities.

4. Show that the linear differential equation

$$ax^3 y'''(x) + (b+3a)x^2 y''(x) + (a+b+c)xy'(x) + dy(x) = 0$$

with $a, b, c, d \in \mathbb{R}$ has characteristic polynomial

$$p(x) = ax^3 + bx^2 + cx + d.$$

Remark: This solves the inverse problem of constructing an equidimensional linear differential equation that has a desired characteristic polynomial.

5. Solve the general damped oscillator problem, which is defined as the following initial value problem:

$$\begin{cases} y''(x) + \beta y'(x) + \omega^2 y(x) = f(x) \\ y(0) = y_0 \ \wedge \ y'(x)(0) = y_1 \end{cases}$$

with $\beta, \omega \in (0, +\infty)$ and $y_0, y_1 \in \mathbb{R}$. Distinguish between the following cases:

(a) *Case 1:* $\beta < 2\omega$ (underdamped oscillator)

(b) *Case 2:* $\beta = 2\omega$ (critically damped oscillator)

(c) *Case 3:* $\beta > 2\omega$ (overdamped oscillator)

Remark: It is easier to solve the combined case $\beta \neq 2\omega$, allowing the use of exponentials of complex numbers for the underdamped subcase. This gives a common solution form for both cases $\beta < 2\omega$ and $\beta > 2\omega$, but for the underdamped case, additional work is then needed to convert the exponentials involving complex numbers into trigonometric functions. This approach will be more economical than attempting to handle the underdamped case from scratch.