Lecture Notes on College Algebra

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CA0: Review of Intermediate Algebra

Intermediate Algebra review

Basic Identities

$$\boxed{\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2\\ (a-b)^2 &= a^2 - 2ab + b^2\end{aligned}}$$

Perfect squares.

$$\boxed{(a+b)(a-b) = a^2 - b^2}$$

Square Difference

$$\boxed{(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca}$$

$$\boxed{\begin{aligned}(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3\\ (a-b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3\end{aligned}}$$

Perfect cube

$$\boxed{\begin{aligned}a^3 + b^3 &= (a+b)(a^2 - ab + b^2)\\ a^3 - b^3 &= (a-b)(a^2 + ab + b^2)\end{aligned}}$$

Sum/Diff of cubes

Note that:

$$(-a+b)^2 = (b-a)^2$$
$$(-a-b)^2 = (-(a+b)^2) = (a+b)^2$$
$$(-a+b)^3 = (b-a)^3$$
$$(-a-b)^3 = -(a+b)^3 .$$

EXERCISES

① Simplify the expressions:

a) $(2x+3)^2$
b) $(-5x+2)^2$
c) $(3-2x)^2$
d) $(x^2+2x^3)^2$
e) $(2x+3)(2x-3)$
f) $(x^2+x)(x^2-x)$
g) $(x^2+3x+2)^2$
h) $(x^3+5x-2)^2$

② Simplify the following expressions

a) $(x+1)^3+(2x-1)^2$
b) $(x^2-2x)^3+x(-x+2)^2$
c) $2(2x-1)^2-3(x+2)(-x+2)-(-x+2)^2$
d) $(x^3+2)^2-(x^3-2)(x^3+2)+(1-2x^3)^2$
e) $(2x+1)^3+(2x-1)^3$
f) $(3x-2)^3-(3x+2)^3$

▸ Fast multiplication: $\boxed{(x+a)(x+b)=x^2+(a+b)x+ab}$

③ Simplify the following:

a) $(x+2)(x+3)$
b) $(x-3)(x+4)$
c) $(x+5)(x+7)$
d) $(x-3)(x-6)$
e) $(x+2)(x-3)(x+1)$
f) $(x-3)(x-4)(x-2)$

Factoring

Case 1: Common Factors.

④ Factor the following:

a) $(x^3+2x^2)(x^2-x)$
b) $(2x+1)^2(3x-2)-(x-4)(2x+1)-(2x+1)^2$
c) $(3x-2)^3(x+2)+(2x+1)(3x-2)^2$
d) $3(x-1)(x-2)^2-(x-1)^2(2-x)+2(1-x)(x-1)$
e) $2(2x+1)(3-2x)^2+(1+2x)^2(2x-3)^3$
f) $(5x+3)^4(2x+3)^3+3(5x+3)^3(2x+3)^4$

Case 2: Difference of squares

$$\boxed{a^2-b^2=(a+b)(a-b)}$$

⑤ Factor the following:

a) $(2x+3)^2-(4x-1)^2$
b) $(4x^2+3x+3)^2-(3-4x^2)^2$
c) $28(x+3)^2-7(1-2x)^2$
d) $(2x^2-8)-(4-2x)^3$
e) $(3x-6)^2-2(x^2-4)-5(4-x^2)$
f) $(x^2-16)^2+4(x+4)^2$.

Case 3: Perfect Square

$$\boxed{a^2 \pm 2ab + b^2 = (a \pm b)^2}$$

$\mapsto$ 4th-order factorization

$$a^4 + a^2b^2 + b^4 = a^4 + \underline{2a^2b^2} + b^4 - \underline{a^2b^2} =$$

$$= (a^2+b^2)^2 - a^2b^2 =$$
$$= [(a^2+b^2) - ab][(a^2+b^2) + ab]$$

EXERCISES

⑥ Factor the expression:

a) $(x+1)^2 + 6(x+1) + 9$
b) $(2x-3)^2 - 4(2x-3) + 9$
c) $-8(x+3) + 1 + 16(x+3)^2$
d) $x^4 + x^2 + 1$
e) $4x^4 - 21x^2y^2 + 9y^4$
f) $16x^4 + 4$
g) $4a^4 - 13a^2 + 1$
h) $4x^4 - 37x^2y^2 + 9y^4$
i) $9x^8 - 15x^4 + 1$.

<u>Case 4</u> : $a^3-b^3=(a-b)(a^2+ab+b^2)$

$a^3+b^3=(a+b)(a^2-ab+b^2)$

Other cases:

$$a^4-b^4=(a^2-b^2)(a^2+b^2)$$
$$=(a-b)(a+b)(a^2+b^2)$$
$$a^6-b^6=(a^3-b^3)(a^3+b^3)$$
$$=(a-b)(a^2+ab+b^2)(a+b)(a^2-ab+b^2)$$

etc.

EXERCISES

(7) Factor the expressions:

a) $8x^3-27$

b) $27(x+1)^3-1$

c) $8(x+1)^3+(x-1)^3$

d) $3x^4-3$

e) ab^4-a^4b

f) $40x^4-5x$

g) $3(x+1)^3+81(x-1)^3$

h) $16ax^4-81(x-1)^4a$

i) $64x^6-1$

8. Factor the expressions by grouping:

a) $x^3+y^3+x^2-y^2$
b) $x^2+2xy+y^2+x^3+y^3$
c) $x^3-y^3-3ax+3ay$
d) $xy(x+y)+y^2(x+y)-x^3-y^3$
e) $x^3-xy(x-y)-y^3$

f) $a(x^4-1)+bx(x^2-1)$
g) $(x^3-y^3)-(x^2-y^2)-(x-y)^2$
h) $a^3+b^3-a-b-a^2b-ab^2$
i) $3(x^4-16)+x(x^3-27)$

CA1: Review of Sets

BRIEF REVIEW OF SETS

Definitions

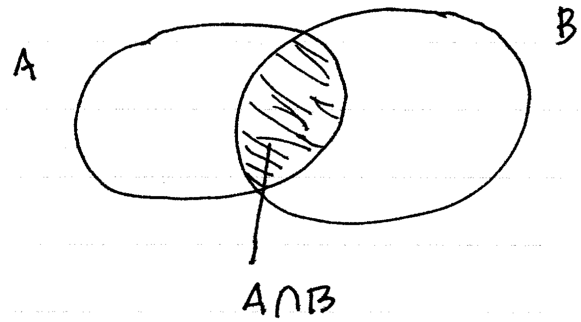
- A set A is a collection of elements x. An element can be a number, a point, or another set.
- A set can be defined by listing its elements:
  e.g. $A = \{2, 3, 5, 9, 12\}$
- Special sets:
  a) $\mathbb{R}$ = the set of all real numbers.
  b) $\emptyset = \{\} =$ the empty set (it has no elements).
- Notation:
  a) $x \in A$: x is an element of A (x belongs to A)
  b) $x \notin A$: x is not an element of A
  c) $A = B$: A and B have the same elements
  d) $A \subseteq B$: The elements of A all also belong to B.

Set operations

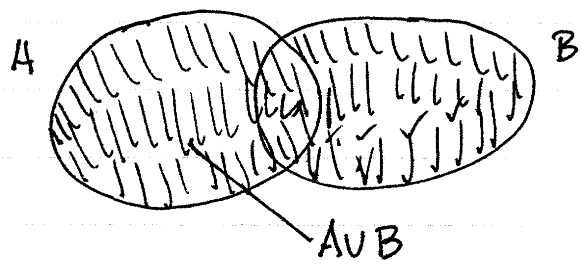
Let A, B be two sets. They can be combined into defining a new set via the following operations:

a) Intersection : $C = A \cap B$

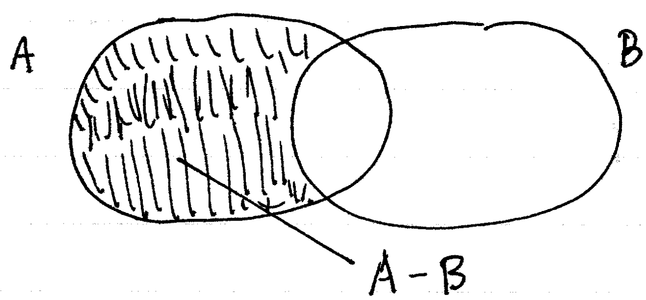
C has all the elements that belong to both sets A and B.



b) Union : $C = A \cup B$
C has all the elements of A and all the elements of B



c) Difference: $C = A - B$
C has all the elements of A except for any elements of A that also belong to B.



EXAMPLE

For $A = \{1,2,3,5,9,10\}$ and $B = \{3,4,5,10,11\}$, evaluate $A \cup B$, $A \cap B$, $A - B$, $B - A$, $(A-B) \cap (B-A)$

Solution

$$A \cup B = \{1,2,3,5,9,10\} \cup \{3,4,5,10,11\}$$
$$= \{1,2,3,4,5,9,10,11\}$$
$$A \cap B = \{1,2,3,5,9,10\} \cap \{3,4,5,10,11\}$$
$$= \{3,5,10\}$$
$$A - B = \{1,2,3,5,9,10\} - \{3,4,5,10,11\}$$
$$= \{1,2,\ \ 9\}$$
$$B - A = \{3,4,5,10,11\} - \{1,2,3,5,9,10\}$$
$$= \{4,11\}$$
$$(A-B) \cap (B-A) = \{1,2,9\} \cap \{4,11\} = \emptyset.$$

EXERCISES

1. Identify the following statements as TRUE or FALSE:

a) $3 \in \{1,2,4\}$

b) $5 \in \{2,5,6\}$

c) $2 \notin \{1,3,7,9\}$

d) $5 \in \mathbb{R}$

e) $3 \notin \mathbb{R}$

f) $\sqrt{2} \notin \emptyset$

g) $\sqrt{5} \in \emptyset$

h) $\{1,2,5\} \subseteq \{1,2,3,5\}$

i) $\{2,4,6\} \subseteq \{1,2,4,5\}$

j) $\emptyset \subseteq \mathbb{R}$

k) $\mathbb{R} \subseteq \mathbb{R}$

l) $\mathbb{R} \subseteq \emptyset$

2. Evaluate the sets $A \cap B$, $A \cup B$, $A - B$, and $B - A$, with the sets A and B defined as follows:

a) $A = \{1,2,3,5\}$ and $B = \{2,4,6\}$

b) $A = \{2,3,5,9\}$ and $B = \{3,5\}$

c) $A = \{3,5,8\}$ and $B = \{2,4,6\}$

d) $A = \emptyset$ and $B = \{1,3,7\}$

e) $A = \emptyset$ and $B = \mathbb{R}$

f) $A = \mathbb{R}$ and $B = \mathbb{R}$

g) $A = \emptyset$ and $B = \emptyset$.

3. Evaluate the set $D = (A \cap B) - C$ with the sets A, B, and C defined as:

a) $A = \{1,3,8,9\}$, $B = \{2,3,4,8\}$, and $C = \{1,3,4\}$

b) $A = \{2,3,4,5\}$, $B = \{4,5,7\}$, and $C = \{4,5,6\}$

c) $A = \{1,2,3,4\}$, $B = \{2,4,6\}$, and $C = \{1,2,3\}$

(4) Evaluate the set $D = A \cup (B - C)$ with the sets A, B, and C defined as:

a) $A = \{1, 2\}$, $B = \{2, 3, 5\}$, and $C = \{1, 2, 3\}$

b) $A = \{2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{1, 2, 5\}$

c) $A = \{2, 3, 7\}$, $B = \emptyset$, and $C = \{1, 2\}$

d) $A = \emptyset$, $B = \{2, 4\}$, and $C = \{4, 5\}$

e) $A = \{1\}$, $B = \{2, 5, 7\}$, and $C = \emptyset$.

CA2: Equations and Inequalities

EQUATIONS AND INEQUALITIES

▼ Terminology

- An equation is an expression of the form $f(x)=g(x)$ which may or may not be true for some values of x.
- A solution set S of an equation $f(x)=g(x)$ is the set of all real numbers $x\in\mathbb{R}$ for which the equation is true.

▶ example: For the equation $x^2=4$, the solution set is $S=\{-2,2\}$.

- An identity is an equation $f(x)=g(x)$ with solution set $S=\mathbb{R}$. (i.e. the equation is always true for all real numbers $x\in\mathbb{R}$).
- An equation $f(x)=g(x)$ is inconsistent if it's solution set is $S=\emptyset$. (i.e. the equation is always false for all real numbers $x\in\mathbb{R}$. OR equivalently, the equation is never true for any real number $x\in\mathbb{R}$).

▶ examples

a) The equation $(x+1)^2=x^2+2x+1$ is an identity.

b) The equation $x^2+4=0$ is inconsistent.
(because $x^2+4\geqslant 0+4=4>0$ for all $x\in\mathbb{R}$).

▼ Basic Logic Notation

- A predicate $p(x)$ is a statement about x which is either true or false depending on the value of x.

▶ example : Any equation is also a predicate. For example $p(x): 2x+1=3x$ is a predicate.

↳ Logical or/and/implications

$p(x) \wedge q(x)$: $p(x)$ and $q(x)$ are both true.

$p(x) \vee q(x)$: At least one of $p(x)$ or $q(x)$ is true ($p(x)$ OR $q(x)$).

$p(x) \Rightarrow q(x)$: If $p(x)$ is true, then $q(x)$ is true.

$p(x) \Leftrightarrow q(x)$: $p(x)$ is true if and only if $q(x)$ is true. (i.e. if $p(x)$ is true then $q(x)$ is true AND if $q(x)$ is true then $p(x)$ is true).

▶ examples

a) $x>2 \Rightarrow x>1$ TRUE

b) $x>1 \Rightarrow x>2$ FALSE

c) $x>2 \Leftrightarrow x>1$ FALSE

d) $2x=2 \Leftrightarrow x=1$ TRUE.

Basic properties of equations.

1) Let $x, y, a \in \mathbb{R}$. Then
$x = y \Leftrightarrow x + a = y + a$
(i.e.: we can add a number to both sides of an equation).

2) $x + a = y \Leftrightarrow x = y - a$
(i.e.: we can move a term on the other side of the equation but we must change its sign.)

3) Let $x, y, a \in \mathbb{R}$ and assume that $a \neq 0$. Then
$x = y \Leftrightarrow ax = ay$.
(i.e.: we can multiply a <u>non-zero</u> number to both sides of an equation)
▸ Note this property requires that <u>$a \neq 0$.</u> !!

4) Let $x, y \in \mathbb{R}$. Then
$xy = 0 \Leftrightarrow x = 0 \vee y = 0$
(i.e. A product xy is zero if and only if $x = 0$ or $y = 0$).

5) Let $x, y \in \mathbb{R}$. Then
$x^2 = y^2 \Leftrightarrow x = y \vee x = -y$
(i.e. $x^2 = y^2$ if and only if $x = y$ or $x = -y$).
▸ Note that
$x = y \Rightarrow x^2 = y^2$ is TRUE, but
$x^2 = y^2 \Rightarrow x = y$ is FALSE!

6) Let $x, y \in \mathbb{R}$. Then
$$x^2+y^2=0 \Leftrightarrow x=0 \wedge y=0$$

Solving an equation

To solve an equation $f(x)=g(x)$, we use the above properties 1-6 to construct an argument of the form:

$$\begin{aligned} f(x)=g(x) &\Leftrightarrow f_1(x)=g_1(x) \Leftrightarrow \\ &\Leftrightarrow f_2(x)=g_2(x) \Leftrightarrow \\ &\Leftrightarrow \cdots \Leftrightarrow \\ &\Leftrightarrow x=x_1 \vee x=x_2 \vee \cdots \vee x=x_n \end{aligned}$$

It follows that the solution set is
$$S=\{x_1, x_2, \ldots, x_n\}$$

- It is very important that EVERY step in the argument must be valid in BOTH directions (i.e. $\Leftrightarrow$ instead of only $\Rightarrow$ or $\Leftarrow$)
- When $\Rightarrow$ fails (but $\Leftarrow$ works): Every number you find is a solution but you may have more solutions out there that you have failed to find.
- When $\Leftarrow$ fails (but $\Rightarrow$ works): All of your solutions are among the numbers you found, but some of your numbers may not satisfy the equations (extraneous "solutions").

V Polynomial Equations

- A polynomial equation is an equation of the form
  $$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$
  with $a_0, a_1, \ldots, a_n \in \mathbb{R}$.
- n = degree of the equation.

① Linear Equations $\longrightarrow$ $\boxed{\begin{array}{l} ax+b=0 \\ \text{with } a,b \in \mathbb{R} \end{array}}$

Solution:

- $a \neq 0 \Rightarrow$ unique solution $x = -b/a$
- $a = 0 \wedge b \neq 0 \Rightarrow$ inconsistent ($S = \emptyset$)
- $a = 0 \wedge b = 0 \Rightarrow$ identity ($S = \mathbb{R}$).

EXAMPLES

a) $\dfrac{x-2}{3} - \dfrac{x+1}{2} = \dfrac{1-3x}{6} \Leftrightarrow$

$\Leftrightarrow 6\left[\dfrac{x-2}{3} - \dfrac{x+1}{2}\right] = 6 \cdot \dfrac{1-3x}{6} \Leftrightarrow$

$\Leftrightarrow 2(x-2) - 3(x+1) = 1 - 3x \Leftrightarrow$

$\Leftrightarrow 2x - 4 - 3x - 3 = 1 - 3x \Leftrightarrow$

$\Leftrightarrow (2-3+3)x = 4+3+1 \Leftrightarrow$

$\Leftrightarrow 2x = 8 \Leftrightarrow x = 4 \longleftarrow \underline{S = \{4\}}$.

b) $\frac{x+1}{2} = x - \frac{2x+3}{4} \Leftrightarrow 2(x+1) = 4x - (2x+3) \Leftrightarrow$

$\Leftrightarrow 2x+2 = 4x-2x-3 \Leftrightarrow (2-4+2)x = -2-3 \Leftrightarrow$

$\Leftrightarrow 0x = -5 \leftarrow$ inconsistent $\rightarrow$ $\underline{S = \varnothing}$

c) $3-5x-2(4-5x) = -4(x-1)+3(3x-2)-3 \Leftrightarrow$

$\Leftrightarrow 3-5x-8+10x = -4x+4+9x-6-3 \Leftrightarrow$

$\Leftrightarrow (-5+10+4-9)x = -3+8+4-6-3 \Leftrightarrow$

$\Leftrightarrow 0x = 0 \leftarrow$ identity $\longrightarrow$ $\underline{S = \mathbb{R}}$.

EXERCISE

① Solve the equations

a) $5x-2(3-x) = -6-3(-x-1)$

b) $-3+2(5+4x) = 9-3(4-2x)$

c) $2x - \frac{3-2x}{6} = 1 - \frac{5-x}{4}$

d) $\frac{2x}{5} - \frac{x-3}{15} = -1 - \frac{x+1}{10}$

e) $x + \frac{3-x}{3} = 1 + \frac{2x}{3}$

f) $\frac{x-5}{2} + \frac{14}{4} = \frac{7x}{2} - 3(x-3)$

g) $\frac{x+2}{6} - \frac{5-x}{2} = \frac{2x-7}{6} + \frac{x-3}{3}$.

② Completed square equations

$$\boxed{(ax+b)^2 = c}$$

a) If $c>0$, then
$(ax+b)^2 = c \Leftrightarrow ax+b = \sqrt{c} \vee ax+b = -\sqrt{c}$
$\Leftrightarrow \ldots$

b) If $c=0$, then
$(ax+b)^2 = 0 \Leftrightarrow ax+b = 0 \Leftrightarrow \ldots$

c) If $c<0$, then
$(ax+b)^2 = c$ is <u>inconsistent in $\mathbb{R}$.</u>

EXAMPLES

a) $(2x-1)^2 - 5 = 0 \Leftrightarrow (2x-1)^2 = 5 \Leftrightarrow$
$\Leftrightarrow 2x-1 = \sqrt{5} \vee 2x-1 = -\sqrt{5} \Leftrightarrow$
$\Leftrightarrow 2x = 1+\sqrt{5} \vee 2x = 1-\sqrt{5} \Leftrightarrow$
$\Leftrightarrow x = \dfrac{1+\sqrt{5}}{2} \vee x = \dfrac{1-\sqrt{5}}{2}$

b) $(3x+2)^2 = 0 \Leftrightarrow 3x+2 = 0 \Leftrightarrow 3x = -2 \Leftrightarrow x = -\dfrac{2}{3}$

c) $(5x+3)^2 + 2 = 0 \Leftrightarrow (5x+3)^2 = -2 < 0$
thus inconsistent in $\mathbb{R}$.

EXERCISES

(2) Solve the equations

a) $(x+1)^2 = 5$

b) $(2x+3)^2 - 7 = 0$

c) $(3x-1)^2 + 2 = 5$

d) $(5x-2)^2 + 1 = -2$

e) $(3x+2)^2 = 0$

f) $(3-2x)^2 = 3$

(3) Quadratic Equations $\to$ $ax^2+bx+c=0$ with $a, b, c \in \mathbb{R}$ and $a \neq 0$.

- 1 Calculate the discriminant
$$\Delta = b^2 - 4ac$$
- 2 Distinguish among the following cases:

a) $\Delta > 0 \Rightarrow$ Two solutions
$$x_1 = \frac{-b+\sqrt{\Delta}}{2a} \vee x_2 = \frac{-b-\sqrt{\Delta}}{2a}$$

b) $\Delta = 0 \Rightarrow$ One solution $x = -\frac{b}{2a} = x_1 = x_2$

c) $\Delta < 0 \Rightarrow$ Equation is inconsistent in $\mathbb{R}$. and has 2 solutions in $\mathbb{C}$: $x_{1,2} = (-b \pm i\sqrt{-\Delta})/2a$

$\to$ If the quadratic has two solutions x_1 and x_2 or one double solution (when $\Delta = 0 \Rightarrow x_1 = x_2 = -b/2a$) then they satisfy:

$$\boxed{x_1 + x_2 = -\frac{b}{a}}$$
$$\boxed{x_1 x_2 = \frac{c}{a}}$$

<u>Proof</u>

$$x_1 + x_2 = \frac{-b+\sqrt{\Delta}}{2a} + \frac{-b-\sqrt{\Delta}}{2a} =$$
$$= \frac{-b+\sqrt{\Delta}-b-\sqrt{\Delta}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

$$x_1 x_2 = \frac{-b+\sqrt{\Delta}}{2a} \cdot \frac{-b-\sqrt{\Delta}}{2a} = \frac{(-b)^2-(\sqrt{\Delta})^2}{4a^2} =$$
$$= \frac{b^2-\Delta}{4a^2} = \frac{b^2-(b^2-4ac)}{4a^2} =$$
$$= \frac{b^2-b^2+4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a} \qquad \square$$

Truncated Forms

1) $ax^2 = 0 \Leftrightarrow x = 0$
2) $ax^2 + bx = 0 \Leftrightarrow x(ax+b) = 0$
$\Leftrightarrow x = 0 \vee ax + b = 0$
(linear equations)
3) $ax^2 + c = 0 \Leftrightarrow x^2 = -\frac{c}{a}$

$$\Leftrightarrow \begin{cases} x = \pm\sqrt{-c/a}, & \text{if } -c/a \geq 0 \\ \text{inconsistent in } \mathbb{R}, & \text{if } -c/a < 0. \end{cases}$$

EXAMPLES

a) $2x^2+x-6=0$

Solution

$$\Delta = b^2-4ac = 1^2-4\cdot 2\cdot(-6) = 1+48 = 49 = 7^2 \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{-b\pm\sqrt{\Delta}}{2a} = \frac{-1\pm 7}{2\cdot 2} = \frac{-1\pm 7}{4} =$$

$$= \begin{cases} -8/4 = -2 \\ 6/4 = 3/2. \end{cases}$$

b) $x^2-6x+9=0$

Solution

$$\Delta = b^2-4ac = (-6)^2-4\cdot 1\cdot 9 = 36-36 = 0 \Rightarrow$$

$\Rightarrow$ unique solution $x = \dfrac{-b}{2a} = \dfrac{-(-6)}{2\cdot 1} = 3$

c) $2x^2-5x+4=0$

Solution

$$\Delta = b^2-4ac = (-5)^2-4\cdot 2\cdot 4 = 25-32 = -7 < 0 \Rightarrow$$

$\Rightarrow$ inconsistent in $\mathbb{R}$.

d) $3x^2 - 2 = 0 \Leftrightarrow 3x^2 = 2 \Leftrightarrow x^2 = \frac{2}{3} \Leftrightarrow$

$\Leftrightarrow x = \pm \frac{\sqrt{2}}{\sqrt{3}} = \frac{\pm\sqrt{6}}{3}$

e) $2x^2 + 5x = 0 \Leftrightarrow x(2x+5) = 0 \Leftrightarrow$
$\Leftrightarrow x = 0 \vee 2x + 5 = 0 \Leftrightarrow$
$\Leftrightarrow x = 0 \vee 2x = -5 \Leftrightarrow$
$\Leftrightarrow x = 0 \vee x = -5/2.$

↳ Fast factorization : $\boxed{x^2 + (a+b)x + ab = (x+a)(x+b)}$

f) $x^2 + 7x + 10 = 0 \Leftrightarrow (x+2)(x+5) = 0 \quad [2+5=7 \wedge 2 \cdot 5 = 10]$
$\Leftrightarrow x + 2 = 0 \vee x + 5 = 0 \Leftrightarrow$
$\Leftrightarrow x = -2 \vee x = -5$
thus $S = \{-2, -5\}$.

- Fast factorization circumvents the application of the quadratic formula. However, do not spend too much time looking for the fast factorization. The quadratic formula is also very efficient technique.

EXERCISES

③ Solve the equations

a) $2x^2-3x+1=0$

b) $x^2-4x+4=0$

c) $x^2+2x+4=0$

d) $2x^2-x-3=0$

e) $x^2=3x$

f) $x(2x+1)=x+4$

g) $x(x+1)=4$

h) $x^2-6x+9=0$

i) $x(2x+1)-x^2=0$

j) $(x-1)(x+2)=4$

④ Polynomial equations of high order

Form: $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$
with $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$,
$a_n \neq 0$, $n \in \mathbb{N}$ with $n \geq 3$.

Such equations can be solved by factoring:

- 1 Factor the equation to 1st and 2nd order factors.
- 2 Use the property:

$$a_1 a_2 a_3 \cdots a_n = 0 \Leftrightarrow a_1 = 0 \vee a_2 = 0 \vee \cdots \vee a_n = 0$$

- 3 Solve the resulting equations.

EXAMPLES

a) $(x+3)^2(x^2-4)^3(x^2-1)=0 \Leftrightarrow$

$\Leftrightarrow x+3=0 \lor x^2-4=0 \lor x^2-1=0 \Leftrightarrow$

$\Leftrightarrow x=-3 \lor x^2=4 \lor x^2=1 \Leftrightarrow$

$\Leftrightarrow x=-3 \lor x=2 \lor x=-2 \lor x=1 \lor x=-1.$

b) $x^3=10-2(x-1)^2 \Leftrightarrow$

$\Leftrightarrow x^3=10-2(x^2-2x+1) \Leftrightarrow$

$\Leftrightarrow x^3=10-2x^2+4x-2 \Leftrightarrow$

$\Leftrightarrow x^3+2x^2-4x-8=0 \Leftrightarrow$

$\Leftrightarrow x^2(x+2)-4(x+2)=0 \Leftrightarrow$

$\Leftrightarrow (x^2-4)(x+2)=0 \Leftrightarrow x^2-4=0 \lor x+2=0 \Leftrightarrow$

$\Leftrightarrow x=2 \lor x=-2 \lor x=-2 \Leftrightarrow x=2 \lor x=-2.$

c) $(x+2)(x^2-4)+(x-2)(x^2+5x+6)=0 \Leftrightarrow$

$\Leftrightarrow (x+2)(x+2)(x-2)+(x-2)(x+2)(x+3)=0 \Leftrightarrow$

$\Leftrightarrow (x+2)(x-2)[(x+2)+(x+3)]=0 \Leftrightarrow$

$\Leftrightarrow (x+2)(x-2)(2x+5)=0 \Leftrightarrow$

$\Leftrightarrow x+2=0 \lor x-2=0 \lor 2x+5=0 \Leftrightarrow$

$\Leftrightarrow x=-2 \lor x=2 \lor x=-5/2.$

d) $(x^3-8)(x^2+4x+4)+(x^2-4)(x^2+5x+6)=0 \Leftrightarrow$

$\Leftrightarrow (x-2)(x^2+2x+4)(x+2)^2+(x-2)(x+2)(x+2)(x+3)=0$

$\Leftrightarrow (x-2)(x+2)^2[(x^2+2x+4)+(x+3)]=0 \Leftrightarrow$

$$\Leftrightarrow (x-2)(x+2)^2(x^2+2x+4+x+3)=0 \Leftrightarrow$$

$$\Leftrightarrow (x-2)(x+2)^2(x^2+3x+7)=0 \Leftrightarrow$$

$$\Leftrightarrow x-2=0 \vee x+2=0 \vee x^2+3x+7=0 \qquad (1)$$

To solve $x^2+3x+7=0$:

$$\Delta = b^2-4ac = 3^2-4\cdot 1\cdot 7 = 9-28<0 \Rightarrow$$

$\Rightarrow$ no real solutions

It follows that

$$(1) \Leftrightarrow x-2=0 \vee x+2=0 \Leftrightarrow$$

$$\Leftrightarrow x=2 \vee x=-2$$

thus $S=\{-2,2\}$.

$\hookrightarrow$ Note that we use equation labelling to interrupt the main line of our argument, to solve all quadratic factors, and then restart and finish it.

EXERCISES

④ Solve the equations

a) $(2x+1)(x-2)(x+3)=0$

b) $(x-1)(x+1)^2(1-3x)=0$

c) $(3x-1)(x+2)(x^2+1)=0$

d) $(2x-1)^3(2x^2+1)(x^2-1)=0$

⑤ Solve the equations

a) $5x^3-20x=0$

b) $x^3=x^2+6x$

c) $(3x-1)(x-2)^2=9(3x-1)$

d) $x^3-x^2-x+1=0$

e) $(x^2-4)^2-(x+2)^2(5x-4)=0$

f) $3(x-1)^2-2(x-1)(x+1)=(x+1)^2$

g) $(x-3)(2x+1)^2-(x^2-9)(x+3)=0$

h) $x^5+x^4+x^3+x^2+x+1=0$

i) $(x+1)^4-x^4=4x^3$.

Special cases / tricks

① Binomial Equations

Let $\kappa \in \mathbb{Z}$ be an integer, let $p \in (0, +\infty)$ be a positive number and $n \in (-\infty, 0)$ be a negative number. Then

Odd Binomial: $[f(x)]^{2k+1} = a \Leftrightarrow f(x) = \sqrt[2k+1]{a}$

Even Binomial: $[f(x)]^{2k} = p \Leftrightarrow f(x) = \sqrt[2k]{p} \vee f(x) = -\sqrt[2k]{p}$

$[f(x)]^{2k} = 0 \Leftrightarrow f(x) = 0$

$[f(x)]^{2k} = n \longleftarrow$ inconsistent.

EXAMPLES

a) $(2x+1)^3 - 8 = 0$

Solution

$$(2x+1)^3 - 8 = 0 \Leftrightarrow (2x+1)^3 = 8 \Leftrightarrow 2x+1 = \sqrt[3]{8} \Leftrightarrow 2x+1 = 2$$
$$\Leftrightarrow 2x = 2-1 \Leftrightarrow 2x = 1 \Leftrightarrow x = 1/2$$

thus $S = \{1/2\}$.

b) $(1-3x)^4 - 16 = 0$

Solution

$$(1-3x)^4-16=0 \Leftrightarrow (1-3x)^4=16 \Leftrightarrow$$
$$\Leftrightarrow 1-3x=\sqrt[4]{16} \vee 1-3x=-\sqrt[4]{16} \Leftrightarrow$$
$$\Leftrightarrow 1-3x=2 \vee 1-3x=-2 \Leftrightarrow$$
$$\Leftrightarrow -3x=-1+2 \vee -3x=-1-2 \Leftrightarrow$$
$$\Leftrightarrow -3x=1 \vee -3x=-3 \Leftrightarrow x=-1/3 \vee x=1$$

thus $S=\{-1/3, 1\}$.

c) $(x^2+x)^4=0$

Solution

$$(x^2+x)^4=0 \Leftrightarrow x^2+x=0 \Leftrightarrow x(x+1)=0 \Leftrightarrow$$
$$\Leftrightarrow x=0 \vee x+1=0 \Leftrightarrow x=0 \vee x=-1$$

thus $S=\{0,-1\}$.

d) $(x-3)^6+2=0$

Solution

$(x-3)^6+2=0 \Leftrightarrow (x-3)^6=-2 \leftarrow$ inconsistent

thus $S=\emptyset$.

2 → Auxilliary Substitution

Sometimes, equations can be solved only via auxilliary substitution, as in the following example:

EXAMPLE

a) $(x+1)^4 - 4(x+1)^2 + 3 = 0$

Solution

Let $y = (x+1)^2$. Then the equation yields:

$$y^2 - 4y + 3 = 0 \Leftrightarrow (y-3)(y-1) = 0 \Leftrightarrow y-3=0 \vee y-1=0$$

$$\Leftrightarrow y = 3 \vee y = 1 \Leftrightarrow$$

$$\Leftrightarrow (x+1)^2 = 3 \vee (x+1)^2 = 1 \Leftrightarrow$$

$$\Leftrightarrow x+1 = \sqrt{3} \vee x+1 = -\sqrt{3} \vee x+1 = 1 \vee x+1 = -1$$

$$\Leftrightarrow x = -1+\sqrt{3} \vee x = -1-\sqrt{3} \vee x = 0 \vee x = -2$$

and therefore

$$S = \{-1+\sqrt{3}, -1-\sqrt{3}, 0, 2\}.$$

b) $(x^2+2x)^2 - 5(x^2+2x) + 4 = 0$

Solution

Let $y = x^2+2x$. Then the equation yields:

$$y^2 - 5y + 4 = 0 \Leftrightarrow (y-4)(y-1) = 0 \Leftrightarrow y-4=0 \vee y-1=0 \Leftrightarrow$$

$$\Leftrightarrow x^2+2x-4 = 0 \vee x^2+2x-1 = 0 \quad (1).$$

Solve: $x^2+2x-4=0$.

$$\Delta = b^2 - 4ac = 2^2 - 4 \cdot 1 \cdot (-4) = 4+16 = 20 = 5 \cdot 4 \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-2 \pm 2\sqrt{5}}{2 \cdot 1} = -1 \pm \sqrt{5}$$

Solve: $x^2+2x-1=0$

$\Delta = b^2-4ac = 2^2-4\cdot 1\cdot(-1) = 4+4=8=(2\sqrt{2})^2 \Rightarrow$

$\Rightarrow x_{1,2} = \dfrac{-b\pm\sqrt{\Delta}}{2a} = \dfrac{-2\pm 2\sqrt{2}}{2} = -1\pm\sqrt{2}$

Therefore:

$(1) \Leftrightarrow x=-1+\sqrt{5} \vee x=-1+\sqrt{5} \vee x=-1-\sqrt{2} \vee x=-1+\sqrt{2}$.

so $S=\{-1+\sqrt{5}, -1-\sqrt{5}, -1-\sqrt{2}, -1+\sqrt{2}\}$.

$\hookrightarrow$ Note that it is not convenient to solve these 2 quadratics equations simultaneously, so we stop the main argument, label the last step as equation (1), solve the two quadratic equations separately and then we use the equation label to restart the argument.

3 → Sum of squares

Some special equations can be solved via using the following properties:

$$\boxed{\begin{array}{l} a^2+b^2=0 \Leftrightarrow a=0 \wedge b=0 \\ a^2+b^2+c^2=0 \Leftrightarrow a=0 \wedge b=0 \wedge c=0 \end{array}}$$

EXAMPLES

a) $(x+2)^6+(x^3-4x)^4=0$

Solution

$(x+2)^6+(x^3-4x)^4=0 \Leftrightarrow [(x+2)^3]^2+[(x^3-4x)^2]^2=0$

$\Leftrightarrow (x+2)^3=0 \wedge (x^3-4x)^2=0 \Leftrightarrow x+2=0 \wedge x^3-4x=0$

$\Leftrightarrow x+2=0 \wedge x(x^2-4)=0 \Leftrightarrow$

$\Leftrightarrow x+2=0 \wedge x(x-2)(x+2)=0 \Leftrightarrow$

$\Leftrightarrow x=-2 \wedge (x=0 \vee x-2=0 \vee x+2=0) \Leftrightarrow$

$\Leftrightarrow x=-2 \wedge (x=0 \vee x=2 \vee x=-2) \Leftrightarrow$

$\Leftrightarrow x=-2.$

b) $(x^2-9)^2+(x-3)^2(x^2+4x+3)^2=0$

Solution

$(x^2-9)^2 + (x-3)^2(x^2+4x+3)^2 = 0 \Leftrightarrow$

$\Leftrightarrow (x^2-9)^2 + [(x-3)(x^2+4x+3)]^2 = 0 \Leftrightarrow$

$$\Leftrightarrow \begin{cases} x^2-9=0 \\ (x-3)(x^2+4x+3)=0 \end{cases} \quad (1).$$

Solve: $x^2-9=0 \Leftrightarrow x^2=9 \Leftrightarrow x=3 \vee x=-3 \quad (2)$

Solve: $(x-3)(x^2+4x+3)=0 \Leftrightarrow (x-3)(x+1)(x+3)=0$

$\Leftrightarrow x-3=0 \vee x+1=0 \vee x+3=0 \Leftrightarrow$

$\Leftrightarrow x=3 \vee x=-1 \vee x=-3 \quad (3)$

From (1), (2), and (3):

$$(1) \Leftrightarrow \begin{cases} x=3 \vee x=-3 \\ x=3 \vee x=-1 \vee x=-3 \end{cases} \Leftrightarrow x=3 \vee x=-3$$

Thus $S=\{3,-3\}$.

↳ Note that braces are used as an abbreviation for AND $(\wedge)$:

$$\begin{cases} f_1(x)=f_2(x) \\ g_1(x)=g_2(x) \end{cases}$$

is the same thing as:

$f_1(x)=f_2(x) \wedge g_1(x)=g_2(x)$.

EXERCISES

(6) Solve the equations:

a) $(3x+2)^4 = 16$

b) $(x^2+3x)^3 = -8$

c) $(2x+5)^5 = 32$

d) $(2x-1)^5 + 3 = 0$

e) $(x+2)^6 - 2 = 0$

f) $(x^2-2x)^7 + 1 = 0$

g) $16(5x+3)^4 - 81 = 0$

h) $2(x^2-1)^3 - 54 = 0$

(7) Solve the equations

a) $x^4 - 5x^2 + 6 = 0$

b) $3x^6 + 5x^3 + 2 = 0$

c) $2x^4 - 7x^2 - 4 = 0$

(8) Solve the following equations

a) $(x-1)^6 - 9(x-1)^3 + 8 = 0$

b) $(x^2+x)^2 - 3(x^2+x) + 2 = 0$

c) $(x^4-1)^2 + 2(x^4-1) + 1 = 0$

d) $(x^2-3x)^2 + 5(x^2-3x) + 6 = 0$

↳ Use an auxilliary substitution

(9) Solve the equations:

a) $(3x-9)^2 + (2x-6)^2 = 0$

b) $(x^2+7x+10)^2 + (x^2-5x+6)^2 = 0$

c) $(x+1)^2 + (x^2-1)^4 = 0$

↳ Use "sum of squares" technique.

▼ Rational Equations

- A rational equation is an equation that has an unknown x in the denominator of at least one fraction.

▶ Solution

- 1 Find the LCM (Least Common Multiple) of the denominators
- 2 From the condition $LCM(x) \neq 0$ find the domain $A \subseteq \mathbb{R}$ of the equation.
- 3 Multiply both sides of the equation with the LCM and solve the resulting polynomial equation
- 4 Accept the solutions that belong to the domain A and reject any solutions that do not belong to the domain A

EXAMPLES

a) $$\frac{4x}{x^2-x} = \frac{4}{x^2-1} - \frac{x}{x+1} \quad (1)$$

Require:

$$\begin{cases} x^2-x \neq 0 \\ x^2-1 \neq 0 \\ x+1 \neq 0 \end{cases} \Leftrightarrow \begin{cases} x(x-1) \neq 0 \\ (x-1)(x+1) \neq 0 \\ x+1 \neq 0 \end{cases} \Leftrightarrow \begin{cases} x \neq 0 \\ x \neq 1 \\ x \neq -1 \end{cases}$$

thus domain: $A = \mathbb{R} - \{0, -1, 1\}$.

$$(1) \Leftrightarrow \frac{4}{x-1} = \frac{4}{x^2-1} - \frac{x}{x+1} \leftarrow \text{LCM} = x^2-1 = (x-1)(x+1)$$

$$\Leftrightarrow 4(x+1) = 4 - x(x-1) \Leftrightarrow$$

$$\Leftrightarrow 4x+4 = 4 - x^2 + x \Leftrightarrow 4x = -x^2 + x \Leftrightarrow$$

$$\Leftrightarrow x^2 + (4-1)x = 0 \Leftrightarrow x^2 + 3x = 0 \Leftrightarrow$$

$$\Leftrightarrow x(x+3) = 0 \Leftrightarrow x = 0 \lor x+3 = 0 \Leftrightarrow$$

$$\Leftrightarrow x = 0 \lor x = -3$$

Reject $x = 0$ since $0 \notin A$

Accept $x = -3$ since $-3 \in A$.

Solution set: $S = \{-3\}$.

b) $$\frac{x-14}{x^2-4} + \frac{3}{x-2} = \frac{4}{x+2} \quad (1)$$

Require:

$$\begin{cases} x^2-4 \neq 0 \\ x-2 \neq 0 \\ x+2 \neq 0 \end{cases} \Leftrightarrow \begin{cases} (x-2)(x+2) \neq 0 \\ x-2 \neq 0 \\ x+2 \neq 0 \end{cases} \Leftrightarrow \begin{cases} x \neq 2 \\ x \neq -2 \end{cases}$$

thus domain : $A = \mathbb{R} - \{2, -2\}$.

$LCM = x^2 - 4 = (x-2)(x+2)$

$(1) \Leftrightarrow (x-14) + 3(x+2) = 4(x-2) \Leftrightarrow$

$\Leftrightarrow x - 14 + 3x + 6 = 4x - 8 \Leftrightarrow$

$\Leftrightarrow 4x - 8 = 4x - 8 \Leftrightarrow 0x = 0 \leftarrow$ identity.

Solution set: $\underline{S = \mathbb{R} - \{-2, 2\}}$.

EXERCISES

(10) Solve the equations

a) $$\frac{x}{x-3} + 3 = \frac{3}{x-3}$$

b) $$\frac{1}{x+1} + \frac{1}{x-1} = \frac{2x}{x^2-1}$$

c) $$\frac{1}{2-x} + \frac{2}{x+1} + \frac{3}{x^2-x-2} = 0$$

d) $$\frac{1}{x} - \frac{x}{1-x} = \frac{6x+5}{x^2-x}$$

e) $$\frac{13}{x+1} - \frac{1}{1-x} = \frac{5x-3}{x^2-1}$$

f) $$\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x-1} + \frac{1}{x-2} = 0$$

g) $$\frac{2}{x(x+2)} = \frac{-1}{x^2+5x+6}$$

h) $$\frac{x+1}{x-2} + \frac{x-1}{x+2} = \frac{2x^2+4}{x^2-4}$$

Parametric Linear Equations

These are equations where in addition to the unknown x, there is another <u>parameter</u> a. The goal is to find x in terms of a. In doing so, it is necessary to distinguish the values of a for which the equation has a unique solution from the values of a for which the equation is either inconsistent or an identity.

