Lecture Notes on Calculus 3

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Trigonometric identities

Trigonometric identities

$$\boxed{a \pm b} \Downarrow \qquad\qquad \boxed{2a} \Downarrow$$

$$\left.\begin{aligned}
\sin(a \pm b) &= \sin a \cos b \pm \sin b \cos a \\
\cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\
\tan(a \pm b) &= \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b} \\
\cot(a \pm b) &= \frac{\cot a \cot b \mp 1}{\cot b \pm \cot a} \quad \textbf{(!!)}
\end{aligned}\right\} \Longrightarrow
\begin{aligned}
\sin(2a) &= 2 \sin a \cos a \\
\cos(2a) &= \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a \\
\tan(2a) &= \frac{2 \tan a}{1 - \tan^2 a} \\
\cot(2a) &= \frac{\cot^2 a - 1}{2 \cot a}
\end{aligned}$$

- ▶ $\sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b$
- ▶ $\cos(a+b)\cos(a-b) = \cos^2 a - \sin^2 b$

$$\boxed{3a} \Longrightarrow \quad \begin{aligned} \sin(3a) &= -4\sin^3 a + 3 \sin a \\ \cos(3a) &= +4\cos^3 a - 3\cos a \end{aligned} \qquad \tan(3a) = \frac{3 \tan a - \tan^3 a}{1 - 3\tan^2 a}$$

In terms of

$$\boxed{\cos 2a} \Downarrow \qquad\qquad \boxed{\tan(a/2)} \Downarrow$$

$$\sin^2 a = \frac{1 - \cos(2a)}{2} \quad \cos^2 a = \frac{1 + \cos(2a)}{2} \quad \sin a = \frac{2\tan(a/2)}{1 + \tan^2(a/2)} \quad \cos a = \frac{1 - \tan^2(a/2)}{1 + \tan^2(a/2)}$$

$$\tan^2 a = \frac{1 - \cos(2a)}{1 + \cos(2a)} \quad \cot^2 a = \frac{1 + \cos(2a)}{1 - \cos(2a)} \quad \tan a = \frac{2\tan(a/2)}{1 - \tan^2(a/2)} \quad \cot a = \frac{1 - \tan^2(a/2)}{2\tan(a/2)}$$

Transformation to

$$\boxed{\text{sum}} \Downarrow \qquad\qquad \boxed{\text{product}} \Downarrow$$

$$\left.\begin{aligned}
2\sin a \cos b &= \sin(a-b) + \sin(a+b) \\
2\cos a \cos b &= \cos(a-b) + \cos(a+b) \\
2\sin a \sin b &= \cos(a-b) - \cos(a+b)
\end{aligned}\right\} \Longrightarrow
\begin{aligned}
\sin a \pm \sin b &= 2 \sin \frac{a \pm b}{2} \cos \frac{a \mp b}{2} \\
\cos a + \cos b &= 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2} \\
\cos a - \cos b &= 2 \sin \frac{a+b}{2} \sin \frac{b-a}{2} \quad \textbf{(!!)} \\
\tan a \pm \tan b &= \frac{\sin(a \pm b)}{\cos a \cos b} \\
\cot a \pm \cot b &= \frac{\sin(b \mp a)}{\sin a \sin b} \quad \textbf{(!!)}
\end{aligned}$$

Also note the factorizations:

- ▶ $1 \pm \sin a = \sin(\pi/2) \pm \sin a = 2 \sin \dfrac{(\pi/2) \pm a}{2} \cos \dfrac{(\pi/2) \mp a}{2}$
- ▶ $\sin a \pm \cos b = \sin a \pm \sin(\pi/2 - b) = 2 \sin \dfrac{a \pm (\pi/2 - b)}{2} \cos \dfrac{a \mp (\pi/2 - b)}{2}$
- ▶ $1 + \cos a = 2\cos^2(a/2)$
- ▶ $1 - \cos a = 2\sin^2(a/2)$

CAL3.1: Vectors in 3d

VECTORS IN $\mathbb{R}^3$

Cartesian product

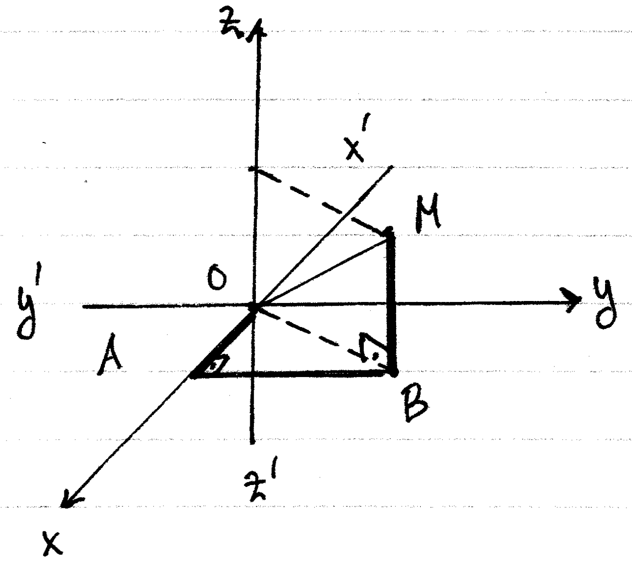
Let A, B, C be three sets. We define the Cartesian products

$$A \times B = \{(a,b) \mid a \in A \land b \in B\}$$
$$A \times B \times C = \{(a,b,c) \mid a \in A \land b \in B \land c \in C\}$$

where (a,b) is an <u>ordered pair</u> and (a,b,c) is an <u>ordered triplet</u>

- The behavior of ordered pairs and ordered triplets is covered by the following axioms:
$$(a_1, a_2) = (b_1, b_2) \Leftrightarrow a_1 = b_1 \land a_2 = b_2$$
$$(a_1, a_2, a_3) = (b_1, b_2, b_3) \Leftrightarrow a_1 = b_1 \land a_2 = b_2 \land a_3 = b_3$$
- Geometrical three-dimensional space can be represented as
$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) \mid x \in \mathbb{R} \land y \in \mathbb{R} \land z \in \mathbb{R}\}$$
with $\mathbb{R}$, the set of all real numbers.
An element $(x,y,z) \in \mathbb{R}^3$ represents a point in space with cartesian coordinates x, y, z, as defined below.
- Likewise, geometrical two-dimensional space can be represented as
$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R} \land y \in \mathbb{R}\}$$
An element $(x,y) \in \mathbb{R}^2$ represents a point on a plane with coordinates (x,y).

Cartesian coordinates



The cartesian coordinate system consists of three lines $(x'x)$, $(y'y)$, $(z'z)$ such that

$$\begin{cases} (x'x) \perp (y'y) \perp (z'z) \perp (x'x) \\ (x'x) \cap (y'y) \cap (z'z) = \{O\} \end{cases}$$

where O is the origin of the coordinate system.

Let $M(x, y, z)$ be a point with coordinates (x, y, z).
We define x, y, z as follows:
Let B be the projection of M to the xy plane.
Let A be the projection of B to the $x'x$ axis.
Then, we define:

$$\begin{cases} x = \overline{OA} \\ y = \overline{AB} \\ z = \overline{BM} \end{cases}$$

The bar indicates using directional distance. For example, $\overline{OA}$ is positive or negative depending on whether A is on the Ox or the Ox' ray.

