Homework 01: First-order ODEs

1. The logistic population model is intended to model population growth under finite resources. If $y(t)$ is the population at time t, λ is the population growth rate, and N is the carrying capacity, then according to the logistic model, $y(t)$ is governed by

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \lambda y(N - y)$$

   (a) Using the initial condition $y(0) = y_0$, show that

$$y(t) = \frac{Ny_0}{y_0 + (N - y_0)\exp(-\lambda N t)}$$

   Show the validity of this result regardless of whether or not y_0 is a fixed point.

   (b) Show that $y(t)$ has an inflection point at $y = N/2$, using directly the differential equation instead of the explicit solution.

   (c) Assuming initialization at $y_0 \in (0, N/2)$, find the time t at which the solution reaches the inflection point

2. Consider an ordinary differential equation of the form

$$M(x, y) + N(x, y)y' = 0$$

   such that

$$\forall \lambda \in (0, +\infty) : \begin{cases} M(\lambda x, \lambda y) = \lambda^a M(x, y) \\ N(\lambda x, \lambda y) = \lambda^a N(x, y) \end{cases}$$

   with $a \in \mathbb{R}$. Show that the substitution $u = y/x$ reduces this differential equation to the separable form

$$\frac{1}{x} + \frac{N(1, u)}{M(1, u) + uN(1, u)} \frac{\mathrm{d}u}{\mathrm{d}x} = 0$$

3. Consider the initial value problem

$$\begin{cases} y' - 2xy = 1 \\ y(0) = y_0 \end{cases}$$

   Show that its unique solution is given by

$$y(x) = \exp(x^2)\left[\frac{\pi}{2}\operatorname{erf}(x) + y_0\right]$$

   with $\operatorname{erf}(x)$ the error function, defined as

$$\operatorname{erf}(x) = \frac{2}{\pi}\int_0^x \exp(-t^2)\,\mathrm{d}t$$

4. A Bernoulli ordinary differential equation is an equation of the form

$$y' + p(x)y = q(x)y^n$$

   with $n \in \mathbb{N}$.

   (a) Show that the substitution $u = y^{1-n}$ reduces the Bernoulli equation to a linear ordinary differential equation of the form

$$u' + (1 - n)p(x)u = (1 - n)q(x)$$

   (b) Use this substitution to solve the following Bernoulli initial value problem:

$$\begin{cases} y' + xy = xy^2 \\ y(0) = y_0 \end{cases}$$