<u>Solution</u>

- 1 Simplify equation to
$$A(a)\,x = B(a).$$
- 2 For $A(a) \neq 0$, unique solution $x = \dfrac{B(a)}{A(a)}$
- 3 For $A(a) = 0$, consider what happens on a case by case basis.
(i.e. equation is either identity or inconsistent).

EXAMPLE

a) $a^2(x-1) = 4(x-a+1) \Leftrightarrow$

$\Leftrightarrow a^2x - a^2 = 4x - 4a + 4 \Leftrightarrow$

$\Leftrightarrow (a^2-4)x = a^2 - 4a + 4 \Leftrightarrow$

$\Leftrightarrow (a-2)(a+2)x = (a-2)^2 \quad (1)$

Distinguish cases:

- Case 1: $a \in \mathbb{R} - \{2, -2\}$

  (1) has a unique solution
  $$x = \frac{(a-2)^2}{(a-2)(a+2)} = \frac{a-2}{a+2}$$

- Case 2: $a = 2$

  $(1) \Leftrightarrow 0x = 0 \leftarrow$ identity

- Case 3: $a = -2$

  $(1) \Leftrightarrow 0x = (-2-2)^2 \Leftrightarrow 0x = 16 \leftarrow$ inconsistent.

Solution set:
$$S = \begin{cases} \{(a-2)/(a+2)\} & , a \in \mathbb{R} - \{2, -2\} \\ \mathbb{R} & , a = 2 \\ \varnothing & , a = -2. \end{cases}$$

EXERCISES

(11) Solve the equations with respect to x:

a) $a^2x + 2 = 2ax + x + a$
b) $2a + 3x = a^2x + 1$
c) $2a^2x - 5 = 4a - x$
d) $4ax + a^2x = 3x + 2$
e) $a^2(x-1) + a(x+2) - 6x + 15 = 0$
f) $a^3 + a^2x + a^2 + ax + a + x = 0$

(12) Solve the equations with respect to x:

a) $$\frac{x+a}{a+1} + \frac{a+1}{a-1} = \frac{(a+1)^2}{a^2-1}$$

b) $$\frac{x-2}{a-2} + \frac{x-2}{a+2} = 1$$

c) $$x - 2 = \frac{3}{a} + \frac{x+1}{a^2}$$

d) $$\frac{x+2}{3a} - \frac{1}{6a} + \frac{a}{6} + \frac{x}{2a} = 0$$

Inequalities – Terminology.

- An <u>inequality</u> is an expression of the form $f(x) < g(x)$ or $f(x) \leq g(x)$ or $f(x) > g(x)$ or $f(x) \geq g(x)$. which is either true or false depending on the value of the variable x.
- The <u>solution set S</u> of an inequality is the set of all real numbers $x \in \mathbb{R}$ for which the inequality is true.

Intervals

The solution sets of inequalities are written as unions of intervals, which are defined as follows:

$x \in [a, b] \iff a \leq x \leq b$	$x \in [a, +\infty) \iff a \leq x$
$x \in (a, b] \iff a < x \leq b$	$x \in (a, +\infty) \iff a < x$
$x \in [a, b) \iff a \leq x < b$	$x \in (-\infty, b] \iff x \leq b$
$x \in (a, b) \iff a < x < b$	$x \in (-\infty, b) \iff x < b$

The set of all real numbers can also be written as
$\mathbb{R} = (-\infty, +\infty)$.

Basic properties of inequalities

1) Let $x, y, a \in \mathbb{R}$. Then

$$x > y \Leftrightarrow x + a > y + a$$
$$x \geq y \Leftrightarrow x + a \geq y + a$$
$$x < y \Leftrightarrow x + a < y + a$$
$$x \leq y \Leftrightarrow x + a \leq y + a$$

(i.e.: We can add any number to both sides of an inequality)

2) Let $x, y \in \mathbb{R}$ and $p \in (0, +\infty)$. Then

$x > y \Leftrightarrow px > py$	$x < y \Leftrightarrow px < py$
$x \geq y \Leftrightarrow px \geq py$	$x \leq y \Leftrightarrow px \leq py$

(i.e: We can multiply a positive number to both sides of an inequality.)

3) Let $x, y \in \mathbb{R}$ and $n \in (-\infty, 0)$. Then

$x > y \Leftrightarrow nx < ny$	$x < y \Leftrightarrow nx > ny$
$x \geq y \Leftrightarrow nx \leq ny$	$x \leq y \Leftrightarrow nx \geq ny$

(i.e: We can multiply a negative number to both sides of an inequality, but then the direction of the inequality must be reversed).

↳ The general strategy for solving inequalities is to first move every term to the same side, then <u>simplify</u> and <u>factor</u> the resulting expression.

Polynomial inequalities

1) Linear Inequalities $\longrightarrow$ $ax+b \gtrless 0$

Consider, for example, the inequality $ax+b>0$.

a) For $a>0$:

$ax+b>0 \Leftrightarrow ax>-b \Leftrightarrow x>-b/a$.

thus $S=(-b/a, +\infty)$.

b) For $a<0$:

$ax+b>0 \Leftrightarrow ax>-b \Leftrightarrow \underline{\underline{x<-b/a}}$

thus $S=(-\infty, -b/a)$.

c) For $a=0$: the inequality is an identity or it is inconsistent, which we determine on a case by case basis.

EXAMPLES

a) $3(x-2)-5(x+1) \geqslant 3-2(3-x) \Leftrightarrow$

$\Leftrightarrow 3x-6-5x-5 \geqslant 3-6+2x \Leftrightarrow$

$\Leftrightarrow -2x-11 \geqslant -3+2x \Leftrightarrow -2x-2x \geqslant 11-3 \Leftrightarrow$

$\Leftrightarrow -4x \geqslant 8 \underset{(!!)}{\Leftrightarrow} x \leqslant \dfrac{8}{-4} \Leftrightarrow x \leqslant -2$

therefore $S=(-\infty, -2]$

- Note that because of $\leqslant$, -2 is included in S.

b) $x+3-\dfrac{3x-5}{2} > 2-\dfrac{x}{2} \Leftrightarrow$

$\Leftrightarrow 2(x+3)-(3x-5) > 4-x \Leftrightarrow$

$\Leftrightarrow 2x+6-3x+5 > 4-x \Leftrightarrow -x+11 > 4-x$

$\Leftrightarrow 0x > 4-11 \Leftrightarrow 0x > -7 \longleftarrow$ always true.

therefore $S = \mathbb{R}$.

(i.e. the inequality is an identity; it is satisfied by all real numbers $x \in \mathbb{R}$).

c) $\dfrac{x-3}{4}-\dfrac{x+5}{2} < -1-\dfrac{10+x}{4} \Leftrightarrow$

$\Leftrightarrow (x-3)-2(x+5) < -4-(10+x) \Leftrightarrow$

$\Leftrightarrow x-3-2x-10 < -4-10-x \Leftrightarrow$

$\Leftrightarrow -x-13 < -14-x \Leftrightarrow 0x < 13-14 \Leftrightarrow 0x < -1. \longleftarrow$ always false

therefore $S = \emptyset$.

(i.e. the inequality is inconsistent. It can never be satisfied).

$\hookrightarrow$ Note that when the inequality has a fractions, we first eliminate all fractions by multiplying both sides of the inequality with a large enough positive number.

EXERCISES

(13) Solve the following inequalities

1) $-3x+1>0$
2) $0x>4$
3) $0x<4$
4) $0x\geq 2$
5) $0x>-4$
6) $0x<-4$
7) $0x>0$
8) $0x\geq 0$
9) $0x\leq 2$
10) $0x\leq 0$
11) $-x-2<0$
12) $1>3x$
13) $4(2x-1)\leq x-2$
14) $3(2x+7)-4(15-x)\leq 29+12x$
15) $2(4x+5)-3(x+3)\leq -5x-9(1-x)$
16) $-6(x-2)-(5-3x)<9(x+3)-2x$
17) $2(x+1)\geq 4-(x+3)-3(2-x)$
18) $13-3(x-2)<4(x+3)-7(x-3)$
19) $1-\dfrac{3-x}{3}\geq\dfrac{19}{21}-\dfrac{1-x}{7}$
20) $\dfrac{x-3}{2}-\dfrac{x-5}{4}>1-\dfrac{4-x}{3}$
21) $\dfrac{x+1}{3}-\dfrac{5x-16}{6}\geq\dfrac{x+8}{12}$
22) $\dfrac{10x-1}{24}-\dfrac{2x-1}{8}<\dfrac{2x+5}{4}-\dfrac{x+3}{2}$
23) $\dfrac{2}{5}-\dfrac{3-x}{2}<\dfrac{x-1}{10}-\dfrac{3-2x}{5}$
24) $\dfrac{x+1}{16}-\dfrac{1+x}{2}\geq\dfrac{x-1}{16}-\dfrac{2x+1}{4}$

2) Quadratic Inequalities $\boxed{ax^2+bx+c \gtrless 0}$

- •1 Calculate the discriminant $\Delta = b^2-4ac$ and the two zeroes x_1 and x_2 (if they exist) given by: $x_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

- •2 The expression $f(x) = ax^2+bx+c$ has the same sign as the coefficient "a" for all values of x EXCEPT when $\Delta > 0$ and $x_1 < x < x_2$ (i.e. when x is located between the zeroes x_1 and x_2). We use this rule to construct a sign chart.
- •3 From the sign chart we deduce the solution set.

Sign charts

x		x_1		x_2	
ax^2+bx+c	+	0	−	0	+

($a>0$ and $\Delta>0$)

x		x_1		x_2	
ax^2+bx+c	−	0	+	0	−

($a<0$ and $\Delta>0$).

x		$x_1=x_2$	
ax^2+bx+c	+	0	+

($a>0$ and $\Delta=0$)

x		$x_1=x_2$	
ax^2+bx+c	−	0	−

($a<0$ and $\Delta=0$)

x	
ax^2+bx+c	+

($a>0$ and $\Delta<0$)

x	
ax^2+bx+c	−

($a<0$ and $\Delta<0$).

EXAMPLES

a) Solve $-3x^2+6x+2 \geq 0$.

Solution

$$\Delta = b^2-4ac = 6^2-4(-3)\cdot 2 = 36+24 = 60 = 2^2\cdot 3\cdot 5 =$$
$$= 2^2\cdot 15 \Rightarrow \sqrt{\Delta} = 2\sqrt{15} \Rightarrow$$
$$\Rightarrow x_{1,2} = \frac{-b\pm\sqrt{\Delta}}{2a} = \frac{-6\pm 2\sqrt{15}}{2(-3)} = \frac{-3\pm\sqrt{15}}{-3} =$$
$$= 1 \pm \frac{\sqrt{15}}{3}.$$

x		$1-(\sqrt{15})/3$		$1+(\sqrt{15})/3$	
$-3x^2+6x+2$	$-$	0	$+$	0	$-$

and therefore $S = \left[1-\frac{\sqrt{15}}{3}, 1+\frac{\sqrt{15}}{3}\right]$

b) $-3x^2+x-2>0$

Solution

$$\Delta = b^2-4ac = 1^2-4(-3)(-2) = 1-24 = -23 < 0$$

x	
$-3x^2+x-2$	$-$

Therefore $S = \emptyset$ (i.e. the equation is inconsistent).

c) $x^2+x+3>0$

<u>Solution</u>

$\Delta = b^2-4ac = 1^2-4\cdot1\cdot3 = 1-12 = -11<0$

x	
x^2+x+3	$+$

therefore $S=\mathbb{R}$. (i.e. equation is an identity).

d) $x^2+4x+4\leqslant 0$

<u>Solution</u>

$\Delta = b^2-4ac = 4^2-4\cdot1\cdot4 = 16-16=0 \Rightarrow$

$\Rightarrow x_1=x_2=\dfrac{-b}{2a} = \dfrac{-4}{2\cdot1} = -2$

x		-2	
x^2+4x+4	$+$	0	$+$

therefore $S=\{2\}$ (!)

e) $x^2+6x+9>0$

<u>Solution</u>

$\Delta = b^2-4ac = 6^2-4\cdot1\cdot9 = 36-36=0 \Rightarrow$

$\Rightarrow x_1=x_2=\dfrac{-b}{2a} = \dfrac{-6}{2\cdot1} = -3$

x		-3	
x^2+6x+9	$+$	0	$+$

therefore $S = (-\infty, -3) \cup (-3, +\infty) = \mathbb{R} - \{-3\}$.

Factorizable quadratic inequalities

An alternative technique is available if the quadratic has an obvious factorization. Recall the following obvious factorizations:

$$x^2 - a^2 = (x-a)(x+a)$$

$$ax^2 + bx = x(ax+b).$$

$$x^2 + (a+b)x + ab = (x+a)(x+b).$$

To use this method we require the following templates for the sign of the linear factor $ax+b$.

x		$-b/a$	
$ax+b$	$-$	0	$+$

$(a>0) \leftrightarrow$ increasing

x		$-b/a$	
$ax+b$	$+$	0	$-$

$(a<0) \leftrightarrow$ decreasing

EXAMPLES

a) $x^2+5x+6>0$.

Solution

$$x^2+5x+6>0 \Leftrightarrow (x+2)(x+3)>0$$

x		-3		-2	
$x+2$	$-$		$-$	0	$+$
$x+3$	$-$	0	$+$		$+$
ineq	$+$	0	$-$	0	$+$

and therefore
$S = (-\infty, -3) \cup (-2, +\infty)$.

- 1 Identify the zeroes of every factor and sort them from smallest to largest.
- 2 Write the zeroes and signs for each factor.
- 3 Multiply signs of all factors to determine the sign of the inequality.

b) $3x - 2x^2 \leq 0$

<u>Solution</u>

$$3x - 2x^2 \leq 0 \iff x(3-2x) \leq 0 \leftarrow \text{Zeroes: } 0,\ 3/2$$

x		0		$3/2$	
x	$-$	0	$+$		$+$
$3-2x$	$+$		$+$	0	$-$
ineq	$-$	0	$+$	0	$-$

and therefore:
$S = (-\infty, 0] \cup [3/2, +\infty)$.

3) Higher-order inequalities

These are inequalities of the form

$$a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \gtrless 0$$

- 1 Move everything to the left-hand side and factor to linear and quadratic factors.
- 2 Find the zeroes of every factor.
- 3 Make a sign chart for each factor and for their product.
- 4 See where the inequality is satisfied.

EXAMPLES

a) Solve: $x^2(x-2) \leq 2x(x-2)^2$.

Solution

$x^2(x-2) \leq 2x(x-2)^2 \Leftrightarrow x^2(x-2) - 2x(x-2)^2 \leq 0 \Leftrightarrow$

$\Leftrightarrow x(x-2)(x-2(x-2)) \leq 0 \Leftrightarrow$

$\Leftrightarrow x(x-2)(x-2x+4) \leq 0 \Leftrightarrow x(x-2)(-x+4) \leq 0.$ (1)

Zeroes: $0, 2, 4$

x		0		2		4	
x	$-$	0	$+$		$+$		$+$
$x-2$	$-$		$-$	0	$+$		$+$
$4-x$	$+$		$+$		$+$	0	$-$
ineq	$+$	0	$-$	0	$+$	0	$-$

thus

$(1) \Leftrightarrow x \in [0,2] \cup [4, +\infty)$

and therefore

$S = [0,2] \cup [4, +\infty)$.

Let $K \in \mathbb{N}$. Then:

a) Even powers: $(ax+b)^{2K}$ and $(ax^2+bx+c)^{2K}$ are ALWAYS positive.

b) Odd powers: $(ax+b)^{2K+1}$ and $(ax^2+bx+c)^{2K+1}$ have the same sign they would have had without the odd power. Therefore:

$(ax+b)^{2K+1}$ has the same sign as $(ax+b)$.

$(ax^2+bx+c)^{2K+1}$ has the same sign as (ax^2+bx+c).

b) Solve: $(x^2-1)^2(x^2+x-1)^3 > 0$

Solution

Zeroes of $f_1(x) = x^2-1 = (x-1)(x+1)$: -1 and $+1$.

Zeroes of $f_2(x) = x^2+x-1$.

$\Delta = b^2-4ac = 1^2-4\cdot 1\cdot(-1) = 1+4 = 5 \Rightarrow$

$\Rightarrow x_{1,2} = \dfrac{-b\pm\sqrt{\Delta}}{2a} = \dfrac{-1\pm\sqrt{5}}{2} \longrightarrow \dfrac{-1-\sqrt{5}}{2}$ and $\dfrac{-1+\sqrt{5}}{2}$.

- Now, we must sort the zeroes.

We claim that: $\dfrac{-1-\sqrt{5}}{2} < -1 < \dfrac{-1+\sqrt{5}}{2} < 1$.

We can show this easily with a calculator or via the following argument:

$$\frac{-1-\sqrt{5}}{2} < \frac{-1-\sqrt{4}}{2} = \frac{-1-2}{2} = \frac{-3}{2} < -1$$

$$\frac{-1+\sqrt{5}}{2} > \frac{-1+\sqrt{4}}{2} = \frac{-1+2}{2} = \frac{1}{2} > -1$$

$$\frac{-1+\sqrt{5}}{2} < \frac{-1+\sqrt{9}}{2} = \frac{-1+3}{2} = \frac{2}{2} = 1.$$

From the above it follows that

$$\frac{-1-\sqrt{5}}{2} < -1 < \frac{-1+\sqrt{5}}{2} < 1$$

- Now we construct the sign table:

x		$(-1-\sqrt{5})/2$		-1		$(-1+\sqrt{5})/2$		1	
$(x^2-1)^2$	+		+	0	+		+	0	+
$(x^2+x-1)^3$	+	0	−		−	0	+		+
ineq	+	0	−	0	−	0	+	0	+

It follows that

$$S = \left(-\infty, \frac{-1-\sqrt{5}}{2}\right) \cup \left(\frac{-1+\sqrt{5}}{2}, 1\right) \cup (1, +\infty).$$

c) Solve: $x^7 < x^3$.

Solution

$$x^7 < x^3 \Leftrightarrow x^7 - x^3 < 0 \Leftrightarrow x^3(x^4-1) < 0 \Leftrightarrow$$

$$\Leftrightarrow x^3(x^2-1)(x^2+1) < 0 \Leftrightarrow$$

$$\Leftrightarrow x^3(x-1)(x+1)(x^2+1) < 0.$$

Zeroes: $0, -1, +1$.

x		-1		0		$+1$	
x^3	$-$		$-$	0	$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$		$+$
x^2+1	$+$		$+$		$+$		$+$
ineq	$-$	0	$+$	0	$-$	0	$+$

therefore: $S = (-\infty, -1) \cup (0, 1)$.

↳ Note that incomplete quadratic factors of the form ax^2+b with $a>0$ and $b>0$ are always positive. (Because $\Delta < 0$ and $a>0$).

EXERCISES

(14) Solve the inequalities:

a) $x^2+3x+2 \leq 0$
b) $x^2+5x+6 > 0$
c) $x^2+8x-33 \geq 0$
d) $2x^2-20x+50 \leq 0$
e) $-2x^2+x-1 \geq 0$
f) $x^2+x+2 < 0$
g) $x^2-4x+8 > 0$
h) $x^2+28x+196 \leq 0$
i) $x^2-22x+121 > 0$
j) $x^2-26x+169 < 0$
k) $x^2-4 < 0$
l) $x^2-5 \leq 0$
m) $x^2+3 > 0$
n) $x^2+2 \leq 0$
o) $x^2+3x > 0$
p) $2x^2-5x < 0$
q) $x^2+x-3 \geq 0$
r) $x^2+2x-7 < 0$

(15) Solve the inequalities

a) $(3x-1)(x-1)^3(2-x)(2x+5)^4 \geq 0$
b) $5x(x^2-4x+3)(x^2-10x+25)(x^2+x+1) \leq 0$
c) $(2x^4-x^2)(x^2-3)^2(2-x)^3 < 0$
d) $(2x^2-5x-3)^2(x^3-x^2-x) \geq 0$

(16) Solve the inequalities:

a) $x^3+4x > 5x^2$
b) $x^3+x \leq x^2+1$
c) $x^3 < 8$
d) $x^8 > x^2$
e) $x^9 \leq 4x^5$
f) $x^8+64x^2 \leq 18x^5$

g) $2x(x+1)^2 - 3x^2(x+1) \leq 0$

h) $(2x+1)^4(2x-1)^3 > (2x+1)^3(2x-1)^4$

i) $x(x+1)^2(x+2) \geq (x^2+2x)(x^2+3x+2)$

j) $3(2x+3)^2(x-1)^3 \leq 2(2x+3)^3(x-1)^2$

k) $x(x+1)^5 < x(x+1)^3$

l) $(x+3)^5(2x-1)^3 \geq (x+3)^3(2x-1)^5$

m) $x^3(x^2+3x)^7 \leq x^5(x^2+3x)^5$.

Rational Inequalities

Form : $\dfrac{P(x)}{Q(x)} \gtrless 0$

with P, Q polynomials.

Method : The method entails the same steps as with polynomial inequalities. However, the zeroes of numerator factors must be distinguished from the zeroes of denominator factors.

- Denominator zeroes are shown with the ǂ symbol instead of ϕ in the last entry of your sign table because at these zeroes, the expression is undefined.
- Denominator zeroes are to be excluded from the solution set.

examples

1) $\dfrac{x-5}{x-3} \geqslant \dfrac{x-2}{x-1}$ (1)

Solution:

$$(1) \Leftrightarrow \frac{x-5}{x-3} - \frac{x-2}{x-1} \geqslant 0 \Leftrightarrow \frac{(x-5)(x-1)-(x-2)(x-3)}{(x-3)(x-1)} \geqslant 0$$

$$\Leftrightarrow \frac{(x^2-6x+5)-(x^2-5x+6)}{(x-3)(x-1)} \geq 0$$

$$\Leftrightarrow \frac{(-6+5)x+(5-6)}{(x-3)(x-1)} \geq 0$$

$$\Leftrightarrow \frac{-x-1}{(x-3)(x-1)} \geq 0. \quad (2)$$

Zeroes: $-1, 3, 1$

x		-1		1		3	
$-x-1$	$+$	0	$-$		$-$		$-$
$x-3$	$-$		$-$		$-$	0	$+$
$x-1$	$-$		$-$	0	$+$		$+$
$f(x)$	$+$	0	$-$	$\|$	$+$	$\|$	$-$

$(2) \Leftrightarrow x \in (-\infty, -1] \cup (1,3)$

Note that -1 is a zero of $f(x)$ but 1 and 3 are not, so they are not included in the solution.

→ CAUTION: If the fraction has cancellations then you must find the domain of the inequality before solving it:

example: $\dfrac{(x+1)(x^2+4x+4)}{(x^2+5x+6)} \geq 0. \quad (1)$

$$(1) \Leftrightarrow \frac{(x+1)(x+2)^2}{(x+2)(x+3)} \geq 0 \Leftrightarrow \frac{(x+1)(x+2)}{x+3} \geq 0$$

▶ Domain:

$x^2+5x+6 \neq 0 \Leftrightarrow \underline{x \in \mathbb{R}-\{-2,-3\} = A}$

x		-3		-2		-1	
$x+1$	$-$		$-$		$-$	0	$+$
$x+2$	$-$		$-$	0	$+$		$+$
$x+3$	$-$	0	$+$		$+$		$+$
$f(x)$	$-$	$\|\|$	$+$	$\|\|$	$-$	0	$+$

thus

$(1) \Leftrightarrow \underline{x \in (-3,-2) \cup [-1,+\infty)}$

-2 looks like a numerator zero but it cannot solve the original inequality because the domain

$A = \mathbb{R}-\{-2,-3\}$

of the inequality EXCLUDES -2 !!

EXERCISES

(17) Solve the inequalities:

a) $\dfrac{2-x}{3x+1} \geq 0$ b) $\dfrac{-(1-x)(3+x)(-3+x)}{(x+2)^2(x+1)^3} \geq 0$

c) $\dfrac{-x^2(3-x)(x^2+3x+2)(x^2-3)}{3(x+1)} \geq 0$

(18) Solve the inequalities

a) $\dfrac{2x-1}{x^2+4x+3} \leq \dfrac{1}{5}$

b) $\dfrac{x+1}{x-1} < \dfrac{2x+3}{x+1}$

c) $\dfrac{x^2+14}{x^2+6x+8} \leq 1$

d) $\dfrac{x+1}{x^2+x-2} \leq \dfrac{x}{x^2-1}$

e) $\dfrac{(x+1)^3-1}{(x-1)^3-1} \leq 1$

f) $\dfrac{x-10}{x^2+5} < \dfrac{1}{2}$

g) $\dfrac{6x^2-3x+8}{x^2-5x+6} \leq 6$

h) $\dfrac{x}{1+x^2} > 10$

System of inequalities

$$\begin{cases} f_1(x) \gtrless g_1(x) \\ f_2(x) \gtrless g_2(x) \\ \cdots \\ f_n(x) \gtrless g_n(x) \end{cases}$$

- 1 Find the solution sets $S_1, S_2, \ldots, S_n$ for each inequality separately.
- 2 The solution set S for the system is the intersection $S = S_1 \cap S_2 \cap \cdots \cap S_n$

EXAMPLE

$x+1 \leq (2x+1)^2 < 10x+5$

Solution

$x+1 \leq (2x+1)^2 \Leftrightarrow x+1 \leq 4x^2+4x+1 \Leftrightarrow$

$\Leftrightarrow 4x^2+3x \geq 0 \Leftrightarrow x(4x+3) \geq 0.$ (1)

x		$-3/4$		0	
x	$-$		$-$	0	$+$
$4x+3$	$-$	0	$+$		$+$
	$+$	0	$-$	0	$+$

$(1) \Leftrightarrow x \in (-\infty, -3/4] \cup [0, +\infty).$

and

$(2x+1)^2 < 10x+5 \Leftrightarrow (2x+1)^2 - 5(2x+1) < 0 \Leftrightarrow$

$\Leftrightarrow (2x+1)[(2x+1)-5] < 0 \Leftrightarrow (2x+1)(2x-4) < 0$ (2)

$\Leftrightarrow (2x+1)(x-2) < 0$ (2)

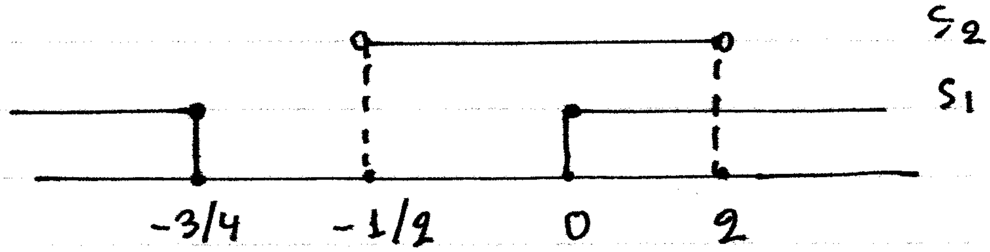
x		$-1/2$		2	
$2x+1$	$-$	0	$+$		$+$
$x-2$	$-$		$-$	0	$+$
	$+$	0	$-$		$+$

$(1) \Leftrightarrow x \in (-\infty, -1/2)$

$(2) \Leftrightarrow x \in (-1/2, 2)$.

Thus $S_1 = (-\infty, -3/4] \cup [0, +\infty)$

$S_2 = (-1/2, 2)$



It follows that the solution set is

$$S = S_1 \cap S_2 =$$
$$= [(-\infty, -3/4] \cup [0, +\infty)] \cap [(-1/2, 2)]$$
$$= [0, 2)$$

↳ In general, to calculate intersections of solution sets we use a geometrical construction as shown in the above example.

EXERCISES

(20) Solve the following systems of inequalities.

a) $-3 \leq \dfrac{x+5}{2-x} < 4$

b) $-2x < \dfrac{3x-1}{4} \leq x^2-1$

c) $5x-1 < (x+1)^2 \leq 7x-3$

d) $\begin{cases} 3x^3+2x > 5x^2 \\ x^3+4x > x^2 \end{cases}$

e) $\begin{cases} 2x-5 < 3x-7 \\ 2x^2 \leq 9 \end{cases}$

f) $\begin{cases} \dfrac{x-1}{3x+2} > 0 \\ (x^2-9)(x^2+x+5) \leq 0 \end{cases}$

g) $\begin{cases} \dfrac{x-1}{x+1} < \dfrac{1}{2} \\ \dfrac{(x-1)(x-2)}{(x+1)(x+2)} > 2 \\ \dfrac{x^2-4x+1}{x^2+x-1} \leq 2 \end{cases}$

Absolute Values

- Let $x \in \mathbb{R}$ be given. We define the absolute value $|x|$ of x as:

$$\boxed{|x| = \begin{cases} x & , \text{ if } x \geq 0 \\ -x & , \text{ if } x < 0 \end{cases}}$$

examples: $|3| = 3$, $|-7| = 7$, $|0| = 0$

Properties of absolute value

$\lvert x \rvert \geq 0$	$\lvert x \rvert - \lvert y \rvert \leq \lvert x+y \rvert \leq \lvert x \rvert + \lvert y \rvert$
$\lvert -x \rvert = \lvert x \rvert$	$\lvert x \rvert - \lvert y \rvert \leq \lvert x-y \rvert \leq \lvert x \rvert + \lvert y \rvert$
$-\lvert x \rvert \leq x \leq \lvert x \rvert$	$\lvert xy \rvert = \lvert x \rvert \lvert y \rvert$, $\left\lvert \frac{x}{y} \right\rvert = \frac{\lvert x \rvert}{\lvert y \rvert}$
$\lvert x \rvert^2 = x^2$	

Equations with absolute values

Let $a, x \in \mathbb{R}$, and $p \in [0, +\infty)$, and $n \in (-\infty, 0)$. Then:

1) $|x| = |a| \Leftrightarrow x = a \vee x = -a$
2) $|x| = p \Leftrightarrow x = p \vee x = -p$
3) $|x| = n$ is inconsistant.

We use the above 3 properties to solve equations with absolute values as in the following examples.

EXERCISES

(23) If $a<3<b$, show that

a) $A=|3-a|+|3-b|-|a-b|$
b) $B=|a-3|+|b|+|a-b|$
c) $C=|a-4|-|b-2|$
d) $D=|a-b|-|5-a|-|1-b|$
e) $E=|a^3-5a^2|-|b^2-ab|+|2b+a-1|$
f) $F=|b^2-9|+|2b+1|-|3b-ab|$.
g) $G=|b^3-4b|+|2a^3b-2a^2b^2|+|b+2ab-3b^2|$

(24) Simplify the following expressions.

a) $A=\dfrac{x^2+2|x|}{|x|+2}$ b) $B=\dfrac{|x|^3+3x^2}{2|x|+6}$

c) $C=\dfrac{x^2+4|x|+4}{|x|+2}$ d) $D=\dfrac{x^2-1}{|x|+1}$

EXAMPLES

a) $(x^2-|x|)(3|x|+1)=0 \quad (1)$

Solution

Let $y=|x| \Rightarrow x^2=|x|^2=y^2$, thus

$$(1) \Leftrightarrow (y^2-y)(3y+1)=0 \Leftrightarrow y(y-1)(3y+1)=0 \Leftrightarrow$$
$$\Leftrightarrow y=0 \vee y-1=0 \vee 3y+1=0 \Leftrightarrow$$
$$\Leftrightarrow y=0 \vee y=1 \vee y=-1/3 \Leftrightarrow$$
$$\Leftrightarrow |x|=0 \vee |x|=1 \vee |x|=-1/3 \Leftrightarrow$$
$$\Leftrightarrow x=0 \vee x=1 \vee x=-1$$

thus $S=\{0,1,-1\}$.

Note that $|x|=-1/3$ has no solutions since $-1/3<0$.

b) $|x+3|+2=0 \quad (1)$

Solution

$(1) \Leftrightarrow |x+3|=-2<0 \leftarrow$ inconsistent.

Thus $S=\emptyset$.

c) $|2x-1|-5=0 \quad (1)$

Solution

$$(1) \Leftrightarrow |2x-1|=5 \Leftrightarrow 2x-1=\pm 5 \Leftrightarrow 2x=1\pm 5$$
$$\Leftrightarrow x=\frac{1\pm 5}{2}=\begin{cases} 6/2=3 \\ -4/2=-2 \end{cases}, \text{ thus } S=\{3,-2\}.$$

d) $|2x+3| = |x+9|$

Solution

$|2x+3| = |x+9| \Leftrightarrow 2x+3 = x+9 \vee 2x+3 = -(x+9) \Leftrightarrow$
$\Leftrightarrow (2-1)x = 9-3 \vee 2x+3 = -x-9 \Leftrightarrow$
$\Leftrightarrow x = 6 \vee 2x+x = -3-9 \Leftrightarrow x = 6 \vee 3x = -12$
$\Leftrightarrow x = 6 \vee x = -4.$
thus $S = \{6, -4\}$.

e) $|x-4| = 5-2x \quad (1)$

Solution

Require $5-2x \geq 0 \Leftrightarrow 5 \geq 2x \Leftrightarrow x \leq 5/2$
thus domain: $A = (-\infty, 5/2]$.
$(1) \Leftrightarrow x-4 = 5-2x \vee x-4 = -(5-2x) \Leftrightarrow$
$\Leftrightarrow x+2x = 4+5 \vee x-4 = -5+2x \Leftrightarrow$
$\Leftrightarrow 3x = 9 \vee x-2x = 4-5 \Leftrightarrow$
$\Leftrightarrow x = 3 \vee -x = -1 \Leftrightarrow$
$\Leftrightarrow x = 3 \vee x = 1.$ $\leftarrow$ accept $x=1$, reject $x=3$

$\hookrightarrow$ For equations of the form $|f(x)| = g(x)$ we require
$g(x) \geq 0 \Leftrightarrow x \in A$
and reject solutions that do not belong to A.

f) $|x+3|-|2-x| = x+5 \quad (1)$

<u>Solution</u>

x		-3		2	
$x+3$	$-$	0	$+$		$+$
$2-x$	$+$		$+$	0	$-$

Distinguish 3 cases:

<u>Case 1:</u> If $x \in (-\infty, -3)$ then

$|x+3| = -(x+3)$ and $|2-x| = 2-x$.

$$(1) \Leftrightarrow -(x+3)-(2-x) = x+5 \Leftrightarrow$$
$$\Leftrightarrow -x-3-2+x = x+5 \Leftrightarrow$$
$$\Leftrightarrow -x-5 = 5 \Leftrightarrow x = -5-5 = -10 \leftarrow \text{accepted}$$

$-10 \in (-\infty, -3)$.

<u>Case 2</u>: If $x \in [-3, 2)$ then

$|x+3| = x+3$ and $|2-x| = 2-x$.

$$(1) \Leftrightarrow (x+3)-(2-x) = x+5 \Leftrightarrow$$
$$\Leftrightarrow x+3-2+x = x+5 \Leftrightarrow 2x+1 = x+5 \Leftrightarrow$$
$$\Leftrightarrow 2x-x = 5-1 \Leftrightarrow x = 4 \leftarrow \text{rejected}$$

$4 \notin [-3, 2)$.

<u>Case 3</u>: If $x \in [2, +\infty)$ then

$|x+3| = x+3$ and $|2-x| = -(2-x)$

$$(1) \Leftrightarrow (x+3)+(2-x) = x+5 \Leftrightarrow$$
$$\Leftrightarrow x+3+2-x = x+5 \Leftrightarrow 5 = x+5 \Leftrightarrow x = 0$$

rejected

$0 \notin [2, +\infty)$.

Thus $S = \{-10\}$.

EXERCISES

(25) Solve the equations

a) $\dfrac{2+|-5x|}{|x|-1} = 3$

b) $\dfrac{3+|x|}{|2x|+1} = 4$

c) $2x^2 + 5|x| + 7 = 0$

d) $(|2x|-3)(|x^3|-x^2) = 0$

(26) Solve the equations

a) $|2x| = |x-1|$

b) $|3x-2| = |2-x|$

c) $|x^2-1| = |2-x|$

d) $|2x-3| = x$

e) $|2x-1| = 4$

f) $|2x^2-5x-1| = 4x$

g) $|x^2-1| = 2x+1$

h) $x^3-x^2+|x-1| = 0$

(27) Solve the equations

a) $|3x| + |2-x| - x + 1 = 0$

b) $|x-3| - 3|x-1| + |x| = 5$

c) $2|x+1| - 3|x-1| = 1$

d) $|x^2-4x+3| - 2|3-x^2| = 1$

Inequalities with absolute values

The solution of inequalities with absolute values is based on the following properties:

1) If $a>0$, then

$|x| \leq a \Leftrightarrow -a \leq x \leq a$

$|x| < a \Leftrightarrow -a < x < a$

$|x| \geq a \Leftrightarrow x \geq a \vee x \leq -a$

$|x| > a \Leftrightarrow x > a \vee x < -a$

2) If $a<0$, then

$$\left.\begin{array}{l} |x| \leq a \longleftarrow \text{Inconsistent} \\ |x| < a \longleftarrow \text{Inconsistent} \\ |x| \geq a \longleftarrow \text{Identity} \\ |x| > a \longleftarrow \text{Identity} \end{array}\right\} \begin{array}{l} \text{because} \\ |x| \geq 0 \end{array}$$

3) For the case $a=0$:

$|x| \leq 0 \Leftrightarrow x=0$

$|x| < 0 \longleftarrow \text{Inconsistent}$

$|x| \geq 0 \longleftarrow \text{Identity}$

$|x| > 0 \Leftrightarrow x \neq 0$

We apply these properties as in the following examples.

EXAMPLES

a) $|2x-3| \leq 5$

Solution

$$|2x-3| \leq 5 \Leftrightarrow -5 \leq 2x-3 \leq 5 \Leftrightarrow \begin{cases} 2x-3 \leq 5 \\ -5 \leq 2x-3 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2x \leq 8 \\ 2x \geq -5+3 = -2 \end{cases} \Leftrightarrow \begin{cases} x \leq 4 \\ x \geq -1 \end{cases} \Leftrightarrow x \in [-1, 4]$$

thus $S = [-1, 4]$

b) $|1-2x| > 7$

Solution

$$|1-2x| > 7 \Leftrightarrow 1-2x > 7 \vee 1-2x < -7 \Leftrightarrow$$

$$\Leftrightarrow 1-7 > 2x \vee 1+7 < 2x \Leftrightarrow -6 > 2x \vee 8 < 2x$$

$$\Leftrightarrow x < -3 \vee x > 4$$

thus $S = (-\infty, -3) \cup (4, +\infty)$.

c) $|x-5| < -2$

Solution

Since $\forall x \in \mathbb{R} : |x-5| \geq 0$, it follows that $|x-5| < -2$ is inconsistent. Thus $S = \emptyset$.

d) $|21-4x| \geq -3$

Solution

$\forall x \in \mathbb{R}: |21-4x| \geq 0$ thus
$\forall x \in \mathbb{R}: |21-4x| \geq -3$ thus
solution set $S = \mathbb{R}$.

(!) e) $|x-2| \geq |x+3|$

Solution

$|x-2| \geq |x+3| \Leftrightarrow (x-2)^2 \geq (x+3)^2 \Leftrightarrow$
$\Leftrightarrow x^2-4x+4 \geq x^2+6x+9 \Leftrightarrow$
$\Leftrightarrow -4x+4 \geq 6x+9 \Leftrightarrow -4x-6x \geq -4+9 \Leftrightarrow$
$\Leftrightarrow -10x \geq 5 \Leftrightarrow 10x \leq -5 \Leftrightarrow x \leq -\frac{1}{2}$
thus $S = (-\infty, -1/2]$.

↳ For inequalities of the form $|f(x)| \lesseqgtr |g(x)|$ we can raise squares because BOTH sides of the inequality are guaranteed to be positive.

f) $|x-3| > 2x+1$ (1)

↳ We CANNOT square both sides because we do NOT know whether $2x+1$ is positive or negative.

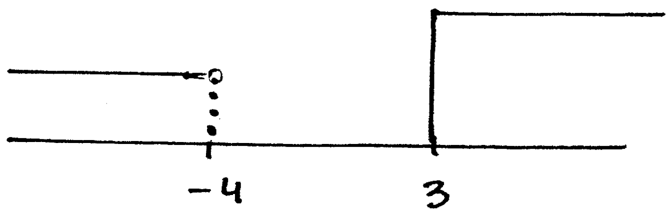
Solution

Distinguish two cases:

Case 1: $x-3 \geq 0 \Leftrightarrow x \geq 3 \Leftrightarrow x \in [3, +\infty)$

Then $|x-3| = x-3$.

$(1) \Leftrightarrow x-3 > 2x+1 \Leftrightarrow x-2x > 3+1 \Leftrightarrow -x > 4$
$\Leftrightarrow x < -4 \Leftrightarrow x \in (-\infty, -4)$.

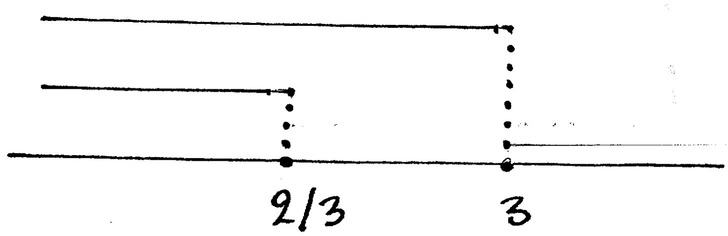


thus $S_1 = (-\infty, -4) \cap [3, +\infty) = \emptyset$.

Case 2: $x-3 < 0 \Leftrightarrow x < 3 \Leftrightarrow x \in (-\infty, 3)$

then $|x-3| = -(x-3)$.

$(1) \Leftrightarrow -(x-3) > 2x+1 \Leftrightarrow -x+3 > 2x+1 \Leftrightarrow$
$\Leftrightarrow -x-2x > 1-3 \Leftrightarrow -3x > -2 \Leftrightarrow 3x < 2 \Leftrightarrow$
$\Leftrightarrow x < 2/3 \Leftrightarrow x \in (-\infty, 2/3)$.



thus $S_2 = (-\infty, 2/3) \cap (-\infty, 3) = (-\infty, 2/3)$.

It follows that the solution set is:

$S = S_1 \cup S_2 = \emptyset \cup (-\infty, 2/3)$
$= (-\infty, 2/3)$.

g) $|x+3|-|1-x|-2x>7 \quad (1)$

Solution

x		-3		1	
$x+3$	$-$	0	$+$		$+$
$1-x$	$+$		$+$	0	$-$

Distinguish three cases:

Case 1 : For $x \in (-\infty,-3)$, we have

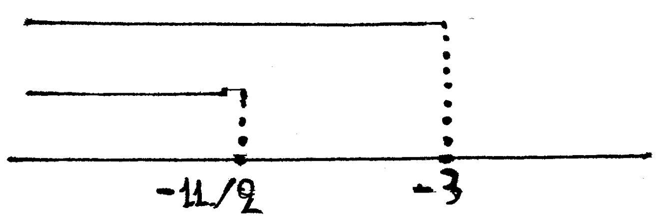
$|x+3| = -(x+3)$ and $|1-x| = 1-x$

$(1) \Leftrightarrow -(x+3)-(1-x)-2x>7 \Leftrightarrow$

$\Leftrightarrow -x-3-1+x \quad -2x>7 \Leftrightarrow$

$\Leftrightarrow -2x-4>7 \Leftrightarrow -2x>7+4 \Leftrightarrow -2x>11$

$\Leftrightarrow x<-11/2.$



thus $S_1 = (-\infty,-11/2) \cap (-\infty,-3) = (-\infty,-11/2)$

Case 2: For $x \in [-3,1)$, we have

$|x+3| = x+3$ and $|1-x| = 1-x$

$(1) \Leftrightarrow (x+3)-(1-x)-2x>7 \Leftrightarrow$

$\Leftrightarrow x+3-1+x-2x>7 \Leftrightarrow$

$\Leftrightarrow 0x+2>7 \Leftrightarrow 0x>7-2 \Leftrightarrow 0x>5 \leftarrow$ inconsistent

Thus: $S_2 = \emptyset \cap [-3,1) = \emptyset$.