<u>Terminology</u>: x-axis is the line $x'Ox$
y-axis is the line $y'Oy$
z-axis is the line $z'Oz$

Likewise:

xy plane: plane defined by $(x'x)$ and $(y'y)$
yz plane: plane defined by $(y'y)$ and $(z'z)$
zx plane: plane defined by $(z'z)$ and $(x'x)$

Distance Formula

(1) $\boxed{M(x,y,z) \Rightarrow OM = \sqrt{x^2+y^2+z^2}}$

Proof

$OA \perp AB \Rightarrow O\hat{A}B$ right triangle with $\hat{A} = 90°$
$\Rightarrow OB^2 = OA^2 + AB^2 = x^2+y^2$
$OB \perp BM \Rightarrow O\hat{B}M$ right triangle with $\hat{B} = 90°$
$\Rightarrow OM^2 = OB^2 + BM^2 = OB^2 + z^2 =$
$= (x^2+y^2) + z^2$
$\Rightarrow OM = \sqrt{x^2+y^2+z^2}$ □

(2) $\boxed{\left.\begin{array}{l} A(x_1,y_1,z_1) \\ B(x_2,y_2,z_2) \end{array}\right\} \Rightarrow AB = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}$

Proof

Using A as the origin of a coordinates system where the axises are parallel and similarly oriented with the coordinate system around O, the coordinates of B are $(x_2-x_1, y_2-y_1, z_2-z_1)$. Using the previous result we immediately calculate AB □.

The sphere (A, r) with center A and radius r is defined as:

$(A, r) = \{ M \in \mathbb{R}^3 \mid AM = r \}$

and therefore:

$(A, r) : (x - x_A)^2 + (y - y_A)^2 + (z - z_A)^2 = r^2$

with $A(x_A, y_A, z_A)$.

EXAMPLES

Find all $\lambda \in \mathbb{R}$ such that $AB = a$ with $A(1, \lambda+1, \lambda-1)$ and $B(\lambda, \lambda+1, -1)$

Solution

Since,

$$AB^2 = (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 =$$
$$= (1-\lambda)^2 + [(\lambda+1)-(\lambda+1)]^2 + [(\lambda-1)-(-1)]^2 =$$
$$= (1-\lambda)^2 + 0^2 + (\lambda - 1 + 1)^2 = (1-\lambda)^2 + \lambda^2 =$$
$$= 1 - 2\lambda + \lambda^2 + \lambda^2 = 2\lambda^2 - 2\lambda + 1 \Rightarrow$$
$$\Rightarrow AB = \sqrt{2\lambda^2 - 2\lambda + 1}$$

it follows that

$$AB = a \Leftrightarrow \sqrt{2\lambda^2 - 2\lambda + 1} = a \qquad [\text{Require } a \geq 0]$$
$$\Leftrightarrow 2\lambda^2 - 2\lambda + 1 = a^2 \Leftrightarrow 2\lambda^2 - 2\lambda + 1 - a^2 = 0 \qquad (1)$$

The corresponding discriminant is:

$$\Delta = (-2)^2 - 4 \cdot 2 \cdot (1 - a^2) = 4 - 8(1-a^2) = 4 - 8 + 8a^2$$
$$= 8a^2 - 4 = 4(2a^2 - 1) = 4(\sqrt{2}a - 1)(\sqrt{2}a + 1)$$

and note the sign of Δ; where we require $a \geq 0$

a		$-1/\sqrt{2}$		0		$1/\sqrt{2}$	
$\sqrt{2}a - 1$	$-$		$-$		$-$	0	$+$
$\sqrt{2}a + 1$	$-$	0	$+$		$+$		$+$
Δ					$-$	0	$+$

We distinguish between the following cases

<u>Case 1</u>: Assume that $a \in (0, 1/\sqrt{2})$. Then $\Delta < 0$, and therefore Eq. (1) has no real solutions.

<u>Case 2</u>: Assume that $a = 1/\sqrt{2}$. Then $\Delta = 0$, and therefore Eq. (1) has one real solution:

$\lambda = -(-2)/(2 \cdot 2) = 1/2$

<u>Case 3</u>: Assume that $a \in (1/\sqrt{2}, +\infty)$. Then $\Delta > 0$, and therefore Eq. (1) has two real solutions

$$\lambda_{1,2} = \frac{-(-2) \pm \sqrt{4(2a^2-1)}}{2 \cdot 2} = \frac{2 \pm 2\sqrt{2a^2-1}}{2 \cdot 2}$$
$$= \frac{1 \pm \sqrt{2a^2-1}}{2}$$

We conclude that

$AB = a \Leftrightarrow \lambda \in S$

with

$$S = \begin{cases} \{[1+\sqrt{2a^2-1}]/2, [1-\sqrt{2a^2-1}]/2\} & , \text{if } a \in (1/\sqrt{2}, +\infty) \\ \{1/2\} & , \text{if } a = 1/\sqrt{2} \\ \emptyset & , \text{if } a \in (0, 1/\sqrt{2}) \end{cases}$$

8) Find the center and radius of the sphere

$(c): x^2+y^2+z^2-4x+2y+6z=11.$

<u>Solution</u>

Since

$(c): x^2+y^2+z^2-4x+2y+6z=11 \Leftrightarrow$

$\Leftrightarrow (x^2-4x+4)+(y^2+2y+1)+(z^2+6z+9)=11+4+1+9$

$\Leftrightarrow (x-2)^2+(y+1)^2+(z+3)^2=25$

$\Leftrightarrow (x-2)^2+(y-(-1))^2+(z-(-3))^2=5^2$

it follows that (c) has center $A(2,-1,-3)$ and radius $r=5$.

EXERCISES

(1) Find the distance AB between the points A and B with coordinates:

a) $A(\sqrt{2}+\sqrt{3}, 2, \sqrt{2})$ and $B(\sqrt{2}-\sqrt{3}, 3, 1)$

b) $A(0, \sqrt{3}, \sqrt{5})$ and $B(2, 1+\sqrt{3}, \sqrt{2})$

c) $A(a, b, a)$ and $B(-b, a, b)$

d) $A(ab, b^2, bc)$ and $B(ac, bc, c^2)$

e) $A(at, bt, ct)$ and $B(bt, ct, at)$

(2) Find all $\lambda \in \mathbb{R}$ such that for $A(\lambda, \lambda+1, \lambda+2)$ and $B(\lambda-1, 2\lambda, 2)$ satisfy $AB = 1$.

(3) Let $\overset{\Delta}{ABC}$ be a triangle with $A(a, b, c)$, $B(b, c, a)$, and $C(c, a, b)$. Show that $\overset{\Delta}{ABC}$ is an equilateral triangle.