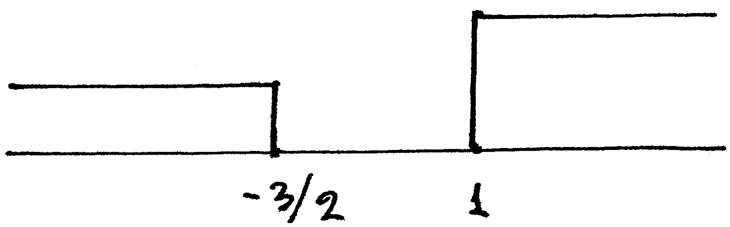
Case 3: For $x \in [1, +\infty)$, we have
$|x+3| = x+3$ and $|1-x| = -(1-x)$

$(1) \Leftrightarrow (x+3) + (1-x) - 2x > 7 \Leftrightarrow$

$\Leftrightarrow x+3+1-x-2x > 7 \Leftrightarrow$

$\Leftrightarrow -2x+4 > 7 \Leftrightarrow -2x > 7-4 \Leftrightarrow -2x > 3 \Leftrightarrow$

$\Leftrightarrow x < -3/2$



thus $S_3 = (-\infty, -3/2) \cap [1, +\infty) = \emptyset$.
It follows that the solution set is
$S = S_1 \cup S_2 \cup S_3 = (-\infty, -11/2) \cup \emptyset \cup \emptyset = (-\infty, -11/2)$

EXERCISES

(28) Solve the inequalities

a) $|x|<5$

b) $|x|\geq 3$

c) $|x|<-1$

d) $|x|\geq 0$

e) $|x|<0$

f) $|x|>0$

g) $|x|>-2$

h) $|x|\geq -3$

i) $3|x|-2>|x|+8$

j) $2(|x|-1)\geq 3|x|-2$

k) $2|x|-|-2x|\geq 3-|3x|$

l) $\dfrac{2+|x|}{|x|+1}>2$

(29) Solve the inequalities

a) $|-3x|\geq -2$

b) $|-2x|\geq |2x|$

c) $|1+2x|\leq 3$

d) $|x-3|>1$

e) $|x-30|-6>0$

f) $|3x+2|-4\leq 0$

g) $|x-5|-4\leq 0$

h) $|2x-3|+3<0$

i) $-3|7-x|+5\leq 0$

j) $|x^2+3|\geq 4x$

(30) Solve the inequalities

a) $|2x|+|x+1|<3$

b) $|x+1|-|x-1|\geq 1/2$

c) $|2x-1|+|x-3|+3<-|x+1|$

d) $|x^2-1|-3x\geq 0$.

e) $|2x+3|-|x+5|<0$

f) $|x+1|-3x>0$

g) $|x^2-1|\leq |2x+1|$.

Quadratic parametric equations

- 1 Simplify to the form
  $A(a)x^2 + B(a)x + C(a) = 0$
- 2 Make sign chart for the discriminant
  $\Delta(a) = B^2(a) - 4A(a)C(a)$
- 3 Distinguish the cases:
  $A(a) = 0 \longleftarrow$ Linear equation
  $\Delta(a) > 0 \longleftarrow$ 2 solutions
  $\Delta(a) = 0 \longleftarrow$ 1 solution
  $\Delta(a) < 0 \longleftarrow$ no real solutions.

EXAMPLE

$(a+4)x^2 + (a+2)x - 2 = 0 \qquad (1)$

Solution

For $a+4=0 \Leftrightarrow a=-4$

$(1) \Leftrightarrow (-4+2)x - 2 = 0 \Leftrightarrow -2x-2=0 \Leftrightarrow 2x=-2$

$\Leftrightarrow x=-1.$

Assume $a \neq -4$.

$$\begin{aligned}
\Delta(a) &= B^2(a) - 4A(a)C(a) = \\
&= (a+2)^2 - 4(a+4)\cdot(-2) = \\
&= a^2 + 4a + 4 + 8(a+4) = \\
&= a^2 + 4a + 4 + 8a + 32 = a^2 + 12a + 36 = \\
&= (a+6)^2
\end{aligned}$$

a		-6	
$(a+6)^2$	$+$	0	$+$
$\Delta(a)$	$+$	0	$+$

For $a \in \mathbb{R} - \{-6, -4\} \Rightarrow \Delta(a) > 0 \Rightarrow$

$\Rightarrow$ 2 solutions:

$$x_{1,2} = \frac{-B(a) \pm \sqrt{\Delta(a)}}{2A(a)} = \frac{-(a+2) \pm \sqrt{(a+6)^2}}{2(a+4)} =$$

$$= \frac{-(a+2) \pm |a+6|}{2(a+4)} = \frac{-(a+2) \pm (a+6)}{2(a+4)}$$

with

$$x_1 = \frac{-(a+2)-(a+6)}{2(a+4)} = \frac{-a-2-a-6}{2(a+4)} = \frac{-2a-8}{2(a+4)}$$

$$= \frac{-2(a+4)}{2(a+4)} = -1$$

$$x_2 = \frac{-(a+2)+(a+6)}{2(a+4)} = \frac{-a-2+a+6}{2(a+4)} = \frac{4}{2(a+4)} =$$

$$= \frac{2}{a+4}$$

For $a = -6$:

$(1) \Leftrightarrow (-6+4)x^2 + (-6+2)x - 2 = 0 \Leftrightarrow$

$\Leftrightarrow -2x^2 - 4x - 2 = 0 \Leftrightarrow -2(x^2+2x+1) = 0$

$\Leftrightarrow -2(x+1)^2 = 0 \Leftrightarrow x+1 = 0 \Leftrightarrow x = -1.$

Solution set:

$$S = \begin{cases} \{-1, 2/(a+4)\}, & a \in \mathbb{R} - \{-6, -4\} \\ \{-1\}, & a \in \{-6, -4\} \end{cases}$$

b) $(a+4)x^2 + (a+1)x + 1 = 0 \quad (1)$

Solution

For $a+4=0 \Leftrightarrow a=-4$

(1) $\Leftrightarrow (-4+1)x+1=0 \Leftrightarrow -3x+1=0 \Leftrightarrow 3x=1 \Leftrightarrow$

$\Leftrightarrow x=1/3$.

Assume that $a \neq -4$.

$\Delta(a) = (a+1)^2 - 4(a+4)\cdot 1 =$

$= a^2+2a+1-4a-16 =$

$= a^2-2a-15 = (a+3)(a-5)$

a		-3		5	
$a+3$	$-$	0	$+$		$+$
$a-5$	$-$		$-$	0	$+$
$\Delta(a)$	$+$	0	$-$	0	$+$

For $a \in (-\infty, -3) \cup (5, +\infty)$, two solutions:

$$x_{1,2} = \frac{-B(a) \pm \sqrt{\Delta(a)}}{2A(a)} =$$

$$= \frac{-(a+1) \pm \sqrt{a^2-2a-15}}{2(a+4)}$$

For $a=-3$:

$(-3+4)x^2+(-3+1)x+1=0 \Leftrightarrow$

$\Leftrightarrow x^2-2x+1=0 \Leftrightarrow (x-1)^2=0 \Leftrightarrow x-1=0$

$\Leftrightarrow x=1$.

For $a=5$:

$(5+4)x^2+(5+1)x+1=0 \Leftrightarrow$

$\Leftrightarrow 9x^2+6x+1=0 \Leftrightarrow (3x+1)^2=0 \Leftrightarrow$

$\Leftrightarrow 3x+1=0 \Leftrightarrow 3x=-1 \Leftrightarrow x=-1/3$.

For $a \in (-3,5) \Rightarrow \Delta(a)<0 \Rightarrow$ no real solutions.

Solution set:

$$S=\begin{cases} \left\{ \dfrac{-(a+1)+\sqrt{a^2-2a-15}}{2(a+4)}, \dfrac{-(a+1)-\sqrt{a^2-2a-15}}{2(a+4)} \right\} \\ \text{if } a\in(-\infty,-4)\cup(-4,-3)\cup(-3,+\infty) \\ \{1/3\}, \text{ if } a=-4 \\ \{1\}, \text{ if } a=-3 \\ \{-1/3\}, \text{ if } a=5 \\ \varnothing, \text{ if } a\in(-3,5) \end{cases}$$

EXERCISES

(19) Solve the following equations with respect to x

a) $(3a-1)x^2+2x+4a-1=0$
b) $3ax^2-(a+1)x+3=0$
c) $(1-a)^2x^2+(a-1)x-a(a+1)=0$
d) $(a-2)x^2+2(a+3)x+(2a-18)=0$
e) $(2a-1)x^2-2(a+1)x+a+1=0$
f) $(a^2-1)x^2-4(a+2)x+3=0$

V Equations with radicals

- These are equations wherein the unknown x appears at least under one square root.
- The square roots can be eliminated by squaring both sides of the equation using the property

$$\boxed{\forall a, b \in [0, +\infty): (a = b \Leftrightarrow a^2 = b^2)}$$

We therefore have to satisfy the constraint that both sides of the equation must be positive or zero before raising the square. This constraint imposes a domain A that must be used to accept or reject the solutions found.

- We distinguish the following cases:

1 $\longrightarrow$ $\boxed{\sqrt{f(x)} = \sqrt{g(x)}}$

•1 We require:

$$\begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \end{cases} \Leftrightarrow \cdots \Leftrightarrow x \in A \longleftarrow \text{Domain}$$

•2 Solve the equation:

$$\sqrt{f(x)} = \sqrt{g(x)} \Leftrightarrow f(x) = g(x) \Leftrightarrow \cdots \Leftrightarrow x \in S_0$$

•3 Accept/reject solutions: $S = S_0 \cap A$.

EXAMPLE

Solve: $\sqrt{3x+1} = \sqrt{2-x}$

Solution

Require: $\begin{cases} 3x+1 \geq 0 \\ 2-x \geq 0 \end{cases} \Leftrightarrow \begin{cases} 3x \geq -1 \\ 2 \geq x \end{cases} \Leftrightarrow \begin{cases} x \geq -1/3 \\ x \leq 2 \end{cases}$

$\Leftrightarrow x \in [-1/3, 2]$ thus: $A = [-1/3, 2]$.

Note that

$\sqrt{3x+1} = \sqrt{2-x} \Leftrightarrow 3x+1 = 2-x \Leftrightarrow 3x+x = 2-1 \Leftrightarrow$

$\Leftrightarrow 4x = 1 \Leftrightarrow x = 1/4 \leftarrow$ accepted (because $1/4 \in A$).

It follows that $S = \{1/4\}$.

② $\rightarrow$ $\boxed{\sqrt{f(x)} = g(x)}$

Note that we need $f(x) \geq 0$. Furthermore, the equation has no solutions with $g(x) < 0$ since $f(x) \in \mathbb{R} \Rightarrow \sqrt{f(x)} \geq 0$, so we can go ahead and require $g(x) \geq 0$ to justify squaring both sides of the equation. On the other hand, squaring the equation gives $f(x) = [g(x)]^2$ and since for any $x \in S$ $[g(x)]^2 \geq 0 \Rightarrow f(x) \geq 0$, it follows that the resulting solutions are guaranteed to satisfy $f(x) \geq 0$. Consequently, it is enough to require <u>only</u> $g(x) \geq 0$.

Solution Method

- 1 Require $g(x) \geq 0 \Leftrightarrow \cdots \Leftrightarrow x \in A$.
- 2 Solve:
  $\sqrt{f(x)} = g(x) \Leftrightarrow f(x) = [g(x)]^2 \Leftrightarrow \cdots \Leftrightarrow x \in S_0$
- 3 Solution set: $S = S_0 \cap A$.

EXAMPLE

Solve $\sqrt{x^2 - 2x + 6} + 3 = 2x$.

Solution

$$\sqrt{x^2-2x+6} + 3 = 2x \Leftrightarrow \sqrt{x^2-2x+6} = 2x - 3. \quad (1)$$

Require: $2x - 3 \geq 0 \Leftrightarrow 2x \geq 3 \Leftrightarrow x \in [3/2, +\infty)$

thus domain: $A = [3/2, +\infty)$.

$$(1) \Leftrightarrow x^2 - 2x + 6 = (2x-3)^2 \Leftrightarrow x^2 - 2x + 6 = 4x^2 - 12x + 9$$

$$\Leftrightarrow (4-1)x^2 + (-12+2)x + (9-6) = 0 \Leftrightarrow$$

$$\Leftrightarrow 3x^2 - 10x + 3 = 0 \quad (2)$$

$$\Delta = b^2 - 4ac = (-10)^2 - 4 \cdot 3 \cdot 3 = 100 - 36 = 64 = 8^2 \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-10) \pm 8}{2 \cdot 3} = \frac{10 \pm 8}{6} =$$

$$= \begin{cases} 18/6 = 3 \in A & \leftarrow \text{accept} \\ 2/6 = 1/3 \notin A & \leftarrow \text{reject.} \end{cases}$$

It follows that $S = \{3\}$.

(3) $\rightarrow$
$$\boxed{\begin{array}{l}\sqrt{f(x)}+\sqrt{g(x)}=0\\ \sqrt{f(x)}+\sqrt{g(x)}+\sqrt{h(x)}=0\end{array}}$$

We use the following property:
$\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n}=0 \Leftrightarrow a_1=0 \wedge a_2=0 \wedge \cdots \wedge a_n=0.$
Applying this property does NOT require us to impose a domain A on the unknown x.

EXAMPLE

Solve: $\sqrt{x^2-9}+\sqrt{x^2+5x+6}=0$

Solution

$$\sqrt{x^2-9}+\sqrt{x^2+5x+6}=0 \Leftrightarrow \begin{cases} x^2-9=0 \\ x^2+5x+6=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (x-3)(x+3)=0 \\ (x+2)(x+3)=0 \end{cases} \Leftrightarrow \begin{cases} x-3=0 \vee x+3=0 \\ x+2=0 \vee x+3=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=3 \vee x=-3 \\ x=-2 \vee x=-3 \end{cases} \Leftrightarrow x \in \{3,-3\} \cap \{-2,-3\} = \{-3\}.$$

thus $S=\{-3\}$.

(4) $\longrightarrow$ $$\boxed{\begin{array}{l} \sqrt{f(x)} + \sqrt{g(x)} = h(x) \\ \sqrt{f(x)} + \sqrt{g(x)} = \sqrt{h(x)} \end{array}}$$

The solution method is the same for both types of equations. Without loss of generality consider the equation:

$$\sqrt{f(x)} + \sqrt{g(x)} = h(x).$$

▶ Method

•1 Require $\begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \Leftrightarrow \cdots \Leftrightarrow x \in A_1 \\ h(x) \geq 0 \end{cases}$

•2 Solve:

$$\sqrt{f(x)} + \sqrt{g(x)} = h(x) \Leftrightarrow \left(\sqrt{f(x)} + \sqrt{g(x)}\right)^2 = [h(x)]^2$$

$$\Leftrightarrow f(x) + 2\sqrt{f(x)g(x)} + g(x) = [h(x)]^2 \Leftrightarrow$$

$$\Leftrightarrow 2\sqrt{f(x)g(x)} = [h(x)]^2 - f(x) - g(x) \qquad (1).$$

(type 2 equation)

•3 Require: $[h(x)]^2 - f(x) - g(x) \geq 0 \Leftrightarrow \cdots \Leftrightarrow x \in A_2$.

•4 Solve:

$$(1) \Leftrightarrow 4f(x)g(x) = \left([h(x)]^2 - f(x) - g(x)\right)^2 \Leftrightarrow$$

$$\Leftrightarrow \cdots \Leftrightarrow x \in S_0.$$

•5 For the solution set:

$$S = S_0 \cap A_1 \cap A_2.$$

EXAMPLE

Solve: $\sqrt{x+6} = \sqrt{5(x+2)} - \sqrt{x+1}$

Solution

$$\sqrt{x+6} = \sqrt{5(x+2)} - \sqrt{x+1} \Leftrightarrow$$
$$\Leftrightarrow \sqrt{x+6} + \sqrt{x+1} = \sqrt{5(x+2)} \qquad (1)$$

▸ Require: $\begin{cases} x+6 \geq 0 \\ x+1 \geq 0 \\ 5(x+2) \geq 0 \end{cases} \Leftrightarrow \begin{cases} x \geq -6 \\ x \geq -1 \\ x+2 \geq 0 \end{cases} \Leftrightarrow \begin{cases} x \geq -6 \\ x \geq -1 \\ x \geq -2 \end{cases} \Leftrightarrow x \geq -1$

thus $A_1 = [-1, +\infty)$.

$$(1) \Leftrightarrow (\sqrt{x+6} + \sqrt{x+1})^2 = 5(x+2) \Leftrightarrow$$
$$\Leftrightarrow (x+6) + 2\sqrt{(x+6)(x+1)} + (x+1) = 5(x+2) \Leftrightarrow$$
$$\Leftrightarrow 2\sqrt{(x+6)(x+1)} = 5(x+2) - (x+6) - (x+1) \Leftrightarrow$$
$$\Leftrightarrow 2\sqrt{(x+6)(x+1)} = 5x+10-x-6-x-1 \Leftrightarrow$$
$$\Leftrightarrow 2\sqrt{(x+6)(x+1)} = 3x+3 \qquad (2)$$

▸ Require $3x+3 \geq 0 \Leftrightarrow x+1 \geq 0 \Leftrightarrow x \geq -1$

thus $A_2 = [-1, +\infty)$.

$$(2) \Leftrightarrow 4(x+6)(x+1) = (3x+3)^2 \Leftrightarrow 4(x^2+7x+6) = 9(x+1)^2$$
$$\Leftrightarrow 4x^2+28x+24 = 9(x^2+2x+1) \Leftrightarrow$$
$$\Leftrightarrow 4x^2+28x+24 = 9x^2+18x+9 \Leftrightarrow$$
$$\Leftrightarrow (9-4)x^2 + (18-28)x + (9-24) = 0 \Leftrightarrow$$
$$\Leftrightarrow 5x^2 - 10x - 15 = 0 \Leftrightarrow x^2 - 2x - 3 = 0$$
$$\Leftrightarrow (x+1)(x-3) = 0 \Leftrightarrow x+1 = 0 \vee x-3 = 0 \Leftrightarrow$$
$$\Leftrightarrow x = -1 \vee x = 3.$$

It follows that the solution set reads:

$$S = \{-1,3\} \cap A_1 \cap A_2 = \{-1,3\} \cap [-1,+\infty) \cap [-1,+\infty) = \{-1,3\}.$$

thus, both solutions are accepted.

EXERCISES

(21) Solve the equations

a) $\sqrt{3x^2+2x+1} - 13 = 5x$
b) $x - \sqrt{x^2-7} = 7$
c) $x - \sqrt{4-x^2} = 1$
d) $x - 2\sqrt{x^2+x+3} = -x-2$
e) $13 - \sqrt{4x^2+7x-8} = 2x$
f) $\sqrt{x-2} + \sqrt{x^2-2x} = 0$
g) $3\sqrt{x-1} + \sqrt{x^2-2x+1} + \sqrt{x^3-x^2} = 0$
h) $\sqrt{2x+1} + \sqrt{x+1} = 1$
i) $\sqrt{x-2} - \sqrt{3x} = -\sqrt{7-x}$
j) $\sqrt{x+1} - \sqrt{2x+9} = \sqrt{4-x}$
k) $\sqrt{2x+1} = 1 - \sqrt{x+1}$

(22) Solve the equations

a) $2x^2-7x = 3\sqrt{2x^2-7x+7} - 3$ } substitution
b) $\sqrt{2+\sqrt{x-5}} = \sqrt{13-x}$

$$\left.\begin{array}{l} \text{c) } \sqrt{x^2-6x+5} = x-a \\ \text{d) } \sqrt{x^2+1} = x-a \\ \text{e) } \sqrt{x+a} = a-\sqrt{x} \end{array}\right\} \text{parametric}$$

CA3: Systems of Equations

SYSTEMS OF EQUATIONS

Linear 2x2 systems

- To solve the linear system

$$\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$$

we calculate:

$$D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2-a_2b_1$$

$$D_x=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1b_2-c_2b_1$$

$$D_y=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2-a_2c_1$$

a) If $D\neq 0 \Rightarrow$ unique solution $\begin{cases} x=D_x/D \\ y=D_y/D \end{cases}$

b) If $D=0$ and ($D_x\neq 0$ or $D_y\neq 0$), then the system is inconsistent.

c) Otherwise the system can be reduced to one equation or shown to be inconsistent.

EXAMPLES

a) $\begin{cases} 2x+3y=8 \\ 5x-2y=1 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = 2\cdot(-2)-3\cdot5=-4-15=-19$$

$$Dx=\begin{vmatrix} 8 & 3 \\ 1 & -2 \end{vmatrix} = 8\cdot(-2)-3\cdot1=-16-3=-19$$

$$Dy=\begin{vmatrix} 2 & 8 \\ 5 & 1 \end{vmatrix} = 2\cdot1-5\cdot8=2-40=-38$$

thus there is a unique solution:

$$\left.\begin{aligned} x&=\frac{Dx}{D}=\frac{-19}{-19}=1 \\ y&=\frac{Dy}{D}=\frac{-38}{-19}=2 \end{aligned}\right\} \Rightarrow S=\{(1,2)\}.$$

b) $\begin{cases} (a+1)x+(a-1)y=4a+2 \\ 2ax+(a-1)y=7a-1 \end{cases}$

Solution

$$D=\begin{vmatrix} a+1 & a-1 \\ 2a & a-1 \end{vmatrix} = (a+1)(a-1)-2a(a-1)=$$

$$=(a-1)(a+1-2a)=(a-1)(-a+1)=-(a-1)(a-1)$$

$$=-(a-1)^2.$$

Distinguish two cases:

Case 1: $D \neq 0 \Leftrightarrow -(a-1)^2 = 0 \Leftrightarrow a-1 \neq 0 \Leftrightarrow \underline{a \neq 1}$

$$D_x = \begin{vmatrix} 4a+2 & a-1 \\ 7a-1 & a-1 \end{vmatrix} = (4a+2)(a-1) - (7a-1)(a-1) =$$

$$= (a-1)\left[(4a+2) - (7a-1)\right] =$$

$$= (a-1)(4a+2-7a+1) = (a-1)(-3a+3) =$$

$$= -3(a-1)(a-1) = -3(a-1)^2.$$

and

$$D_y = \begin{vmatrix} a+1 & 4a+2 \\ 2a & 7a-1 \end{vmatrix} = (a+1)(7a-1) - 2a(4a+2) =$$

$$= 7a^2 - a + 7a - 1 - 8a^2 - 4a =$$

$$= (7-8)a^2 + (-1+7-4)a - 1 =$$

$$= -a^2 + 2a - 1 = -(a^2 - 2a + 1) = -(a-1)^2$$

thus we have a unique solution:

$$x = \frac{D_x}{D} = \frac{-3(a-1)^2}{-(a-1)^2} = 3$$

$$y = \frac{D_y}{D} = \frac{-(a-1)^2}{-(a-1)^2} = 1.$$

Case 2: $D = 0 \Leftrightarrow (a-1)^2 = 0 \Leftrightarrow a-1 = 0 \Leftrightarrow a = 1$

For $a=1$, the system reads

$$\begin{cases} (1+1)x + (1-1)y = 4\cdot 1 + 2 \\ 2\cdot 1\cdot x + (1-1)y = 7\cdot 1 - 1 \end{cases} \Leftrightarrow \begin{cases} 2x = 6 \\ 2x = 6 \end{cases} \Leftrightarrow 2x = 6$$

$\Leftrightarrow x = 3. \Leftrightarrow (x, y) = (3, y)$

Solution set $S = \{(3, y) \mid y \in \mathbb{R}\}$.

It follows that:

$$S = \begin{cases} \{(1,2)\} & , \text{ if } a \neq 1 \\ \{(3,y) \mid y \in \mathbb{R}\} & , \text{ if } a = 1. \end{cases}$$

EXERCISES

① Solve the following systems:

a) $\begin{cases} 5x - 7 = -y \\ 10x + 2y = 13 \end{cases}$ b) $\begin{cases} x - 2y = 1 \\ 3x + y = 0 \end{cases}$

c) $\begin{cases} 2x - 3y = 15 \\ -6x + 9y = -45 \end{cases}$

② Solve the systems with respect to x and y:

a) $\begin{cases} ax + (a+1)y = 3a+2 \\ 2x + (2a-1)y = 8 \end{cases}$ b) $\begin{cases} 2ax + ay = 4 \\ ax + (a-1)y = 2 \end{cases}$

c) $\begin{cases} 2ax + (a-3)y = a-1 \\ (a-3)x + 2ay = a - a^2 \end{cases}$

d) $\begin{cases} (a-1)x - y = a+1 \\ (8a+5)x + (a+5)y = -5 \end{cases}$

e) $\begin{cases} (a^2-1)x - (a-1)y = a \\ (a-1)^2 x + (a-1)y = a+1 \end{cases}$

▼ Linear nxn systems

Consider an nxn linear system of equations of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

The prefered method for solving this system is the method of determinants.

Definition of nxn determinants

- An $n \times n$ matrix $A \in M_n(\mathbb{R})$ is a collection of n^2 numbers $A_{ab} \in \mathbb{R}$ arranged in n rows and n columns as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

Remember:
A_{rc} : row, column
A_{vh} : vertical, horizontal

- A_{ab} = element at row a and column b.

- For a 2x2 matrix $A \in M_2(\mathbb{R})$ with

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

we have defined the determinant of A as

$$\det A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

For more general nxn matrices $A \in M_n(\mathbb{R})$ we define the determinant $\det A$ recursively as follows:

- Let $A \in M_n(\mathbb{R})$ be an nxn matrix. The <u>minor</u> $M_{ab}(A)$ is defined as an $(n-1) \times (n-1)$ matrix obtained from A by deleting
a) The "a" row of A AND
b) The "b" row of A.
The determinant $\det A$ can then be expanded in terms of the determinants of the minors of A, using either:
a) Expansion accross row "a"; for $a = 1, 2, \dots, n$

$$\boxed{\det A = \sum_{b=1}^{n} (-1)^{a+b} \det(M_{ab}(A)) A_{ab}}$$

b) Expansion accross column "b" for $b = 1, 2, 3, \dots, n$

$$\boxed{\det A = \sum_{a=1}^{n} (-1)^{a+b} \det(M_{ab}(A))\, A_{ab}}$$

- Each expansion yields determinants of smaller matrices, so we keep expanding until we obtain 2×2 determinants.
- It can be shown that any one of the above expansions gives the same result.

EXAMPLES

- Definition of minors.

For $A = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 1 & 5 & 7 & 2 \\ 3 & 1 & 5 & 2 \\ 1 & 4 & 7 & 3 \end{bmatrix} \Rightarrow$

Note that $A_{23} = 7$

$$\Rightarrow M_{23}(A) = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 1 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

- Evaluation of 3x3 determinants

$$\begin{vmatrix} 3 & 1 & 2 \\ 1 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix} \longrightarrow = \quad \boxed{\text{sign of } (-1)^{a+b} \longleftrightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}}$$

$$= \underset{(-1)^{2+1}}{-1} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \underset{(-1)^{2+2}}{+5} \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} \underset{(-1)^{2+3}}{-1} \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} =$$

$$= -1(1\cdot 1 - 2\cdot 3) + 5(3\cdot 1 - 2\cdot 2) - 1(3\cdot 3 - 2\cdot 1) =$$

$$= -(1-6) + 5(3-4) - (9-2) =$$

$$= -(-5) + 5\cdot(-1) - 7 = 5 - 5 - 7 = -7$$

- Take advantage of zeroes.

$$\begin{vmatrix} 4 & 1 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 1 & 3 & 1 \end{vmatrix} \longrightarrow =$$

$$= 4\cdot(-2) \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 4(-2)(2\cdot 1 - 3\cdot 3)$$

$$= 4(-2)(2-9) = 4(-2)(-7) = (-8)(-7)$$

$$= 56.$$

Cramer's rule

Given the $n \times n$ linear system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

we define the determinant D given by:

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

If $D \neq 0$, then the system has a unique solution that can be evaluated as follows:

- 1 Let D_a be the determinants in which the "a" column of D is replaced with $b_1, b_2, \dots, b_n$, so that

$$D_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{vmatrix}, \ldots$$

and $$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{vmatrix}$$

•2 The unique solution is given by

$$\boxed{x_a = \frac{D_a}{D}}$$

↳ This method does not work when $D = 0$.
For that case we use more advanced techiniques that you will learn in Linear Algebra.

EXAMPLE

Solve the system.

$$\begin{cases} 2x+y+z=4 \\ y+2z=2 \\ x-z=0 \end{cases}$$

Solution

We note that

$$\begin{cases} 2x+y+z=4 \\ y+2z=2 \\ x-z=0 \end{cases} \Leftrightarrow \begin{cases} 2x+1y+1z=4 \\ 0x+1y+2z=2 \\ 1x+0y-1z=0 \end{cases}$$

and also that

$$D=\begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} - 0 + 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} =$$

$$= 2(1\cdot(-1)-0\cdot 2)+1(1\cdot 2-1\cdot 1)=$$

$= 2(-1)+1(2-1) = -2+1=-1 \neq 0 \Rightarrow$ the system has a unique solution.

Furthermore:

$$D_1=\begin{vmatrix} 4 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & -1 \end{vmatrix} = (-1)\begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} =$$

$$= (-1)(4\cdot 1 - 2\cdot 1) = (-1)(4-2) = (-1)\cdot 2 = -2\ ,$$

and

$$D_2 = \begin{vmatrix} 2 & 4 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} - 0 + 1\begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} =$$

$$= 2(2\cdot(-1) - 0) + 1(4\cdot 2 - 2\cdot 1) = 2(-2) + 1(8-2) =$$

$$= -4 + 6 = 2\ ,$$

and

$$D_3 = \begin{vmatrix} 2 & 1 & 4 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = (1\cdot 2 - 1\cdot 4) =$$

$$= 2 - 4 = -2.$$

It follows that

$$\left.\begin{aligned} x &= \frac{D_1}{D} = \frac{-2}{-1} = 2 \\ y &= \frac{D_2}{D} = \frac{2}{-1} = -2 \\ z &= \frac{D_3}{D} = \frac{-2}{-1} = 2 \end{aligned}\right\} \Rightarrow \underline{(x,y,z) = (2,-2,2).}$$

Thus $S = \{(2,2,-2)\}$

EXERCISES

(3) Solve the following linear systems of equations:

a) $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

b) $\begin{cases} x + 2y + 3z = -3 \\ -2x + y - z = 6 \\ 3x - 3y + 2z = -11 \end{cases}$

c) $\begin{cases} 14 + 3x + z = 4y - 2x \\ 2y = 10 + x + 2z \\ x + y + z = 1 - 2x \end{cases}$

d) $\begin{cases} 3(x+y+z) = 1 - 2z \\ 3(x+3z) = 2 - 5y \\ 5(x+2y) = 4 - 17z + y \end{cases}$

2nd order systems

1) Linear + Quadratic equation

Method: Solve the linear equation first and substitute the solution to the quadratic equation.

EXAMPLE

$$\begin{cases} 2x^2+xy-y^2=0 \\ x+3y=7 \end{cases}$$

Solution

$$\begin{cases} 2x^2+xy-y^2=0 \\ x+3y=7 \end{cases} \Leftrightarrow \begin{cases} 2x^2+xy-y^2=0 \\ x=7-3y \end{cases}$$

$$\Leftrightarrow \begin{cases} 2(7-3y)^2+(7-3y)y-y^2=0 & (1) \\ x=7-3y \end{cases}$$

We note that

$(1) \Leftrightarrow 2(49-42y+9y^2)+(7-3y)y-y^2=0 \Leftrightarrow$

$\Leftrightarrow 98-84y+18y^2+7y-3y^2-y^2=0 \Leftrightarrow$

$\Leftrightarrow (18-3-1)y^2+(-84+7)y+98=0$

$\Leftrightarrow 14y^2-77y+98=0$

$\Delta=b^2-4ac=(-77)^2-4\cdot14\cdot98=5929-5488$

$=441=21^2 \Rightarrow$

$$\Rightarrow y_1 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{-(-77)+21}{2\cdot 14} = \frac{77+21}{2\cdot 14} =$$

$$= \frac{98}{2\cdot 14} = \frac{7}{2} \quad \text{and}$$

$$y_2 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{-(-77)-21}{2\cdot 14} = \frac{77-21}{2\cdot 14} =$$

$$= \frac{56}{2\cdot 14} = \frac{4}{2} = 2.$$

It follows that the system gives:

$$\begin{cases} x = 7-3y \\ y = 2 \vee y = 7/2 \end{cases} \Leftrightarrow \begin{cases} x = 7-3y \\ y = 2 \end{cases} \vee \begin{cases} x = 7-3y \\ y = 7/2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 7-6 = 1 \\ y = 2 \end{cases} \vee \begin{cases} x = 7-21/2 = -7/2 \\ y = 7/2 \end{cases}$$

thus

$$S = \{(1,2), (-7/2, 7/2)\}.$$

EXERCISES

(4) Solve the following systems.

a) $\begin{cases} 3x^2+4y^2+12x=7 \\ x+2y=3 \end{cases}$

b) $\begin{cases} 2x^2-3xy+5y^2=1 \\ 3x-2y=2 \end{cases}$

c) $\begin{cases} 2x^2+y^2=17 \\ 6x-4y=0 \end{cases}$

d) $\begin{cases} x^2+xy+2y^2=4 \\ x+3y=4. \end{cases}$

2) The Fundamental system

$$\boxed{\begin{cases} x+y=a \\ xy=b \end{cases} \Leftrightarrow \begin{cases} x=\rho_1 \\ y=\rho_2 \end{cases} \vee \begin{cases} x=\rho_2 \\ y=\rho_1 \end{cases}}$$

where ρ_1, ρ_2 are the zeroes of

$$\boxed{f(x) = x^2 - ax + b}$$

If $\rho_1 = \rho_2$, then the system has a unique solution $(x,y) = (\rho, \rho)$.

EXAMPLES

$$\begin{cases} x+y=5 \\ xy=6 \end{cases} \quad (1)$$

Solution

Let $f(x) = z^2 - 5z + 6$

$\Delta = b^2 - 4ac = (-5)^2 - 4 \cdot 1 \cdot 6 = 25 - 24 = 1 \Rightarrow$

$$\Rightarrow z_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-5) \pm 1}{2 \cdot 1} = \frac{5 \pm 1}{2} =$$

$$= \begin{cases} 6/2 = 3 \\ 4/2 = 2 \end{cases}, \text{ therefore}$$

$$(1) \Leftrightarrow \begin{cases} x=3 \\ y=2 \end{cases} \vee \begin{cases} x=2 \\ y=3 \end{cases}, \text{ thus } S = \{(3,2),(2,3)\}$$

EXERCISES

(5) Solve the following systems

a) $\begin{cases} x+y=-5 \\ xy=4 \end{cases}$

d) $\begin{cases} x+y=1 \\ xy=1 \end{cases}$

b) $\begin{cases} x+y=3 \\ xy=2 \end{cases}$

e) $\begin{cases} x+y=7 \\ xy=-2 \end{cases}$

c) $\begin{cases} x+y=4 \\ xy=4 \end{cases}$

f) $\begin{cases} x+y=2 \\ xy=4 \end{cases}$

3) Symmetric systems

A symmetric system is a system of the form

$$\begin{cases} f_1(x,y)=0 \\ f_2(x,y)=0 \end{cases}$$

such that $f_1(x,y)=f_1(y,x)$ and
$f_2(x,y)=f_2(y,x)$.

Method: We use the Cauchy identities:

$$\boxed{\begin{aligned} a^2+b^2 &= (a+b)^2-2ab \\ a^3+b^3 &= (a+b)^3-3ab(a+b) \end{aligned}}$$

to rewrite the system in terms of $x+y$ and xy. Then let $a=x+y$ and $b=xy$ to solve for a,b. Then we solve the resulting fundamental systems to find x,y.

EXAMPLES

a) $\begin{cases} x^3+y^3=9 \\ xy(x+y)=6 \end{cases}$

Solution

$$\begin{cases} x^3+y^3=9 \\ xy(x+y)=6 \end{cases} \Leftrightarrow \begin{cases} (x+y)^3-3xy(x+y)=9 \\ xy(x+y)=6 \end{cases} \quad (1)$$

Let $a=x+y$ and $b=xy$. Then

$$(1) \Leftrightarrow \begin{cases} a^3-3ab=9 \\ ab=6 \end{cases} \Leftrightarrow \begin{cases} a^3-3\cdot 6=9 \\ ab=6 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a^3=18+9=27 \\ ab=6 \end{cases} \Leftrightarrow \begin{cases} a=3 \\ ab=6 \end{cases} \Leftrightarrow \begin{cases} a=3 \\ 3b=6 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a=3 \\ b=2 \end{cases} \Leftrightarrow \begin{cases} x+y=3 \\ xy=2 \end{cases} \quad (2)$$

Let $f(z)=z^2-3z+2=(z-2)(z-1)=0 \Leftrightarrow$
$\Leftrightarrow z_1=2 \vee z_2=1$, thus

$$(2) \Leftrightarrow \begin{cases} x=2 \\ y=1 \end{cases} \vee \begin{cases} x=1 \\ y=2 \end{cases}.$$

b) $\begin{cases} 3x^2+3y^2-xy=33 \\ x^2+y^2+xy=19 \end{cases}$

Solution

$$\begin{cases} 3x^2+3y^2-xy=33 \\ x^2+y^2+xy=19 \end{cases} \Leftrightarrow \begin{cases} 3(x^2+y^2)-xy=33 \\ x^2+y^2+xy=19 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 3[(x+y)^2-2xy]-xy=33 \\ (x+y)^2-2xy+xy=19 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 3(x+y)^2 - 6xy - xy = 33 \\ (x+y)^2 - xy = 19 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 3(x+y)^2 - 7xy = 33 \\ (x+y)^2 - xy = 19 \end{cases} \qquad (1)$$

Let $a = (x+y)^2$ and $b = xy$. Then

$$(1) \Leftrightarrow \begin{cases} 3a - 7b = 33 \\ a - b = 19 \end{cases} \Leftrightarrow \begin{cases} 3a - 7b = 33 \\ -3a + 3b = -57 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a - b = 19 \\ -4b = -24 \end{cases} \Leftrightarrow \begin{cases} a - 6 = 19 \\ b = 6 \end{cases} \Leftrightarrow \begin{cases} a = 25 \\ b = 6 \end{cases}$$

$$\Leftrightarrow \begin{cases} (x+y)^2 = 25 \\ xy = 6 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x+y = 5 \\ xy = 6 \end{cases} \vee \begin{cases} x+y = -5 \\ xy = 6 \end{cases} \qquad (2)$$

Since $f_1(z) = z^2 - 5z + 6 = (z-2)(z-3) = 0 \Leftrightarrow$
$\Leftrightarrow z = 2 \vee z = 3$ and
$f_2(z) = z^2 + 5z + 6 = (z+2)(z+3) = 0 \Leftrightarrow z = -2 \vee z = -3$
it follows that

$$(2) \Leftrightarrow \begin{cases} x = 2 \\ y = 3 \end{cases} \vee \begin{cases} x = 3 \\ y = 2 \end{cases} \vee \begin{cases} x = -2 \\ y = -3 \end{cases} \vee \begin{cases} x = -3 \\ y = -2. \end{cases}$$

EXERCISES

⑥ Solve the following systems

a) $\begin{cases} x^2+y^2=17 \\ xy=14 \end{cases}$ b) $\begin{cases} x+y+xy=23 \\ xy(x+y)=126 \end{cases}$

c) $\begin{cases} x^2+y^2+x+y=44 \\ 3(x^2+y^2)-4xy=87 \end{cases}$ d) $\begin{cases} x+y=1 \\ \dfrac{1}{x}+\dfrac{1}{y}=-\dfrac{1}{6} \end{cases}$

e) $\begin{cases} x+y=13 \\ \dfrac{x}{y}+\dfrac{y}{x}=\dfrac{97}{36} \end{cases}$ f) $\begin{cases} 2x^2+2y^2-xy=32 \\ x^2+y^2+3xy=44 \end{cases}$

⑦ Solve the following systems:

a) $\begin{cases} x^3+y^3=35 \\ x+y=5 \end{cases}$ b) $\begin{cases} x+xy+y=11 \\ x^2y+xy^2=30 \end{cases}$

c) $\begin{cases} x^3+y^3=7 \\ xy(x+y)=-2 \end{cases}$ d) $\begin{cases} (x+y)xy=30 \\ (x+y)(x^2+y^2)=65 \end{cases}$

→ The following systems become symmetric after a change of variables.

(8) Solve the following systems

a) $\begin{cases} x+y^2=7 \\ xy^2=12 \end{cases}$

b) $\begin{cases} x^2-y=23 \\ x^2y=50 \end{cases}$

c) $\begin{cases} x^2+y^2=(5/2)xy \\ x-y=(1/4)xy \end{cases}$

d) $\begin{cases} x^2-xy+y^2=7 \\ x-y=1. \end{cases}$

4) Homogeneous systems

A homogeneous 2nd-order system is a system of the form

$$\begin{cases} a_1 x^2 + b_1 xy + c_1 y^2 = d_1 \\ a_2 x^2 + b_2 xy + c_2 y^2 = d_2 \end{cases}$$

with $|d_1| + |d_2| \neq 0$. To solve this system:

- 1 Examine if it has solutions $(0,k)$ and $(k,0)$
- 2 Now assume $xy \neq 0$. Define $y = \lambda x$
- 3 Rewrite:

$$a_1 x^2 + b_1 xy + c_1 y^2 = d_1 \Leftrightarrow$$
$$x^2 (a_1 + b_1 \lambda + c_2 \lambda^2) = d_1 \Leftrightarrow$$
$$x^2 = \frac{d_1}{a_1 + b_1 \lambda + c_1 \lambda^2}$$

and similarly

$$a_2 x^2 + b_2 x y + c_2 y^2 = d_2 \Leftrightarrow \cdots$$

$$\Leftrightarrow \quad x^2 = \frac{d_2}{a_2 + b_2 \lambda + c_2 \lambda^2}$$

- 4 Solve for λ:

$$\frac{d_1}{a_1 + b_1 \lambda + c_1 \lambda^2} = \frac{d_2}{a_2 + b_2 \lambda + c_2 \lambda^2}$$

EXAMPLE

$$\begin{cases} x^2+xy+y^2=19 & (1) \\ 2x^2+3xy-y^2=17 \end{cases}$$

Solution

Case 1 : For $x=0$:

$$(1) \Leftrightarrow \begin{cases} y^2 = 19 \\ -y^2 = 17 \end{cases} \leftarrow \text{inconsistent.}$$

Case 2 : For $y=0$:

$$(1) \Leftrightarrow \begin{cases} x^2 = 19 \\ 2x^2 = 17 \end{cases} \Leftrightarrow \begin{cases} x^2 = 19 \\ x^2 = 17/2 \end{cases} \leftarrow \text{inconsistent.}$$

Case 3: For $xy \neq 0$. Let $y=ax$.
We note that

$$x^2+xy+y^2=19 \Leftrightarrow x^2(1+a+a^2)=19 \Leftrightarrow$$
$$\Leftrightarrow x^2 = \frac{19}{1+a+a^2} \quad \text{and}$$

$$2x^2+3xy-y^2=17 \Leftrightarrow x^2(2+3a-a^2) = 17 \Leftrightarrow$$
$$\Leftrightarrow x^2 = \frac{17}{2+3a-a^2}$$

Solve:

$$\frac{19}{1+a+a^2} = \frac{17}{2+3a-a^2} \Leftrightarrow$$

$$\Leftrightarrow 19(2+3a-a^2) - 17(1+a+a^2) = 0 \Leftrightarrow$$

$$\Leftrightarrow 38 + 57a - 19a^2 - 17 - 17a - 17a^2 = 0 \Leftrightarrow$$

$$\Leftrightarrow (-19-17)a^2 + (57-17)a + (38-17) = 0 \Leftrightarrow$$

$$\Leftrightarrow -36a^2 + 40a + 21 = 0 \Leftrightarrow$$

$$\Leftrightarrow 36a^2 - 40a - 21 = 0.$$

$$\Delta = b^2 - 4ac = (-40)^2 - 4 \cdot 36 \cdot (-21) =$$

$$= 1600 + 3024 = 4624 = 68^2 \Rightarrow$$

$$\Rightarrow \alpha_1 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{-(-40)+68}{2 \cdot 36} = \frac{40+68}{72} =$$

$$= \frac{108}{72} = \frac{3}{2} \quad \text{and}$$

$$\alpha_2 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{-(-40)-68}{2 \cdot 36} = \frac{40-68}{72} =$$

$$= \frac{-28}{72} = \frac{-7}{18}$$

It follows that:

$$(1) \Leftrightarrow \begin{cases} x^2+xy+y^2 = 19 \\ y = (3/2)x \end{cases} \vee \begin{cases} x^2+xy+y^2 = 19 \\ y = -(7/18)x. \end{cases} \quad (2)$$

We note that; for $y = (3/2)x$:

$$x^2+xy+y^2 = 19 \Leftrightarrow x^2 + (3/2)x^2 + (9/4)x^2 = 19$$

$\Leftrightarrow 4x^2 + 6x^2 + 9x^2 = 19 \Leftrightarrow 19x^2 = 19 \Leftrightarrow x^2 = 1$

and for $y = -(7/18)x$:

$x^2 + xy + y^2 = 19 \Leftrightarrow x^2 - (7/18)x^2 + (7/18)^2 x^2 = 19$

$\Leftrightarrow 18^2 x^2 - 7 \cdot 18 x^2 + 7^2 x^2 = 19 \cdot 18^2 \Leftrightarrow$

$\Leftrightarrow 324x^2 - 126x^2 + 49x^2 = 6156 \Leftrightarrow$

$\Leftrightarrow 247x^2 = 6156 \Leftrightarrow 13x^2 = 324 \Leftrightarrow 13x^2 = 18^2$

and therefore:

$$(2) \Leftrightarrow \begin{cases} x^2 = 1 \\ y = (3/2)x \end{cases} \vee \begin{cases} 13x^2 = 18^2 \\ y = -(7/18)x \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 1 \\ y = 3/2 \end{cases} \vee \begin{cases} x = -1 \\ y = -3/2 \end{cases} \vee \begin{cases} x = 18/\sqrt{13} \\ y = -7/\sqrt{13} \end{cases} \vee \begin{cases} x = -18/\sqrt{13} \\ y = 7/\sqrt{13} \end{cases}$$

It follows that

$S = \{(1, 3/2), (-1, -3/2), (18/\sqrt{13}, -7/\sqrt{13}), (-18/\sqrt{13}, 7/\sqrt{13})\}$.

EXERCISES

(9) Solve the following systems

a) $\begin{cases} x^2 + 2xy - y^2 = 1 \\ 2x^2 - xy + 3y^2 = 12 \end{cases}$

b) $\begin{cases} x^2 - xy + y^2 = 1 \\ 3x^2 - 2xy - 2y^2 = -3 \end{cases}$

c) $\begin{cases} 2x^2 + 3xy + 5y^2 = 8 \\ 4x^2 - 7xy + 10y^2 = 16 \end{cases}$

CA4: Functions

FUNCTIONS

Preliminary Concepts

- An ordered pair (a,b) is a collection of two elements a and b in which a is the first element and b is the second element.
- By definition: (equality of ordered pairs).

  $$(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2 \wedge b_1 = b_2.$$

example

For $a \neq b$: $(a,b) \neq (b,a)$

$\{a,b\} = \{b,a\}$

i.e. in set equality the order with which the elements are listed is not important. In ordered-pair equality, the order with which the elements are listed is taken into account.

- Let A, B be two sets. The cartesian product $A \times B$ is defined as

  $$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

example

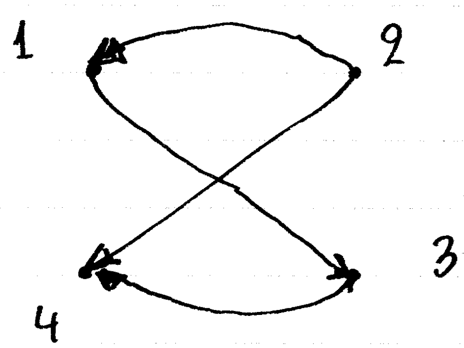
For $A = \{1,3\}$ and $B = \{2,4,5\}$
$A \times B = \{(1,2),(1,4),(1,5),(3,2),(3,4),(3,5)\}$
$B \times A = \{(2,1),(2,3),(4,1),(4,3),(5,1),(5,3)\}$

- Any subset $R \subseteq A \times B$ with $R \neq \emptyset$ is called a relation between elements of A and elements of B.

A relation can be represented by a cartesian graph or Venn diagram as in the following example:

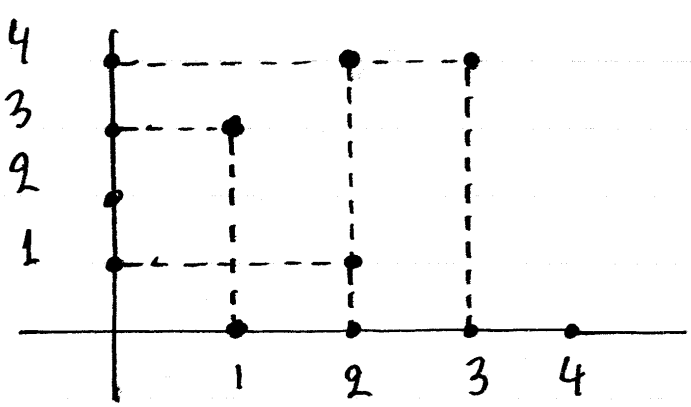
example

$R = \{(1,3),(2,4),(3,4),(2,1)\} \subseteq \mathbb{R} \times \mathbb{R}.$



Venn Diagram

Each ordered pair corresponds to an arrow.



Cartesian graph

Each ordered pair corresponds to an point on the axis system.

Quantified statements

- All definitions and theorems concerning relations, mapping, and functions require that we use the notation of quantified statements.
- Let $p(x)$ be some statement about x that is TRUE or FALSE depending on the value of the variable x. We now define the following statements:
  a) $\forall x \in A : p(x)$
  "For all $x \in A$, $p(x)$ is true"
  b) $\exists x \in A : p(x)$
  "There is at least one $x \in A$, such that $p(x)$ is true".
- We define $S = \{x \in A \mid p(x)\}$ as the set of all elements x of A for which $p(x)$ is true. It follows that:

$$\forall x \in A : p(x) \Leftrightarrow S = \{x \in A \mid p(x)\} = A$$
$$\exists x \in A : p(x) \Leftrightarrow S = \{x \in A \mid p(x)\} \neq \emptyset$$

EXAMPLE

$\forall a, b \in \mathbb{R} : \exists x \in \mathbb{R} : a + x = b$

"For all real numbers a, b there is another real number x such that $a + x = b$"

↳ This is an example of a well-known rule of regular algebra rewritten as a quantified statement.

Definitions of mappings

1) Algebraic Definition.
A mapping $f: A \to B$ is a relation $f \subseteq A \times B$ that satisfies the following properties:

- $\forall (a_1, b_1), (a_2, b_2) \in f : (a_1 = a_2 \Rightarrow b_1 = b_2)$
- $\forall a \in A : \exists b \in B : (a, b) \in f$.

2) Venn Diagram definition
A mapping $f: A \to B$ is a relation in whose Venn diagram every element of A has one and only one outgoing arrow.

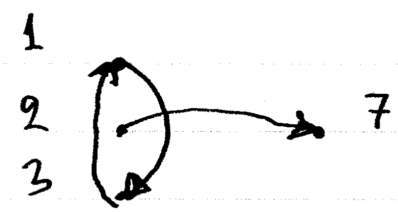
3) Cartesian graph definition
A mapping $f: A \to B$ is a relation whose graph has points such that no two points share the same x-coordinate.
(also see: vertical line test).

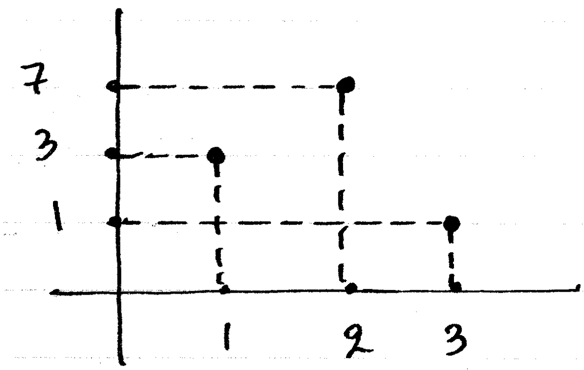
examples

The previous example was not a mapping.
The following is a mapping: $f: \{1, 2, 3\} \to \mathbb{R}$

$$f = \{(1, 3), (2, 7), (3, 1)\}$$

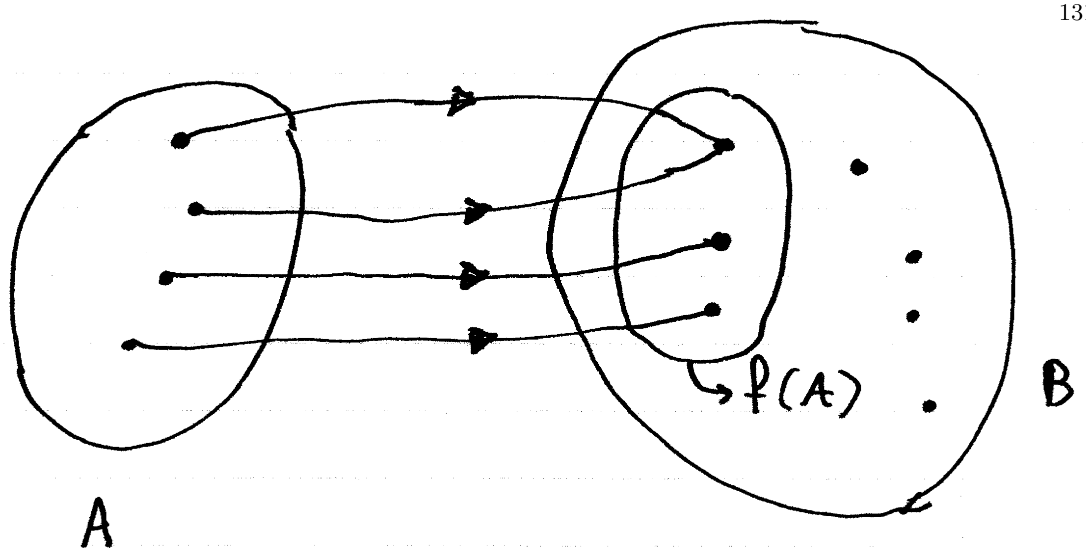
Venn Diagram





Cartesian Graph.

- Thus a mapping $f: A \to B$ maps every element $x \in A$ to a <u>unique</u> element of B, which we shall denote as $f(x)$. Obviously $f(x) \in B$. We call $f(x)$ the <u>image</u> of x under the mapping f.
  The set A is the <u>domain</u> of the mapping f and we write
  $$A = \mathrm{dom} f$$
- The <u>range $f(A)$</u> of the mapping f is the set of all elements of B which are an image of some element in A:
  $$f(A) = \{ f(x) \mid x \in A \}$$
  Obviously $f(A) \subseteq B$. It is possible that some elements of B are NOT images of any element of A (see schematic)



example

For $f = \{(1,3), (2,7), (3,1)\}$
Domain $A = dom(f) = \{1,2,3\}$

$$\left.\begin{array}{l} f(1) = 3 \\ f(2) = 7 \\ f(3) = 1 \end{array}\right\} \Rightarrow f(A) = \{1,3,7\} \leftarrow \text{Range.}$$

Inverse Relations and Mappings

- Let $R \subseteq A \times B$ be a relation. We define the inverse relation as
$$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$
The Venn diagram of R^{-1} is obtained from the Venn diagram of R by reversing the direction of all arrows!!
- The inverse f^{-1} of a mapping $f : A \to B$ is a relation, of course, but there is no guarantee that it will also be a mapping.

- Let $f : A \to B$ be a mapping.
  We say that

$$\boxed{f \text{ one-to-one} \Leftrightarrow \forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)}$$

- $$\boxed{\left.\begin{array}{l} f \text{ mapping} \\ f \text{ one-to-one} \end{array}\right\} \Rightarrow \begin{array}{l} f^{-1} \text{ is also} \\ \text{a mapping} \end{array}}$$

<u>example</u>

For $f = \{(1,3), (2,7), (3,1)\}$
$f^{-1} = \{(3,1), (7,2), (1,3)\}$ is also a mapping.

For $f = \{(1,3), (2,4), (3,4)\}$
the inverse $f^{-1} = \{(3,1), (4,2), (4,3)\}$
is NOT a mapping.

- If $f : A \to B$ is a one-to-one mapping then f^{-1} is also the mapping $f^{-1} : f(A) \to B$. Then, the range of f is the domain of f^{-1}.

i.e: $\operatorname{dom}(f^{-1}) = f(A) = f(\operatorname{dom} f)$

EXERCISES

① Write out $A\times B$ and $B\times A$ for A, B defined as follows:

a) $A=\{1,3,7\}$ and $B=\{2,5\}$

b) $A=\{1,2\}$ and $B=\{1,2,3\}$

c) $A=\{2,9\}$ and $B=\{1,7\}$

d) $A=\{3\}$ and $B=\{2,6\}$

e) $A=\{5\}$ and $B=\{4\}$

f) $A=\emptyset$ and $B=\{2,3,4\}$

g) $A=\emptyset$ and $B=\emptyset$.

② Write out the following statements in complete English sentences.

a) $\forall x,y\in\mathbb{R}: x+y=y+x$

b) $\forall x,y,z\in\mathbb{R}: x(y+z)=xy+xz$

c) $\exists x\in\mathbb{R}: 3x+1=5$

d) $\forall x\in\mathbb{R}-\{0\}: \exists y\in\mathbb{R}: xy=1$

e) $\exists a\in\mathbb{R}: \forall x\in\mathbb{R}: \dfrac{1}{1+x^2}<a$

f) $\forall x_1,x_2\in A: (f(x_1)=f(x_2)\Rightarrow x_1=x_2)$
(definition of "f is one-to-one")

g) $\exists x_1,x_2\in A: (f(x_1)=f(x_2)\wedge x_1\neq x_2)$
(definition of "f is not one-to-one")

(3) Make Venn diagrams for the following relations. Which of these relations are mappings? If yes, show the domain and range. Is the inverse relation a mapping?

a) $f = \{(1,2),(2,2),(3,1),(4,5)\}$
b) $f = \{(2,3),(1,5),(3,4),(2,5)\}$
c) $f = \{(2,1),(3,5),(5,3)\}$
d) $f = \{(3,7)\}$
e) $f = \{(1,5),(2,3),(3,7),(2,4)\}$
f) $f = \{(2,3),(3,2)\}$
g) $f = \{(1,3),(2,1),(3,2),(4,4),(5,6)\}$
h) $f = \{(2,4),(4,1),(1,1),(3,6),(5,5)\}$
i) $f = \{(3,7),(5,5),(6,2),(1,9),(2,7)\}$

▼ Functions – Basic Concepts

- A real-valued function (or just function) f is a mapping $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and $A \neq \emptyset$
- $A = \mathrm{dom}(f)$ ⟵ Domain of f

  $f(A) = \{f(x) \mid x \in A\}$ ⟵ Range of f.

↳ To properly define a function f, we must define
 a) The domain $A = \mathrm{dom}(f)$
 b) The formula $y = f(x)$.

examples : How to write a function definition
a) Let $f: [0, +\infty) \to \mathbb{R}$ be a function with
 $f(x) = \sqrt{x}$
b) Let $f(x) = \sqrt{x}$, $\forall x \in [0, +\infty)$.

↳ When the domain A is not given, we assume by default the widest possible subset of $\mathbb{R}$ for which the formula $y = f(x)$ yields a real number.

- Method : To find the default domain of a function f we introduce the necessary constraints such that

a) There is NO DIVISION BY ZERO
b) There is NO ROOT OF A NEGATIVE NUMBER.

EXAMPLES

a) For $f(x)=x^2+3x+1$, evaluate $f(1+\sqrt{2})$ and $f(2a-1)$.

Solution

$$f(1+\sqrt{2}) = (1+\sqrt{2})^2+3(1+\sqrt{2})+1 =$$
$$= (1+2\sqrt{2}+2)+3(1+\sqrt{2})+1 =$$
$$= 1+2\sqrt{2}+2+3+3\sqrt{2}+1 = 7+5\sqrt{2}$$
$$f(2a-1) = (2a-1)^2+3(2a-1)+1 =$$
$$= 4a^2-4a+1+6a-3+1 =$$
$$= 4a^2+2a-1.$$

b) Find the default domain for $f(x)=x^3(x^2+1)^4$.

Solution

No constraints, thus $A=\mathbb{R}$.

↳ For polynomial functions of the form
$f(x)=a_n x^n + a_{n-1}x^{n-1}+\dots+a_1 x+a_0$
there are no constraints, therefore the default domain is always $A=\mathbb{R}$.

c) Find the default domain for $f(x) = \dfrac{x^2-9}{x^2+4x+3}$

Solution

We require $x^2+4x+3 \neq 0$

Solve: $x^2+4x+3=0 \Leftrightarrow (x+1)(x+3)=0 \Leftrightarrow$

$\Leftrightarrow x+1=0 \vee x+3=0 \Leftrightarrow x=-1 \vee x=-3.$

It follows that $A = \mathbb{R} - \{-1,-3\}.$

$= (-\infty,-3) \cup (-3,-1) \cup (-1,+\infty).$

$\hookrightarrow$ Note that this function can be simplified to:

$$f(x) = \frac{x^2-9}{x^2+4x+3} = \frac{(x-3)(x+3)}{(x+3)(x+1)} = \frac{x-3}{x+1}$$

However, the default domain of the simplified formula is wider: $A = \mathbb{R} - \{-1\}$. Thus, to find the correct default domain, you must NOT try to simplify or otherwise modify the formula for $f(x)$ before writing down the constraints.

d) Find the default domain of

$$f(x) = \sqrt{\frac{2x+1}{1+3x}} \quad \text{and} \quad g(x) = \frac{\sqrt{2x+1}}{\sqrt{1+3x}}$$

Solution

- **For $f(x)$**

Require $\dfrac{2x+1}{1+3x} \geqslant 0$. (1)

x		$-1/2$		$-1/3$	
$2x+1$	$-$	0	$+$		$+$
$1+3x$	$-$		$-$	0	$+$
ineq	$+$	0	$-$	$\|$	$+$

Thus $(1) \iff x \in (-\infty, -1/2] \cup (-1/3, +\infty)$
and therefore $A = (-\infty, -1/2] \cup (-1/3, +\infty)$.

- **For $g(x)$**

Require: $\begin{cases} 2x+1 \geqslant 0 \\ 1+3x > 0 \end{cases} \iff \begin{cases} 2x \geqslant -1 \\ 3x > -1 \end{cases} \iff \begin{cases} x \geqslant -1/2 \\ x > -1/3 \end{cases}$

$\iff x > -1/3$

therefore $A = (-1/3, +\infty)$.

EXERCISES

(4) Find the default domain for the following functions

a) $f(x) = x^2(x+1)^3$

b) $f(x) = \dfrac{3}{x-1}$

c) $f(x) = \dfrac{x-2}{x^2-4}$

d) $f(x) = \dfrac{3x^2}{x^2-4x}$

e) $f(x) = \dfrac{2x^2-3}{x^2+5x+6}$

f) $f(x) = \dfrac{x^2-4}{x^2-5x+6}$

g) $f(x) = \dfrac{x+2}{x-3}$

compare l) $f(x) = \sqrt{\dfrac{x^2-4}{x^2-9}}$

h) $f(x) = \sqrt{3-x}$

i) $f(x) = \sqrt{x^2-x-6}$

j) $f(x) = \sqrt{x^3+5x^2-6x}$

k) $f(x) = \sqrt{x^4-x^3+x-1}$

l) $f(x) = \sqrt{\dfrac{x+2}{x-3}}$

m) $f(x) = \dfrac{\sqrt{x+2}}{\sqrt{x-3}}$

(5) Find the default domain for the following functions

a) $f(x) = \dfrac{-3|x-1|}{|x+4}$

b) $f(x) = \dfrac{x-1}{|x-2|-2}$

c) $f(x) = \sqrt{|3x-1|-2}$

d) $f(x) = \dfrac{\sqrt{x}}{|x|}$

e) $f(x) = \sqrt{\dfrac{|x|-2}{|x|+1}}$

f) $f(x) = \dfrac{x-1}{\sqrt{2-|x+2|}}$

▼ Algebra with functions

- Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be two functions.
  We also define
  $$C = \{x \in B \mid g(x) = 0\}$$
  We now give the following definitions:

1) Equality:
$$\boxed{f = g \Leftrightarrow \begin{cases} \mathrm{dom}(f) = \mathrm{dom}(g) \\ \forall x \in \mathrm{dom}(f) : f(x) = g(x) \end{cases}}$$

2) Sum $f+g$
$$\boxed{\begin{array}{l} \mathrm{dom}(f+g) = \mathrm{dom}(f) \cap \mathrm{dom}(g) = A \cap B \\ \forall x \in A \cap B : (f+g)(x) = f(x) + g(x) \end{array}}$$

3) Product fg
$$\boxed{\begin{array}{l} \mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g) = A \cap B \\ \forall x \in A \cap B : (fg)(x) = f(x) g(x) \end{array}}$$

4) Scalar Product cf with $c \in \mathbb{R}$
$$\boxed{\begin{array}{l} \mathrm{dom}(cf) = \mathrm{dom}(f) \\ \forall x \in A : (cf)(x) = cf(x) \end{array}}$$

5) Division f/g
$$\boxed{\begin{array}{l} \mathrm{dom}(f/g) = [\mathrm{dom}(f) \cap \mathrm{dom}(g)] - \{x \in B \mid g(x) = 0\} \\ \qquad = (A \cap B) - C \\ \forall x \in \mathrm{dom}(f/g) : (f/g)(x) = \dfrac{f(x)}{g(x)} \end{array}}$$

EXAMPLES

a) Given the functions:

$f = \{(1,3),(2,7),(3,6),(4,1)\}$

$g = \{(3,1),(5,2),(4,9),(6,3)\}$

define the function $h = f+g$.

Solution

$$\mathrm{dom}(h) = \mathrm{dom}(f+g) = \mathrm{dom}(f) \cap \mathrm{dom}(g) =$$
$$= \{1,2,3,4\} \cap \{3,5,4,6\} = \{3,4\}.$$

$h(3) = (f+g)(3) = f(3)+g(3) = 6+1 = 7$

$h(4) = (f+g)(4) = f(4)+g(4) = 1+9 = 10$

It follows that

$h = f+g = \{(3,7),(4,10)\}$.

b) Given the functions $f(x) = \sqrt{9-x^2}$ and $g(x) = \sqrt{x^2-1}$, define the functions $h = \dfrac{f}{g}$ and $\varphi = f+g$

Solution

• **Domain of f**

Require $9-x^2 \geqslant 0 \Leftrightarrow (3-x)(3+x) \geqslant 0$
$\Leftrightarrow x \in [-3,3]$

x		-3		3	
$3-x$	$+$		$+$	0	$-$
$3+x$	$-$	0	$+$		$+$
	$-$	0	$+$	0	$-$

thus $\text{dom}(f) = [-3,3]$.

• **Domain of g**

Require $x^2-1 \geqslant 0 \Leftrightarrow (x-1)(x+1) \geqslant 0 \Leftrightarrow$
$\Leftrightarrow x \in (-\infty,-1] \cup [1,+\infty)$

x		-1		$+1$	
$x-1$	$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$
	$+$	0	$-$	0	$+$

• **Defining $h = f/g$**

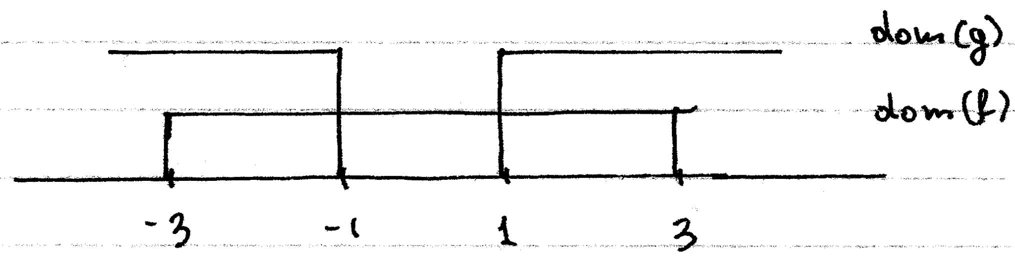
▷ First we find the domain of h.

Solve $g(x) = 0 \Leftrightarrow \sqrt{x^2-1} = 0 \Leftrightarrow x^2-1=0 \Leftrightarrow x^2 = 1$
$\Leftrightarrow x = 1 \vee x = -1 \Leftrightarrow x \in \{-1,1\}$.

It follows that

$$\begin{aligned}
\text{dom}(h) = \text{dom}(f/g) &= (\text{dom} f \cap \text{dom} g) - \{x \in \mathbb{R} \mid g(x) = 0\} \\
&= ([-3,3] \cap ((-\infty,-1] \cup [1,+\infty)) - \{-1,1\} = \\
&= ([-3,-1] \cup [1,3]) - \{-1,1\} = \\
&= [-3,-1) \cup (1,3]
\end{aligned}$$

On the 2nd line, to find the intersection we use:



$\mathrm{dom}\, f \cap \mathrm{dom}\, g = [-3,-1] \cup [1,3]$.

- The formula for h is:

$$h(x) = \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{9-x^2}}{\sqrt{x^2-1}}, \quad \forall x \in [-3,-1] \cup [1,3]$$

↳ Note again that to define a new function we must determine BOTH the domain and the formula of the new function

- <u>Defining $\varphi = f+g$</u>

$$\mathrm{dom}(\varphi) = \mathrm{dom}(f+g) = \mathrm{dom}(f) \cap \mathrm{dom}(g) =$$
$$= \dots = [-3,-1] \cup [1,3]$$

$$\varphi(x) = (f+g)(x) = f(x) + g(x) =$$
$$= \sqrt{9-x^2} + \sqrt{x^2-1}, \quad \forall x \in [-3,-1] \cup [1,3].$$

↳ In problems with multiple requests, it is sufficient to calculate $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ once, and then reuse the result, as we have done above.

c) Given the functions $f_1 : A \to \mathbb{R}$, $f_2 : A \to \mathbb{R}$, $g_1 : B \to \mathbb{R}$, and $g_2 : B \to \mathbb{R}$ with $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$, and $A \cap B \neq \emptyset$, show that

$$f_1 = f_2 \wedge g_1 = g_2 \Rightarrow f_1 + g_1 = f_2 + g_2$$

Solution

Assume that $f_1 = f_2$ and $g_1 = g_2$. Then

$$f_1 = f_2 \Rightarrow \forall x \in A : f_1(x) = f_2(x) \quad (1)$$

$$g_1 = g_2 \Rightarrow \forall x \in B : g_1(x) = g_2(x) \quad (2)$$

It follows that

$$\left.\begin{array}{l} \mathrm{dom}(f_1+g_1) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, g_1 = A \cap B \\ \mathrm{dom}(f_2+g_2) = \mathrm{dom}\, f_2 \cap \mathrm{dom}\, g_2 = A \cap B \end{array}\right\} \Rightarrow$$

$$\Rightarrow \mathrm{dom}(f_1+g_1) = \mathrm{dom}(f_2+g_2) \quad (3)$$

and

$$\begin{aligned} (f_1+g_1)(x) &= f_1(x) + g_1(x) = && [\text{by def}] \\ &= f_2(x) + g_2(x) = && [\text{use (1) and (2)}] \\ &= (f_2+g_2)(x),\ \forall x \in A \cap B. && (4) \end{aligned}$$

From (3) and (4): $f_1 + g_1 = f_2 + g_2$.

Note that to show that two functions f, g are equal (i.e. $f = g$), we have to show that:

1) Both functions have the same domain A

$$\mathrm{dom}(f) = \mathrm{dom}(g) = A$$

2) The formulas for f and g always agree:

$$\forall x \in A : f(x) = g(x)$$

d) Given the functions $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ with $A \cap B \neq \emptyset$ and the numbers $a, b \in \mathbb{R}$, show that $(af)(bg) = (ab)(fg)$.

<u>Solution</u>

$$\operatorname{dom}[(af)(bg)] = \operatorname{dom}(af) \cap \operatorname{dom}(bg) =$$
$$= \operatorname{dom}(f) \cap \operatorname{dom}(g) = A \cap B \quad (1)$$

$$\operatorname{dom}[(ab)(fg)] = \operatorname{dom}(fg) = \operatorname{dom}(f) \cap \operatorname{dom}(g) = A \cap B \quad (2)$$

From (1) and (2):

$$\operatorname{dom}[(af)(bg)] = \operatorname{dom}[(ab)(fg)]. \quad (3)$$

We also note that

$$[(af)(bg)](x) = (af)(x) \cdot (bg)(x) = [af(x)][bg(x)]$$
$$= ab\, f(x)\, g(x) = ab\,(fg)(x) =$$
$$= [(ab)(fg)](x), \ \forall x \in A \cap B \quad (4)$$

From (3) and (4): $(af)(bg) = (ab)(fg)$.

EXERCISES

(6) Define the functions $h_1 = f+g$, $h_2 = fg$, $h_3 = f/g$, and $h_4 = g/f$, when f and g are defined as:

a) $f = \{(1,3),(2,4),(3,1),(4,7)\}$
   $g = \{(2,0),(3,2),(4,1),(5,3)\}$

b) $f = \{(1,1),(3,2),(2,5),(5,6)\}$
   $g = \{(2,0),(3,1),(4,3),(5,2),(0,2)\}$

c) $f = \{(1,3),(3,1)\}$
   $g = \{(1,0),(0,1),(2,2),(3,4)\}$

d) $f = \{(2,3),(3,6),(4,2),(5,1),(6,2)\}$
   $g = \{(1,5),(3,2),(6,3),(2,4)\}$

(7) Let f, g, h be functions with $f: A \to \mathbb{R}$, $g: A \to \mathbb{R}$, and $h: B \to \mathbb{R}$. Show that

a) $f = g \Rightarrow f+h = g+h$
b) $f = g \Rightarrow fh = gh$
c) $(-f)(-g) = fg$.

(8) Find the default domain for the functions f, g and define the functions $h_1 = f+g$, $h_2 = fg$, $h_3 = f/g$ with f and g given by:

a) $f(x) = \sqrt{1-x2}$ and $g(x) = 1/x$

b) $f(x) = x^2\sqrt{1-x}$ and $g(x) = \dfrac{2x}{\sqrt{1-x}}$

c) $f(x) = \sqrt{4-x^2}$ and $g(x) = \dfrac{3x+1}{\sqrt{x^2+x}}$

d) $f(x) = \sqrt{2x+3}$ and $g(x) = \sqrt{x^2+x-2}$

9. Consider the functions f, g, h with $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$, and $h: B \to \mathbb{R}$ where $A \cap B \neq \emptyset$. Show that

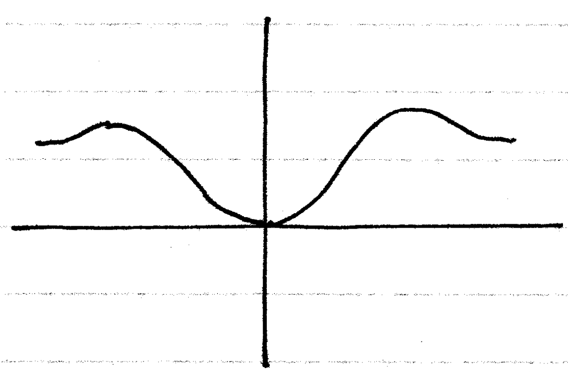
$$f(ag + bh) = a(fg) + b(fh)$$

▼ Odd and even functions

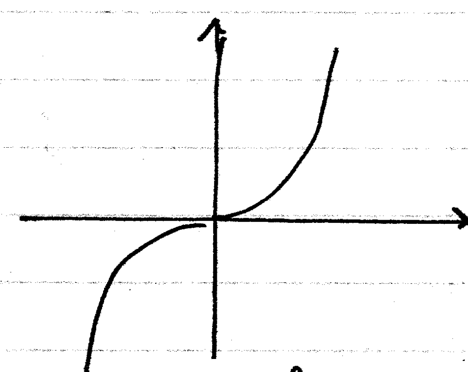
- Let $f: A \to \mathbb{R}$ be a function. We say that

$$\boxed{\begin{array}{l} f \text{ even} \iff \forall x \in A : (-x \in A \land f(-x) = f(x)) \\ f \text{ odd} \iff \forall x \in A : (-x \in A \land f(-x) = -f(x)) \end{array}}$$

- A prerequisite for the function f to be odd or even is that it has a domain A that is symmetric around the origin (i.e. $\forall x \in A : -x \in A$). If the domain is not symmetric, then the function can be neither even nor odd.
- An even function has graph that is symmetric accross the y-axis.
- An odd function has graph such that the $x<0$ part is obtained by first reflecting the $x>0$ part accross the y-axis and then the x-axis.



f even



f odd.

EXAMPLES

Which of the following functions are odd or even?

a) $f(x) = |3x+2| + |3x-2|$

Solution

Domain $A = \mathbb{R}$ is symmetric.

$f(-x) = |3(-x)+2| + |3(-x)-2| =$

$= |-3x+2| + |-3x-2| =$

$= |3x-2| + |3x+2| =$

$= |3x+2| + |3x-2| = f(x)\ ,\ \forall x \in \mathbb{R} \Rightarrow$

$\Rightarrow f$ even.

b) $f(x) = \dfrac{5x^3}{|x|-1}$

Solution

Domain:

Solve $|x|-1=0 \Leftrightarrow |x|=1 \Leftrightarrow x=1 \vee x=-1$

thus $A = \mathbb{R} - \{-1, 1\}$ which is symmetric.

$f(-x) = \dfrac{5(-x)^3}{|-x|-1} = \dfrac{-5x^3}{|x|-1} = -f(x),\ \forall x \in A \Rightarrow$

$\Rightarrow f$ odd.

c) $f(x) = \dfrac{x^2}{x+1}$

<u>Solution</u>

- Domain.

Solve $x+1=0 \Leftrightarrow x=-1$,
thus $A = \mathbb{R} - \{-1\}$.

- Note that A is not symmetric because $-1 \notin A$ and $+1 \in A$. Therefore f is not even and f is not odd.

$\updownarrow\!\!\rightarrow$ To establish that a function is even or odd, you have to first find the domain and show that it is symmetric before you calculate $f(-x)$. If the domain is not symmetric, then the function is neither odd nor even.

d) If f is an odd function and g an even function, show that fg is an odd function.

<u>Solution</u>

Define $A = \text{dom}(f)$ and $B = \text{dom}(g)$.
$\text{dom}(fg) = \text{dom}(f) \cap \text{dom}(g) = A \cap B$

- Symmetry.

f odd $\Rightarrow A$ symmetric
g even $\Rightarrow B$ symmetric

$$
\begin{aligned}
-x \in A \cap B &\Rightarrow -x \in A \wedge -x \in B \Rightarrow && [\text{set intersection}]\\
&\Rightarrow x \in A \wedge x \in B \Rightarrow && [A, B \text{ symmetric}]\\
&\Rightarrow x \in A \cap B && [\text{set intersection}]
\end{aligned}
$$

therefore $A \cap B$ is symmetric.

- Parity

$$
\begin{aligned}
(fg)(-x) &= f(-x)g(-x) = [-f(x)]\,g(x) = \\
&= -f(x)g(x) = -(fg)(x), \ \forall x \in A \cap B \Rightarrow
\end{aligned}
$$

$\Rightarrow fg$ odd.

↳ In general to prove that a set A is symmetric we must prove

$$-x \in A \Rightarrow x \in A$$

or equivalently

$$x \in A \Rightarrow -x \in A$$

To do that we use the following properties:

$$
\begin{aligned}
x \in A \cap B &\Leftrightarrow x \in A \wedge x \in B\\
x \in A \cup B &\Leftrightarrow x \in A \vee x \in B\\
x \in A - B &\Leftrightarrow x \in A \wedge x \notin B
\end{aligned}
$$

as well as any pre-existing symmetry assumptions.

EXERCISES

(10) Which of the following functions are odd or even? Show that they are odd or even, when applicable, or show that they are neither odd nor even.

a) $f(x) = |x+4| + |x-4|$

b) $f(x) = 2x^3 - 3x$

c) $f(x) = 2x^4 - 3x^2 + 5$

d) $f(x) = \sqrt{1-x^2}$

e) $f(x) = \dfrac{x-1}{x+1}$

f) $f(x) = \dfrac{3x|x|}{2|x|+1}$

g) $f(x) = \dfrac{5x^5 - 4x^3}{x^4+3}$

h) $f(x) = \dfrac{x^2+4}{x^2+5x+6}$

i) $f(x) = \sqrt{1-x} + \sqrt{1+x}$

j) $f(x) = \dfrac{2x+1}{2x-1} + \dfrac{2x-1}{2x+1}$

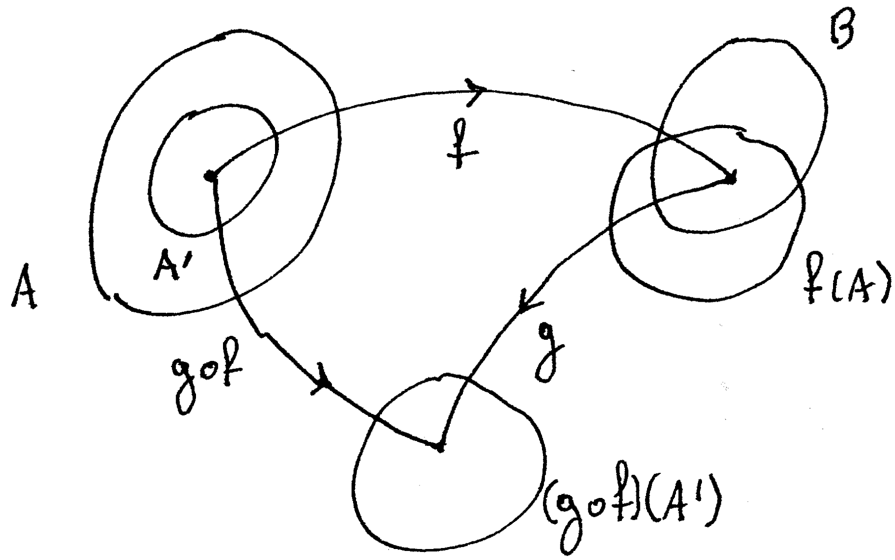
(11) If f, g are even functions, show that $f+g$ and fg are also even.

(12) If f, g are odd functions, show that $f+g$ and fg are also odd.

▼ Function composition

- Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ be two functions. We assume that $f(A) \cap B \neq \emptyset$. Let A' be the subset of A whose elements are mapped by f into the intersection $f(A) \cap B$. Thus A' is given by
$$A' = \{x \in A \mid f(x) \in B\}.$$
We may therefore define the function $g \circ f: A' \to \mathbb{R}$ as follows:

$$\boxed{\begin{array}{l} \mathrm{dom}(g\circ f) = \{x \in \mathrm{dom}(f) \mid f(x) \in \mathrm{dom}(g)\} = A' \\ \forall x \in A': \ (g\circ f)(x) = g(f(x)) \end{array}}$$



- We note that the belonging condition for $g \circ f$ is

$$\boxed{x \in \mathrm{dom}(g\circ f) \Leftrightarrow \begin{cases} x \in \mathrm{dom}(f) \\ f(x) \in \mathrm{dom}(g) \end{cases}}$$

<u>Method</u>: To define $g \circ f$:

- 1 Find the domain $dom(g \circ f)$ by solving:

$$x \in dom(g \circ f) \Leftrightarrow \begin{cases} x \in dom(f) \\ f(x) \in dom(g) \end{cases} \Leftrightarrow \dots$$

- 2 Find the formula of $g \circ f$:

$$(g \circ f)(x) = g(f(x)) = \dots$$

<u>example</u>

For $f(x) = \sqrt{1-x^2}$ and $g(x) = x^2 + 3x + 2$

$x \in dom(f) \Leftrightarrow 1 - x^2 \geq 0 \Leftrightarrow \dots \Leftrightarrow x \in [-1, 1]$

thus $dom(f) = [-1, 1]$

Also $dom(g) = \mathbb{R}$.

(a) To find $g \circ f$:

$$x \in dom(g \circ f) \Leftrightarrow \begin{cases} x \in dom(f) \\ f(x) \in dom(g) \end{cases} \Leftrightarrow \begin{cases} x \in [-1, 1] \\ \sqrt{1-x^2} \in \mathbb{R} \end{cases}$$

$$\Leftrightarrow x \in [-1, 1]$$

thus $dom(g \circ f) = [-1, 1]$ and

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{1-x^2}) =$$

$$= (\sqrt{1-x^2})^2 + 3\sqrt{1-x^2} + 2$$

$$= 1 - x^2 + 3\sqrt{1-x^2} + 2 =$$

$$= 3 - x^2 + 3\sqrt{1-x^2}.$$

(b) To find $f \circ g$:

$$x \in \mathrm{dom}(f \circ g) \Leftrightarrow \begin{cases} x \in \mathrm{dom}(g) \\ g(x) \in \mathrm{dom}(f) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x \in \mathbb{R} \\ x^2+3x+2 \in [-1,1] \end{cases} \Leftrightarrow -1 \leq x^2+3x+2 \leq 1$$

$$\Leftrightarrow \begin{cases} x^2+3x+3 \geqslant 0 & (1) \\ x^2+3x+1 \leqslant 0 & (2) \end{cases}$$

For (1): $\left.\begin{array}{l}\Delta = 9-4\cdot1\cdot3 = 9-12 < 0 \\ a = 1 > 0\end{array}\right\} \Rightarrow$

$\Rightarrow x^2+3x+3 > 0, \forall x \in \mathbb{R}$

$\Rightarrow$ (1) identity.

For (2): $\Delta = 9-4\cdot1\cdot1 = 9-4 = 5 \Rightarrow$

$\Rightarrow x_{1,2} = \dfrac{-3 \pm \sqrt{5}}{2}$

(2) $\Leftrightarrow x \in [x_1, x_2]$

Thus

$\mathrm{dom}(f \circ g) = \left[\dfrac{-3-\sqrt{5}}{2}, \dfrac{-3+\sqrt{5}}{2}\right]$ and

$(f \circ g)(x) = f(x^2+3x+2) =$

$= \sqrt{1-(x^2+3x+2)^2}$

EXAMPLES

a) Given $f = \{(1,4),(2,2),(3,1),(4,1)\}$ and

$g = \{(2,3),(3,2),(4,5)\}$

define $f \circ g$ and $g \circ f$

Solution

We note that $\mathrm{dom}(f) = \{1,2,3,4\}$ and $\mathrm{dom}(g) = \{2,3,4\}$.

• $f \circ g$ definition:

$g(2) = 3 \in \mathrm{dom}(f) \Rightarrow (f \circ g)(2) = f(g(2)) = f(3) = 1$

$g(3) = 2 \in \mathrm{dom}(f) \Rightarrow (f \circ g)(3) = f(g(3)) = f(2) = 2$

$g(4) = 5 \notin \mathrm{dom}(f) \Rightarrow 5 \notin \mathrm{dom}(f \circ g)$.

It follows that $f \circ g = \{(2,1),(3,2)\}$.