(4) Write the equation of a sphere with center C and radius r given by:

a) $C(1, 2, 3)$ and $r = 5$

b) $C(-2, -3, -6)$ and $r = \sqrt{5}$

c) $C(\pi, e, \sqrt{2})$ and $r = \sqrt{\pi}$

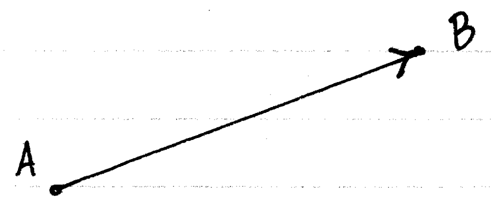
(5) Find the radius and center for the following spheres:

a) $(c): x^2+y^2+z^2-12x+14y-8z+1=0$
b) $(c): x^2+y^2+z^2+2x-6y-10z+34=0$
c) $(c): 4x^2+4y^2+4z^2-4x+8y+16z-13=0$
d) $(c): x^2+y^2+z^2+8x-4y-22z+77=0$

(6) Find the equation of the sphere with center C and passing through the point A with
a) $C(1,2,1)$ and $A(3,2,-1)$
b) $C(2,0,3)$ and $A(-1,-1,-2)$
c) $C(0,1,-1)$ and $A(2,2,3)$

Geometric vectors

- Let A, B be two points. The vector $\vec{AB}$ is a segment AB in which we define a direction from A to B.
  We say that: A is the initial point of $\vec{AB}$
  B is the terminal point of $\vec{AB}$



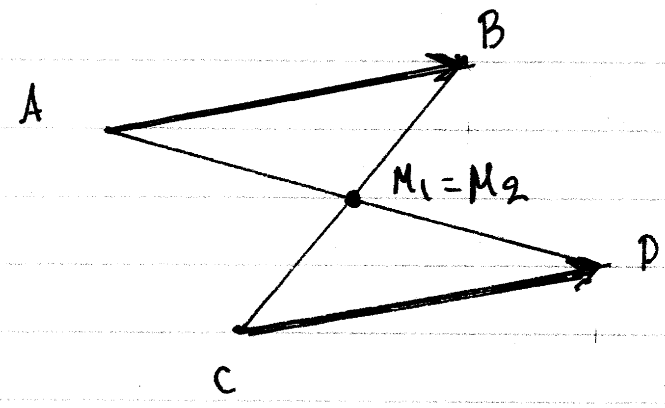
Geometric vector equivalence

Given the vectors $\vec{AB}$ and $\vec{CD}$, we say that $\vec{AB} = \vec{CD}$ if and only if the vectors $\vec{AB}$ and $\vec{CD}$ have the same length, are parallel to each other and have the same direction.

More rigorously, we give the following definition:

Def: Let $\vec{AB}$ and $\vec{CD}$ be two vectors and let
M_1 = midpoint of AD
M_2 = midpoint of BC
Then:
$$\vec{AB} = \vec{CD} \Longleftrightarrow M_1 = M_2$$

which is illustrated in the following figure:



Recall the definition of the midpoint:

$$\boxed{M = \text{midpoint of } AB \Leftrightarrow \begin{cases} x_M = (1/2)(x_A + x_B) \\ y_M = (1/2)(y_A + y_B) \\ z_M = (1/2)(z_A + z_B) \end{cases}}$$

- Coordinate representation of geometric vectors.

Choose a coordinate system and assume that $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ with respect to the choosen coordinate system. We define the coordinate representation of the vector $\vec{AB}$ as

$$\vec{AB} = (x_B - x_A, y_B - y_A, z_B - z_A)$$

Note that this representation is dependent on our choice of coordinate system.

It follows that

$$\vec{AA} = (0, 0, 0) = \mathbf{0}$$

with $\mathbf{0}$ the zero vector.

We will now show that equivalent geometric vectors have the same coordinate representation.

Prop: Let $\vec{AB}$ and $\vec{CD}$ be two vectors with $\vec{AB} = (a_1, a_2, a_3)$ and $\vec{CD} = (b_1, b_2, b_3)$

Then:

$$\vec{AB} = \vec{CD} \iff a_1 = b_1 \land a_2 = b_2 \land a_3 = b_3$$
$$\iff (a_1, a_2, a_3) = (b_1, b_2, b_3)$$

Proof

We note that

$$\vec{AB} = (a_1, a_2, a_3) \implies \begin{cases} a_1 = x_B - x_A \\ a_2 = y_B - y_A \\ a_3 = z_B - z_A \end{cases}$$

$$\vec{CD} = (b_1, b_2, b_3) \implies \begin{cases} b_1 = x_D - x_C \\ b_2 = y_D - y_C \\ b_3 = z_D - z_C \end{cases}$$

Let M be the midpoint of AD and let N be the midpoint of BC. Then:

$$\begin{cases} x_M = (1/2)(x_A + x_D) \\ y_M = (1/2)(y_A + y_D) \\ z_M = (1/2)(z_A + z_D) \end{cases} \land \begin{cases} x_N = (1/2)(x_B + x_C) \\ y_N = (1/2)(y_B + y_C) \\ z_N = (1/2)(z_B + z_C) \end{cases}$$

It follows that

$$a_1 = b_1 \iff x_B - x_A = x_D - x_C \iff x_A + x_D = x_B + x_C \iff$$
$$\iff (1/2)(x_A + x_D) = (1/2)(x_B + x_C) \iff$$
$$\iff x_M = x_N$$

and similarly, we have:

$a_2 = b_2 \Leftrightarrow y_M = y_N$
$a_3 = b_3 \Leftrightarrow z_M = z_N$

It follows that

$$\begin{aligned}
\vec{AB} = \vec{CD} &\Leftrightarrow M = N \Leftrightarrow \\
&\Leftrightarrow x_M = x_N \wedge y_M = y_N \wedge z_M = z_N \\
&\Leftrightarrow a_1 = b_1 \wedge a_2 = b_2 \wedge a_3 = b_3 \\
&\Leftrightarrow (a_1, a_2, a_3) = (b_1, b_2, b_3) \qquad \square
\end{aligned}$$

Vector operations

Vector operations are defined in terms of a particular coordinate system but result in a vector or number that is independent of our choice of coordinate system.

1 → Vector addition/subtraction

Def: Let $u, v \in \mathbb{R}^3$ be two vectors with $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in some coordinate system. We define the vectors $u+v$ and $u-v$ such that

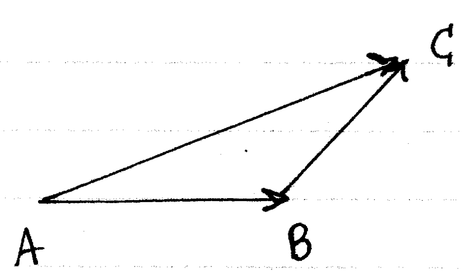
$$u+v = (u_1+v_1, u_2+v_2, u_3+v_3)$$

$$u-v = (u_1-v_1, u_2-v_2, u_3-v_3)$$

The following statement shows that the result of adding two vectors is independent of the choice of coordinate system.

Prop: Given three points A, B, C:

$$\vec{AB} + \vec{BC} = \vec{AC}$$



Proof

$$\vec{AB}+\vec{BC} = (x_B-x_A, y_B-y_A, z_B-z_A)+(x_C-x_B, y_C-y_B, z_C-z_B)$$
$$= (x_B-x_A+x_C-x_B,\ y_B-y_A+y_C-y_B,\ z_B-z_A+z_C-z_B) =$$
$$= (x_C-x_A,\ y_C-y_A,\ z_C-z_A) =$$
$$= \vec{AC} \qquad \square$$

▶ Properties of vector addition

$\forall u, v \in \mathbb{R}^3 : \ u+v = v+u$	(Commutative)
$\forall u, v, w \in \mathbb{R}^3 : \ u+(v+w) = (u+v)+w$	(Associative)
$\forall u \in \mathbb{R}^3 : \ u + \mathbf{0} = u$	(Neutral element)