• $g \circ f$ definition

$f(1) = 4 \in \mathrm{dom}(g) \Rightarrow (g \circ f)(1) = g(f(1)) = g(4) = 5$

$f(2) = 2 \in \mathrm{dom}(g) \Rightarrow (g \circ f)(2) = g(f(2)) = g(2) = 3$

$f(3) = 1 \notin \mathrm{dom}(g) \Rightarrow 3 \notin \mathrm{dom}(g \circ f)$

$f(4) = 1 \notin \mathrm{dom}(g) \Rightarrow 4 \notin \mathrm{dom}(g \circ f)$

It follows that $g \circ f = \{(1,5),(2,3)\}$

↳ To evaluate $f \circ g$ for a discrete problem, first we write down $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$. Then, for each element $x \in \mathrm{dom}(g)$ we do the following:

- 1 Calculate $g(x)$.
- 2 If $g(x) \in \text{dom}(f)$, then we can go ahead and calculate $(f \circ g)(x)$.
- 3 If $g(x) \notin \text{dom}(f)$, then $x \notin \text{dom}(f \circ g)$; in other words, $(f \circ g)(x)$ cannot be calculated and that x is not in the domain of $f \circ g$.

b) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Show that f even and g odd $\Rightarrow g \circ f$ even $\wedge$ $f \circ g$ even.

Solution

- For $g \circ f$.

Since $\text{dom}(f) = \mathbb{R}$ and $\text{dom}(g) = \mathbb{R}$, it follows that:

$$\text{dom}(g \circ f) = \{x \in \text{dom}(f) \mid f(x) \in \text{dom}(g)\} = \\ = \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\} = \mathbb{R}.$$

which is symmetric. Furthermore:

$$\begin{aligned} (g \circ f)(-x) &= g(f(-x)) && [\text{def.}] \\ &= g(f(x)) && [f \text{ even}] \\ &= (g \circ f)(x), \forall x \in \mathbb{R} && [\text{def.}] \end{aligned}$$

$\Rightarrow g \circ f$ even.

- For $f \circ g$

$$\begin{aligned} \text{dom}(f \circ g) &= \{x \in \text{dom}(g) \mid g(x) \in \text{dom}(f)\} \\ &= \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\} = \mathbb{R} \end{aligned}$$

which is symmetric. Furthermore:

$$\begin{aligned} (f \circ g)(-x) &= f(g(-x)) && [\text{def}] \\ &= f(-g(x)) && [g \text{ odd}] \end{aligned}$$

$= f(g(x))$ [f even]

$= (f \circ g)(x), \forall x \in \mathbb{R}$ [def]

$\Rightarrow f \circ g$ even.

EXERCISES

(13) Find $f \circ g$ and $g \circ f$ for the following functions

a) $f(x) = x^2+1$, $g(x) = \sqrt{3-x}$

b) $f(x) = 2x+1$, $g(x) = x^2+2$

c) $f(x) = \sqrt{4-x^2}$, $g(x) = \sqrt{1-x^2}$

d) $f(x) = \dfrac{x+2}{x+1}$, $g(x) = \dfrac{1}{x}$

• (14) Let f, g, h be three functions. Show that

$$f = g \Rightarrow f \circ h = g \circ h$$

• (15) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Show that

a) f even and g even $\Rightarrow f \circ g$ even

b) f odd and g odd $\Rightarrow f \circ g$ odd

c) f even and g odd $\Rightarrow f \circ g$ even

(16) Let $f = \{(1,3), (2,4), (3,1), (4,2)\}$

$g = \{(2,4), (3,1), (4,2)\}$

Define $f \circ g$ and $g \circ f$.

(17) Let $f = \{(1,2), (3,2), (2,4), (4,4)\}$

$g = \{(1,3), (2,1), (3,5), (4,2)\}$

Define $f \circ g$ and $g \circ f$.

Functions and Monotonicity

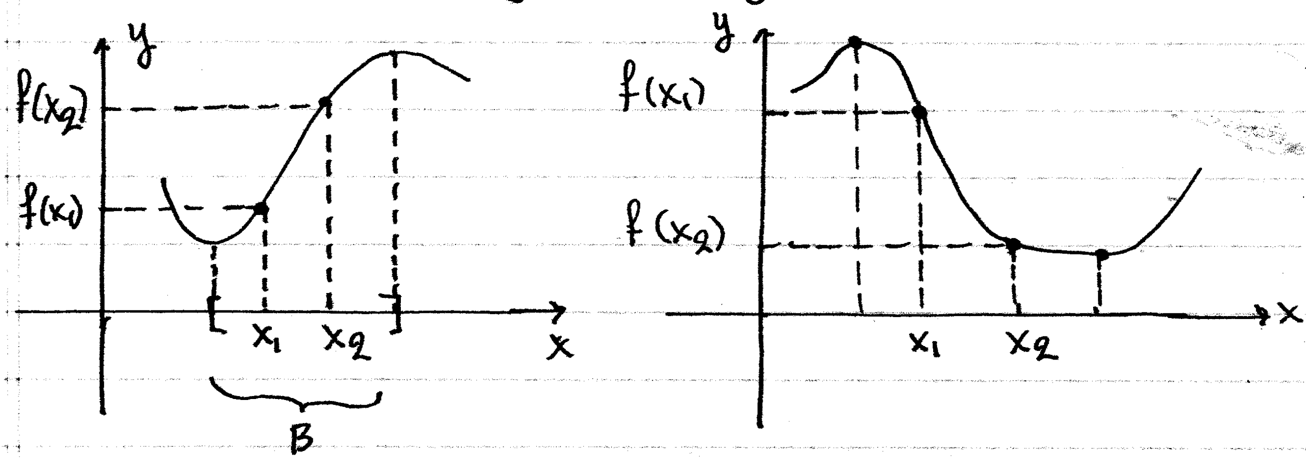
Let f be a function with $f: A \to \mathbb{R}$ and let $B \subseteq A$. We make the following definitions:

$$\boxed{\begin{array}{l} f \nearrow B \Leftrightarrow \forall x_1, x_2 \in B : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2)) \\ f \searrow B \Leftrightarrow \forall x_1, x_2 \in B : (x_1 < x_2 \Rightarrow f(x_1) > f(x_2)) \end{array}}$$

We read:

$f \nearrow B$: f is strictly increasing in B

$f \searrow B$: f is strictly decreasing in B.



Monotonicity can be determined directly from the definition with 2 methods:

1) Analytic Method
2) Synthetic Method.

In Calculus, monotonicity can also be determined using Differential Calculus.

Analytic Method

To show $f \nearrow B$ or $f \searrow B$.

- 1 Let $x_1, x_2 \in B$ be given with $x_1 < x_2$.
- 2 Calculate and <u>factor</u> $\Delta f(x_1, x_2) = f(x_2) - f(x_1)$
- 3 Determine the sign of each factor of Δf and then conclude whether $\Delta f > 0$ or $\Delta f < 0$.
- 4 Finish the argument.

EXAMPLES

a) Show that $f(x) = 3x + 5$ is strictly increasing in $\mathbb{R}$.

<u>Solution</u>

$dom(f) = \mathbb{R}$.

Let $\underline{x_1, x_2 \in \mathbb{R}}$ be given with $\underline{x_1 < x_2}$.

$$\Delta f(x_1, x_2) = f(x_2) - f(x_1) = (3x_2 + 5) - (3x_1 + 5) =$$
$$= 3(x_2 - x_1)$$

Since $x_1 < x_2 \Rightarrow x_2 - x_1 > 0 \Rightarrow$
$$\Rightarrow 3(x_2 - x_1) > 0 \Rightarrow$$
$$\Rightarrow f(x_2) - f(x_1) > 0 \Rightarrow$$
$$\Rightarrow \underline{f(x_1) < f(x_2)}$$

- Thus: $\forall x_1, x_2 \in \mathbb{R} : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2)) \Rightarrow f \nearrow \mathbb{R}$.

b) Show that $f(x) = \dfrac{2x}{x-1}$ is strictly decreasing in $(1, +\infty)$.

Solution

Let $x_1, x_2 \in (1, +\infty)$ be given with $x_1 < x_2$.
Then:

$$\Delta f(x_1, x_2) = f(x_2) - f(x_1) = \frac{2x_2}{x_2 - 1} - \frac{2x_1}{x_1 - 1} =$$

$$= \frac{2x_2(x_1-1) - 2x_1(x_2-1)}{(x_1-1)(x_2-1)} =$$

$$= \frac{2x_1x_2 - 2x_2 - 2x_1x_2 + 2x_1}{(x_1-1)(x_2-1)} =$$

$$= \frac{-2x_2 + 2x_1}{(x_1-1)(x_2-1)} = \frac{2(x_1 - x_2)}{(x_1-1)(x_2-1)}$$

Since $x_1 < x_2 \Rightarrow x_1 - x_2 < 0$.

$x_1 \in (1, +\infty) \Rightarrow x_1 > 1 \Rightarrow x_1 - 1 > 0$

$x_2 \in (1, +\infty) \Rightarrow x_2 > 1 \Rightarrow x_2 - 1 > 0$

therefore $\Delta f(x_1, x_2) < 0 \Rightarrow f(x_2) - f(x_1) < 0 \Rightarrow$
$\Rightarrow \underline{f(x_1) > f(x_2)}$

Thus:

$\forall x_1, x_2 \in (1, +\infty): (x_1 < x_2 \Rightarrow f(x_1) > f(x_2)) \Rightarrow$
$\Rightarrow f \searrow (1, +\infty)$.

c) Show that $f(x) = x^2 + 5x + 6$ is strictly increasing in $(-5/2, +\infty)$.

Solution

Let $x_1, x_2 \in (-5/2, +\infty)$ be given with $x_1 < x_2$

Then

$$\Delta f(x_1, x_2) = f(x_2) - f(x_1) = (x_2^2 + 5x_2 + 6) - (x_1^2 + 5x_1 + 6)$$
$$= (x_2^2 - x_1^2) + 5(x_2 - x_1) =$$
$$= (x_2 - x_1)(x_2 + x_1) + 5(x_2 - x_1) =$$
$$= (x_2 - x_1)(x_2 + x_1 + 5)$$

Since $x_1 < x_2 \Rightarrow x_2 - x_1 > 0 \quad (1)$

$$\left.\begin{array}{l} x_1 \in (-5/2, +\infty) \Rightarrow x_1 > -5/2 \\ x_2 \in (-5/2, +\infty) \Rightarrow x_2 > -5/2 \end{array}\right\} \Rightarrow$$

$\Rightarrow x_1 + x_2 > -5/2 - 5/2 = -5 \Rightarrow x_1 + x_2 + 5 > 0 \quad (2)$

From (1) and (2):

$\Delta f(x_1, x_2) > 0 \Rightarrow f(x_2) - f(x_1) > 0 \Rightarrow f(x_1) < f(x_2)$

It follows that:

$\forall x_1, x_2 \in (-5/2, +\infty) : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2)) \Rightarrow$

$\Rightarrow f \uparrow (-5/2, +\infty)$.

↳ For quadratics $f(x) = ax^2 + bx + c$, monotonicity changes at the axis of symmetry at $x = -b/2a$.

↳ In addition to the usual properties, it is good to know the following additional properties:

1) We can add two inequalities if they have the same direction:

$$\left.\begin{array}{l} a > b \\ x > y \end{array}\right\} \Rightarrow a + x > b + y$$

2) We can multiply two inequalities if they have the same direction AND all sides are POSITIVE!

$$\left.\begin{array}{l} a > b > 0 \\ x > y > 0 \end{array}\right\} \Rightarrow ax > by$$

3) We can raise an inequality to a positive power if both sides of the inequality are positive

$$\left.\begin{array}{l} a > b > 0 \\ p > 0 \end{array}\right\} \Rightarrow a^p > b^p > 0$$

e.g. $a > b > 0 \Rightarrow \sqrt{a} > \sqrt{b} > 0$ for $p = 1/2$.

4) We can raise an inequality to a negative power if both sides of the inequality are positive but then the direction of the inequality is reversed.

$$\left.\begin{array}{l} a > b > 0 \\ n < 0 \end{array}\right\} \Rightarrow 0 < a^n < b^n$$

e.g. $a > b > 0 \Rightarrow 0 < \frac{1}{a} < \frac{1}{b}$ for $n = -1$.

We rely on these properties heavily for the synthetic method. We also need the following previously mentioned properties:

5) $x < y \Rightarrow x + a < y + a$

6) $\left.\begin{array}{l} x < y \\ p > 0 \end{array}\right\} \Rightarrow px < py$ 7) $\left.\begin{array}{l} x < y \\ n < 0 \end{array}\right\} \Rightarrow nx > ny$.

to add/multiply a constant to both sides of an inequality.

Synthetic Method

To show that $f \nearrow B$ or $f \searrow B$:

- •1 Let $x_1, x_2 \in B$ be given with $x_1 < x_2$.
- •2 Use a sequence of deductions to show that

  $$x_1 < x_2 \Rightarrow \dots \Rightarrow \dots \Rightarrow f(x_1) < f(x_2)$$

  or

  $$x_1 < x_2 \Rightarrow \dots \Rightarrow \dots \Rightarrow f(x_1) > f(x_2)$$

  using the above properties of inequalities.
- •3 Wrap up the argument.

EXAMPLES

a) For $f(x) = 3-(1-2x)^2$ show that $f \searrow (1/2, +\infty)$

Solution

Let $x_1, x_2 \in (1/2, +\infty)$ be given with $x_1 < x_2$. Then:

$$x_1 < x_2 \Rightarrow -2x_1 > -2x_2 \Rightarrow 1-2x_1 > 1-2x_2 \overset{*}{\Rightarrow}$$

$$\Rightarrow \underbrace{0 <}_{(!)} 2x_1 - 1 < 2x_2 - 1 \quad [\text{because } x_1 > 1/2 \land x_2 > 1/2]$$

$$\Rightarrow (2x_1-1)^2 < (2x_2-1)^2 \overset{**}{\Rightarrow} (1-2x_1)^2 < (1-2x_2)^2$$

$$\Rightarrow -(1-2x_1) > -(1-2x_2)^2 \Rightarrow 3-(1-2x_1)^2 > 3-(1-2x_2)^2$$

$$\Rightarrow f(x_1) > f(x_2).$$

Thus: $\forall x_1, x_2 \in (1/2, +\infty) : (x_1 < x_2 \Rightarrow f(x_1) > f(x_2))$

$$\Rightarrow f \searrow (1/2, +\infty).$$

* We multiply inequality with -1 to ensure that both sides are positive before going ahead and squaring it.

** Here we use $x^2 = (-x)^2$.

↳ In the above solution you should be able to identify which inequality property is used at every step.

b) For $f(x) = 3x + 1 + \sqrt{1-x^2}$, show that $f \nearrow (-1, 0)$

<u>Solution</u>

Let $\underline{x_1, x_2 \in (-1, 0)}$ be given such that $\underline{x_1 < x_2}$. Then

$x_1 < x_2 \Rightarrow 3x_1 < 3x_2 \Rightarrow 3x_1 + 1 < 3x_2 + 1 \qquad (1)$

Also note that

$$x_1 < x_2 \Rightarrow -x_1 > -x_2 > 0 \Rightarrow (-x_1)^2 > (-x_2)^2 \Rightarrow x_1^2 > x_2^2$$
$$\Rightarrow -x_1^2 < -x_2^2 \Rightarrow 1 - x_1^2 < 1 - x_2^2 \qquad (2)$$

and

$$x_1 \in (-1, 0) \Rightarrow -1 < x_1 < 0 \Rightarrow 1 > -x_1 > 0 \Rightarrow 1 > (-x_1)^2 \Rightarrow$$
$$\Rightarrow 1 > x_1^2 \Rightarrow 1 - x_1^2 > 0 \qquad (3)$$

and similarly

$$x_2 \in (-1, 0) \Rightarrow \cdots \Rightarrow 1 - x_2^2 > 0. \qquad (4)$$

From (2), (3), (4), it follows that

$$0 < 1 - x_1^2 < 1 - x_2^2 \Rightarrow \sqrt{1 - x_1^2} < \sqrt{1 - x_2^2} \qquad (5)$$

From (1) and (5), adding the inequalities:

$3x_1+1+\sqrt{1-x_1^2} < 3x_2+1+\sqrt{1-x_2^2} \Rightarrow$

$\Rightarrow \underline{f(x_1) < f(x_2)}$

Thus $\forall x_1, x_2 \in (-1,0): (x_1<x_2 \Rightarrow f(x_1)<f(x_2))$

$\Rightarrow f \nearrow (-1,0)$.

↳ Note that before we raise an inequality to any power we have to ensure/check that both sides of the inequality are positive.
Thus in the above:

$x_1<x_2 \Rightarrow x_1^2<x_2^2$ is WRONG

since $x_1<0$ and $x_2<0$. Be careful!!

↳ Note that it was necessary to interrupt the main line of the argument:

$x_1<x_2 \Rightarrow \cdots \Rightarrow \sqrt{1-x_1^2} < \sqrt{1-x_2^2}$

to show that $1-x_1^2>0$ and $1-x_2^2>0$.
Note the careful use of equation labels to interrupt and restart our main argument.

c) For $f(x)=\dfrac{1}{x^2-2}$, show that $f \nearrow (-\infty,-\sqrt{2})$

<u>Solution</u>

Let $\underline{x_1,x_2 \in (-\infty,-\sqrt{2})}$ be given with $\underline{x_1<x_2}$.
Then

$x_1 < x_2 \Rightarrow -x_1 > -x_2 > 0 \Rightarrow (-x_1)^2 > (-x_2)^2 \Rightarrow x_1^2 > x_2^2$

$\Rightarrow x_1^2 - 2 > x_2^2 - 2 \qquad (1)$

Also note that

$x_1 \in (-\infty, -\sqrt{2}) \Rightarrow x_1 < -\sqrt{2} \Rightarrow -x_1 > \sqrt{2} \Rightarrow (-x_1)^2 > 2 \Rightarrow$

$\Rightarrow x_1^2 > 2 \Rightarrow x_1^2 - 2 > 0. \quad (2)$

and similarly $x_2 \in (-\infty, -\sqrt{2}) \Rightarrow x_2^2 - 2 > 0 \quad (3).$

From (1), (2), and (3):

$$x_1^2 - 2 > x_2^2 - 2 > 0 \Rightarrow \frac{1}{x_1^2 - 2} < \frac{1}{x_2^2 - 2} \Rightarrow \underline{f(x_1) < f(x_2)}$$

It follows that

$\forall x_1, x_2 \in (-\infty, -\sqrt{2}) : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2))$

$\Rightarrow f \nearrow (-\infty, -\sqrt{2}).$

EXERCISES

(18) Use the analytic method to determine the monotonicity of the following functions.

a) $f(x) = 3x+2$ on $\mathbb{R}$

b) $f(x) = 5-4x$ on $\mathbb{R}$

c) $f(x) = x^2-4x+5$ on $(-\infty, 2)$

d) $f(x) = \dfrac{3x+1}{x+2}$ on $(-2, +\infty)$

e) $f(x) = \dfrac{x+8}{3x+1}$ on $(-\infty, -1/3)$

f) $f(x) = (2x+5)^2-3$ on $(-\infty, -5/2)$

g) $f(x) = (x-1)(2x+1)$ on $(1, +\infty)$

(19) Use the synthetic method to determine the monotonicity of the following functions

a) $f(x) = 5x-3$ on $\mathbb{R}$

b) $f(x) = 2-7x$ on $\mathbb{R}$

c) $f(x) = (2x+3)^2+1$ on $(0, +\infty)$

d) $f(x) = (2-5x)^3-2$ on $(0, +\infty)$

e) $f(x) = \dfrac{-2}{2x^2+3}$ on $(0, +\infty)$

f) $f(x) = \sqrt{2x-1}$ on $(1, +\infty)$

g) $f(x) = 2 - 3\sqrt{4-x^2}$ on $(0, 2)$

h) $f(x) = -1 + 2\sqrt{9-(x+1)^2}$ on $(-4, -1)$

i) $f(x) = 3x+2+\sqrt{x+1}$ on $(0, +\infty)$

j) $f(x) = (2x-1)\sqrt{2x+1}$ on $(1, +\infty)$

(20) Let $f(x) = -1/x$.

a) Show that $f \nearrow (-\infty, 0)$ and $f \nearrow (-\infty, 0)$

b) Now, show that the statement $f \nearrow (-\infty, 0) \cup (0, +\infty)$ is FALSE.

$\hookrightarrow$ To show that $f \nearrow A$ is FALSE, it is sufficient to find a counterexample, that is, to find some $x_1, x_2 \in A$ with $x_1 < x_2$ and $f(x_1) \geq f(x_2)$. In other words:

$f \nearrow A$ is false $\Leftrightarrow \exists x_1, x_2 \in A : (x_1 < x_2 \land f(x_1) \geq f(x_2))$

$f \searrow A$ is false $\Leftrightarrow \exists x_1, x_2 \in A : (x_1 < x_2 \land f(x_1) \leq f(x_2))$

This exercise shows that the general claim

$$f \nearrow A_1 \land f \nearrow A_2 \Rightarrow f \nearrow A_1 \cup A_2$$

is not always true, by demonstrating a counterexample.

(21) Let $f(x) = \dfrac{ax+b}{cx+d}$. Show that given $D = ad - bc$,

a) If $D > 0$, then $f \nearrow (-\infty, -d/c)$ and $f \nearrow (-d/c, +\infty)$

b) If $D < 0$, then $f \searrow (-\infty, -d/c)$ and $f \searrow (-d/c, +\infty)$

(22) Given the functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ and $g:\mathbb{R}\longrightarrow\mathbb{R}$, show that

a) $f\nearrow\mathbb{R}$ and $g\nearrow\mathbb{R}\Rightarrow f+g\nearrow\mathbb{R}$

b) $f\nearrow\mathbb{R}$ and $g\searrow\mathbb{R}\Rightarrow f\circ g\searrow\mathbb{R}$

c) $f\searrow\mathbb{R}$ and $g\searrow\mathbb{R}\Rightarrow g\circ f\nearrow\mathbb{R}$

d) f odd and $f\nearrow[0,+\infty)\Rightarrow f\nearrow\mathbb{R}$

e) f even and $f\nearrow(0,+\infty)\Rightarrow f\searrow(-\infty,0)$.

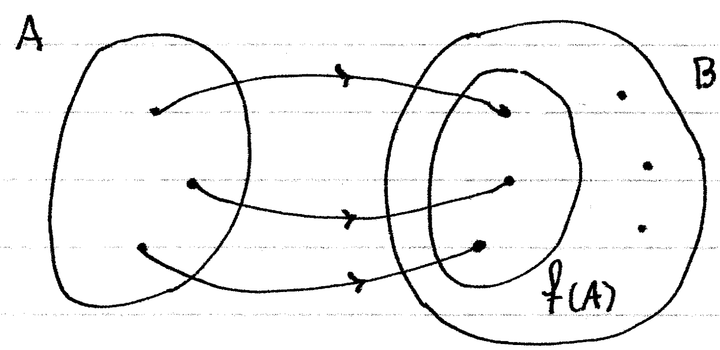
Inverse Functions

- Let $f : A \longrightarrow \mathbb{R}$ be a function with range $f(A)$. In order for f to have an inverse function, it has to satisfy the "one-to-one" property.

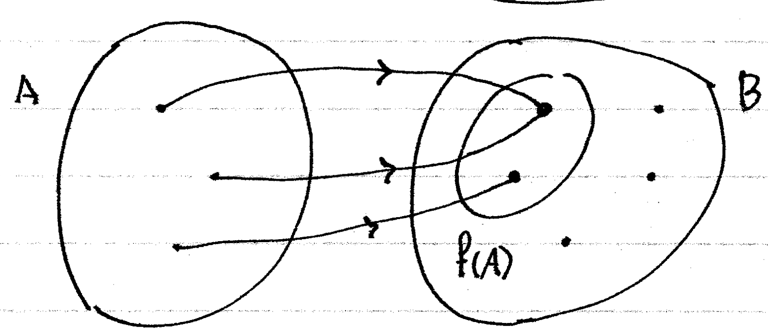
One-to-one functions

- $\boxed{f \text{ one-to-one} \Leftrightarrow \forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)}$

▸ Venn diagram interpretation: In a one-to-one function, every point in the range $f(A)$ of f receives only one arrow.



f is one-to-one



f is NOT one-to-one.

▸ Negated definition:

f NOT one-to-one $\Leftrightarrow \exists x_1, x_2 \in A : (f(x_1) = f(x_2) \wedge x_1 \neq x_2)$

EXAMPLES

a) Show that $f(x) = \dfrac{2x-1}{3x+2}$ is one-to-one.

Solution

- Domain: $A = \mathbb{R} - \{-2/3\}$ (Require $3x+2 \neq 0$).
- Let $x_1, x_2 \in A$ be given such that $f(x_1) = f(x_2)$.

  Then:

$$f(x_1) = f(x_2) \Rightarrow \frac{2x_1 - 1}{3x_1 - 1} = \frac{2x_2 - 1}{3x_2 - 1} \Rightarrow$$

$$\Rightarrow (2x_1 - 1)(3x_2 - 1) = (3x_1 - 1)(2x_2 - 1) \Rightarrow$$

$$\Rightarrow 6x_1x_2 - 2x_1 - 3x_2 + 1 = 6x_1x_2 - 3x_1 - 2x_2 + 1 \Rightarrow$$

$$\Rightarrow -2x_1 - 3x_2 = -3x_1 - 2x_2 \Rightarrow 3x_1 - 2x_1 = 3x_2 - 2x_2$$

$$\Rightarrow \underline{x_1 = x_2}.$$

Thus $\forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2) \Rightarrow$

$\Rightarrow f$ one-to-one.

↳ To show that a function f is one-to-one

- 1 Let $x_1, x_2 \in A$ be given with $f(x_1) = f(x_2)$.
- 2 Show: $f(x_1) = f(x_2) \Rightarrow \cdots \Rightarrow \cdots \Rightarrow x_1 = x_2$
- 3 Conclude argument.

b) Show that $f(x) = 2x^2 + 6x - 7$ is NOT one-to-one.

Solution

- Domain: $A = \mathbb{R}$ (no restrictions)
- Solve $f(x) = -7 \Leftrightarrow 2x^2 + 6x - 7 = -7 \Leftrightarrow 2x^2 + 6x = 0$

$$\Leftrightarrow 2x(x+3) = 0 \Leftrightarrow$$
$$\Leftrightarrow 2x = 0 \vee x+3 = 0 \Leftrightarrow$$
$$\Leftrightarrow x = 0 \vee x = -3$$

It follows that for $x_1 = 0$ and $x_2 = -3$:

$$\begin{cases} f(x_1) = f(x_2) = -7 \\ x_1 \neq x_2 \end{cases} \Rightarrow f \text{ not one-to-one.}$$

$\hookrightarrow$ To show that a function f is NOT one-to-one, it is enough to find just one counterexample $x_1, x_2 \in A$ (i.e. specific choices for x_1 and x_2) such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$.

EXERCISES

(23) Show that the following functions are one-to-one

a) $f(x) = ax+b$ with $a, b \in \mathbb{R}$ and $a \neq 0$.

b) $f(x) = a/x$ with $a \in \mathbb{R}$ and $a \neq 0$

c) $f(x) = \dfrac{ax+b}{cx+d}$ with $a, b, c, d \in \mathbb{R}$ and $D = ad-bc \neq 0$.

(24) Show that $f(x) = ax^2+bx+c$ with $a, b, c \in \mathbb{R}$ and $a \neq 0$ is NOT one-to-one.
(Hint: Solve $f(x) = c$)

(25) Let $f: A \longrightarrow \mathbb{R}$ be a function. Show that

a) $f \nearrow A \Rightarrow f$ one-to-one

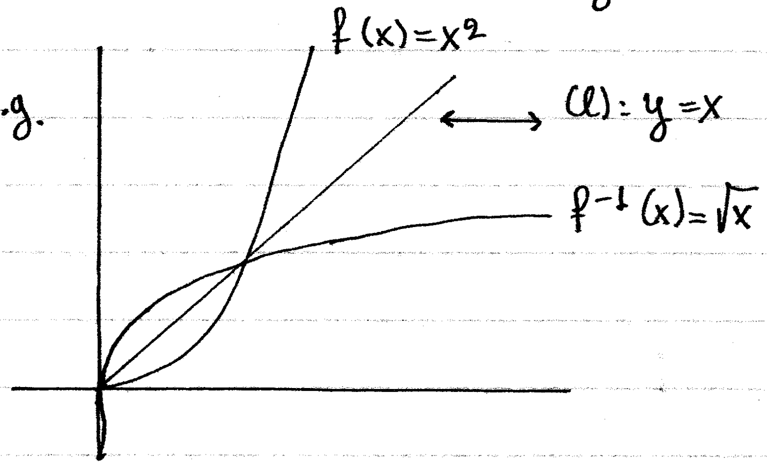
b) $f \searrow A \Rightarrow f$ one-to-one

c) f even $\Rightarrow f$ not one-to-one.

Definition of the inverse function

- Let $f: A \to \mathbb{R}$ be a one-to-one function with range $f(A)$. Then there is a unique function $f^{-1}: f(A) \to \mathbb{R}$ such that
$$f^{-1}(x) = y \Leftrightarrow f(y) = x$$
We call f^{-1} the <u>inverse</u> of f.
- Note that the range $f(A)$ of f is the domain of its inverse f^{-1}.
- It can be shown that
$$\forall x \in A: f^{-1}(f(x)) = x$$
$$\forall x \in f(A): f(f^{-1}(x)) = x$$
- The graph of f^{-1} is the reflection of the graph of f accross the line $(\ell): y = x$.

e.g.



<u>Method</u> : To find the inverse of a function $f: A \to \mathbb{R}$ we work as follows:

- 1 We setup the equation
  $f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow \cdots$
- 2 It may be necessary to require restrictions on y to evaluate $f(y)$. If that is the case, then do so.
- 3 Solve for y. During the process, it may be necessary to require restrictions on x to ensure that at least one solution exists. These restrictions define the domain of the inverse function f^{-1}.
- 4 When you show that, under possible restrictions on x, that your equation has a unique solution $y = y_0(x)$, you implicitly prove that both f is one-to-one and that $f^{-1}(x) = y_0(x)$. Thus you have the formula of the inverse function.
- 5 If applicable, check the constraints on y from step 2. They may or may not introduce further restrictions on the variable x and therefore on the domain of the inverse function.

EXAMPLES

a) Find the inverse function of $f(x) = \dfrac{x+3}{2x-5}$

Solution

$$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow \frac{y+3}{2y-5} = x \quad (\text{Require } \underline{2y-5 \neq 0})$$

$$\Leftrightarrow y+3 = x(2y-5) \Leftrightarrow y+3 = 2xy - 5x \Leftrightarrow (1-2x)y = -3-5x \quad (1)$$

For $1-2x = 0$: $x = 1/2$, and therefore

$(1) \Leftrightarrow 0y = -3-5\cdot(1/2) \Leftrightarrow 0y = -3-5/2 \longleftarrow$ inconsistent

thus $\underline{x = 1/2 \notin \mathrm{dom}(f^{-1})}$.

For $1-2x \neq 0$:

$$(1) \Leftrightarrow y = \frac{-3-5x}{1-2x}$$

Now we must check the requirement $2y-5 \neq 0$.

We note that:

$$2y-5 = 2\cdot\left(\frac{-3-5x}{1-2x}\right) - 5 = \frac{2(-3-5x)-5}{1-2x} =$$

$$= \frac{-6-10x-5(1-2x)}{1-2x} = \frac{-6-10x-5+10x}{1-2x} =$$

$$= \frac{-11}{1-2x} \neq 0$$

thus $2y-5 \neq 0$ is satisfied.

Thus $f^{-1}(x) = \dfrac{-3-5x}{1-2x}$ with $\mathrm{dom}(f^{-1}) = \mathbb{R} - \{1/2\}$.

↳ In the above example we see that the domain of f^{-1} coincides with the widest possible domain. However, this is not always true, as seen in the next example.

b) Find the inverse function of $f(x) = 2 + \dfrac{\sqrt{3x+1}}{3}$

Solution

$$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow 2 + \frac{\sqrt{3y+1}}{3} = x \Leftrightarrow$$

$$\Leftrightarrow 6 + \sqrt{3y+1} = 3x \Leftrightarrow \sqrt{3y+1} = 3x - 6 \Leftrightarrow \sqrt{3y+1} = 3(x-2) \quad (1)$$

Require $3(x-2) \geq 0 \Leftrightarrow x \geq 2$, otherwise equation (1) is inconsistent. For $x \geq 2$:

$$(1) \Leftrightarrow 3y + 1 = 9(x-2)^2 \Leftrightarrow 3y = 9(x-2)^2 - 1 \Leftrightarrow$$

$$\Leftrightarrow y = 3(x-2)^2 - \frac{1}{3}$$

It follows that

$$f^{-1}(x) = 3(x-2)^2 - 1/3 \quad \text{with } \mathrm{dom}(f^{-1}) = [2, +\infty)$$

↳ In this example we see that the domain $\mathrm{dom}(f^{-1})$ is restricted from the widest possible domain of the polynomial formula for $f^{-1}(x)$ which is $\mathbb{R}$.

Thus, to determine the domain of the inverse function f^{-1}, it is necessary to keep track of all constraints, as I suggested in the methodology.

c) Find the inverse function of $f(x) = 4x - 3$.

<u>Solution</u>

$$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow 4y - 3 = x \Leftrightarrow 4y = x + 3 \Leftrightarrow$$
$$\Leftrightarrow y = \frac{x+3}{4}$$

It follows that:

$$f^{-1}(x) = \frac{x+3}{4} \quad \text{with } \operatorname{dom}(f^{-1}) = \mathbb{R} \text{ (no constraints)}.$$

EXERCISES

(26) Find the inverse function f^{-1} for the following functions:

a) $f(x) = 3x+2$

b) $f(x) = 1-2x$

c) $f(x) = 2(1-x)+3(2x+1)$

d) $f(x) = x(x+2)-(x^2-5)$

e) $f(x) = \dfrac{2}{3x}$

f) $f(x) = \dfrac{3}{2x-1}$

g) $f(x) = \dfrac{2}{x-4}$

h) $f(x) = \dfrac{2x+3}{x-1}$

i) $f(x) = \dfrac{x+4}{3x-1}$

j) $f(x) = 3+\sqrt{x-1}$

k) $f(x) = -1-2\sqrt{2-3x}$

l) $f(x) = 2+\dfrac{\sqrt{1-3x}}{4}$

m) $f(x) = \dfrac{2-\sqrt{3x+2}}{5}$

n) $f(x) = \sqrt{\dfrac{2x+1}{3x-2}}$

o) $f(x) = \sqrt{\dfrac{x-3}{4x+5}}$

$\hookrightarrow$ To confirm $f^{-1}(x)$ with computer algebra, either simplify $f^{-1}(f(x))$ or evaluate it for a few values of x and confirm that $f^{-1}(f(x)) = x$

Calculating the range $f(A)$

- Let $f: A \to \mathbb{R}$ be a function. Recall that the range $f(A)$ of f is given by

$$f(A) = \{f(x) \mid x \in A\}$$

It follows that $y \in f(A)$ if and only if the equation $y = f(x)$ has at least one solution $x = x_0$ with $x_0 \in A$. (if there are more solutions it is not necessary for ALL to belong to A. One is sufficient).

Method: To find the range of a function we work as follows:

- 1 Find the ~~ra~~ domain A.
- 2 Solve the equation $y = f(x)$ with respect to x until we obtain a parametric equation of the form

$$a(y)x + b(y) = 0 \qquad \text{OR} \qquad \text{(Case 1)}$$
$$a(y)x^2 + b(y)x + c(y) = 0. \qquad \text{(Case 2)}$$

- 3 Find the solvability set S for which the simplified equation has a solution.

Case 1: $a(y)x + b(y) = 0$

a) For $a(y) = 0$, check on a case by case basis whether the equation is inconsistent.

b) For $a(y) \neq 0$, $y \in S$.

Case 2: $a(y)x^2 + b(y)x + c(y) = 0$

a) For $a(y) = 0$, check on a case by case basis whether the equation has a solution.

b) For $a(y) \neq 0$, require that
$$b^2(y) - 4a(y)c(y) \geq 0.$$

• 4 Find which elements of S also belong to $f(A)$.

Case 1: If $A = \mathbb{R}$

then, $f(A) = S \leftarrow$ you are done

Case 2: If $A = \mathbb{R} - \{x_0, x_1, x_2, \ldots, x_n\}$

then from the simplified equation
$$a(y)x + b(y) = 0 \quad \text{OR}$$
$$a(y)x^2 + b(y)x + c(y) = 0$$
we set $x = x_0, x = x_1, \ldots, x = x_n$, solve for y, and find $y = y_0, y = y_1, \ldots, y = y_n$. These y values give solutions x that do not belong to A and must be therefore excluded. It follows that
$$f(A) = S - \{y_0, y_1, y_2, \ldots, y_n\}$$

Case 3: If A is a union of intervals then solve for x all the way and then solve

$x \in A \Leftrightarrow$ system of inequalities in terms of y

$\Leftrightarrow y \in f(A)$

(bite the bullet case).

Functions whose range is obvious

1) For $f(x) = ax + b$ with $a \neq 0$

$A = \mathrm{dom}(f) = \mathbb{R}$

$f(A) = \mathbb{R}$.

2) For $f(x) = \dfrac{ax+b}{cx+d}$ with $D = ad - bc \neq 0$, $c \neq 0$.

$A = \mathrm{dom}(f) = \mathbb{R} - \{-d/c\}$

$f(A) = \mathbb{R} - \{\frac{a}{c}\}$.

examples

a) For $f(x) = x^2 + 2x + 3$, $A = \mathrm{dom}(f) = \mathbb{R}$.

Solve $y = f(x) \Leftrightarrow y = x^2 + 2x + 3 \Leftrightarrow$

$\Leftrightarrow \underline{x^2 + 2x + (3 - y) = 0}$ (1)

Solvability set:

(1) has a solution $\Leftrightarrow \Delta(y) \geqslant 0 \Leftrightarrow$

$\Leftrightarrow 2^2 - 4 \cdot 1 \cdot (3 - y) \geqslant 0 \Leftrightarrow 4 - 12 + 4y \geqslant 0$

$\Leftrightarrow 4y \geqslant 8 \Leftrightarrow y \geqslant 2 \Leftrightarrow y \in [2, +\infty) = S$

Since $A = \mathbb{R} \Rightarrow f(A) = S = [2, +\infty)$.

b) For $f(x) = \dfrac{x^2+x+1}{x^2+5x+6}$

Domain: Require $x^2+5x+6 \neq 0 \Leftrightarrow (x+2)(x+3) \neq 0$
$\Leftrightarrow x \in \mathbb{R} - \{-2,-3\}$
thus $A = \text{dom}(f) = \mathbb{R} - \{-2,-3\}$.

Solve:

$$y = f(x) \Leftrightarrow y = \frac{x^2+x+1}{x^2+5x+6} \Leftrightarrow \cdots \Leftrightarrow$$

$$\Leftrightarrow (y-1)x^2 + (5y-1)x + (6y-1) = 0 \qquad (1)$$

Solvability condition:

For $y-1=0 \Leftrightarrow y=1$, eq. (1) gives $4x+5=0 \Leftrightarrow$
$\Leftrightarrow x = -5/4 \in A$

thus $\underline{\underline{1 \in f(A)}}$ (2)

For $y-1 \neq 0$, eq (1) has a solution $\Leftrightarrow$

$\Leftrightarrow \Delta(y) \geq 0 \Leftrightarrow (5y-1)^2 - 4(y-1)(6y-1) \geq 0 \Leftrightarrow \cdots \Leftrightarrow$

$\Leftrightarrow y^2 + 18y - 3 \geq 0 \Leftrightarrow \cdots \Leftrightarrow$

$\Leftrightarrow \underline{y \in (-\infty, -9-2\sqrt{21}] \cup [-9+2\sqrt{21}, +\infty)}$ (3)

Possible exclusions:

For $x=-2$, eq. (1) gives

$4(y-1) - 2(5y-1) + (6y-1) = 0 \Leftrightarrow \cdots \Leftrightarrow$

$\Leftrightarrow 0y = 3 \leftarrow$ inconsistent, so no exclusion.

For $x=-3$, eq. (1) gives:

$$9(y-1)-3(5y-1)+(6y-1)=0 \Leftrightarrow \cdots \Leftrightarrow$$
$$\Leftrightarrow 0y=7 \leftarrow \text{inconsistent, so no exclusions.}$$

From (2), (3), as there are no exclusions, it follows that

$$f(A)=(-\infty,-9-2\sqrt{21}]\cup[-9+2\sqrt{21},+\infty)\cup\{1\}$$
$$=(-\infty,-9-2\sqrt{21}]\cup[-9+2\sqrt{21},+\infty),$$

because $-9+2\sqrt{21}<1$.

c) $f(x)=3-(2x+1)^2$

Domain: $A=\mathrm{dom}(f)=\mathbb{R}$.

Solve: $y=f(x) \Leftrightarrow y=3-(2x+1)^2 \Leftrightarrow$

$$\Leftrightarrow \underline{(2x+1)^2=3-y} \quad (1)$$

Solvability:

Eq. (1) has a solution $\Leftrightarrow 3-y\geq 0 \Leftrightarrow y\leq 3$

$$\Leftrightarrow y\in(-\infty,3]$$

Since $A=\mathbb{R} \Rightarrow f(A)=(-\infty,3]$. (no exclusions)

↳ For functions with roots, we follow the general methodology and also note that

1) The equation $y = f(x)$ usually simplifies to

$$\sqrt{a(x)} = b(y)$$

At this step we require $\underline{b(y) \geq 0}$. Then we continue solving for x to get

$$a(y)x + b(y) = 0$$

OR $a(y)x^2 + b(y)x + c(y) = 0$.

and continue as usual.

2) The domain A is usually complicated so we may have to solve for x all the way e.g.

$$y = f(x) \Leftrightarrow \cdots \Leftrightarrow x = g_1(y) \vee x = g_2(y).$$

Then in addition to solvability conditions we also require

$$g_1(y) \in A \quad \vee \quad g_2(y) \in A$$

$g_1(y) \in A$ → system of inequalities

$g_2(y) \in A$ → another system of inequalities

→ Take <u>union</u> of solution sets

examples

a) For $f(x) = \sqrt{x+3} + 2$

Domain: Require $x+3 \geq 0 \Leftrightarrow x \in [-3, +\infty)$
thus $A = dom(f) = [-3, +\infty)$

Solve:

$y = f(x) \Leftrightarrow y = \sqrt{x+3} + 2 \Leftrightarrow$
$\Leftrightarrow \sqrt{x+3} = y - 2 \quad (1)$

▸ Require $y - 2 \geq 0 \Leftrightarrow y \in [2, +\infty)$

$(1) \Leftrightarrow x + 3 = (y-2)^2 \Leftrightarrow x = -3 + (y-2)^2$

Require

$x \in A \Leftrightarrow -3 + (y-2)^2 \in [-3, +\infty) \Leftrightarrow$
$\Leftrightarrow -3 + (y-2)^2 \geq -3$
$\Leftrightarrow (y-2)^2 \geq 0 \leftarrow$ Always true

Thus

$y \in f(A) \Leftrightarrow y \in [2, +\infty)$, so $f(A) = [2, +\infty)$.

b) For $f(x) = 2 - \sqrt{1 - x^2}$

Domain: Require $1 - x^2 \geq 0 \Leftrightarrow \cdots \Leftrightarrow x \in [-1, 1]$
thus $A = dom(f) = [-1, 1]$

Solve:

$y = f(x) \Leftrightarrow y = 2 - \sqrt{1 - x^2} \Leftrightarrow \sqrt{1 - x^2} = 2 - y \quad (1)$

▸ Require $2 - y \geq 0$

$$(1) \Leftrightarrow 1 - x^2 = (2-y)^2 \Leftrightarrow x^2 = 1-(2-y)^2. \quad (2)$$

Eq. (2) has a solution $\Leftrightarrow$
$1-(2-y)^2 \geq 0 \Leftrightarrow \cdots \Leftrightarrow \underline{y \in [1,3]}$

Trick $\left\{ \begin{array}{l} \text{Since } 1-x^2 = (2-y)^2 \geq 0, \text{ this solution is} \\ \text{in the domain } A \text{ so there are no exclusions!!} \end{array} \right.$

Thus:

$$y \in f(A) \Leftrightarrow \begin{cases} 2-y \geq 0 \\ y \in [1,3] \end{cases} \Leftrightarrow \begin{cases} y \in (-\infty, 2] \\ y \in [1,3] \end{cases}$$

so

$$f(A) = (-\infty, 2] \cap [1,3] = [1,2].$$

c) For $f(x) = \sqrt{x^2-9} - 2$

Domain: Require $x^2 - 9 \geq 0 \Leftrightarrow \cdots \Leftrightarrow x \in (-\infty, -3] \cup [3, +\infty)$
thus $A = \mathrm{dom}(f) = (-\infty, -3] \cup [3, +\infty)$

Solve

$$y = f(x) \Leftrightarrow y = \sqrt{x^2-9} - 2 \Leftrightarrow y+2 = \sqrt{x^2-9} \quad (1)$$

Require $y+2 \geq 0 \Leftrightarrow \underline{y \in [-2, +\infty)}$

$(1) \Leftrightarrow (y+2)^2 = x^2 - 9 \Leftrightarrow x^2 = 9 + (y+2)^2 \quad (2)$

Eq. (2) has a solution $\Leftrightarrow \underline{9 + (y+2)^2 \geq 0}$

Since $x^2 - 9 = (y+2)^2 \geq 0$, the solutions of (2) will belong to A so there are no exclusions.

Thus

$$y \in f(A) \Leftrightarrow \begin{cases} y \in [-2, +\infty) \\ 9 + (y+2)^2 \geq 0 \leftarrow \text{identity} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow y \in [-2, +\infty)$$

thus

$$f(A) = [-2, +\infty).$$

EXERCISES

(27) Find the range and domain of the following functions

a) $f(x) = 3x - 1$

b) $f(x) = 1 - (2x+3)^2$

c) $f(x) = x^2 + 5x + 6$

d) $f(x) = x^2 - 10x + 9$

e) $f(x) = \dfrac{2x+5}{x-3}$

f) $f(x) = 2 - \sqrt{3x+2}$

g) $f(x) = 2 + \sqrt{1-2x}$

h) $f(x) = 1 - 2\sqrt{4-x^2}$

i) $f(x) = \sqrt{(x+1)^2 - 9}$

j) $f(x) = \sqrt{x^2 + 3x + 2}$

(28) Same with the following functions

a) $f(x) = \dfrac{x^2 + x - 2}{x^2 + 1}$

b) $f(x) = \dfrac{x^2 - 1}{2x + 1}$

c) $f(x) = \dfrac{(x+1)^2}{x^2 + 3x + 2}$

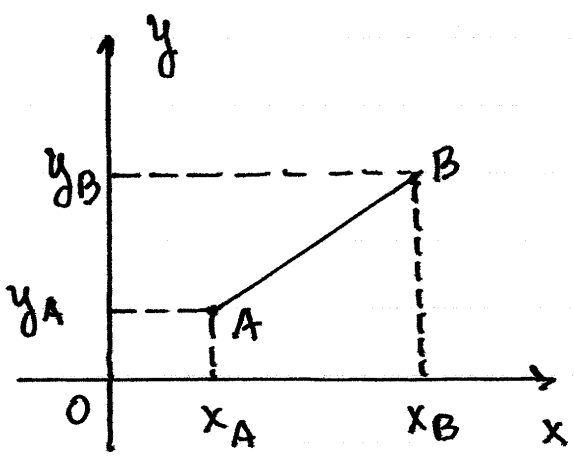
d) $f(x) = \sqrt{\dfrac{x-1}{x+2}}$

e) $f(x) = \dfrac{\sqrt{x}}{\sqrt{x} - 1}$

CA5: Graphing functions

GRAPHING FUNCTIONS

Coordinate system



Let A, B be two points on the plane with coordinates
$A(x_A, y_A)$
$B(x_B, y_B)$

- The slope $m(AB)$ is defined as

$$\boxed{m(AB) = \frac{y_B - y_A}{x_B - x_A}}$$

when $x_A \neq x_B$.

- The distance (AB) between A and B is given by

$$\boxed{(AB) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}}$$

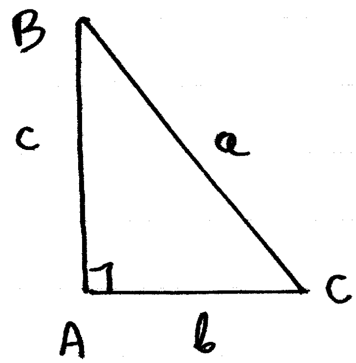
(distance formula)

Proof of distance formula

The distance formula is derived from the Pythagorean theorem:

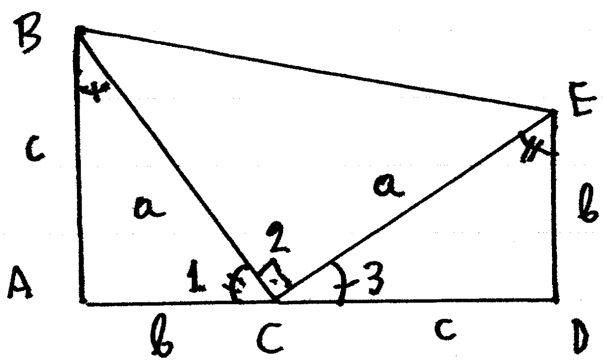
- Let $\overset{\triangle}{ABC}$ be a triangle. Then

$$\boxed{\hat{A} = 90^\circ \Rightarrow a^2 = b^2 + c^2}$$



with $a = (BC)$
$b = (CA)$
$c = (AB)$

Proof (by President Garfield)



Extend AC to D such that $CD = AB$
Let $DE \perp AD$ with $DE = AC$.
Draw CE, BE

Let $C_1 = A\hat{C}B$, $C_2 = B\hat{C}E$, $C_3 = E\hat{C}D$.

Note that

$AB = CD$ and $AC = ED$ and $\hat{A} = \hat{D} \Rightarrow \overset{\triangle}{ABC} = \overset{\triangle}{DCE} \Rightarrow$

$\Rightarrow CE = BC$ and $\hat{C}_3 = \hat{B}$

The area of the trapezoid

$$(ABED) = \frac{1}{2}(AB + DE)AD = \frac{1}{2}(c+b)(b+c) =$$
$$= \frac{(b+c)^2}{2}$$

The area of the three triangles:

$$(ABC) = (CDE) = \frac{1}{2} AB \cdot AC = \frac{bc}{2}.$$

Note that

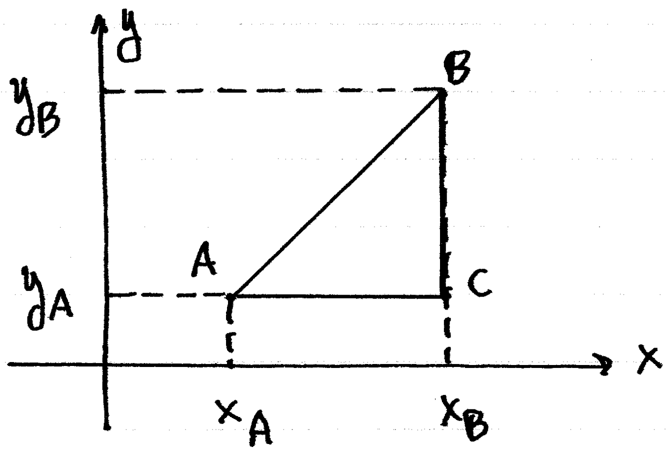
$$\hat{C}_2 = 180 - \hat{C}_1 - \hat{C}_3 = 180 - \hat{C}_1 - \hat{B} = \hat{A} = 90^\circ \Rightarrow$$

$$\Rightarrow (BCE) = \frac{1}{2} BC \cdot CE = \frac{a^2}{2}.$$

Since

$$(ABED) = (ABC) + (CDE) + (BCE) \Rightarrow$$
$$\Rightarrow \frac{(b+c)^2}{2} = \frac{bc}{2} + \frac{bc}{2} + \frac{a^2}{2} \Rightarrow$$
$$\Rightarrow a^2 = (b+c)^2 - bc - bc =$$
$$= b^2 + 2bc + c^2 - 2bc$$
$$= b^2 + c^2 \quad \square$$

- To show the distance formula note that:



$AC = x_B - x_A$ and $BC = y_B - y_A$.
Since
$\hat{C} = 90° \Rightarrow AB^2 = AC^2 + BC^2 =$
$= (x_B - x_A)^2 + (y_B - y_A)^2 \Rightarrow$

$\Rightarrow AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$ $\square$

- The <u>midpoint M</u> between two points A and B is defineds as the unique point such that
  a) $AM = MB$
  b) $AM + MB = AB$

It can be shown that for $A(x_A, y_A)$ and $B(x_B, y_B)$, the midpoint M has coordinates

$$\boxed{x_M = \frac{x_A + x_B}{2} \quad \text{and} \quad y_M = \frac{y_A + y_B}{2}}$$

EXAMPLES

a) Find the slope, distance, and midpoint between $A(2,-3)$ and $B(-1,-5)$.

Solution

Slope: $$m(AB) = \frac{y_B - y_A}{x_B - x_A} = \frac{(-5)-(-3)}{(-1)-2} = \frac{-5+3}{-1-2} =$$

$$= \frac{-2}{-3} = \frac{2}{3}$$

Distance: $$AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2} =$$

$$= \sqrt{[2-(-1)]^2 + [(-3)-(-5)]^2} =$$

$$= \sqrt{(2+1)^2 + (-3+5)^2} =$$

$$= \sqrt{3^2 + 2^2} = \sqrt{9+4} = \sqrt{13}.$$

Midpoint: $$\left.\begin{array}{l} x_M = \dfrac{x_A + x_B}{2} = \dfrac{2+(-1)}{2} = \dfrac{1}{2} \\[2ex] y_M = \dfrac{y_A + y_B}{2} = \dfrac{(-3)+(-5)}{2} = \dfrac{-8}{2} = -4 \end{array}\right\} \Rightarrow$$

$$\Rightarrow M(1/2, -4).$$

b) If $M(2a+1, a+2)$ is the midpoint between $A(a-1, 3a-1)$ and B, find all $a \in \mathbb{R}$ such that $AB = 8$.

Solution

First, we calculate AM:

$$AM^2 = (x_A - x_M)^2 + (y_A - y_M)^2 =$$
$$= [(a-1) - (2a+1)]^2 + [(3a-1) - (a+2)]^2$$
$$= (a-1-2a-1)^2 + (3a-1-a-2)^2 =$$
$$= (-a-2)^2 + (2a-3)^2 =$$
$$= (a^2 + 4a + 4 + 4a^2 - 12a + 9 =$$
$$= 5a^2 - 8a + 13 \qquad (1)$$

From (1):

$$AB = 8 \Leftrightarrow AM = 4 \Leftrightarrow AM^2 = 16 \Leftrightarrow 5a^2 - 8a + 13 = 16 \Leftrightarrow$$
$$\Leftrightarrow 5a^2 - 8a - 3 = 0$$

$$\Delta = b^2 - 4ac = (-8)^2 - 4 \cdot 5 \cdot (-3) = 64 + 60 = 124 = 2^2 \cdot 31$$

$$\Rightarrow a_{1,2} = \frac{-(-8) \pm 2\sqrt{31}}{2 \cdot 5} = \frac{8 \pm 2\sqrt{31}}{2 \cdot 5} =$$
$$= \frac{4 \pm \sqrt{31}}{5}.$$

EXERCISES

① Find the slope, distance, and midpoint between the following points:

a) $A(2,1)$, $B(4,3)$
b) $A(0,0)$, $B(2,5)$
c) $A(-1,-2)$, $B(3,-4)$
d) $A(3,-4)$, $B(-1,-2)$
e) $A(-2,-2)$, $B(-1,1)$

② Let $A(a,a+1)$, $B(2a-1,a-1)$.
Find all values of a such that
a) $AB = 1$
b) The slope $m(AB) = -2$

③ If $M(1,3)$ is the midpoint between $A(-1,1)$ and B, find the coordinates of B.

④ If $M(3a-1,a+1)$ is the midpoint between $A(a,a-1)$ and B, find all $a \in \mathbb{R}$ such that the distance $AB = 2$.

Curves represented by equations.

- The curve $(C): f(x,y) = g(x,y)$ consists of all the points of the plane that satisfy the equation
$$f(x,y) = g(x,y).$$
It follows that

$$\boxed{(x,y) \in (C) \iff f(x,y) = g(x,y)}$$

- We now consider 3 curves:
1) The line 2) Parabola 3) circle.

The line

- Every line (ℓ) can be represented as
$$(\ell): Ax + By + C = 0$$
with $|A| + |B| > 0$.
For $B \neq 0$, the slope m of the line (ℓ) is given by

$$\boxed{m = \frac{-A}{B}}$$

such that for any points $P_1, P_2 \in (\ell)$
$m(P_1P_2) = m = -A/B$.

- The equation of the line can be found as follows:

1) Given a point and slope:

$$\boxed{\left.\begin{array}{l} M(x_0, y_0) \in (\ell) \\ m = \text{slope of } (\ell) \end{array}\right\} \Rightarrow (\ell): y - y_0 = m(x - x_0)}$$

2) Given two points:

$$\boxed{\left.\begin{array}{l} A(x_1, y_1) \in (\ell) \\ B(x_2, y_2) \in (\ell) \end{array}\right\} \Rightarrow (\ell): \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}}$$

3) Vertical line:

$$\boxed{\left.\begin{array}{l} M(x_0, y_0) \in (\ell) \\ (\ell) \parallel y'y \end{array}\right\} \Rightarrow (\ell): x = x_0}$$

- <u>Distance of point from a line</u>

The distance of the point $A(x_0, y_0)$ from the line $(\ell): Ax + By + C = 0$ is given by

$$\boxed{d(A,(\ell)) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2+B^2}}}$$

▶ <u>Relative position between two lines</u>

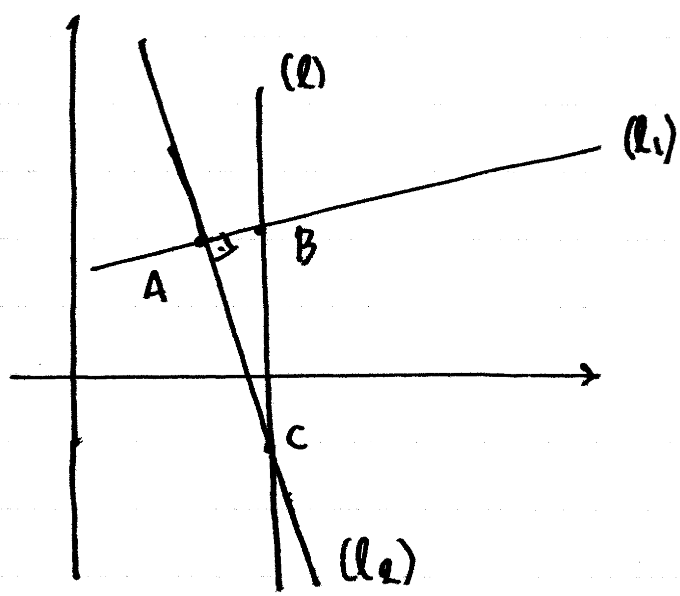
- Consider two lines

$$(\ell_1): \ y = m_1 x + b_1$$
$$(\ell_2): \ y = m_2 x + b_2$$

It can be shown that

$$\boxed{\begin{array}{l} (\ell_1) // \ell_2 \Leftrightarrow m_1 = m_2 \\ (\ell_1) \perp (\ell_2) \Leftrightarrow m_1 m_2 = -1 \end{array}} \quad \begin{array}{l} \text{(parallel)} \\ \text{(perpendicular)} \end{array}$$

- Proof of perpendicular condition:



Let $A(x_0, y_0) = (\ell_1) \cap (\ell_2)$.
Introduce the line
$(\ell): \ x = x_0 + 1$
and let
$B = (\ell) \cap (\ell_1)$
$C = (\ell) \cap (\ell_2)$
with $B(x_0+1, y_1)$
and $C(x_0+1, y_2)$

To find $y_1 - y_0$; and $y_2 - y_0$:

$$m(AB) = m_1 \Leftrightarrow \frac{y_1 - y_0}{(x_0+1) - x_0} = m_1 \Leftrightarrow y_1 - y_0 = m_1$$

$$m(AC) = m_2 \Leftrightarrow \frac{y_2 - y_0}{(x_0+1) - x_0} = m_2 \Leftrightarrow y_2 - y_0 = m_2$$

thus:

$$AB = \sqrt{((x_0+1) - x_0)^2 + (y_1 - y_0)^2} = \sqrt{1 + m_1^2}$$

$$AC = \sqrt{((x_0+1) - x_0)^2 + (y_2 - y_0)^2} = \sqrt{1 + m_2^2}$$

$$BC = \cdots = y_2 - y_1 = m_2 - m_1$$

It follows that

$$\underline{(l_1) \perp (l_2)} \Leftrightarrow \hat{A} = 90^\circ \Leftrightarrow BC^2 = AB^2 + AC^2 \Leftrightarrow$$

$$\Leftrightarrow (m_2 - m_1)^2 = \left(\sqrt{1 + m_1^2}\right)^2 + \left(\sqrt{1 + m_2^2}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow m_2^2 - 2m_1 m_2 + m_1^2 = (1 + m_1^2) + (1 + m_2^2) \Leftrightarrow$$

$$\Leftrightarrow -2m_1 m_2 = 2 \Leftrightarrow \underline{m_1 m_2 = -1}$$

EXERCISES

5. Find the equation of the line
$(\ell): Ax + By + C = 0$
such that

a) (ℓ) goes through $A(1,3)$, $B(2,5)$
b) (ℓ) goes through $A(2,3)$, $B(2,4)$
c) (ℓ) goes through $A(1,4)$, $B(4,1)$
d) (ℓ) goes through $A(-2,3)$, $B(-3,-2)$
e) (ℓ) goes through $A(2,3)$ with slope -1
f) (ℓ) goes through $A(-1,2)$ with slope 4
g) $(\ell) \parallel (\ell_1): 3x + 2y + 4 = 0$ and goes through $A(-1,5)$
h) $(\ell) \parallel (\ell_2): x - 3y + 1 = 0$ and goes through $A(3,-2)$
i) $(\ell) \perp (\ell_1): 2x + 5y + 4 = 0$ and goes through $A(-2,3)$
j) $(\ell) \perp (\ell_1): 3x - y + 2 = 0$ and goes through $A(1,-1)$

6. Find $a \in \mathbb{R}$ such that $(\ell_1): ax + 5y + 7 = 0$ and $(\ell_2): 2(a-1)x - 3y + 9 = 0$ are parallel.

(7) Find $a \in \mathbb{R}$ such that the lines
$(l_1): (a+3)x + y - 7 = 0$ and $(l_2): (1-a)x - 4y + 12 = 0$
are perpendicular.

(8) Find $a \in \mathbb{R}$ such that $(l_1): (a-1)x - y + (a-2) = 0$
and $(l_2): (3a-7)x - y - 2a + 5 = 0$ are
parallel. Then find a line perpendicular
to (l_1) and (l_2) going through the point
$A(-1,1)$

(9) Find $a \in \mathbb{R}$ such that the line

$$(l): (a-2)x - (a-1)y + (3a-2)(a-1) = 0$$

is:

a) Parallel to the x-axis

b) Parallel with $(q): 4x - y + 3 = 0$

c) Perpendicular to $(l_1): 2x + y - 5 = 0$

EXAMPLES

a) Find the equation of the line going through $A(1,2)$ and $B(-2,-3)$.

Solution

$$(AB): \frac{y-y_A}{x-x_A} = \frac{y_B-y_A}{x_B-x_A} \Leftrightarrow \frac{y-2}{x-1} = \frac{(-3)-2}{(-2)-1} \Leftrightarrow$$

$$\Leftrightarrow \frac{y-2}{x-1} = \frac{-5}{-3} \Leftrightarrow \frac{y-2}{x-1} = \frac{5}{3} \Leftrightarrow$$

$$\Leftrightarrow 3(y-2) = 5(x-1) \Leftrightarrow 3y-6 = 5x-5 \Leftrightarrow 5x-3y+6-5=0$$

$$\Leftrightarrow 5x-3y+1=0$$

thus $(AB): 5x-3y+1=0$.

b) Find the equation of the line through $A(3,-4)$ with slope -2.

Solution

$$(\ell): y-(-4) = (-2)(x-3) \Leftrightarrow y+4 = -2(x-3) \Leftrightarrow$$

$$\Leftrightarrow y+4 = -2x+6 \Leftrightarrow 2x+y+4-6=0 \Leftrightarrow 2x+y-2=0.$$

thus:

$(\ell): 2x+y-2=0$.

c) Find all values $a \in \mathbb{R}$ such that the lines

$(l_1): 3x + (a+1)y + 2a = 0$

$(l_2): ax - (2a-1)y + 1 = 0$

are perpendicular.

Solution

Let $m_1 =$ slope of (l_1)

$m_2 =$ slope of (l_2).

Then $m_1 = \dfrac{-3}{a+1}$ and $m_2 = \dfrac{a}{2a-1}$.

It follows that

$$(l_1) \perp (l_2) \Leftrightarrow m_1 m_2 = -1 \Leftrightarrow \frac{-3}{a+1} \frac{a}{2a-1} = -1$$

$$\Leftrightarrow \frac{-3a}{(a+1)(2a-1)} = -1 \Leftrightarrow \frac{3a}{(a+1)(2a-1)} = 1 \quad (1)$$

• Domain of equation: $A = \mathbb{R} - \{-1, 1/2\}$.

$$(1) \Leftrightarrow 3a = (a+1)(2a-1) \Leftrightarrow 3a = 2a^2 - a + 2a - 1 \Leftrightarrow$$

$$\Leftrightarrow 2a^2 + (-1+2-3)a - 1 = 0 \Leftrightarrow 2a^2 - 2a - 1 = 0.$$

$$\Delta = (-2)^2 - 4 \cdot 2 \cdot (-1) = 4 + 8 = 12 = 4 \cdot 3 \Rightarrow$$

$$\Rightarrow a_{1,2} = \frac{2 \pm 2\sqrt{3}}{2 \cdot 2} = \frac{1 \pm \sqrt{3}}{2} \leftarrow \text{both accepted (they belong to } A\text{)}.$$

Thus:

$$(l_1) \perp (l_2) \Leftrightarrow a = \frac{1+\sqrt{3}}{2} \vee a = \frac{1-\sqrt{3}}{2}.$$

The circle

- A circle (c) with center A and radius r is the set of all points M on the plane such that AM = r. Thus

$$(c) = \{M \mid AM = r\}$$

This circle can be represented as

$$\boxed{(c): (x-x_A)^2 + (y-y_A)^2 = r^2}$$

with $A(x_A, y_A)$.

- The curve $(c): x^2+y^2+Ax+By = C$ is a circle if and only if

$$\boxed{C+(A/2)^2+(B/2)^2 > 0}$$

Proof

$$(c): x^2+y^2+Ax+By = C \Leftrightarrow$$

$$\Leftrightarrow x^2+Ax+\left(\frac{A}{2}\right)^2 + y^2+By+\left(\frac{B}{2}\right)^2 = C+(A/2)^2+(B/2)^2$$

$$\Leftrightarrow \left(x+\frac{A}{2}\right)^2+\left(y+\frac{B}{2}\right)^2 = C+(A/2)^2+(B/2)^2$$

We see that for (c) to be a circle, the RHS has to be positive.

↳ We conclude from the argument above that if (c) is a circle then it has center $O(-A/2, -B/2)$ and radius r given by

$$\boxed{r = \sqrt{C + (A/2)^2 + (B/2)^2}}$$

EXERCISES

(10) Write the equation $(c): x^2 + y^2 + Ax + By = C$ for the circle with center O and radius r such that

a) $O(1,-1)$, $r = 2$

b) $O(3,-2)$, $r = 3$

c) $O(-1,-4)$, $r = 1/2$

d) $O(1/2, 1/3)$, $r = 1/3$

e) $O(a, a-1)$, $r = a+2$ with $a+2 > 0$.

(11) Find the center and radius for the following circles, if they are indeed circles:

a) $(c): x^2 + y^2 + 3x + 2y = 5$

b) $(c): x^2 + y^2 - 2x + 4y = 0$

c) $(c): x^2+y^2+3x+y=1$
d) $(c): x^2+y^2+6x+4y=-2$
e) $(c): x^2+y^2+x+y+10=0$
f) $(c): x^2+y^2-x+2y+9=0$

(12) For what values $a \in \mathbb{R}$ are the following curves circles?

a) $x^2+y^2+(a+1)x+(a-1)y=a$
b) $x^2+y^2+(2a-1)x+(3a+2)y=a+1$
c) $x^2+y^2+(a+2)x+(a-1)y=1-a^2$

(13) Find $a \in \mathbb{R}$ such that the circle
$$(c): x^2+y^2+(2a+1)x+(2a-1)y=a+3$$
passes through the point $A(1,2)$.
What is the radius and center of the circle?

(14) Find $a \in \mathbb{R}$ such that the radius r of the circle
$$(c): x^2+y^2+(a-1)x+(a+1)y=2a-1$$
satisfies $1 \leq r < 2$.

▼ Graphing linear and quadratic functions.

- Let $f: A \to \mathbb{R}$ be a function. The graph of the function f is a curve (c) given by

$$\boxed{(c): \begin{cases} y = f(x) \\ x \in A \end{cases}}$$

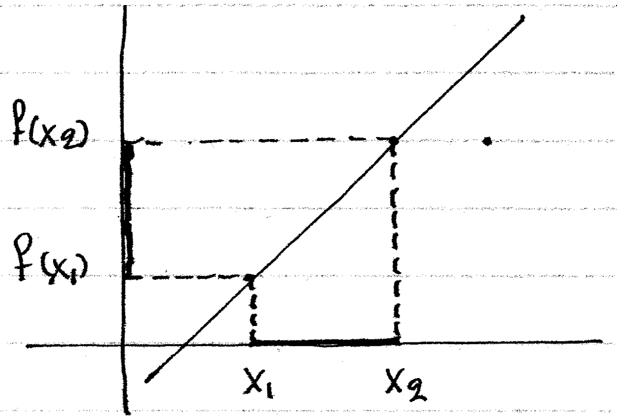
The linear function $\boxed{f(x) = ax+b}$

- Domain : $A = \mathbb{R}$
- Range : $f(A) = \mathbb{R}$
- Monotonicity : $f \uparrow \mathbb{R} \Leftrightarrow a > 0$
  $f \downarrow \mathbb{R} \Leftrightarrow a < 0$
- Graph : The graph of $f(x) = ax+b$ is a line with slope a.

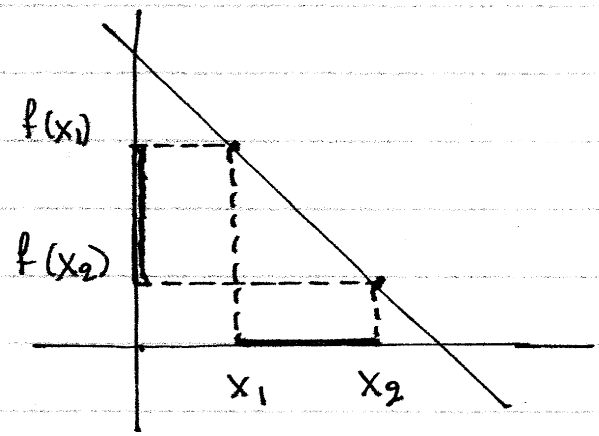
▸ To draw the graph of f it is sufficient to find two points on the graph. The line is then uniquely defined from these two points.

▸ Range on restricted intervals:

$$\boxed{\begin{array}{l} a > 0 \Rightarrow f([x_1, x_2]) = [f(x_1), f(x_2)] \\ a < 0 \Rightarrow f([x_1, x_2]) = [f(x_2), f(x_1)] \end{array}}$$



$a>0$



$a<0$

- Range on unrestricted intervals

$$a>0 \Rightarrow \begin{cases} f([x_1,+\infty)) = [f(x_1),+\infty) \\ f((-\infty,x_2]) = (-\infty, f(x_2)] \end{cases}$$

$$a<0 \Rightarrow \begin{cases} f([x_1,+\infty)) = (-\infty, f(x_1)] \\ f((-\infty,x_2]) = [f(x_2),+\infty) \end{cases}$$

Linear functions with absolute values.

Before working with a linear functions that has absolute values we rewrite it in the form:

$$f(x)=\begin{cases} a_1x+b_1 & , x\in A_1 \\ a_2x+b_2 & , x\in A_2 \\ \vdots & \quad \vdots \\ a_nx+b_n & , x\in A_n \end{cases}$$

with $A_1, A_2, \dots, A_n$ intervals.

- Domain: $A = A_1 \cup A_2 \cup \cdots \cup A_n = \mathbb{R}$
- Range: $f(A) = f(A_1) \cup f(A_2) \cup \cdots \cup f(A_n)$
- Monotonicity: Depends on the intervals $A_1, A_2, \dots, A_n$
- Graph: A sequance of line segments with different slopes per segment.

EXAMPLES

a) For $f(x) = |x+1| + |x-2| - 1 \leftarrow \begin{cases} \text{Domain} \\ \text{Range} \\ \text{Monotonicity} \\ \text{Graph.} \end{cases}$

x		-1		2	
$x+1$	$-$	0	$+$		$+$
$x-2$	$-$		$-$	0	$+$

$\forall x \in (-\infty, -1]$: $f(x) = -(x+1) - (x-2) - 1 =$
$= -x - 1 - x + 2 - 1 = -2x$

$\forall x \in [-1, 2]$: $f(x) = (x+1) - (x-2) - 1 =$
$= x + 1 - x + 2 - 1 = 2$

$\forall x \in [2, +\infty)$: $f(x) = (x+1) + (x-2) - 1 = 2x - 2$

therefore

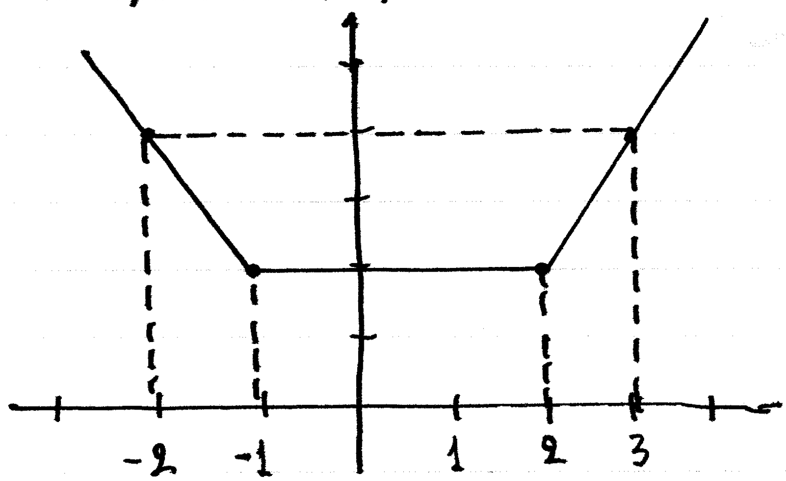
$$f(x) = \begin{cases} -2x & , \quad x \in (-\infty, -1] \\ 2 & , \quad x \in [-1, 2] \\ 2x - 2 & , \quad x \in [2, +\infty) \end{cases}$$

Domain : $A = \mathbb{R}$

Range : $\left.\begin{aligned} f((-\infty, -1]) &= [2, +\infty) \\ f([-1, 2]) &= \{2\} \\ f([2, +\infty)) &= [2, +\infty) \end{aligned}\right\} \Rightarrow$

$\Rightarrow f(A) = [2, +\infty) \cup \{2\} \cup [2, +\infty)$
$= [2, +\infty)$

- Monotonicity: $f \searrow (-\infty,-1]$
  f const. $[-1,2]$
  $f \nearrow [2,+\infty)$
- Graph.
  Use $f(-2) = -2\cdot(-2) = 4$
  $f(-1) = 2$
  $f(2) = 2$
  $f(3) = 2\cdot 3 - 2 = 4$



8) For what values of $a \in \mathbb{R}$ is the function $f(x) = 2x(a-1) + a^2(x-1)$ increasing in $\mathbb{R}$?

Solution

We note that

$$f(x) = 2x(a-1) + a^2(x-1) =$$
$$= 2ax - 2x + a^2x - a^2 =$$
$$= (a^2+2a-2)x - a^2, \quad \forall x \in \mathbb{R}.$$

It follows that

$$f \nearrow \mathbb{R} \Leftrightarrow a^2+2a-2 > 0 \quad (1)$$

$$\Delta = 2^2 - 4 \cdot 1 \cdot (-2) = 4 + 8 = 12 = 4 \cdot 3 \Rightarrow$$
$$\Rightarrow a_{1,2} = \frac{-2 \pm 2\sqrt{3}}{2 \cdot 1} = -1 \pm \sqrt{3}$$

Thus

$$f \nearrow \mathbb{R} \Leftrightarrow a = -1+\sqrt{3} \vee a = -1-\sqrt{3}.$$

EXERCISES

(15) For the following functions, write the domain, range, monotonicity, and draw the corresponding graph:

a) $f(x) = 3x - 2$

b) $f(x) = -2x + 4$

c) $f(x) = |2x - 1|$

d) $f(x) = |2 - x| + x$

e) $f(x) = \dfrac{|x|}{x}$

f) $f(x) = |x-3| + |x+2| - 5$

g) $f(x) = |2x+1| + |x-2|$

h) $f(x) = |1-x| + |x-1| + 1$

i) $f(x) = |x| + |x+1| + |x+2|$

j) $f(x) = |x-1| + |2x| + |2x+1|$

(16) For what values $a \in \mathbb{R}$ are the following functions increasing in $\mathbb{R}$?

a) $f(x) = (a+1)x + (a-2)$

b) $f(x) = (a^2+3a+2)x + 2a$

c) $f(x) = a(x+a) + x(a-1)^2$

d) $f(x) = a(a+1)(a+x)$

e) $f(x) = 2x(a^2-1) + a(x-1)$

g) $f(x) = 2(x+a)(a-1) + (x-a)(a+1)^2$

Quadratic function $f(x)=ax^2+bx+c$

- The graph of the quadratic function $f(x)=ax^2+bx+c$ is a curve called parabola which has the following properties:

a) Its vertex is the point $A(-b/2a, -\Delta/4a)$ with $\Delta=b^2-4ac$ the discriminant of the quadratic ax^2+bx+c.

   i) For $a>0 \Rightarrow$ the vertex A is a minimum

   ii) For $a<0 \Rightarrow$ the vertex A is the maximum.

b) It has axis of symmetry the line $(\ell): x=-b/2a$

c) Intersects the y-axis at $C(0,c)$

d) Intersects the x-axis at

   i) $A_1(x_1,0)$ and $A_2(x_2,0)$ with x_1, x_2 the zeroes of the quadratic, when $\Delta>0$.

   ii) Tangent with the x-axis at the vertex, when $\Delta=0$.

   iii) No intersection when $\Delta<0$.

e) For $a>0$: the parabola opens up

   $a<0$: the parabola opens down.

- To justify the above claims, we rewrite the quadratic in the completed square form:

$$\boxed{f(x)=ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a}}$$

Proof

$$f(x)=ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=$$

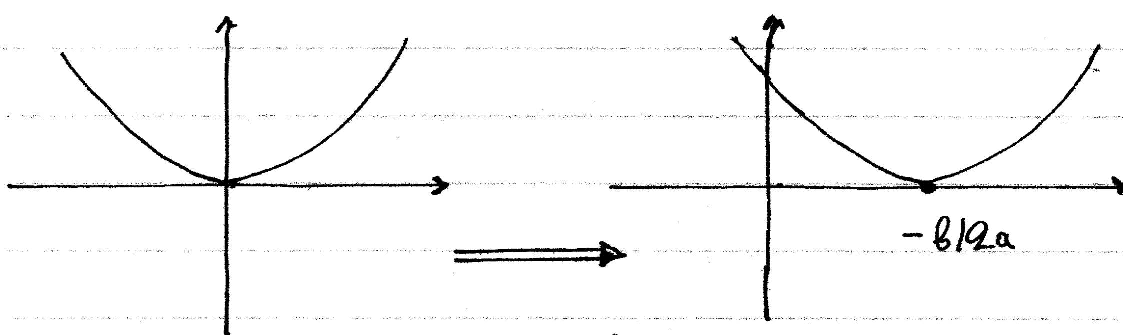
$$=a\left[x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\left(\frac{b}{2a}\right)^2\right]=$$

$$=a\left[\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right]=$$

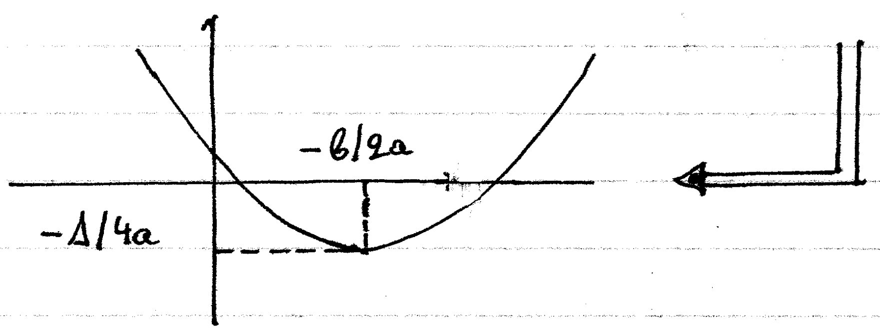
$$=a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]=$$

$$=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}\right]=$$

$$=a\left(x+b/2a\right)^2-\Delta/4a. \qquad \square$$



$f(x)=ax^2$ $\qquad\qquad$ $f(x)=a(x+b/2a)^2$



$f(x)=ax^2+bx+c=a(x+b/2a)^2-\Delta/4a$

- To plot the graph of a quadratic function, we first determine the coordinates of the vertex, and the points where the graph intersects the x-axis (if they exist) and the y-axis. If these are not sufficient, we find additional points by evaluating the function.

EXAMPLE

Graph the function $f(x) = x^2 + 5x + 6$.

Solution

Discriminant $\Delta = b^2 - 4ac = 5^2 - 4\cdot 1\cdot 6 = 25 - 24 = 1$

$$\left.\begin{array}{l} \dfrac{-b}{2a} = \dfrac{-5}{2\cdot 1} = \dfrac{-5}{2} \\ \dfrac{-\Delta}{4a} = \dfrac{-1}{2\cdot 1} = \dfrac{-1}{2} \end{array}\right\} \Rightarrow \text{Vertex } A(-5/2, -1/2)$$

x-axis intercepts:

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-5 \pm \sqrt{1}}{2\cdot 1} = \begin{cases} -6/2 = -3 \\ -4/2 = -2 \end{cases}$$

thus $(-3,0)$, $(-2,0)$.

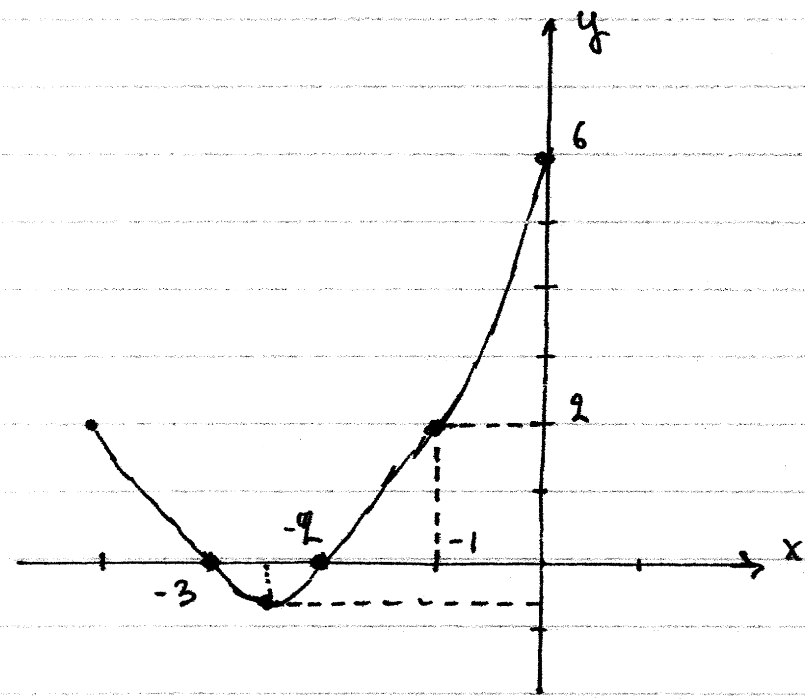
y-axis intercept: $(0,6)$

Also need at $x=-1$: $f(-1) = (-1)^2 + 5\cdot(-1) + 6 =$

$= 1 - 5 + 6 = 2$

Thus, to summarize:

x	-3	$-5/2$	-2	-1	0
y	0	$-1/2$	0	2	6


y
6
2
-2
-1
-3
x

▶ Range of a quadratic function

It can be shown, directly by definition, and also seen via the graph, that for the quadratic function $f(x) = ax^2 + bx + c$, the range $f(\mathbb{R})$ is given by:

$$\boxed{\begin{aligned} a>0 &\Rightarrow f(\mathbb{R}) = [-\Delta/4a, +\infty) \\ a<0 &\Rightarrow f(\mathbb{R}) = (-\infty, -\Delta/4a] \end{aligned}}$$

EXAMPLE

Find the range of $f(x) = 3x^2 - 2x + 5$

Solution

$$\Delta = b^2 - 4ac = (-2)^2 - 4\cdot 3\cdot 5 = 4 - 60 = -56 \Rightarrow$$

$$\Rightarrow \frac{\Delta}{4a} = \frac{-56}{4\cdot 3} = \frac{-56}{12} = \frac{-2^3\cdot 7}{2^2\cdot 3} = \frac{-2\cdot 7}{3} = \frac{-14}{3}$$

$$\Rightarrow f(\mathbb{R}) = [-\Delta/4a, +\infty) = [14/3, +\infty).$$

EXERCISES

(17) For the following functions, write the domain, range, axis of symmetry, vertex, intercepts, and the corresponding completed square form. Then graph the function.

a) $f(x) = 2x^2$

b) $f(x) = -3x^2$

c) $f(x) = x^2/2$

d) $f(x) = -x^2/3$

e) $f(x) = 3x^2+1$

f) $f(x) = -2x^2+3$

g) $f(x) = x^2-2$

h) $f(x) = x^2-2x$

i) $f(x) = -2x^2+3x$

j) $f(x) = 2x^2+4x+2$

k) $f(x) = x^2-4x+3$

l) $f(x) = -x^2+x+2$

m) $f(x) = -2x^2-6x-4$

n) $f(x) = x^2+x+1$

o) $f(x) = x^2+3x+4$

p) $f(x) = -x^2+2x-3$

CA6: Polynomial functions

POLYNOMIAL FUNCTIONS

Basic concepts – Definitions

- A <u>polynomial</u> f is a function $f: \mathbb{R} \to \mathbb{R}$ such that

$$\boxed{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \,, \forall x \in \mathbb{R}}$$

with $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$.