② → Scalar multiplication

Def : Let $\lambda \in \mathbb{R}$ and let $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ be a vector represented in some coordinate system. We define the scalar $\lambda u \in \mathbb{R}^3$ such that

$$\lambda u = (\lambda u_1, \lambda u_2, \lambda u_3)$$

We also define:

$$-u = (-1)u = (-u_1, -u_2, -u_3)$$

▶ Properties of scalar multiplication

1) $-u$ additive inverse of u
$\forall u \in \mathbb{R}^3 : u + (-u) = \mathbf{0}$
2) Distributive over a vector sum
$\forall \lambda \in \mathbb{R} : \forall u, v \in \mathbb{R}^3 : \lambda(u+v) = \lambda u + \lambda v$
3) Distributive over a scalar sum
$\forall \lambda, \mu \in \mathbb{R} : \forall u \in \mathbb{R}^3 : (\lambda + \mu) u = \lambda u + \mu u$
4) Associative property on a mixed product
$\forall \lambda, \mu \in \mathbb{R} : \forall u \in \mathbb{R}^3 : (\lambda \mu) u = \lambda(\mu u) = \mu(\lambda u)$

EXAMPLES

a) Let $u=(1+\sqrt{2}, 2-\sqrt{2}, 3)$ and $v=(1-\sqrt{2}, 3-2\sqrt{2}, 1)$ and define $p=u+v$ and $q=u-v$. Evaluate the vector $w=2p-3q-u$

Solution

$$
\begin{aligned}
w &= 2p-3q-u = 2(u+v)-3(u-v)-u = \\
&= 2u+2v-3u+3v-u = (2-3-1)u+(2+3)v = \\
&= -2u+5v \\
&= -2(1+\sqrt{2}, 2-\sqrt{2}, 3)+5(1-\sqrt{2}, 3-2\sqrt{2}, 1) = \\
&= (-2-2\sqrt{2}, -4+2\sqrt{2}, -6)+(5-5\sqrt{2}, 15-10\sqrt{2}, 5) \\
&= (-2-2\sqrt{2}+5-5\sqrt{2}, -4+2\sqrt{2}+15-10\sqrt{2}, -6+5) \\
&= (3-7\sqrt{2}, 11-8\sqrt{2}, -1)
\end{aligned}
$$

b) Prove: $\forall \lambda,\mu \in \mathbb{R} : \forall u \in \mathbb{R}^3 : (\lambda+\mu)u = \lambda u + \mu u$

Solution

Let $\lambda,\mu \in \mathbb{R}$ and $u \in \mathbb{R}^3$ be given with $u=(u_1,u_2,u_3)$. Then:

$$
\begin{aligned}
(\lambda+\mu)u &= (\lambda+\mu)(u_1,u_2,u_3) = \\
&= ((\lambda+\mu)u_1, (\lambda+\mu)u_2, (\lambda+\mu)u_3) = \\
&= (\lambda u_1+\mu u_1, \lambda u_2+\mu u_2, \lambda u_3+\mu u_3) \\
&= (\lambda u_1, \lambda u_2, \lambda u_3)+(\mu u_1, \mu u_2, \mu u_3) = \\
&= \lambda(u_1,u_2,u_3)+\mu(u_1,u_2,u_3) = \lambda u+\mu u.
\end{aligned}
$$

EXAMPLE

(7) Let $u=(1,3,2)$ and $v=(-2,1,5)$ and define $p=u+\sqrt{2}v$ and $q=2u-\sqrt{2}v$. Evaluate the vectors $w=2p-q$ and $x=p+3q$.

(8) Let $u=(1,-1,3)$ and $v=(2,0,-1)$, and define the vectors $p=u+\sqrt{3}v$ and $q=v-2\sqrt{3}u$ and $r=u+2v$. Evaluate the vectors

a) $w=p+q+r$
b) $x=2p+\sqrt{3}q-r$
c) $y=p-\sqrt{3}q-\sqrt{3}r$
d) $z=\sqrt{3}p+2q-3\sqrt{3}r$

(9) Prove the following properties of vector addition and scalar multiplication

a) $\forall u,v\in\mathbb{R}^3 : u+v=v+u$
b) $\forall u,v,w\in\mathbb{R}^3 : u+(v+w)=(u+v)+w$
d) $\forall \lambda\in\mathbb{R} : \forall u,v\in\mathbb{R}^3 : \lambda(u+v)=\lambda u+\lambda v$
e) $\forall \lambda,\mu\in\mathbb{R} : \forall u\in\mathbb{R}^3 : (\lambda\mu)u=\lambda(\mu u)$.

3 → Scalar Product

Def: Let $u, v \in \mathbb{R}^3$ be two vectors with
$u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$
in some coordinate system. We define:

a) The dot product $u \cdot v \in \mathbb{R}$

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

b) The vector norm $\|u\| \in \mathbb{R}$

$$\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{u \cdot u}$$

▸ Immediate properties of the dot product

$\forall u, v \in \mathbb{R}^3 : u \cdot v = v \cdot u$

$\forall u, v, w \in \mathbb{R}^3 : u \cdot (v + w) = u \cdot v + u \cdot w$

$\forall \lambda \in \mathbb{R} : \forall u, v \in \mathbb{R}^3 : (\lambda u) \cdot v = u \cdot (\lambda v) = \lambda (u \cdot v)$

$\forall u \in \mathbb{R}^3 : \mathbf{0} \cdot u = 0$

▸ Immediate properties of the vector norm

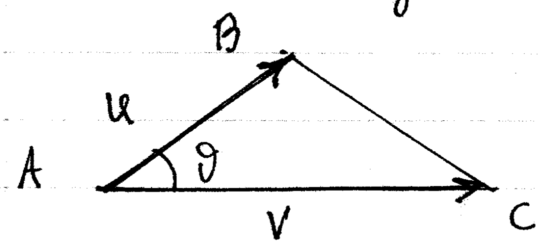
$\forall u \in \mathbb{R}^3 : \|u\|^2 = u \cdot u$

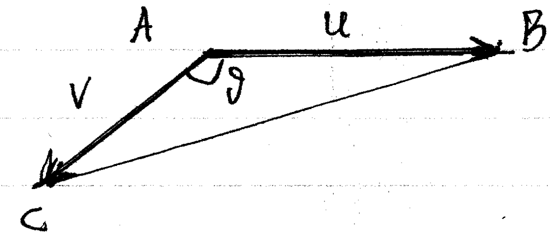
$\forall \lambda \in \mathbb{R} : \forall u \in \mathbb{R}^3 : \|\lambda u\| = |\lambda| \, \|u\|$

We will now use these properties to show that the dot product $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$ is independent of our choice of coordinate system.

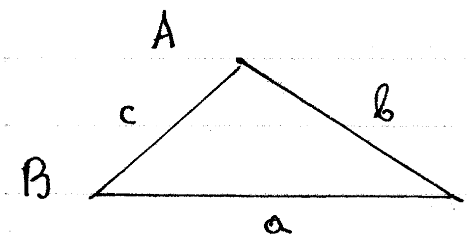
▶ Geometric interpretation of the dot product

Def: Given two vectors $u, v \in \mathbb{R}^3$ with $u = \vec{AB}$ and $v = \vec{AC}$, we define the interior angle $\vartheta(u,v)$ as the angle $\vartheta(u,v) = \hat{A}$ of the triangle $A\hat{B}C$





Our main result is based on the law of cosines from Precalculus:



Define: $a = BC \wedge b = CA \wedge c = AB$

$$\begin{cases} a^2 = b^2 + c^2 - 2bc\cos A \\ b^2 = c^2 + a^2 - 2ca\cos B \\ c^2 = a^2 + b^2 - 2ab\cos C \end{cases}$$

Thm: $\boxed{\forall u, v \in \mathbb{R}^3 : \ u \cdot v = \|u\| \|v\| \cos\vartheta(u,v)}$

Proof

Let $u = \vec{AB}$ and $v = \vec{AC}$ for some points A, B, C and let $\vartheta(u,v)$ be the interior angle of the vectors u, v.

For the triangle $A\hat{B}C$ we have

$$a = BC = \|\vec{BC}\| = \|(\vec{AB} + \vec{BC}) - \vec{AB}\| = \|\vec{AC} - \vec{AB}\| =$$
$$= \|v - u\|$$

$b = CA = \|\vec{AC}\| = \|v\|$

$c = AB = \|\vec{AB}\| = \|u\|$

and therefore, from the law of cosines,

$$\cos\vartheta(u,v) = \cos\hat{A} = \frac{b^2+c^2-a^2}{2bc} = \frac{\|v\|^2+\|u\|^2-\|v-u\|^2}{2\|v\|\|u\|} =$$

$$= \frac{\|u\|^2+\|v\|^2-(v-u)\bullet(v-u)}{2\|u\|\|v\|} =$$

$$= \frac{\|u\|^2+\|v\|^2-v\bullet(v-u)+u\bullet(v-u)}{2\|u\|\|v\|} =$$

$$= \frac{\|u\|^2+\|v\|^2-v\bullet v+v\bullet u+u\bullet v-u\bullet u}{2\|u\|\|v\|}$$

$$= \frac{\|u\|^2+\|v\|^2-\|v\|^2+u\bullet v+u\bullet v-\|u\|^2}{2\|u\|\|v\|}$$

$$= \frac{2u\bullet v}{2\|u\|\|v\|} = \frac{u\bullet v}{\|u\|\|v\|} \Rightarrow$$

$\Rightarrow u\bullet v = \|u\|\|v\|\cos\vartheta(u,v)$. $\square$

▶ Triangle inequalities.

1) $\boxed{\forall u,v \in \mathbb{R}^3 : \ |u \cdot v| \leqslant \|u\| \|v\|}$

Proof

Let $u,v \in \mathbb{R}^3$ be given. Then

$$|u \cdot v| = |\, \|u\| \|v\| \cos\vartheta(u,v)| = |[\|u\| \|v\|]| \cdot |\cos\vartheta(u,v)|$$
$$= \|u\| \|v\| \, |\cos\vartheta(u,v)| \leqslant \|u\| \|v\| . \qquad \square$$

2) $\boxed{\forall u,v \in \mathbb{R}^3 : \ \|u+v\| \leqslant \|u\| + \|v\|}$

Proof

Let $u,v \in \mathbb{R}^3$ be given. Then:

$$\|u+v\|^2 = (u+v) \cdot (u+v) = u \cdot (u+v) + v \cdot (u+v) =$$
$$= u \cdot u + u \cdot v + v \cdot u + v \cdot v =$$
$$= \|u\|^2 + u \cdot v + u \cdot v + \|v\|^2 =$$
$$= \|u\|^2 + 2u \cdot v + \|v\|^2 =$$
$$= \|u\|^2 + 2\|u\| \|v\| \cos\vartheta(u,v) + \|v\|^2 \leqslant$$
$$\leqslant \|u\|^2 + 2\|u\| \|v\| + \|v\|^2 = (\|u\| + \|v\|)^2$$
$$\Rightarrow 0 \leqslant \|u+v\|^2 \leqslant (\|u\| + \|v\|)^2 \Rightarrow$$
$$\Rightarrow \|u+v\| \leqslant \|u\| + \|v\| \qquad \square$$

EXERCISES

(10) Let $u=(1+\sqrt{2}, 1-\sqrt{2}, \sqrt{2})$ and $v=(\sqrt{2}+2, 2+3\sqrt{2}, 1+\sqrt{2})$ be given.

a) Evaluate $u\cdot v$, $\|u\|$, $\|v\|$

b) Use the previous results to calculate

1) $(2u-v)\cdot(u+3v)$ 3) $(u\cdot v+\|u\|)v$

2) $[v\cdot(u+v)]\|u\|$ 4) $[3\|u\|v]\cdot(u-2v)$

(11) Find the interior angle $\vartheta(u,v)$ between the vectors

a) $u=(1,0,3)$ and $v=(2,1,0)$

b) $u=(\sqrt{2}, 1-\sqrt{2}, 0)$ and $v=(\sqrt{2}, 1+\sqrt{2}, 1)$

c) $u=(1+\sqrt{3}, 1-\sqrt{3}, \sqrt{3})$ and $v=(1-\sqrt{3}, 1+\sqrt{3}, -\sqrt{3})$

(12) Let $\hat{ABC}$ be a triangle with $A(1,2,3)$, $B(-4,5,6)$, $C(1,0,1)$. Evaluate $\cos\hat{A}$, $\cos\hat{B}$, $\cos\hat{C}$.

(13) Show that

a) $\|u+v\|^2+\|u-v\|^2 = 2(\|u\|^2+\|v\|^2)$

b) $4(u\cdot v) = \|u+v\|^2 - \|u-v\|^2$

(14) Prove the following properties

a) $\forall u,v,w\in\mathbb{R}^3 : u\cdot(v+w) = u\cdot v + u\cdot w$

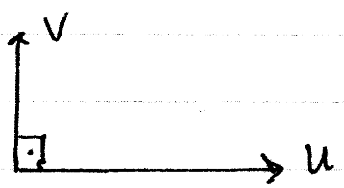
b) $\forall \lambda\in\mathbb{R} : \forall u,v\in\mathbb{R}^3 : (\lambda u)\cdot v = \lambda(u\cdot v)$

c) $\forall \lambda\in\mathbb{R} : \forall u\in\mathbb{R}^3 : \|\lambda u\| = |\lambda|\,\|u\|$

Orthogonality condition

Def: Let $u, v \in \mathbb{R}^3 - \{0\}$ be two vectors. We say that
$u \perp v \Leftrightarrow \theta(u,v) = \pi/2$

Terminology:
$u \perp v$ reads: u is orthogonal to v.



Prop: $\forall u, v \in \mathbb{R}^3 - \{0\} : u \perp v \Leftrightarrow u \cdot v = 0$

Proof

Since

$$\cos\theta(u,v) = \frac{\|u\| \|v\| \cos\theta(u,v)}{\|u\| \|v\|} = \frac{u \cdot v}{\|u\| \|v\|}$$

it follows that

$$u \perp v \Leftrightarrow \theta(u,v) = \pi/2 \Leftrightarrow \cos\theta(u,v) = 0 \Leftrightarrow$$
$$\Leftrightarrow \frac{u \cdot v}{\|u\| \|v\|} = 0 \Leftrightarrow u \cdot v = 0 \qquad \square$$

EXAMPLE

Find all $x \in \mathbb{R}$ such that $u = (0, x, x+1)$ and $v = (x, x-1, x-2)$ are orthogonal

Solution

Since,

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 =$$
$$= 0x + x(x-1) + (x+1)(x-2) =$$
$$= x^2 - x + x^2 + (1-2)x + 1\cdot(-2) =$$
$$= x^2 - x + x^2 - x - 2 = 2x^2 - 2x - 2$$

it follows that

$$u \perp v \Leftrightarrow u \cdot v = 0 \Leftrightarrow 2x^2 - 2x - 2 = 0 \Leftrightarrow$$
$$\Leftrightarrow x^2 - x - 1 = 0$$