- The numbers $a_0, a_1, \ldots, a_n$ are called the <u>coefficients</u> of f.
- The expressions $a_n x^n, a_{n-1} x^{n-1}, \ldots, a_1 x, a_0$ are called the <u>terms</u> of f.
- n = the <u>degree</u> of f and we write:
  $$\deg(f) = n.$$
- The set of all polynomials is denoted as
  $\mathbb{R}[x]$ = all polynomials with <u>real</u> coefficients
  $\mathbb{Q}[x]$ = all polynomials with <u>rational</u> coefficients
  $\mathbb{Z}[x]$ = all polynomials with <u>integer</u> coefficients.
  Thus, it follows that if
  $$[n] = \{1, 2, 3, \ldots, n\}$$
  then

$$\boxed{\begin{aligned} f\in\mathbb{R}[x] &\iff \forall k\in[n] : a_k\in\mathbb{R} \\ f\in\mathbb{Q}[x] &\iff \forall k\in[n] : a_k\in\mathbb{Q} \\ f\in\mathbb{Z}[x] &\iff \forall k\in[n] : a_k\in\mathbb{Z}. \end{aligned}}$$

↳ Properties of degrees

Let $f, g \in \mathbb{R}[x]$. Then

$$\boxed{\begin{aligned} \deg(fg) &= \deg(f) + \deg(g) \\ \deg(f+g) &= \max\{\deg(f), \deg(g)\} \end{aligned}}$$

↳ Polynomial Equality.

- In general, two polynomials $f, g \in \mathbb{R}[x]$ are equal if and only if they have the same degree and same coefficients. In other words, if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
$$g(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

then

$$\boxed{f = g \iff \forall k\in[n] : a_k = b_k}$$

▼ Polynomial Division

- Let $f, g \in \mathbb{R}[x]$ be two polynomials with $\deg(f) \geq \deg(g)$. Then, there exist two unique polynomials $q, r \in \mathbb{R}[x]$ such that

$$\boxed{\forall x \in \mathbb{R} : f(x) = g(x)q(x) + r(x)}$$

▶ Terminology - Remarks

(1) $q(x)$ = quotient of $f(x) \div g(x)$
$r(x)$ = remainder of $f(x) \div g(x)$

(2) Note that

$$\boxed{\begin{array}{l} \deg(q) = \deg(f) - \deg(g) \\ \deg(r) < \deg(g) \end{array}}$$

(3) We say that <u>g divides f</u> if and only if the corresponding remainder is 0:

$$\boxed{g \mid f \Leftrightarrow \exists q \in \mathbb{R}[x] : \forall x \in \mathbb{R} : f(x) = g(x)q(x)}$$

EXAMPLE

Perform the division

$$(2x^4-3x^3+5x^2-7x+1)\div(x^2-2x+3).$$

For $f(x)=2x^4-3x^3+5x^2-7x+1$

$g(x)=x^2-2x+3$

we have

$\deg(q)=\deg(f)-\deg(g)=4-2=2 \Rightarrow$

$\Rightarrow$ let $q(x)=ax^2+bx+c$.

$\deg(r)<\deg(g)=2 \Rightarrow \deg(r)\leq 1 \Rightarrow$

$\Rightarrow$ let $r(x)=dx+e$.

Then

$$\begin{aligned}
f(x) &= g(x)q(x)+r(x)\\
&= (x^2-2x+3)(ax^2+bx+c)+(dx+e)=\cdots=\\
&= ax^4+(b-2a)x^3+(c-2b+3a)x^2+\\
&\quad +(-2c+3b+d)x+(3c+e)\\
&= 2x^4-3x^3+5x^2-7x+1\,,\quad \forall x\in\mathbb{R} \Leftrightarrow
\end{aligned}$$

$$\begin{cases} a=2\\ b-2a=-3\\ c-2b+3a=5\\ -2c+3b+d=-7\\ 3c+e=1 \end{cases} \Leftrightarrow \cdots \Leftrightarrow \begin{cases} a=2\\ b=1\\ c=1\\ d=-8\\ e=-2 \end{cases} \Leftrightarrow$$

$\Leftrightarrow$ $q(x)=2x^2+x+1$ and $r(x)=-8x-2$.

2nd method

$$
\begin{array}{l|l}
x^2-2x+3 & 2x^4-3x^3+5x^2-7x+1 \\
\hline
2x^2+x+1 & \underline{-2x^4+4x^3-6x^2+0x+0} \\
 & \qquad\quad x^3 \quad -x^2-7x+1 \\
 & \qquad \underline{-x^3 \quad +2x^2-3x+0} \\
 & \qquad\qquad\qquad x^2-10x+1 \\
 & \qquad\qquad\quad \underline{-x^2+2x-3} \\
 & \qquad\qquad\qquad\quad -8x-2
\end{array}
$$

thus

quotient $q(x) = 2x^2+x+1$

remainder $r(x) = -8x-2$

EXERCISES

① Do the following divisions.

a) $(-x^2+81):(x-9)$
b) $(x^3+4x^2-11x-30):(-x^2+x+6)$
c) $(18x^3+9x^2-50x-25):(3x-5)$
d) $[(x^2-9)^2-(x+5)(x-3)^2]:(x^2+x-12)$
e) $(x^6-2x):(x^3+1)$
f) $(6x^4-19x^3+15x^2-x-6):(2x^2-3x+2)$
g) $(x^4-3x^2+5x-1):(2x-1)$
h) $(2x^5-11x^4+3x^3+31x^2+2x+5):(2x^3-5x^2-4x+1)$

② If $f(x)=x^2+x-2$, do the division
$[(f(x))^2-f(x+1)]:f(1-x)$

③ If $f(x)=x^2+5x-6$, do the division
$[f(x-2)f(x+2)-f(x)-10]:(x^2-x-2)$

▼ Division with $x-c$

- Let $f \in \mathbb{R}[x]$ be a polynomial and let $c \in \mathbb{C}$ be an arbitrary complex number.

$$\boxed{c \text{ root of } f \iff f(c) = 0}$$

Thm: $x-c$ divides f if and only if c is a root of f.

$$\boxed{x-c \mid f(x) \iff c \text{ root of } f}$$

Proof

$(\Rightarrow)$: Assume $x-c \mid f(x) \Rightarrow$

$\Rightarrow \exists g \in \mathbb{R}[x] : \forall x \in \mathbb{R} : f(x) = (x-c)g(x)$

$\Rightarrow f(c) = (c-c)\, g(c) = 0 \cdot g(c) = 0$

$\Rightarrow c$ root of f.

$(\Leftarrow)$: Assume c root of $f \Rightarrow \underline{f(c) = 0}$. (1)

Let $q, r \in \mathbb{R}[x]$ such that

$\forall x \in \mathbb{R} : f(x) = (x-c)\, q(x) + r(x)$

Note that $\deg(r) < \deg(x-c) = 1 \Rightarrow$

$\Rightarrow \deg(r) = 0 \Rightarrow r(x) = \lambda,\ \forall x \in \mathbb{R}$.

Thus

$$\forall x \in \mathbb{R} : f(x) = (x-c)\,q(x) + \lambda \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} f(c) = (c-c)\,q(c) + \lambda = \lambda \\ f(c) = 0 \end{array}\right\} \Rightarrow \lambda = 0 \Rightarrow$$

$$\Rightarrow \forall x \in \mathbb{R} : f(x) = (x-c)\,q(x) \Rightarrow$$

$$\Rightarrow x - c \mid f(x) \qquad \square$$

↦ Synthetic Division

Let $f \in \mathbb{R}[x]$ with

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

The division $f(x) : (x-c)$ has

a) Remainder:

$$r(x) = b_{-1}$$

b) Quotient

$$q(x) = b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_1 x + b_0$$

such that

$$\boxed{\begin{array}{l} b_{n-1} = a_n \\ b_{k-1} = a_k + c\,b_k \ , \quad k = 0, 1, \dots, n-1 \end{array}}$$

Proof

We note that

$$(x-c)\,q(x)+r(x) = (x-c)\sum_{k=0}^{n-1} b_k x^k + b_{-1} =$$

$$= \sum_{k=0}^{n-1} \left(b_k x^{k+1} - c b_k x^k\right) + b_{-1}$$

$$= \sum_{k=1}^{n} b_{k-1} x^k + \sum_{k=1}^{n-1} c b_k x^k + (b_{-1} - c b_0)$$

$$= b_{n-1} x^n + \sum_{k=1}^{n-1} (b_{k-1} - c b_k) x^k + (b_{-1} - c b_0)$$

$$= a_n x^n + \cdots + a_1 x + a_0 \,, \quad \forall x \in \mathbb{R} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} b_{n-1} = a_n \\ b_{k-1} - c b_k = a_k \,, \quad k = 0, \ldots, n-1 \end{cases} \quad \square$$

▸ Implementation - Example

$(2x^3 - 5x^2 + 6x - 1) : (x-2)$

$$\begin{array}{c|cccc} 2 & 2 & -5 & 6 & -1 \\ & \downarrow & 4 & -2 & 8 \\ \hline & \underbrace{2 \quad -1 \quad 4}_{q(x)} & & & \underbrace{7}_{r(x)} \end{array} \quad \leftarrow \text{Horner scheme}$$

Thus

$(2x^3 - 5x^2 + 6x - 1) = (x-2)(2x^2 - x + 4) + 7$

Note that the last number in the Horner scheme is the remainder $r(x) = b_{-1}$.

Efficient polynomial evaluation

The following result suggests that we may use the Horner scheme to evaluate the polynomial $f(x)$ for specific values of x:

Thm : The remainder of the division $f(x) : (x-c)$ is:

$$\boxed{r(x) = f(c)}$$

Proof

We know that

$$\deg(r) < \deg(x-a) = 1 \Rightarrow \deg(r) = 0 \Rightarrow$$
$$\Rightarrow r(x) = \lambda, \forall x \in \mathbb{R}.$$
$$\Rightarrow \exists q \in \mathbb{R}[x] : f(x) = (x-c)\,q(x) + \lambda$$
$$\Rightarrow f(c) = (c-c)\,q(c) + \lambda =$$
$$= 0\,q(c) + \lambda = \lambda \Rightarrow$$
$$\Rightarrow \forall x \in \mathbb{R} : \ r(x) = f(c). \qquad \square$$

Thus, to evaluate $f(c)$, it is sufficient to do the division $f(x) : (x-c)$ using the Horner scheme. The remainder will be equal to $f(c)$!

EXAMPLE

For $f(x) = 2x^3 - x^2 + 3x + 1$
To find $f(3)$:

$$
\begin{array}{c|cccc}
3 & 2 & -1 & 3 & 1 \\
 & & 6 & 15 & 54 \\
\hline
 & 2 & 5 & 18 & 55
\end{array}
\longrightarrow f(3) = 55 \ !!
$$

EXAMPLE

For what values $a \in \mathbb{R}$ does $x-1$ divide $f(x) = 2x^3 - (a+1)x^2 + (5a-2)x - 7$?

Solution: $x - 1 \mid f(x) \iff f(1) = 0 \iff$

$$\iff 2 \cdot 1^3 - (a+1) \cdot 1^2 + (5a-2) \cdot 1 - 7 = 0$$

$$\iff \cancel{2} - a - 1 + 5a - \cancel{2} - 7 = 0$$

$$\iff 4a - 8 = 0 \iff 4a = 8 \iff a = 2.$$

EXAMPLE

Show that $g(x) = 2x^3 + 3x^2 + x$ divides

$f(x) = (x+1)^{2n} - x^{2n} - 2x - 1$.

Solution:

$$g(x) = 2x^3 + 3x^2 + x = x(2x^2+3x+1) =$$
$$= x(x+1)(2x+1) \text{ , thus}$$

$$g(x) \mid f(x) \Leftrightarrow \begin{cases} x \mid f(x) \\ x+1 \mid f(x) \\ 2x+1 \mid f(x) \end{cases} \Leftrightarrow \begin{cases} f(0) = 0 \\ f(-1) = 0 \\ f(-1/2) = 0 \end{cases} \quad (1)$$

Note that

$$f(0) = (0+1)^{2n} - 0^{2n} - 2 \cdot 0 - 1 = 1 - 1 = 0$$
$$f(-1) = (-1+1)^{2n} - (-1)^{2n} - 2 \cdot (-1) - 1 =$$
$$= 0 - 1 + 2 - 1 = 0$$
$$f(-1/2) = (-1/2+1)^{2n} - (-1/2)^{2n} - 2(-1/2) - 1 =$$
$$= (1/2)^{2n} - (1/2)^{2n} + 1 - 1 = 0$$

Thus from (1): $g(x) \mid f(x)$.

EXAMPLE

Show that $11^{15} + 1$ is a multiple of 12.

Solution:

Define $f(x) = x^{15} + 1 \Rightarrow f(-1) = (-1)^{15} + 1 = 0$

$$\Rightarrow x+1 \mid f(x) \Rightarrow 11+1 \mid f(11) \Rightarrow$$
$$\Rightarrow 12 \mid 11^{15} + 1 .$$

EXERCISES

(4) Perform the following divisions

a) $(-x^2+2x^5+2x-3-2x^4):(x-1)$

b) $(3x^3-19x^2-11x+2):(3x+2)$

c) $(3x^3-x^2-x+1):(x+1)$

d) $(32x^5-243):(2x-3)$

(5) For what values of $\lambda \in \mathbb{R}$:

a) $x+2 \mid f(x)$ with $f(x)=\lambda x^2-(\lambda-1)x-4\lambda$

b) $x+1 \mid f(x)$ with $f(x)=x^3+\lambda x^2+2\lambda x-1$

c) $2x+4 \mid f(x)$ with $f(x)=2x^2-(\lambda-2)x+4$.

(6) Let $f(x)=x^3-\lambda x+1$ and let $g(x)=f(x+1)f(x-2)-f(x)$

For what value of λ does $x-1$ divide $g(x)$?

(7) Let $f(x)=\lambda x^3+(\lambda-1)x^2+2\lambda x+3$ and let $g(x)=f(x-1)f(x+3)$ and let $h(x)=g(x-1)+2f(x+1)$.

For what value of $\lambda \in \mathbb{R}$ does $x+1$ divide $h(x)$?

(8) Consider the polynomial
$$f(x) = ax^4 + bx^2 + c$$
Show that if $x+1$ divides $f(x)$ then $x-1$ also divides $f(x)$.

(9) Find the remainder of the division
$[(3x-7)^{2n+1} - 5(x^2-3)^n + 6x-1] : (x-2)$
with $n \in \mathbb{N}$, $n \geqslant 1$, without doing the division.

(10) Find the remainder of the division
$[(2x-5)^{54} + (3x-8)^{23}] : (x-3)$
without doing the division.

(11) Show that $(x-a)^2 + (x-a)$ divides
$f(x) = (x-a)^{2n} + (x-a+1)^n - 1$
with $n \in \mathbb{N}$, $n \geqslant 1$.

(12) Let $f(x) = ax^n + bx^m + c$ with $n, m \in \mathbb{N}$ and $n \geqslant 1$, $m \geqslant 1$. Show that if $a+b+c=0$ then $x-1$ divides $f(x)$.

$\hookrightarrow$ Exercises 8-12 are short proof-type arguments.

(13) Show that

a) $8^9 - 1$ is a multiple of 7

b) $15^{10} - 1$ is a multiple of 14 and a multiple of 16

c) $5^{2n+1} - 1$ with $n \in \mathbb{N}$, $n \geq 1$ is a multiple of 4.

d) $33^{20} - 3 \cdot 33^{10} + 2$ is a multiple of 34.

(14) Let $f(x) = ax^3 + bx^2 + bx + a$.
Show that

$$x^2 - 1 \mid f(x) \Leftrightarrow a + b = 0.$$

(15) Let $f \in \mathbb{R}[x]$ be a polynomial.
Let $g(x) = f(3x - 5)$. Show that if $x + 2$ divides $f(x)$ then $x - 1$ divides $g(x)$.

(16) Let $f \in \mathbb{R}[x]$ be a polynomial and let

$$g(x) = f(x+1)\, f(x-2) + f(2x).$$

Show that:

$$x^2 - x - 2 \mid f(x) \Rightarrow x - 1 \mid g(x).$$

Rational zero theorem

- Let $a \in \mathbb{Z}$ be an integer. We define the set of divisors of a, Δ_a, as

$$\boxed{\Delta_a = \{x \in \mathbb{Z} \mid x|a\}}$$

Here $x|a$: x divides a.

EXAMPLE

For $a=6$: $\Delta_6 = \{\pm 1, \pm 2, \pm 3, \pm 6\}$
$a=1$: $\Delta_1 = \{\pm 1\}$
$a=5$: $\Delta_5 = \{\pm 1, \pm 5\}$

- Let $a, b \in \mathbb{Z}$ be two integers. The greatest common divisor $GCD(a,b)$ is defined as

$$\boxed{GCD(a,b) = \max(\Delta_a \cap \Delta_b)}$$

- We say that a fraction a/b with $a \in \mathbb{Z}$ and $b \in \mathbb{Z} - \{0\}$ is
  a) irreducible $\Leftrightarrow$ $GCD(a,b) = 1$
  b) reducible $\Leftrightarrow$ $GCD(a,b) \geq 2$

EXAMPLE

For $4/3$:

$$\left.\begin{array}{l}\Delta_4 = \{\pm 1, \pm 2, \pm 4\} \\ \Delta_3 = \{\pm 1, \pm 3\}\end{array}\right\} \Rightarrow \Delta_3 \cap \Delta_4 = \{\pm 1\}$$

$\Rightarrow GCD(4,3) = \max\{\pm 1\} = 1 \Rightarrow$
$\Rightarrow 4/3$ irreducible.

For $8/4$

$$\left.\begin{array}{l}\Delta_8 = \{\pm 1, \pm 2, \pm 4, \pm 8\} \\ \Delta_4 = \{\pm 1, \pm 2, \pm 4\}\end{array}\right\} \Rightarrow$$

$\Rightarrow \Delta_4 \cap \Delta_8 = \{\pm 1, \pm 2, \pm 4\} \Rightarrow$
$\Rightarrow GCD(4,8) = \max\{\pm 1, \pm 2, \pm 4\}$
$= 4 \Rightarrow 8/4$ reducible.

- Let $f \in \mathbb{Z}[x]$ be a polynomial
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
with integer coefficients, $a_k \in \mathbb{Z}$. We associate with f a set of rational numbers $\Delta(f)$ defined as:

$$\boxed{\Delta(f) = \{a/b \mid a \in \Delta_{a_0} \wedge b \in \Delta_{a_n}\}}$$

We now show our main result:

Thm: (Rational zero theorem)

$$\boxed{\left.\begin{array}{l} f \in \mathbb{Z}[x] \\ f(\rho) = 0 \\ \rho \in \mathbb{Q} \end{array}\right\} \Longrightarrow \rho \in \Delta(f)}$$

In words: If f is a polynomial with integer coefficients and $\rho \in \mathbb{Q}$ is a rational root of f (such that $f(\rho) = 0$), then ρ has to be an element of $\Delta(f)$.

An equivalent statement of the rational root theorem is that

$$\boxed{f \in \mathbb{Z}[x] \Rightarrow \{\rho \in \mathbb{Q} \mid f(\rho) = 0\} \subseteq \Delta(f)}$$

An immediate consequence is the integer zero theorem:

Thm: (Integer zero theorem)

$$\boxed{\left.\begin{array}{l} f \in \mathbb{Z}[x] \\ f(\rho) = 0 \\ \rho \in \mathbb{Z} \end{array}\right\} \Rightarrow \rho \in \Delta_{a_0}}$$

or:

$$\boxed{f \in \mathbb{Z}[x] \Rightarrow \{\rho \in \mathbb{Z} \mid f(\rho) = 0\} \subseteq \Delta_{a_0}}$$

Proof

Let $\rho = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{Z} - \{0\}$ such that $GCD(p,q) = 1$. (consequence of $\rho \in \mathbb{Q}$) We assume that $f(\rho) = 0$. It follows that

$$f(p/q) = 0 \Leftrightarrow$$
$$\Leftrightarrow a_n (p/q)^n + a_{n-1}(p/q)^{n-1} + \cdots + a_1 (p/q) + a_0 = 0 \Leftrightarrow$$
$$\Leftrightarrow a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0 \quad (1)$$

From (1):
$$p(a_n p^{n-1} + a_{n-1} p^{n-2} q + \cdots + a_1 q^{n-1}) = -a_0 q^n \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} p \mid a_0 q^n \\ GCD(p,q) = 1 \end{array}\right\} \Rightarrow p \mid a_0 \Rightarrow \underline{p \in \Delta a_0} \quad (2)$$

From (1):
$$q(a_0 q^{n-1} + a_1 p q^{n-2} + \cdots + a_{n-1} p^{n-1}) = -a_n p^n \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} q \mid a_n p^n \\ GCD(p,q) = 1 \end{array}\right\} \Rightarrow q \mid a_n \Rightarrow \underline{q \in \Delta a_n} \quad (3)$$

From (2) and (3):
$$\rho = p/q \in \{a/b \mid a \in \Delta a_0 \land b \in \Delta a_n\} = \Delta(f) \Rightarrow$$

$$\Rightarrow \underline{\rho \in \Delta(f)} \quad \square$$

Method: Factorization of polynomial $f(x)$.

- 1 Find the set $\Delta(f)$ of candidate rational roots of f.
- 2 Test the candidates $\rho \in \Delta(f)$ by calculating $f(\rho)$ by Horner scheme (!!)
- 3 If you find a rational zero, then $x-\rho \mid f(x)$, so we may initiate the factorization
$$f(x) = (x-\rho)\,g(x)$$
from the division $f(x) : (x-\rho)$ which was already done in the previous step.
- 4 Repeat steps 2,3 until factorization is complete.

Then: We can solve $f(x)=0$ or inequalities like $f(x) \geq 0$, $f(x) < 0$, etc.

- If $\rho_1, \rho_2, \dots, \rho_n$ are roots of $f \in \mathbb{R}[x]$ with $\deg f = n$ then

$$\boxed{f(x) = a_n(x-\rho_1)(x-\rho_2)\cdots(x-\rho_n)}$$

EXAMPLE

a) $f(x) = 3x^3 - 22x^2 + 48x - 32 = 0$

$$\left.\begin{array}{l}\Delta_{32} = \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32\} \\ \Delta_3 = \{\pm 1, \pm 3\}\end{array}\right\} \Rightarrow$$

$\Rightarrow \Delta(f) = \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32, \pm 1/3, \pm 2/3, \pm 4/3, \pm 8/3, \pm 16/3, \pm 32/3\}$

Try:

$$\begin{array}{r|rrrr} 1 & 3 & -22 & 48 & -32 \\ & & 3 & -19 & 29 \\ \hline & 3 & -19 & 29 & \boxed{-3} \end{array} \longrightarrow f(1) = \cancel{-3} \neq 0$$

etc.

$$\begin{array}{r|rrrr} 2 & 3 & -22 & 48 & -32 \\ & & 6 & -32 & 32 \\ \hline & 3 & -16 & 16 & \boxed{0} \end{array}$$

Thus $x-2 \mid f(x) \Rightarrow f(x) = (x-2)(3x^2 - 16x + 16)$.

For $\left.\begin{array}{l} g(x) = 3x^2 - 16x + 16 \\ \Delta = 16^2 - 4 \cdot 3 \cdot 16 = 256 - 192 = 64 \end{array}\right\} \Rightarrow$

$$\Rightarrow x_{2,3} = \frac{16 \pm 8}{6} = \begin{cases} 4 \\ 4/3 \end{cases}$$

Thus $\underline{f(x) = 3(x-2)(x-4)(x-4/3)}$.

B) $f(x) = x^4 - 4x^3 + 5x^2 - 4x + 4$

$$\left.\begin{array}{l} \Delta_4 = \{\pm 1, \pm 2, \pm 4\} \\ \Delta_1 = \{\pm 1\} \end{array}\right\} \Rightarrow A(f) = \Delta_4 = \{\pm 1, \pm 2, \pm 4\}$$

Note that $f(1) \neq 0$ and $f(-1) \neq 0$ $(\ldots)$
but

$$\begin{array}{c|ccccc} 2 & 1 & -4 & 5 & -4 & 4 \\ & & 2 & -4 & 2 & -4 \\ \hline & 1 & -2 & 1 & -2 & \boxed{0} \end{array}$$

$$\begin{aligned} f(2) = 0 &\Rightarrow x - 2 \mid f(x) \Rightarrow \\ &\Rightarrow f(x) = (x-2)(x^3 - 2x^2 + x - 2) \\ &\qquad = (x-2)(x^2(x-2) + (x-2)) \\ &\qquad = \underline{(x-2)(x-2)(x^2+1)} \end{aligned}$$

Note that other factorization techniques can still be useful.

EXERCISES

(17) Solve the following equations or inequalities

a) $x^3 - x - 18 = 0$
b) $x^3 - 6x^2 + 11x - 6 \geq 0$
c) $x^4 + x^3 - 31x^2 - 25x + 150 = 0$
d) $x^4 - 6x^3 + 30x - 25 = 0$
e) $x^4 - 3x^3 + 12x - 16 \geq 0$
f) $2x^3 - 5x^2 + x + 2 = 0$
g) $6x^3 - 7x^2 + 1 > 0$
h) $2x^3 - 9x^2 + 7x + 6 \leq 0$
i) $3x^4 - 4x^3 + 1 = 0$
j) $6x^4 + 13x^3 - 2x^2 - 7x + 2 \leq 0$
k) $3x^4 - 8x^3 - 35x^2 - 4x + 20 = 0$

(18) Find $a \in \mathbb{R}$ such that
$$2x^3 + (a-4)x^2 - 5x + 1 - a = 0$$
has solution $x = 2$. Then find all other solutions.

(19) Find $a \in \mathbb{R}$ such that $ax - 1$ divides $f(x) = x^3 - 5x^2 - \dfrac{6}{a}$. Then solve $f(x) = 0$ for those values of a.

(20) Find $a \in \mathbb{R}$ such that
$f(x) = (a-1)x^5 + 3ax^4 - (a+1)x^3 - (a+1)x^2 + 3ax + (a-1)$
is divided by $x-2$.
Then solve the equation $f(x)=0$.

(21) Solve the equation
$(2x^2-x-2)^3 - 2(2x^2-x+3)^2 + 9(2x^2-x+2) + 26 = 0$

(22) Find all the integers $k \in \mathbb{Z}$ such that
the equation $x^3 - x^2 + kx + 4 = 0$
has at least one rational solution.

(23) Show that the equation
$x^4 + x^3 + x^2 + x + 1 = 0$
does not have rational solutions.

CA7: Exponentials and logarithms

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

▼ Definition of powers

- First we recall the following definitions of number sets:

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$	Natural numbers
$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$	Integers
$\mathbb{Q} = \{a/b \mid a \in \mathbb{Z} \wedge b \in \mathbb{N} - \{0\}\}$	Rational numbers
$\mathbb{R}$ = set of real numbers.	Real numbers

- Let $a \in \mathbb{R}$. We give the following incremental definitions of powers:

1) Integer powers $\to$ Let $n \in \mathbb{N}$. Then we define

$$\boxed{a^n = \begin{cases} 1 & , n = 0 \\ \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ times}} & , n > 0 \end{cases}}$$

$$\boxed{a^{-n} = \frac{1}{a^n}, \text{ for } a \neq 0}$$

example: $(-2)^3 = (-2)(-2)(-2) = -8$

$$\left(\frac{-1}{2}\right)^{-2} = \frac{1}{\left(\frac{-1}{2}\right)^2} = \frac{1}{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right)} =$$

$$= \frac{1}{\left(\frac{1}{4}\right)} = 4$$

$3^0 = 1$, 0^0 undefined.

2) Rational powers

- First, recall the definition of roots. Let $n \in \mathbb{N} - \{0\}$. Then, we define:

$$\boxed{\begin{array}{l} x = \sqrt[2n]{a} \Leftrightarrow x^{2n} = a \wedge x > 0 \\ x = \sqrt[2n+1]{a} \Leftrightarrow x^{2n+1} = a \end{array}} \begin{array}{l} \leftarrow a \in (0, +\infty) \\ \leftarrow a \in \mathbb{R}. \end{array}$$

- Note that the equation $x^{2n} = a$ has two solutions and, by convention, we choose the positive solution. The equation $x^{2n+1} = a$ has a unique solution.

examples : $\sqrt{9} = 3$, because $3^2 = 9$
$\sqrt[3]{-8} = 2$, because $(-2)^3 = 8$.

- Let $a \in (0, +\infty)$, $p \in \mathbb{Z}$, and $q \in \mathbb{N} - \{0, 1\}$. Then we define:

$$\boxed{a^{p/q} = (\sqrt[q]{a})^p \ , \ \forall a \in (0, +\infty)}$$

<u>example</u>: $4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$

$27^{5/3} = (\sqrt[3]{27})^5 = 3^5 = 243$

3) <u>Real powers</u>

Let $x \in \mathbb{R}$ and let $x_1, x_2, x_3, \ldots \in \mathbb{Q}$ be a rational sequence approximating x. We indicate that by writing

$$x = \lim (x_n)$$

Then we define:

$$a^x = \lim (a^{x_n}).$$

<u>example</u>

To approximate $2^{\sqrt{3}}$, we note that

$$\sqrt{3} \approx 1.7320508075\ldots$$

and therefore define $2^{\sqrt{3}}$ via the following sequence of approximations:

$$2^{1.7} = 3.249009585\ldots$$

$$2^{1.73} = 3.317278183\ldots$$

$$2^{1.732} = 3.321880096\ldots$$

$$2^{1.73205} = 3.321995226\ldots$$

Properties of powers

• Let $a, b \in (0, +\infty)$ and $x_1, x_2, x \in \mathbb{R}$. It can be shown that:

$a^x > 0$	$(a^{x_1})^{x_2} = a^{x_1 x_2}$
$a^{x_1} a^{x_2} = a^{x_1 + x_2}$	$(ab)^x = a^x b^x$
$\dfrac{a^{x_1}}{a^{x_2}} = a^{x_1 - x_2}$	$\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

• To compare a^x with b^x:

$\left.\begin{matrix} a > b \\ x > 0 \end{matrix}\right\} \Rightarrow a^x > b^x$	$\left.\begin{matrix} a > b \\ x < 0 \end{matrix}\right\} \Rightarrow a^x < b^x$

• To compare a^{x_1} with a^{x_2}:

$\left.\begin{matrix} x_1 < x_2 \\ a > 1 \end{matrix}\right\} \Rightarrow a^{x_1} < a^{x_2}$	$\left.\begin{matrix} x_1 < x_2 \\ 0 < a < 1 \end{matrix}\right\} \Rightarrow a^{x_1} > a^{x_2}$

EXAMPLES

a) Simplify:

$$\left[\left(\sqrt{5\sqrt{5}}\right)^{-1/3}\right]^6 = \left(\sqrt{5\sqrt{5}}\right)^{(-1/3)\cdot 6} = \left(\sqrt{5\sqrt{5}}\right)^{-2}$$
$$= \frac{1}{\left(\sqrt{5\sqrt{5}}\right)^2} = \frac{1}{5\sqrt{5}} = \frac{\sqrt{5}}{5\sqrt{5}\sqrt{5}} = \frac{\sqrt{5}}{5\cdot 5}$$
$$= \frac{\sqrt{5}}{25}$$

b) Simplify:

$$\left(\frac{1}{2}\right)^{-2/3}\left(\frac{1}{4}\right)^{-2/3} = \left(\frac{1}{2}\,\frac{1}{4}\right)^{-2/3} = \left(\frac{1}{8}\right)^{-2/3} =$$
$$= 8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4.$$

c) Compare $(5/3)^{-1/2}$ with 1.

Solution

$$\left.\begin{matrix} 5/3 > 1 \\ -1/2 < 0 \end{matrix}\right\} \Rightarrow (5/3)^{-1/2} < 1^{-1/2} \Rightarrow (5/3)^{-1/2} < 1.$$

d) Compare $(1/3)^{-2/3}$ with $(1/3)^{-4/5}$

Solution

$$\left.\begin{array}{l} 2/3 < 4/5 \Rightarrow -2/3 > -4/5 \\ \qquad 0 < 1/3 < 1 \end{array}\right\} \Rightarrow$$

$$\Rightarrow (1/3)^{-2/3} < (1/3)^{-4/5}.$$

e) Compare $(\sqrt{7})^{\sqrt{3}}$ with $(\sqrt{5})^{\sqrt{2}}$.

Solution

$$\left.\begin{array}{l} 7 > 5 \Rightarrow \sqrt{7} > \sqrt{5} \\ \sqrt{3} > 0 \end{array}\right\} \Rightarrow (\sqrt{7})^{\sqrt{3}} > (\sqrt{5})^{\sqrt{3}} \quad (1)$$

$$\left.\begin{array}{l} 3 > 2 \Rightarrow \sqrt{3} > \sqrt{2} \\ \sqrt{5} > 1 \end{array}\right\} \Rightarrow (\sqrt{5})^{\sqrt{3}} > (\sqrt{5})^{\sqrt{2}} \quad (2)$$

From (1) and (2): $(\sqrt{7})^{\sqrt{3}} > (\sqrt{5})^{\sqrt{2}}$.

$\hookrightarrow$ Note that $1^x = 1,\ \forall x \in \mathbb{R} - \{0\}$ and

$$\left.\begin{array}{l} a > b \Rightarrow a^{1/2} > b^{1/2} \\ \qquad 1/2 > 0 \end{array}\right\} \Rightarrow \sqrt{a} > \sqrt{b}$$

f) Compare $(2/3)^{3/4}$ with $(3/4)^{2/3}$.

Solution

$$\left.\begin{array}{l} 3/4 > 2/3 \\ 0 < 2/3 < 1 \end{array}\right\} \Rightarrow (2/3)^{3/4} < (2/3)^{2/3} \quad (1)$$

$$\left.\begin{array}{l} 2/3 < 3/4 \\ 2/3 > 0 \end{array}\right\} \Rightarrow (2/3)^{2/3} < (3/4)^{2/3} \quad (2)$$

From (1) and (2): $(2/3)^{3/4} < (3/4)^{2/3}$.

The power function

Let $f(x) = a^x$ with $a > 0$

Domain: $A = \mathbb{R}$

Range: $f(A) = (0, +\infty)$

Monotonicity: $a > 1 \Rightarrow f \nearrow \mathbb{R}$

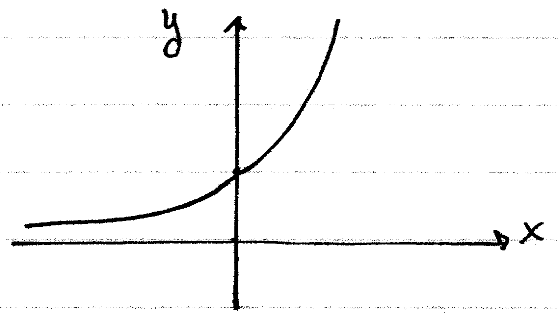
$a = 1 \Rightarrow f$ constant in $\mathbb{R}$

$0 < a < 1 \Rightarrow f \searrow \mathbb{R}$

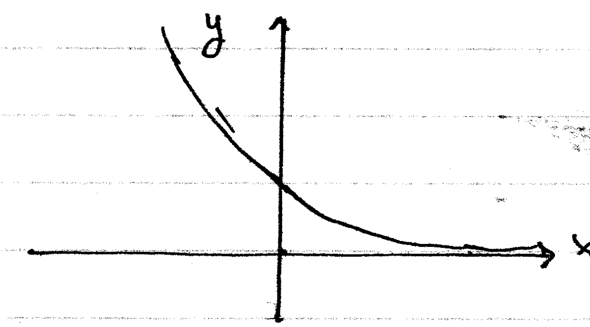
One-to-one: $a \neq 1 \Rightarrow f$ one-to-one

thus: $a^{x_1} = a^{x_2} \Leftrightarrow x_1 = x_2$ (for $a \neq 1$)

Graph: Passes through $(0,1)$.



$a > 1$



$0 < a < 1$.

EXAMPLES

a) Find all $a \in \mathbb{R}$ such that the function $f(x) = (3a+2)^x$ is decreasing in $\mathbb{R}$.

Solution

$$f \searrow \mathbb{R} \Leftrightarrow 0 < 3a+2 < 1 \Leftrightarrow \begin{cases} 3a+2 < 1 \\ 3a+2 > 0 \end{cases} \Leftrightarrow \begin{cases} 3a < -1 \\ 3a > -2 \end{cases}$$

$$\Leftrightarrow \begin{cases} a < -1/3 \\ a > -2/3 \end{cases} \Leftrightarrow -2/3 < a < -1/3 \Leftrightarrow$$

$$\Leftrightarrow a \in (-2/3, -1/3)$$

b) Find the default domain to the function $f(x) = (x^3 - 4x)^{2x+1}$

Solution

We require that

$$x^3 - 4x > 0 \Leftrightarrow x(x^2-4) > 0 \Leftrightarrow x(x-2)(x+2) > 0 \quad (1)$$

x		-2		0		2	
x	$-$		$-$	0	$+$		$+$
$x-2$	$-$		$-$		$-$	0	$+$
$x+2$	$-$	0	$+$		$+$		$+$
ineq	$-$	0	$+$	0	$-$	0	$+$

therefore:

$(1) \Leftrightarrow x \in (-2,0) \cup (2,+\infty)$

It follows that

$$\mathrm{dom}(f) = (-2,0) \cup (2,+\infty)$$

$\hookrightarrow$ In general, for any function $f(x) = a(x)^{b(x)}$ we have to require $\underline{a(x) > 0}$ in addition to any other requirements that may be necessary to evaluate $a(x)$ and $b(x)$.

EXERCISES

(1) Simplify the following arithmetic expressions, using root notation

a) $\dfrac{2 \cdot 2^{-3}}{\sqrt{2}}$ b) $\dfrac{2^{1/2}\, 5^{1/2}}{\sqrt{10}}$

c) $\left(\dfrac{1}{2}\right)^{-3}\left(\dfrac{1}{3}\right)^{-2}$ d) $5^{-2} \cdot 2^{-5}$

e) $4^{1/5} \cdot 8^{1/5}$ (show it equals 2)

f) $\left[\left(\sqrt{2}\right)^{-\sqrt{2}}\right]^{\sqrt{2}}$ (show it equals 1/2)

g) $\left\{\left[\left(\dfrac{2}{3}\right)^{3/2}\right]^{1/3}\right\}^{2}$

h) $\dfrac{\sqrt{2\sqrt{2}}}{\sqrt{2}}$ i) $\left[\left(\sqrt{3\sqrt{3}}\right)^{-1/2}\right]^{8}$

(2) For what values of $a \in \mathbb{R}$ are the following functions increasing in $\mathbb{R}$? decreasing in $\mathbb{R}$?

a) $f(x) = \left(\dfrac{a+1}{a-1}\right)^x$ b) $f(x) = [a(a+2)]^x$

c) $f(x) = \left(\dfrac{a^2}{a+1}\right)^x$

(3) Compare the following numbers with 1

a) $(2/5)^{2/3}$ b) $(3/2)^{2/3}$ c) $(\sqrt{2})^{-3/2}$

d) $(1/3)^{-\sqrt{2}/2}$ e) $(5/4)^{-1/3}$ f) $(\sqrt{3})^{-\sqrt{2}}$

g) $(2-\sqrt{2})^{\sqrt{2}-1}$ h) $(\sqrt{2})^{1-\sqrt{3}}$

(4) Compare the following numbers with each other:

a) $(3/5)^{2/3}$, $(3/5)^{3/4}$
b) $(4/3)^{1/2}$, $(4/3)^{1/3}$
c) $(2/5)^{-2/3}$, $(2/5)^{-3/4}$
d) $(\sqrt{2})^{\sqrt{2}}$, $(\sqrt{2})^{\sqrt{3}}$
e) $(1/2)^{1/3}$, $(1/3)^{1/4}$
f) $(1/3)^{1/2}$, $(1/4)^{1/3}$
g) $(\sqrt{3})^{-\sqrt{2}}$, $\dfrac{1}{\sqrt{2}}$
h) $(\sqrt{5})^{\sqrt{3}}$, $2^{\sqrt{2}}$

$\left.\begin{array}{l} \text{e)} \\ \text{f)} \\ \text{g)} \\ \text{h)} \end{array}\right\}$ harder.

(5) Find the default domain of the following functions:

a) $f(x) = (3x^2 - 10x + 3)^{2x+1}$

b) $f(x) = (x^3 - 2x^2 + 1)^x$

c) $f(x) = \left[\dfrac{x+1}{x-1}\right]^x$

* d) $f(x) = (1/x)^{1/x}$

* e) $f(x) = (x+1)^{1/(x+2)}$

* f) $f(x) = (1-x^2)^{1/x}$

↳ To find the domain of $f(x) = a(x)^{b(x)}$:

$$\boxed{\mathrm{dom}(f) = \mathrm{dom}(a) \cap \mathrm{dom}(b) \cap \{x \in \mathbb{R} \mid a(x) > 0\}}$$

The exponential function

- Let $a \in \mathbb{R}$ be a variable with $a \neq 0$.
  A simple compounding of a with rate r gives
  $$a_1 = (1+r)a$$
  Compounding n times at rate r/n gives:
  $$a_n = \left(1+\frac{r}{n}\right)^n a$$
  The sequence $a_1, a_2, a_3, \ldots$ approximates a number a_∞:
  $$a_\infty = a \cdot \lim_{n \to +\infty} \left(1+\frac{r}{n}\right)^n$$
  Thus we are motivated to define the exponential function
  $$\boxed{\exp(x) = \lim_{n \to +\infty} \left(1+\frac{x}{n}\right)^n}$$

- It can be shown that
  $$\boxed{\forall x \in \mathbb{R} : \exp(x) = e^x}$$

with $e \approx 2.718281828459$

- For $f(x) = \exp(x)$.
  Domain: $A = \mathbb{R}$
  Range: $f(A) = (0, +\infty)$
  Monotonicity: $f \uparrow \mathbb{R}$

- It can also be shown that

$$\boxed{e^x > 1 + x \quad , \forall x \in \mathbb{R}}$$

→ <u>Method</u>: Range of exponential/power functions.

The domain of such functions is usually $A = \mathbb{R}$. Thus after we find the solvability set S for the equation $y = f(x)$, we can then claim that $f(A) = S$.

Note that we use:

$$\boxed{a^{b(y)} = c(y) \text{ has a solution} \Leftrightarrow \underline{c(y) > 0}}$$

with $b(y) = Ay + B$ a linear function.

EXAMPLE

a) $f(x) = 3^{1-2x} - 2$

The equation

$y = f(x) \Leftrightarrow y = 3^{1-2x} - 2 \Leftrightarrow 3^{1-2x} = y+2$

has a unique solution $\Leftrightarrow y+2 > 0$

$\Leftrightarrow y > -2$

Thus $S = [-2, +\infty)$

Since $A = \mathbb{R} \Rightarrow f(A) = S = [-2, +\infty)$.

$\hookrightarrow$ Method: Monotonicity

Usually, the best approach is to work with the definition of monotonicity.

EXAMPLE

$f(x) = 3^{1-2x} - 2 \leftarrow A = \mathbb{R}$

Let $x_1, x_2 \in \mathbb{R}$. with $x_1 < x_2$.

$x_1 < x_2 \Rightarrow -2x_1 > -2x_2 \Rightarrow 1-2x_1 > 1-2x_2$

$\underset{(*)}{\Rightarrow} 3^{1-2x_1} > 3^{1-2x_2}$ (because $3>1$)

$\Rightarrow 3^{1-2x_1} - 2 > 3^{1-2x_2} - 2 \Rightarrow f(x_1) > f(x_2)$

Thus $f \downarrow \mathbb{R}$.