From the quadratic formula

$$(a, b, c) = (1, -1, -1) \Rightarrow \Delta = b^2 - 4ac = (-1)^2 - 4 \cdot 1 \cdot (-1) =$$
$$= 1 + 4 = 5 \Rightarrow$$
$$\Rightarrow x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-1) \pm \sqrt{5}}{2 \cdot 1} = \frac{1 \pm \sqrt{5}}{2}$$

and therefore

$$u \perp v \Leftrightarrow x = \frac{1+\sqrt{5}}{2} \vee x = \frac{1-\sqrt{5}}{2}$$

EXAMPLE

Let $u, v \in \mathbb{R}^3$. Show that

$u+v \perp u-2v \Rightarrow u \cdot v = \|u\|^2 - 2\|v\|^2$

Solution

Assume that $u+v \perp u-2v$. Since

$$\begin{aligned}(u+v)\cdot(u-2v) &= u\cdot(u-2v) + v\cdot(u-2v) = \\ &= u\cdot u + u\cdot(-2v) + v\cdot u + v\cdot(-2v) = \\ &= \|u\|^2 - 2(u\cdot v) + u\cdot v - 2\|v\|^2 = \\ &= \|u\|^2 - u\cdot v - 2\|v\|^2\end{aligned}$$

it follows that

$$\begin{aligned}&u+v \perp u-2v \Rightarrow (u+v)\cdot(u-2v) = 0 \Rightarrow \\ &\Rightarrow \|u\|^2 - u\cdot v - 2\|v\|^2 = 0 \Rightarrow \\ &\Rightarrow u\cdot v = \|u\|^2 - 2\|v\|^2\end{aligned}$$

EXERCISES

(15) Find all $a \in \mathbb{R}$ such that $u \perp v$ when

a) $u = (a+1, a, a-1)$ and $v = (3, 1, 2)$

b) $u = (a^2-1, 3, a+1)$ and $v = (2, a+2, a)$

c) $u = (a, 3a+1, 2a-3)$ and $v = (3, a, a-1)$

(16) Let $u = (a-1, a+1, 2)$ and $v = (2, 0, a)$ be given. Find all $a \in \mathbb{R}$ such that

a) $u \perp v$

b) $u \perp (2u+3v)$

c) $u-v \perp u+v$

d) $2u+v \perp u+2v$

(Hint: It is useful to precalculate $\|u\|^2$, $\|v\|^2$, $u \cdot v$, before doing any of the subquestions).

(17) Let $v = (1, -2, -3)$ and $w = (-3, 2, 0)$. Find all vectors $u \in \mathbb{R}^3$ such that $u \perp v$ and $u \perp w$.
(Hint: You will find that

$$u \perp v \wedge u \perp w \Leftrightarrow u \in \{tp \mid t \in \mathbb{R}\}$$

for some $p \in \mathbb{R}^3$)

(18) Let $u = (x, 0, 1)$, $v = (0, 2, y)$, and $w = (1, z, 1)$. Find all $x, y, z \in \mathbb{R}$ such that $u \perp v \wedge v \perp w \wedge w \perp u$.

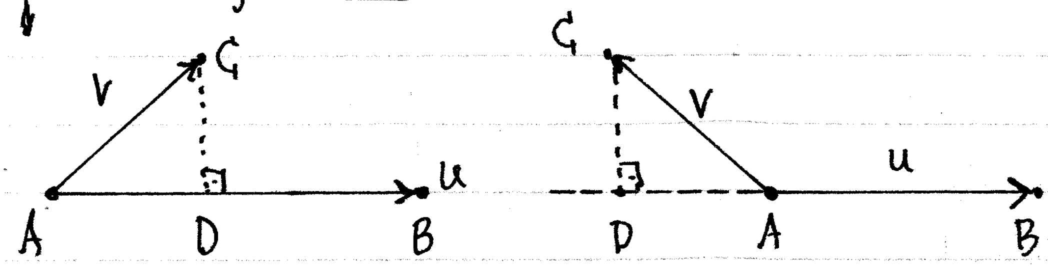
(19) Show that

a) $u \perp v \wedge u \perp w \Rightarrow \forall a, b \in \mathbb{R}: u \perp (av + bw)$

b) $u+v \perp u-v \Rightarrow \|u\| = \|v\|$

c) $\left.\begin{array}{l} u+v \perp u-w \\ u \perp v \wedge u \perp w \end{array}\right\} \Rightarrow \|u\|^2 = v \cdot w$

Projections



Def: Let $u, v \in \mathbb{R}^3 - \{0\}$ be two vectors and write $u = \vec{AB}$ and $v = \vec{AC}$. Let (ℓ) be the line defined by A, B and choose the unique $D \in (\ell)$ such that $CD \perp AB$. Then, we define

a) The projection of v onto u:

$$\mathrm{proj}_u(v) = \vec{AD}$$

b) The component $\mathrm{comp}_u(v)$ of v onto u:

$$\mathrm{proj}_u(v) = [\mathrm{comp}_u(v)] \frac{u}{\|u\|}$$

The dot product can be used to calculate both the projection and the component of v onto u:

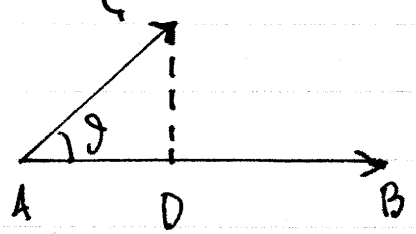
Prop: $\forall u, v \in \mathbb{R}^3 - \{0\} : \mathrm{proj}_u(v) = \dfrac{u \cdot v}{\|u\|^2} u$

$\forall u, v \in \mathbb{R}^3 - \{0\} : \mathrm{comp}_u(v) = \dfrac{u \cdot v}{\|u\|}$

Proof

Let $u, v \in \mathbb{R}^3 - \{0\}$ be given and let $\vartheta = \vartheta(u,v)$.
We distinguish between the following cases:

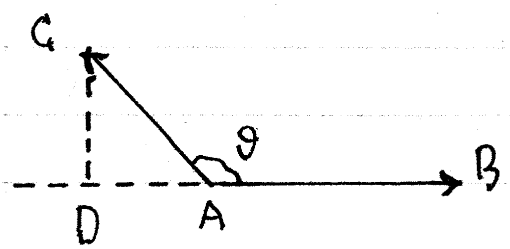
Case 1: Assume $0 \leqslant \vartheta \leqslant \pi/2$. Then



$$\mathrm{proj}_u(v) = \vec{AD} = AD\left[\frac{1}{AB}\,\vec{AB}\right] =$$

$$= \frac{AC\cos\vartheta}{AB}\,\vec{AB} =$$

Case 2: Assume that $\pi/2 < \vartheta \leqslant \pi$. Then



$$\mathrm{proj}_u(v) = \vec{AD} = AD\left[\frac{-1}{AB}\,\vec{AB}\right]$$

$$= \frac{-AC\cos(\pi-\vartheta)}{AB}\,\vec{AB} =$$

$$= \frac{-AC[-\cos(-\vartheta)]}{AB}\,\vec{AB} =$$

$$= \frac{AC\cos(-\vartheta)}{AB}\,\vec{AB} =$$

$$= \frac{AC\cos\vartheta}{AB}\,\vec{AB}$$

In both cases, we have:

$$\mathrm{proj}_u(v) = \frac{AC\cos\vartheta}{AB}\,\vec{AB} = \frac{\|\vec{AC}\|\cos\vartheta}{\|\vec{AB}\|}\,\vec{AB} =$$

$$= \frac{\|v\|\cos\vartheta}{\|u\|}\,u = \frac{\|u\|\,\|v\|\cos\vartheta}{\|u\|^2}\,u =$$

$$= \frac{u \cdot v}{\|u\|^2}\,u$$

Furthermore,

$$\mathrm{proj}_u(v) = \frac{u \cdot v}{\|u\|^2} u = \frac{u \cdot v}{\|u\|} \frac{u}{\|u\|} =$$

$$= \mathrm{comp}_u(v) \frac{u}{\|u\|} \Rightarrow$$

$$\Rightarrow \mathrm{comp}_u(v) = \frac{u \cdot v}{\|u\|}$$

EXAMPLES

a) Let $u = (x-1, 2, 0)$ and $v = (1, 2x+1, 3)$.
Evaluate $\mathrm{proj}_u(v)$ and $\mathrm{comp}_u(v)$.

Solution

Since

$$u \cdot v = (x-1, 2, 0) \cdot (1, 2x+1, 3) =$$
$$= (x-1)1 + 2(2x+1) + 0 \cdot 3 =$$
$$= x - 1 + 4x + 2 = 5x + 1$$

and

$$\|u\|^2 = u \cdot u = (x-1, 2, 0) \cdot (x-1, 2, 0) =$$
$$= (x-1)(x-1) + 2 \cdot 2 + 0 \cdot 0 = (x-1)^2 + 4$$

then

$$\mathrm{proj}_u(v) = \frac{u \cdot v}{\|u\|^2}\, u = \frac{5x+1}{(x-1)^2+4}\,(x-1, 2, 0) =$$
$$= \left(\frac{(5x+1)(x-1)}{(x-1)^2+4}, \frac{2(5x+1)}{(x-1)^2+4}, 0 \right)$$

and

$$\mathrm{comp}_u(v) = \frac{u \cdot v}{\|u\|} = \frac{5x+1}{\sqrt{(x-1)^2+4}}$$

b) Let $u, v \in \mathbb{R}^3 - \{0\}$. Show that $\operatorname{proj}_{(au)}(v) = \operatorname{proj}_u(v)$.

<u>Solution</u>

$$\operatorname{proj}_{(au)}(v) = \frac{(au)\cdot v}{\|au\|^2}\,(au) = \frac{a(u\cdot v)}{[\,|a|\,\|u\|\,]^2}\,(au) =$$

$$= \frac{a^2(u\cdot v)}{|a|^2\|u\|^2}\,u = \frac{a^2(u\cdot v)}{a^2\|u\|^2}\,u =$$

$$= \frac{u\cdot v}{\|u\|^2}\,u = \operatorname{proj}_u(v).$$

EXERCISES

(20) Let $u, v, w \in \mathbb{R}^3 - \{\mathbf{0}\}$. Show that

a) $\text{proj}_u(v+w) = \text{proj}_u(v) + \text{proj}_u(w)$

b) $\forall a \in \mathbb{R}: \text{proj}_u(av) = a\,\text{proj}_u(v)$

c) $\forall a \in \mathbb{R}-\{0\}: \text{proj}_{(au)}(v) = \text{proj}_u(v)$

d) $u \perp v \Rightarrow \text{proj}_u(v) = \mathbf{0}$

e) $\forall a \in \mathbb{R}-\{0\}: \text{comp}_{(au)}(v) = (a/|a|)\,\text{comp}_u(v)$

f) $\text{comp}_u(u) = \|u\|$

g) $\text{proj}_u(u) = u$

(4) → Cross-product

The cross-product can only be defined between 3D vectors, as follows:

Def: Let $u, v \in \mathbb{R}^3$ with $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$. We define the cross-product $u \times v$ such that

$$(u \times v)_1 = u_2 v_3 - u_3 v_2$$
$$(u \times v)_2 = u_3 v_1 - u_1 v_3$$
$$(u \times v)_3 = u_1 v_2 - u_2 v_1$$

An alternate practical definition of the cross-product is using 3×3 determinants. Recall that a 3×3 determinant can be evaluated via the Sarrus rule as follows:

$$\left| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \right| \begin{matrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{matrix} =$$

$$= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2$$
$$- a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3$$

We define the unit vectors

$$e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1)$$

noting that in general any vector can be written as:

$$u = (u_1, u_2, u_3) = u_1 e_1 + u_2 e_2 + u_3 e_3$$

It can be shown that

$$\boxed{\begin{aligned} u \times v &= (u_1, u_2, u_3) \times (v_1, v_2, v_3) = \\ &= \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{aligned}}$$

In terms of unit vectors, we can show that

$e_1 \times e_2 = -e_2 \times e_1 = e_3$	$e_1 \times e_1 = 0$
$e_2 \times e_3 = -e_3 \times e_2 = e_1$	$e_2 \times e_2 = 0$
$e_3 \times e_1 = -e_1 \times e_3 = e_2$	$e_3 \times e_3 = 0$

EXAMPLE

Evaluate $u \times v$ with $u = (1,3,4)$ and $v = (2,1,1)$.

Solution

$$u \times v = (1,3,4) \times (2,1,1) =$$

$$= \left| \begin{matrix} e_1 & e_2 & e_3 \\ 1 & 3 & 4 \\ 2 & 1 & 1 \end{matrix} \right| \begin{matrix} e_1 & e_2 \\ 1 & 3 \\ 2 & 1 \end{matrix} =$$

$$= e_1 3\cdot 1 + e_2 4\cdot 2 + e_3 1\cdot 1 - e_1 4\cdot 1 - e_2 1\cdot 1 - e_3 3\cdot 2$$

$$= 3e_1 + 8e_2 + e_3 - 4e_1 - e_2 - 6e_3 =$$

$$= (3-4)e_1 + (8-1)e_2 + (1-6)e_3 =$$

$$= -e_1 + 7e_2 - 5e_3 = (-1,7,-5)$$

▸ Norm of the cross-product

Thm: $\forall u, v \in \mathbb{R}^3 : \|u \times v\| = \|u\| \|v\| \sin\vartheta(u,v)$

The proof of this result is based on the Lagrange identity, which reads:

$$(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2 = \\ = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}^2$$

Proof

Let $u, v \in \mathbb{R}^3$ be given. Then; for $\vartheta = \vartheta(u,v)$, we have:

$$\begin{aligned}
\|u \times v\|^2 &= \|(u_1,u_2,u_3)\times(v_1,v_2,v_3)\|^2 = \\
&= \|(u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1)\|^2 \\
&= (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2 = \\
&= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2 = \\
&= (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2 \\
&= \|u\|^2\|v\|^2-(u\cdot v)^2 = \\
&= \|u\|^2\|v\|^2-[\|u\|\|v\|\cos\vartheta]^2 = \\
&= \|u\|^2\|v\|^2-\|u\|^2\|v\|^2\cos^2\vartheta = \\
&= \|u\|^2\|v\|^2(1-\cos^2\vartheta) = \|u\|^2\|v\|^2\sin^2\vartheta
\end{aligned}$$

and since $\vartheta \in [0,\pi] \Rightarrow \sin\vartheta \geq 0$.