EXERCISES

(6) Find the range and monotonicity for the following functions.

a) $f(x) = 2 - 5^{3-2x}$
b) $f(x) = 3^{x-1} + 1$
c) $f(x) = \left(\frac{1}{2}\right)^{1-2x} - 3$
d) $f(x) = 2e^{1-x} - 1$
e) $f(x) = \left(\frac{1}{2e}\right)^{x-2} + e$
f) $f(x) = e^{-x^2}$
g) $f(x) = \exp(x^2 - 5x + 6)$

} Monotonicity only! (f, g)

h) $f(x) = 3e - e^{e-x}$
i) $f(x) = (1-e)e^{-x+1}$
j) $f(x) = 5^x + 5^{x+1}$
k) $f(x) = \left(\frac{1}{3}\right)^{1-x} - \left(\frac{1}{3}\right)^{2-x}$

} Use factoring of 5^x or $(1/3)^{-x}$. (j, k)

▼ Logarithmic function

- Consider the function $f(x) = a^x$ with $a \in (0, +\infty) - \{1\}$. We know that f is then one-to-one, consequently the inverse f^{-1} is also a function with the same monotonicity as f.
  We call f^{-1} the <u>logarithmic function</u> with <u>base a</u>:

  $\log_a = f^{-1}$ with $\boxed{\log_a : (0, +\infty) \longrightarrow \mathbb{R}}$

  Thus: $\boxed{y = \log_a x \Leftrightarrow a^y = x}$

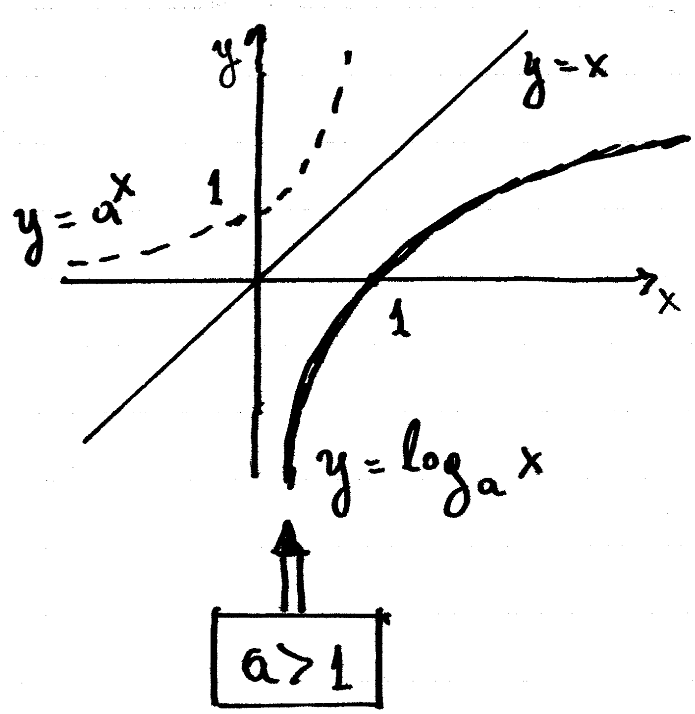
- Immediate consequences of the definition:

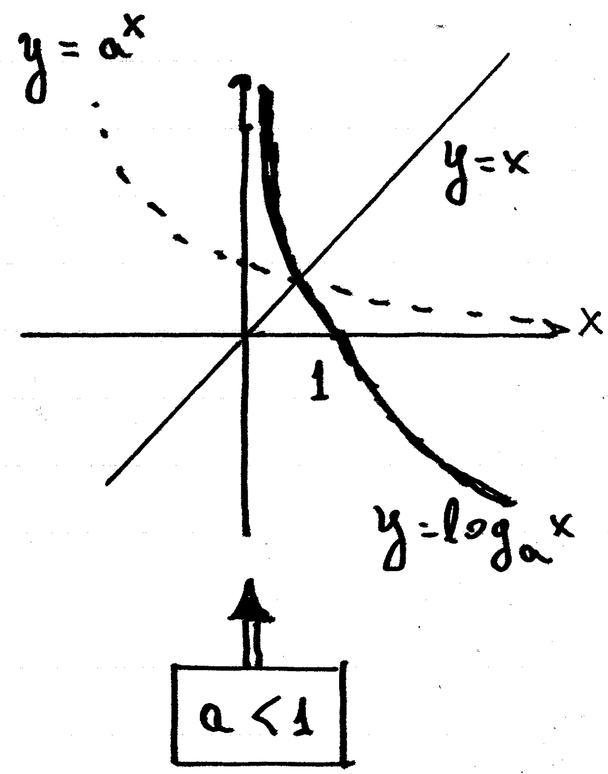
$\log_a 1 = 0$	$\log_a a = 1$
$a^{\log_a x} = x$	$\log_a a^x = x$

Properties of logarithmic function

For $f(x) = \log_a(x)$

- Domain : $A = (0, +\infty)$
- Range : $f(A) = \mathbb{R}$
- Monotonicity: $a > 1 \Leftrightarrow f \nearrow (0, +\infty)$
  $0 < a < 1 \Leftrightarrow f \searrow (0, +\infty)$
- Graph: Because $\log_a$ is the inverse of $g(x) = x^a$, its graph is the mirror image of the graph of g accross the line $(\ell): y = x$





EXAMPLES

a) Find the default domain of the function

$$f(x) = \log_{x^2-4}(2x-1)$$

Solution

Require: $\begin{cases} 2x-1>0 & (1) \\ x^2-4>0 & (2) \\ x^2-4\neq 1 & (3) \end{cases}$

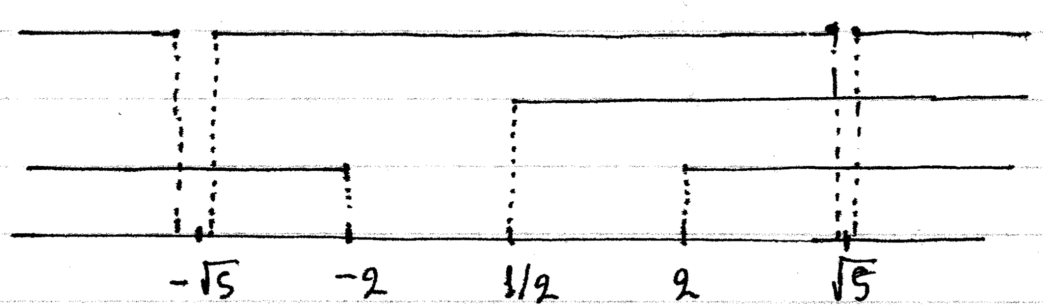
$(1) \Leftrightarrow 2x>1 \Leftrightarrow x>1/2 \Leftrightarrow \underline{x\in(1/2,+\infty)}$

$(2) \Leftrightarrow x^2-4>0 \Leftrightarrow (x-2)(x+2)>0 \Leftrightarrow \underline{x\in(-\infty,-2)\cup(2,+\infty)}$

x		-2		2	
$x-2$	$-$		$-$	0	$+$
$x+2$	$-$	0	$+$		$+$
	$+$	0	$-$	0	$+$

$(3) \Leftrightarrow x^2-4\neq 1 \Leftrightarrow x^2-5\neq 0 \Leftrightarrow (x+\sqrt5)(x-\sqrt5)\neq 0$

$\Leftrightarrow x\neq +\sqrt5 \wedge x\neq -\sqrt5 . \Leftrightarrow \underline{x\in\mathbb{R}-\{-\sqrt5,+\sqrt5\}}.$



Thus: $A=(1/2,+\infty)\cap[(-\infty,-2)\cup(2,+\infty)]\cap[\mathbb{R}-\{-\sqrt5,\sqrt5\}]$

$= (2,\sqrt5)\cup(\sqrt5,+\infty).$

b) Determine the domain and monotonicity of the function f defined by
$f(x) = \sqrt{5} - 3\log_{1/3}(2 - 5e^{-3x})$.

Solution

• Domain

Require: $2 - 5e^{-3x} > 0 \Leftrightarrow -5e^{-3x} > -2 \Leftrightarrow 5e^{-3x} < 2$

$$\Leftrightarrow e^{-3x} < \frac{2}{5} \underset{*}{\Leftrightarrow} \ln(e^{-3x}) < \ln\left(\frac{2}{5}\right) \Leftrightarrow$$

$$\underset{**}{\Leftrightarrow} -3x < \ln(2/5) \Leftrightarrow x > \frac{\ln(2/5)}{3} = \frac{\ln 2 - \ln 5}{3}$$

thus: $A = \left(\frac{\ln(2/5)}{3}, +\infty\right)$

$*$ Here we use $\ln \uparrow (0, +\infty)$ which gives:
$0 < x_1 < x_2 \Leftrightarrow \ln x_1 < \ln x_2$

$**$ Note that $\ln(\exp(x)) = x$, $\forall x \in \mathbb{R}$.

• Monotonicity.

Let $x_1, x_2 \in A$ be given, with $x_1 < x_2$. Then:

$x_1 < x_2 \Rightarrow -3x_1 > -3x_2 \Rightarrow e^{-3x_1} > e^{-3x_2} \Rightarrow$

$\Rightarrow -5e^{-3x_1} < -5e^{-3x_2} \Rightarrow$

$\Rightarrow 2 - 5e^{-3x_1} < 2 - 5e^{-3x_2} \Rightarrow$

$\Rightarrow \log_{1/3}(2 - 5e^{-3x_1}) > \log_{1/3}(2 - 5e^{-3x_2})$

$\Rightarrow -3\log_{1/3}(2 - 5e^{-3x_1}) < -3\log_{1/3}(2 - 5e^{-3x_2})$

$$\Rightarrow \sqrt{5} - 3\log_{1/3}(2-5e^{-3x_1}) < \sqrt{5} - 3\log_{1/3}(2-5e^{-3x_2})$$

$$\Rightarrow \underline{f(x_1) < f(x_2)}.$$

It follows that $f \nearrow \left(\frac{\ln(2/5)}{3}, +\infty\right)$

EXERCISES

(7) Find the default domain of the following functions:

a) $f(x) = \log_3 (x^2+3x+2)$

b) $f(x) = \log_5 (2-|x-1|)$

c) $f(x) = \log_x (x-1)$

d) $f(x) = \log_{x^2-1} (x+1)$

e) $f(x) = \log_{x+2} (5-x)$

$\hookrightarrow$ For the domain of $f(x) = \log_{a(x)} (b(x))$ we require:

$$\begin{cases} b(x) > 0 \\ a(x) > 0 \\ a(x) \neq 1 \end{cases}$$

(8) Determine the domain and monotonicity of the following functions:

a) $f(x) = \log_3 (2x - 1)$

b) $f(x) = 3 - \log_{1/2} (2 - 5x)$

c) $f(x) = \frac{1}{2} \log_2 (3x - 1)$

d) $f(x) = 1 + \log_{1/e} (e - x)$

e) $f(x) = \log_2 (x) + \log_3 (1 + x)$

f) $f(x) = \log_{1/2} (e^x + 1)$

g) $f(x) = 2 - \log_{1/5} (3e^{-x} + 1)$

$\hookrightarrow$ For monotonicity, we use the same method as in exercise 6

Manipulation of Logarithms

Properties

1) $\log_a(x_1 x_2) = \log_a(x_1) + \log_a(x_2)$

2) $\log_a\left(\frac{x_1}{x_2}\right) = \log_a(x_1) - \log_a(x_2)$

3) $\log_a x^k = k \log_a$, $\forall k \in \mathbb{R}$

$$\log_a \sqrt[n]{x} = \frac{1}{n} \log_a x$$

$$\log_a \sqrt{x} = \frac{1}{2} \log_a x$$

4) For $a > 1$: $\log_a x > 0 \Leftrightarrow x > 1$
$\log_a x < 0 \Leftrightarrow x < 1$

For $0 < a < 1$: $\log_a x > 0 \Leftrightarrow x < 1$
$\log_a x < 0 \Leftrightarrow x > 1$

For both cases:
$\log_a x = 0 \Leftrightarrow x = 1$.

▶ Decimal and Natural Logarithms

- We define $\boxed{\log x = \log_{10} x}$ (decimal logarithm)

Thus:
$\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$, $\log 1000 = 3$, etc.

- We also define: $\boxed{\ln(x) = \log_e(x)}$ (natural logarithm).

Note that $\ln \nearrow (0, +\infty)$ since $e > 1$
We can also show that

$$\boxed{a^x = \exp(x \ln a)}$$

$$\boxed{\log_a(x) = \frac{\ln x}{\ln a}}$$

▶ Change of Base

$$\boxed{\log_b(x) = \frac{\log_a(x)}{\log_a(b)}}$$

EXAMPLES

a) Compare $\log_{1/2} 17$ and $\log_{1/2} 21$

Solution

$$\left.\begin{array}{l} 0<1/2<1 \Rightarrow \log_{1/2} \searrow (0,+\infty) \\ 17<21 \end{array}\right\} \Rightarrow \log_{1/2} 17 > \log_{1/2} 21.$$

b) Compare $\log 9$ with $\log 12$

Solution

$$\left.\begin{array}{l} 9<12 \\ \log \nearrow (0,+\infty) \end{array}\right\} \Rightarrow \log 9 < \log 12.$$

c) Compare $\ln 3$ with $\ln 5$

Solution

$$\left.\begin{array}{l} 3<5 \\ \ln \nearrow (0,+\infty) \end{array}\right\} \Rightarrow \ln 3 < \ln 5.$$

d) Show that $\log_2 25 \cdot \log_5 8 = 6$

Solution

$$A = \log_2 25 \log_5 8 = \frac{\ln 25}{\ln 2} \frac{\ln 8}{\ln 5} =$$

$$= \frac{\ln 5^2}{\ln 2} \frac{\ln 2^3}{\ln 5} = \frac{2\ln 5}{\ln 2} \frac{3\ln 2}{\ln 5} =$$

$$= 2 \cdot 3 = 6 = B$$

e) Show that:

e) $\log 2 + \log(2+\sqrt{2}) + \log(2+\sqrt{2+\sqrt{2}}) + \log(2-\sqrt{2+\sqrt{2}}) = 2\log 2$

Solution

$$A = \log 2 + \log(2+\sqrt{2}) + \log\left(2+\sqrt{2+\sqrt{2}}\right) + \log\left(2-\sqrt{2+\sqrt{2}}\right) =$$
$$= \log\left[2(2+\sqrt{2})(2+\sqrt{2+\sqrt{2}})(2-\sqrt{2+\sqrt{2}})\right] =$$
$$= \log\left[2(2+\sqrt{2})(2^2-(\sqrt{2+\sqrt{2}})^2)\right] =$$
$$= \log\left[2(2+\sqrt{2})(4-(2+\sqrt{2}))\right] =$$
$$= \log\left[2(2+\sqrt{2})(2-\sqrt{2})\right] =$$
$$= \log\left[2(2^2-(\sqrt{2})^2)\right] = \log\left[2(4-2)\right] =$$
$$= \log(2 \cdot 2) = \log 2 + \log 2 = 2\log 2 = B.$$

f) Show that:

$$\log_a(b^2\sqrt{b}) \log_{\sqrt{b}}\left(\frac{a^3}{\sqrt{a}}\right) = \frac{25}{2}$$

Solution

$$A = \log_a(b^2\sqrt{b}) \log_{\sqrt{b}}\left(\frac{a^3}{\sqrt{a}}\right) =$$

$$= \log_a\left(b^{2+1/2}\right) \log_{b^{1/2}}\left(a^{3-1/2}\right) =$$

$$= \frac{\ln b^{5/2}}{\ln a} \frac{\ln a^{5/2}}{\ln b^{1/2}} = \frac{(5/2)\ln b}{\ln a} \frac{(5/2)\ln a}{(1/2)\ln b} =$$

$$= \frac{(5/2)(5/2)}{1/2} = \frac{25}{2}$$

g) Write the following expression in terms of $\ln a$, $\ln b$, and $\ln c$:

$$A = \log_9 \left[\frac{9a^3\sqrt{a}}{b^2\sqrt{bc}} \right]$$

<u>Solution</u>

$$A = \log_9 \left[\frac{9a^3\sqrt{a}}{b^2\sqrt{bc}} \right] =$$

$$= \log_9 9 + \log_9 a^3 + \log_9 \sqrt{a} - \log_9 b^2 - \log_9 \sqrt{bc} =$$

$$= 1 + 3\log_9 a + (1/2)\log_9 a - 2\log_9 b - (1/2)[\log_9 b + \log_9 c]$$

$$= 1 + (3 + 1/2)\log_9 a + (-2 - 1/2)\log_9 b + (-1/2)\log_9 c =$$

$$= 1 + (7/2)\log_9 a - (5/2)\log_9 b - (1/2)\log_9 c =$$

$$= 1 + (7/2)\frac{\ln a}{\ln 9} - (5/2)\frac{\ln b}{\ln 9} - (1/2)\frac{\ln c}{\ln 9} =$$

$$= 1 + \frac{7\ln a - 5\ln b - \ln c}{\ln 9}$$

EXERCISES

(9) Compare the numbers

a) $\log_2 5$, $\log_2 3$

b) $\log_{1/3} 11$, $\log_{1/3} 12$

c) $\log 15$, $\log 2$

d) $\ln 2$, $\ln 3$

(10) Show that

a) $\log 3 + 2\log 4 - \log 12 = 2\log 2$

b) $\frac{1}{2}\log 25 + \frac{1}{3}\log 8 + \frac{1}{5}\log 32 = 1 + \log 2$

c) $3\log 2 + \log 5 - \log 4 = 1$

d) $\log_2 3 \, \log_3 4 = 2$

e) $\log_a b \cdot \log_b c \cdot \log_c a = 1$, $\forall a, b, c \in (0,1) \cup (1, +\infty)$

* f) $a^{\log b} = b^{\log a}$, $\forall a, b \in (0, +\infty)$

(11) If $a, b, c \in (0, +\infty)$ and $a \neq b \neq c \neq 0$, and

$$\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$$

show that $a^a b^b c^c = 1$.

(12) Let $x, y \in (0, +\infty)$ with $x^2 + y^2 = 23xy$. Show that
$$\log_a\left[\frac{x+y}{5}\right] = \frac{1}{2}\left(\log_a x + \log_a y\right)$$
(Hint: $x^2 + y^2 = (x+y)^2 - 2xy$)

(13) Write the following in terms of $\ln a$, $\ln b$ and $\ln c$:

a) $\log_3\left[\dfrac{3a^2}{5b\sqrt{c}}\right]$

b) $\log\left[\dfrac{3a^3\sqrt[4]{b^2c}}{5b^2\sqrt[3]{a^2bc^2}}\right]$

(14) If $a, b \in (0,1) \cup (1, +\infty)$, show that

a) $\log_a\left(\dfrac{1}{b^5}\right)\log_b a^2 = -10$

b) $\log_b(a^2)\log_a(b\sqrt{b}) = 3$

(15) If $a, b, c \in (0,1) \cup (1, +\infty)$ show that

a) $\log_a(bc) = \dfrac{1}{\log_b a} + \dfrac{1}{\log_c a}$

b) $\log_{ab}(c) = \dfrac{\log_b(c)}{1 + \log_b(a)}$, c) $\log_a b = -\log_{1/a}(b)$

Logarithmic equations

These are equations that contain a logarithm of the unknown or a logarithm of a function of the unknown.

- 1 Find the domain of the equation using the initial form of the equation.
  Remember that each term $\log_{a(x)} b(x)$ contributes the conditions
  $$\begin{cases} a(x) > 0 \\ a(x) \neq 1 \\ b(x) > 0 \end{cases}$$
- 2 Use the properties of logarithms to reduce the initial equation to one of the following forms:

  1) $\log_x a = b \iff a = x^b \iff \dots$

  2) $\log_a f(x) = b \iff f(x) = a^b \iff \dots$

  3) $\log_a f(x) = \log_a g(x) \iff f(x) = g(x) \iff \dots$

- 3 Accept or reject the solutions based on whether they belong to the domain, found in step 1.

EXAMPLES

a) Solve: $\log_x 64 = 4$

Solution

Domain: Require $\begin{cases} x > 0 \\ x \neq 1 \end{cases} \Leftrightarrow x \in (0,1) \cup (1,+\infty)$

thus $A = (0,1) \cup (1,+\infty)$.

$\log_x 64 = 4 \Leftrightarrow x^4 = 64 \Leftrightarrow x^2 = 8 \lor x^2 = -8 \Leftrightarrow$

$\Leftrightarrow x^2 = 8 \Leftrightarrow x = 2\sqrt{2} \lor x = -2\sqrt{2}$

$\Leftrightarrow x = 2\sqrt{2}$ (Reject $x = -2\sqrt{2}$).

Thus $S = \{2\sqrt{2}\}$.

b) Solve $\log x = -2$

Solution

Domain: Require $x > 0$, thus $A = (0,+\infty)$.

$\log x = -2 \Leftrightarrow x = 10^{-2} \Leftrightarrow x = 0.01 \longleftarrow$ accepted

thus $S = \{0.01\}$.

c) Solve: $\log_{1/2}(x^2 - 4x) = -2$

Solution

Domain: Require $x^2 - 4x > 0 \Leftrightarrow x(x-4) > 0 \Leftrightarrow$

$\Leftrightarrow x \in (-\infty,0) \cup (4,+\infty)$.

x		0		4	
x	−	0	+		+
x−4	−		−	0	+
	+	0	−	0	+

thus $A = (-\infty, 0) \cup (4, +\infty)$.

$\log_{1/2}(x^2 - 4x) = -2 \Leftrightarrow x^2 - 4x = (1/2)^{-2}$

$\Leftrightarrow x^2 - 4x = 4 \Leftrightarrow x^2 - 4x - 4 = 0 \quad (1).$

$\Delta = b^2 - 4ac = (-4)^2 - 4 \cdot 1 \cdot (-4) = 16 + 16 = 32 \Rightarrow (4\sqrt{2})^2$

$\Rightarrow x_{1,2} = \dfrac{-(-4) \pm 4\sqrt{2}}{2 \cdot 1} = +2 \pm 2\sqrt{2}$

$$\left.\begin{array}{l} 2 - 2\sqrt{2} < 0 \Rightarrow 2 - 2\sqrt{2} \in A \\ 2 + 2\sqrt{2} > 2 + 2 = 4 \Rightarrow 2 + 2\sqrt{2} \in A \end{array}\right\} \Rightarrow$$

$\Rightarrow S = \{2 - 2\sqrt{2}, 2 + 2\sqrt{2}\}$.

↳ We use the equation's domain to accept or reject solutions.

d) Solve: $\log_x 81 = (\log_x 3)^2 + 4$

Solution

Domain: Require $\begin{cases} x > 0 \\ x \neq 1 \end{cases}$, thus $A = (0, 1) \cup (1, +\infty)$.

Define $y = \log_x 3$ and note that

$\log_x 81 = \log_x 3^4 = 4 \log_x 3 = 4y$.

It follows that:

$\log_x 81 = (\log_x 3)^2 + 4 \Leftrightarrow 4y = y^2 + 4 \Leftrightarrow$

$\Leftrightarrow y^2 - 4y + 4 = 0 \Leftrightarrow (y-2)^2 = 0 \Leftrightarrow y - 2 = 0 \Leftrightarrow y = 2$

$\Leftrightarrow \log_x 3 = 2 \Leftrightarrow 3 = x^2 \Leftrightarrow x = \sqrt{3} \vee x = -\sqrt{3}$

Since:

$\sqrt{3} \in A$ and $-\sqrt{3} \notin A$

it follows that $S = \{\sqrt{3}\}$.

e) $\ln(x+2) + \ln(x+1) = \ln 6$

Solution

Require: $\begin{cases} x+2>0 \\ x+1>0 \end{cases} \Leftrightarrow \begin{cases} x>-2 \\ x>-1 \end{cases} \Leftrightarrow x > -1$

thus, domain is $A = (-1, +\infty)$. It follows that

$\ln(x+2) + \ln(x+1) = \ln 6 \Leftrightarrow \ln[(x+2)(x+1)] = \ln 6 \Leftrightarrow$

$\Leftrightarrow (x+2)(x+1) = 6 \Leftrightarrow x^2 + 3x + 2 = 6 \Leftrightarrow x^2 + 3x + 2 - 6 = 0$

$\Leftrightarrow x^2 + 3x - 4 = 0 \Leftrightarrow (x+4)(x-1) = 0 \Leftrightarrow$

$\Leftrightarrow x + 4 = 0 \vee x - 1 = 0 \Leftrightarrow x = -4 \vee x = 1$

Since $-4 \notin A$ and $1 \in A$, then $S = \{1\}$.

f) $\ln(\ln(x^2+x)) = 0$

Solution

Require $\begin{cases} x^2+x>0 \\ \ln(x^2+x)>0 \end{cases} \Leftrightarrow \begin{cases} x(x+1)>0 \\ \ln(x^2+x)>\ln 1 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x(x+1)>0 \\ x^2+x>1 \end{cases} \Leftrightarrow \begin{cases} x(x+1)>0 & (1) \\ x^2+x-1>0 & (2) \end{cases}$

We note that for (1):

$x(x+1)>0 \Leftrightarrow \underline{x\in(-\infty,0)\cup(1,+\infty)}$

x		0		1	
x	$-$	0	$+$		$+$
$x+1$	$-$		$-$	0	$+$
	$+$	0	$-$	0	$+$

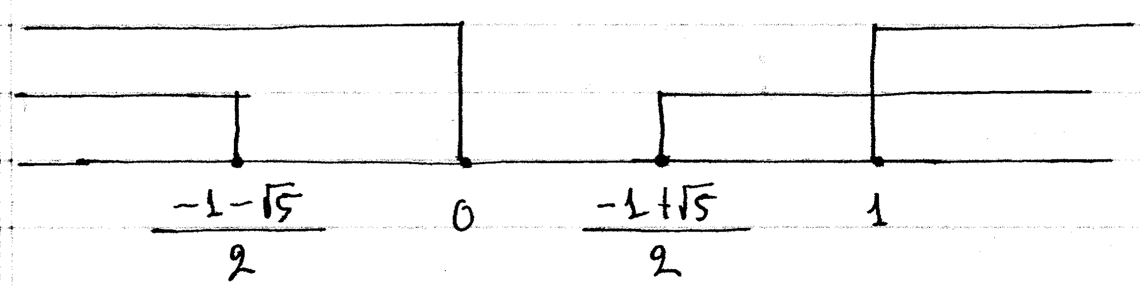
and for (2): $\Delta=b^2-4ac=1^2-4\cdot1\cdot(-1)=1+4=5 \Rightarrow$

$$\Rightarrow x_{1,2}=\frac{-b\pm\sqrt{\Delta}}{2a}=\frac{-1\pm\sqrt5}{2\cdot1}$$

x		$(-1-\sqrt5)/2$		$(-1+\sqrt5)/2$	
x^2+x-1	$+$	0	$-$	0	$+$

thus:

$x^2+x-1>0 \Leftrightarrow \underline{x\in\left(-\infty,\frac{-1-\sqrt5}{2}\right)\cup\left(\frac{-1+\sqrt5}{2},+\infty\right)}.$



It follows that the domain is:

$$A=\left[(-\infty,0)\cup(1,+\infty)\right]\cap\left[\left(-\infty,\frac{-1-\sqrt{5}}{2}\right)\cup\left(\frac{-1+\sqrt{5}}{2},+\infty\right)\right]$$

$$=\left(-\infty,\frac{-1-\sqrt{5}}{2}\right)\cup(1,+\infty).$$

Solving the equation gives:

$$\ln(\ln(x^2+x))=0 \Leftrightarrow \ln(\ln(x^2+x))=\ln 1 \Leftrightarrow$$

$$\Leftrightarrow \ln(x^2+x)=1 \Leftrightarrow \ln(x^2+x)=\ln e \Leftrightarrow$$

$$\Leftrightarrow x^2+x=e \Leftrightarrow x^2+x-e=0$$

$$\Delta=b^2-4ac=1^2-4\cdot1\cdot(-e)=1+4e \Rightarrow$$

$$\Rightarrow x_{1,2}=\frac{-b\pm\sqrt{\Delta}}{2a}=\frac{-1\pm\sqrt{1+4e}}{2}$$

For $x_1=\frac{-1-\sqrt{1+4e}}{2}<\frac{-1-\sqrt{5}}{2} \Rightarrow \frac{-1-\sqrt{1+4e}}{2}\in A.$

For $x_2=\frac{-1+\sqrt{1+4e}}{2}>\frac{-1+\sqrt{1+4\cdot2}}{2}=\frac{-1+\sqrt{1+8}}{2}=$

$$=\frac{-1+\sqrt{9}}{2}=\frac{3-1}{2}=1 \Rightarrow \frac{-1+\sqrt{1+4e}}{2}\in A$$

Since we accept both solutions:

$$S=\left\{\frac{-1-\sqrt{1+4e}}{2},\frac{-1+\sqrt{1+4e}}{2}\right\}.$$

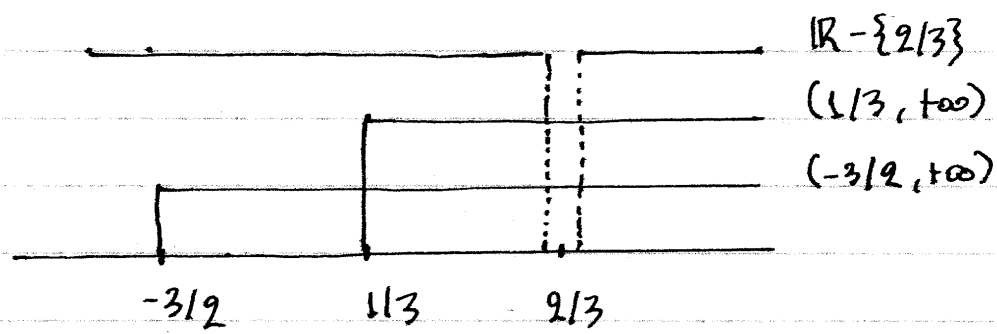
g) $\log_9(2x+3)\log_{3x-1}3 = 1$

<u>Solution</u>

Require $\begin{cases} 2x+3>0 \\ 3x-1>0 \\ 3x-1\neq 1 \end{cases} \Leftrightarrow \begin{cases} 2x>-3 \\ 3x>1 \\ 3x\neq 1+1 \end{cases} \Leftrightarrow \begin{cases} x>-3/2 \\ x>1/3 \\ x\neq 2/3 \end{cases}$

thus domain of equation is:

$$A = (-3/2, +\infty)\cap(1/3, +\infty)\cap(\mathbb{R}-\{2/3\}) =$$
$$= (1/3, 2/3)\cup(2/3, +\infty).$$



Solving the equation:

$$\log_9(2x+3)\log_{3x-1}3 = 1 \Leftrightarrow \frac{\ln(2x+3)}{\ln 9}\,\frac{\ln 3}{\ln(3x-1)} = 1$$

$$\Leftrightarrow \frac{\ln(2x+3)}{\ln(3x-1)}\,\frac{\ln 3}{2\ln 3} = 1 \Leftrightarrow \frac{\ln(2x+3)}{2\ln(3x-1)} = 1 \Leftrightarrow$$

$$\Leftrightarrow \ln(2x+3) = 2\ln(3x-1) \Leftrightarrow \ln(2x+3) = \ln[(3x-1)^2]$$

$$\Leftrightarrow 2x+3 = (3x-1)^2 \Leftrightarrow 2x+3 = 9x^2-6x+1 \Leftrightarrow$$

$$\Leftrightarrow 9x^2+(-6-2)x+1-3 = 0 \Leftrightarrow 9x^2-8x-2 = 0$$

$$\Delta = b^2 - 4ac = (-8)^2 - 4 \cdot 9 \cdot (-2) = 64 + 72 = 136 = 2^3 \cdot 17 \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{-(-8) \pm 2\sqrt{34}}{2 \cdot 9} =$$

$$= \frac{4 \pm \sqrt{34}}{9}$$

$$\begin{array}{r|l} 136 & 2 \\ 68 & 2 \\ 34 & 2 \\ 17 & 17 \\ 1 & \end{array}$$

To accept/reject solutions, we note that

$$\left.\begin{array}{l} \text{for } x_1 = \dfrac{4+\sqrt{34}}{9} > \dfrac{4}{9} > \dfrac{3}{9} = \dfrac{1}{3} \\ \text{and } \dfrac{4+\sqrt{34}}{9} \neq \dfrac{2}{3} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \frac{4+\sqrt{34}}{9} \in A.$$

$$\text{for } x_2 = \frac{4-\sqrt{34}}{9} = \frac{\sqrt{16}-\sqrt{34}}{9} < 0 \Rightarrow$$

$$\Rightarrow \frac{4-\sqrt{34}}{9} \notin A$$

It follows that $S = \left\{ \dfrac{4+\sqrt{34}}{9} \right\}$.

EXERCISES

(16) Solve the equations (1st form)

1) $\log_x\left(\frac{81}{16}\right) = 4$ 2) $\log_x \sqrt{8} = \frac{3}{4}$

3) $\log_x 25 = 8$ 4) $\log_x 16 = \frac{2}{3}$ 5) $\log_x 5 = \frac{1}{3}$

6) $\log_x 16 = -2$ 7) $\log_x \frac{1}{81} = -4$ 8) $\log_x 64 = -2$

(17) Solve the equations (2nd form)

1) $\log_4 x = 3$ 2) $\log x = -3$ 3) $\ln x = 2$

4) $\log_8 x = -\frac{7}{3}$ 5) $\log_8 x = -\frac{7}{3}$ 6) $\log_{2\sqrt[3]{5}} x = -6$

7) $\log_3 (x^2 - x + 3) = 2$ 8) $\log (x^2 - 5x + 16) = 1$

9) $\log_{1/2} (x^2 - 3x) = -1$

(18) Solve the equations (use substitution)

1) $2(\log_x 8)^2 + \log_x 64 + \log_x 8 = 9$

2) $\log_x 256 = (\log_x 4)^2 + 3$

(19) Solve the equations (3rd form)

1) $\log(4x-1) = 2\log 2 + \log(x^2-1)$

2) $\frac{1}{2}\log(x+2) + \log\sqrt{x+3} = 1 + \log\sqrt{3}$

3) $2\log x - \log(x+1) = \log 4 - \log 3$

4) $\log_4(x+2) - \log_4(x-3) = 3$

5) $\log_3 x \cdot \log_9 x = 2$

6) $\log_x 2 + \log_2 x = \frac{5}{2}$

7) $\log[\log(2x^2+x-2)] = 0$

8) $\log[\log(2x^2+x-11)] = 0$

(20) Solve the equations

1) $\log_4(x+12)\log_x 2 = 1$

2) $\log_x(5x^2)[\log_5 x]^2 = 1$

3) $\log_4(\log_3(\log_2 x)) = 0$

Equations with exponentials

① Form : $\boxed{a^{f(x)} = b} \Leftrightarrow \ln a^{f(x)} = \ln b$

$\Leftrightarrow f(x) \ln a = \ln b$

$\Leftrightarrow \ldots$

EXAMPLE

a) Solve : $5^{x^2+x} = 2$.

Solution

$$5^{x^2+x} = 2 \Leftrightarrow \ln 5^{x^2+x} = \ln 2 \Leftrightarrow (x^2+x)\ln 5 = \ln 2$$

$$\Leftrightarrow (\ln 5)x^2 + (\ln 5)x - \ln 2 = 0.$$

$$\Delta = b^2 - 4ac = (\ln 5)^2 - 4(\ln 5)(-\ln 2)$$

$$= (\ln 5 + 4\ln 2)\ln 5 \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-\ln 5 \pm \sqrt{(\ln 5 + 4\ln 2)\ln 5}}{2\ln 5}$$

thus

$$S = \left\{ \frac{-\ln 5 - \sqrt{(\ln 5 + 4\ln 2)\ln 5}}{2\ln 5}, \frac{-\ln 5 + \sqrt{(\ln 5 + 4\ln 2)\ln 5}}{2\ln 5} \right\}$$

② Form : $\boxed{a^{f(x)} = b^{g(x)}} \Leftrightarrow \ln a^{f(x)} = \ln b^{g(x)} \Leftrightarrow$

$\Leftrightarrow f(x)\ln a = g(x)\ln b$

$\Leftrightarrow \ldots$

EXAMPLE

Solve: $3^{2x+1} = 7^{3x-2}$.

Solution

$$3^{2x+1} = 7^{3x-2} \Leftrightarrow \ln 3^{2x+1} = \ln 7^{3x-2} \Leftrightarrow$$
$$\Leftrightarrow (2x+1)\ln 3 = (3x-2)\ln 7 \Leftrightarrow$$
$$\Leftrightarrow (2\ln 3)x + \ln 3 = (3\ln 7)x - 2\ln 7 \Leftrightarrow$$
$$\Leftrightarrow (2\ln 3 - 3\ln 7)x = -\ln 3 - 2\ln 7 \Leftrightarrow$$
$$\Leftrightarrow x = \frac{-\ln 3 - 2\ln 7}{2\ln 3 - 3\ln 7} = \frac{2\ln 7 + \ln 3}{3\ln 7 - 2\ln 3}$$

③ Form $\boxed{f(a^x) = g(a^x)}$ $\longrightarrow$ Let $y = a^x$ and solve $f(y) = g(y)$ first.

EXAMPLE

Solve $e^x - e^{-x} = 2$.

Solution

Let $y = e^x$. Then $e^{-x} = \frac{1}{e^x} = \frac{1}{y}$, and it follows that:

$$e^x - e^{-x} = 2 \Leftrightarrow y - \frac{1}{y} = 2 \Leftrightarrow y^2 - 1 = 2y \Leftrightarrow$$
$$\Leftrightarrow y^2 - 2y - 1 = 0 \quad (1)$$

$$\Delta = b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot (-1) = 4 + 4 = 8 \Rightarrow$$

$$\Rightarrow \quad y_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-2) \pm 2\sqrt{2}}{2 \cdot 1} = -1 \pm \sqrt{2}$$

It follows that

$y = -1 - \sqrt{2} \vee y = -1 + \sqrt{2} \Leftrightarrow$

$\Leftrightarrow e^x = -1 - \sqrt{2} \vee e^x = -1 + \sqrt{2} \Leftrightarrow$

$\Leftrightarrow e^x = -1 + \sqrt{2} \Leftrightarrow \ln e^x = \ln(\sqrt{2} - 1) \Leftrightarrow$

$\Leftrightarrow x = \ln(\sqrt{2} - 1)$

and therefore $S = \{\ln(\sqrt{2} - 1)\}$

↳ Note that the equation $e^x = -1 - \sqrt{2}$ is <u>inconsistent</u> because $-1 - \sqrt{2} < 0$. and $e^x > 0, \forall x \in \mathbb{R}$.

④ <u>Form</u> : $\boxed{A \cdot a^x = B b^x} \Leftrightarrow \ln(A a^x) = \ln(B b^x) \Leftrightarrow$

$\Leftrightarrow \ln A + x \ln a = \ln B + x \ln b \Leftrightarrow$

$\Leftrightarrow \ldots$

EXAMPLE

Solve: $2^{x+4} - 5^{x+2} = 2^{x+2} - 5^x$

<u>Solution</u>

$2^{x+4} - 5^{x+2} = 2^{x+2} - 5^x \Leftrightarrow 2^{x+4} - 2^{x+2} = 5^{x+2} - 5^x \Leftrightarrow$

$\Leftrightarrow 2^{x+2}(2^2 - 1) = 5^x(5^2 - 1) \Leftrightarrow 3 \cdot 2^{x+2} = 24 \cdot 5^x$

$\Leftrightarrow 2^{x+2} = 8 \cdot 5^x \Leftrightarrow \ln(2^{x+2}) = \ln(8 \cdot 5^x) \Leftrightarrow$

$$\Leftrightarrow (x+2)\ln 2 = \ln 8 + x\ln 5 \Leftrightarrow$$
$$\Leftrightarrow (\ln 2)x + 2\ln 2 = 3\ln 2 + x\ln 5 \Leftrightarrow$$
$$\Leftrightarrow (\ln 2 - \ln 5)x = 3\ln 2 - 2\ln 2 \Leftrightarrow$$
$$\Leftrightarrow (\ln 2 - \ln 5)x = \ln 2 \Leftrightarrow$$
$$\Leftrightarrow x = \frac{\ln 2}{\ln 2 - \ln 5}$$

thus $S = \left\{ \dfrac{\ln 2}{\ln 2 - \ln 5} \right\}$

⑤ Form : $\boxed{Aa^{2x} + Ba^x b^x + Cb^{2x} = 0} \Leftrightarrow$

$$\Leftrightarrow A\frac{a^{2x}}{b^{2x}} + B\frac{a^x b^x}{b^{2x}} + C\frac{b^{2x}}{b^{2x}} = 0 \Leftrightarrow$$
$$\Leftrightarrow A\left(\frac{a}{b}\right)^{2x} + B\left(\frac{a}{b}\right)^x + C = 0$$

Let $y = \left(\frac{a}{b}\right)^x$, and the equation yields:

$$Ay^2 + By + C = 0 \Leftrightarrow \dots \Leftrightarrow y = y_1 \vee y = y_2 \Leftrightarrow$$
$$\Leftrightarrow \left(\frac{a}{b}\right)^x = y_1 \vee \left(\frac{a}{b}\right)^x = y_2 \Leftrightarrow \dots$$

EXAMPLE

Solve: $2^{2x+1} + 5\cdot 10^x - 5^{2x} = 0$

Solution

$$2^{2x+1}+5\cdot 10^x-5^{2x}=0 \Leftrightarrow 2\cdot 2^{2x}+5\cdot 2^x\cdot 5^x-5^{2x}=0$$

$$\Leftrightarrow 2\left(\frac{2}{5}\right)^{2x}+5\left(\frac{2}{5}\right)^x-1=0 \qquad (1)$$

Let $y=(2/5)^x$. It follows that

$$(1) \Leftrightarrow 2y^2+5y-1=0 \qquad (2)$$

$$\Delta^2=5^2-4\cdot 2\cdot(-1)=25+8=33 \Rightarrow$$

$$\Rightarrow y_{1,2}=\frac{-b\pm\sqrt{\Delta}}{2a}=\frac{-5\pm\sqrt{33}}{2\cdot 2}=\frac{-5\pm\sqrt{33}}{4}$$

and therefore:

$$(2/5)^x=\frac{-5-\sqrt{33}}{4} \vee (2/5)^x=\frac{-5+\sqrt{33}}{4} \Leftrightarrow$$

$$\Leftrightarrow (2/5)^x=\frac{\sqrt{33}-5}{4} \Leftrightarrow \ln(2/5)^x=\ln\left(\frac{\sqrt{33}-5}{4}\right) \Leftrightarrow$$

$$\Leftrightarrow x\ln(2/5)=\ln(\sqrt{33}-5)-\ln 4 \Leftrightarrow$$

$$\Leftrightarrow x(\ln 2-\ln 5)=\ln(\sqrt{33}-5)-\ln 4 \Leftrightarrow$$

$$\Leftrightarrow x=\frac{\ln(\sqrt{33}-5)-2\ln 2}{\ln 2-\ln 5}$$

Thus: $S=\left\{\frac{\ln(\sqrt{33}-5)-2\ln 2}{\ln 2-\ln 5}\right\}$

EXERCISES

(21) Solve the equations

a) $3^{x^2-5x+11} = 243$

b) $7^{2-|3x|} = 1$

c) $5^{\sqrt{x}} = 625$

d) $4^{x^3-5x^2+6x+3} = 64$

e) $5^{x^4-10x^2+9} = 1$

f) $5^{3x-2} = 7$

g) $2^{2x} = 3^{x+1}$

h) $e^{2x} - 3e^x + 2 = 0$

i) $2^x + \dfrac{6}{2^x} = 5$

(22) Solve the equations

a) $2 \cdot 9^x - 7 \cdot 3^x + 3 = 0$

b) $4^x - 7 \cdot 2^x - 8 = 0$

c) $9^x - 3^{x+1} - 3^x + 3 = 0$

d) $5^{2x-1} + 3 \cdot 5^{x+1} = 80$

e) $2^{2x+1} + 1 = 3 \cdot 2^x$

f) $3 \cdot \left(\frac{3}{2}\right)^x + 2\left(\frac{2}{3}\right)^x = 5$

g) $3^{x+1} - 2^x = 3^{x-1} + 2^{x+3}$

h) $3^{2x+1} - 5 \cdot 6^x + 2 \cdot 4^x = 0$

i) $5 \cdot 3^{2x} + 3 \cdot 25^x = 8 \cdot 15^x$

j) $5^{x-2} - 3 \cdot 2^{x-3} = 7 \cdot 5^{x-3} - 2^x$

k) $3^{x+2} + 9^{x-1} = 1458$