It follows that

$$\|u\times v\| = \sqrt{\|u\|^2\|v\|^2 \sin^2\vartheta} = \sqrt{[\|u\|\,\|v\|\sin\vartheta]^2} =$$
$$= |\,\|u\|\,\|v\|\sin\vartheta\,| = \|u\|\,\|v\|\,|\sin\vartheta| =$$
$$= \|u\|\,\|v\|\sin\vartheta. \qquad \square$$

▸ Algebraic properties of the cross product

We can show that the cross-product satisfies the following properties:

$$\boxed{\begin{array}{l}
\forall u\in\mathbb{R}^3 : \quad u\times u = \mathbf{0} \\
\forall u,v\in\mathbb{R}^3 : \quad u\times v = -v\times u \\
\forall u,v\in\mathbb{R}^3 : \forall\lambda\in\mathbb{R} : \quad (\lambda u)\times v = u\times(\lambda v) = \lambda(u\times v) \\
\forall u,v,w\in\mathbb{R}^3 : \begin{cases} u\times(v+w) = u\times v + u\times w \\ (v+w)\times u = v\times u + w\times u \end{cases} \\
\forall u,v,w\in\mathbb{R}^3 : \quad u\cdot(v\times w) = v\cdot(w\times u) = w\cdot(u\times v) \\
\forall u,v,w\in\mathbb{R}^3 : \quad u\times(v\times w) = (u\cdot w)v - (u\cdot v)w
\end{array}}$$

To prove these properties we use tensor notation, as follows:

a) u_a with $a\in\{1,2,3\}$ will represent the a^{th} component of the vector u

b) Repeating indices are automatically summed over all components when associated with

a product
e.g. $u_a v_a = u_1 v_1 + u_2 v_2 + u_3 v_3$ (dot product)
However, for the vector sum $u+v$ we write
$(u+v)_a = u_a + v_a$
and no summation over $a \in \{1,2,3\}$ is implied.

c) We define the Levi-Cirita tensor ε_{abc}:

$$\boxed{\varepsilon_{abc} = (1/2)(a-b)(b-c)(c-a) = \begin{cases} +1, & \text{if } (a,b,c) \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1, & \text{if } (a,b,c) \in \{(3,2,1),(2,1,3),(1,3,2)\} \\ 0, & \text{if } a=b \vee b=c \vee c=a \end{cases}}$$

and note that the cross-product definition can be rewritten as:

$$\boxed{(u \times v)_a = \varepsilon_{abc} u_b v_c}$$

where summation is implied over b, c.

d) We define the Kronecker tensor δ_{ab} as

$$\boxed{\delta_{ab} = \begin{cases} 1, & \text{if } a=b \\ 0, & \text{if } a \neq b \end{cases}}$$

and note the following basic properties:

1) $$\boxed{\begin{aligned} &\delta_{ab} u_b = u_a \\ &\varepsilon_{abc} u_b u_c = 0_a \end{aligned}}$$

2) $$\boxed{\begin{aligned} &\varepsilon_{abc} = \varepsilon_{bca} = \varepsilon_{cab} \\ &\varepsilon_{abc} = -\varepsilon_{cba} \end{aligned}}$$

3) Relation to Kronecker delta

$$\boxed{\varepsilon_{abc}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ap} & \delta_{aq} & \delta_{ar} \\ \delta_{bp} & \delta_{bq} & \delta_{br} \\ \delta_{cp} & \delta_{cq} & \delta_{cr} \end{vmatrix}}$$

4) Contracted epsilon identities

$$\boxed{\begin{aligned} &\varepsilon_{abc}\varepsilon_{apq} = \delta_{bp}\delta_{cq} - \delta_{bq}\delta_{cp} \\ &\varepsilon_{abp}\varepsilon_{abq} = 2\delta_{pq} \\ &\varepsilon_{abc}\varepsilon_{abc} = 6 \end{aligned}}$$

Note that (4) is consequence of (3). All properties of the cross-product are consequences of (1), (2), (3), (4).

EXAMPLE

a) Prove $u \times (v+w) = u \times v + u \times w$

Proof

$$[u \times (v+w)]_a = \varepsilon_{abc} u_b (v+w)_c$$
$$= \varepsilon_{abc} u_b (v_c + w_c) =$$
$$= \varepsilon_{abc} u_b v_c + \varepsilon_{abc} u_b w_c =$$
$$= (u \times v)_a + (u \times w)_a =$$
$$= (u \times v + u \times w)_a \Rightarrow$$
$$\Rightarrow u \times (v+w) = u \times v + u \times w.$$

b) Prove $u \cdot (v \times w) = v \cdot (w \times u)$

Proof

$$u \cdot (v \times w) = u_a (v \times w)_a = u_a \varepsilon_{abc} v_b w_c =$$
$$= v_b \varepsilon_{abc} w_c u_a = v_b \varepsilon_{bca} w_c u_a =$$
$$= v_b (w \times u)_b = v \cdot (w \times u)$$

c) Prove $u \times (v \times w) = (u \cdot w) v - (u \cdot v) w$

Proof

$$[u \times (v \times w)]_a = \varepsilon_{abc} u_b (v \times w)_c =$$
$$= \varepsilon_{abc} u_b \varepsilon_{cpq} v_p w_q =$$
$$= \varepsilon_{cab} \varepsilon_{cpq} u_b v_p w_q =$$

$$= (\delta_{ap}\delta_{bq} - \delta_{aq}\delta_{bp}) u_b v_p w_q =$$
$$= (\delta_{ap} v_p)(\delta_{bq} w_q) u_b - (\delta_{aq} w_q)(\delta_{bp} v_p) u_b$$
$$= u_b v_a w_b - u_b v_b w_a =$$
$$= (u \cdot w) v_a - (u \cdot v) w_a =$$
$$= [(u \cdot w) v]_a - [(u \cdot v) w]_a =$$
$$= [(u \cdot w) v - (u \cdot v) w]_a$$

d) Let $u, v, w \in \mathbb{R}^3$ with $\|u\| = 4$, $\|v\| = \sqrt{5}$, $\|w\| = 1 + \sqrt{2}$, $\vartheta(u, w) = \pi/6$, and $\vartheta(v, w \times u) = \pi/4$.
Evaluate $u \cdot (v \times w)$.

Solution

$$u \cdot (v \times w) = v \cdot (w \times u) = \|v\| \, \|w \times u\| \cos \vartheta(v, w \times u) =$$
$$= \|v\| \left[\|w\| \|u\| \sin \vartheta(u, w) \right] \cos \vartheta(v, w \times u) =$$
$$= \sqrt{5} \left[(1 + \sqrt{2}) 4 \sin(\pi/6) \right] \cos(\pi/4) =$$
$$= \sqrt{5} \left[(1 + \sqrt{2}) 4 (1/2) \right] (\sqrt{2}/2) =$$
$$= \sqrt{2}\sqrt{5} (1 + \sqrt{2}) = \sqrt{5} (\sqrt{2} + 2).$$

EXERCISES

(21) Let $u=(\sqrt{2}, \sqrt{2}+\sqrt{3}, 1)$, and $v=(2, \sqrt{3}, \sqrt{2})$, and $w=(1, \sqrt{2}, \sqrt{3})$ be given. Evaluate the following:

a) $u \times v$ — c) $u \cdot (v \times w)$

b) $u \times (v+w)$ — d) $u \times (v \times w)$

(22) Let $u=(x, 2y-1, 1)$, $v=(2, 1, z)$, and $w=(1, 1, 2)$. Find all $x, y, z \in \mathbb{R}$ such that $u \times (v \times w) = \mathbf{0}$

(Hint: Use the identity for $u \times (v \times w)$ to expedite your calculations)

(23) Let $u, v \in \mathbb{R}^3$ with $\|u\| = \|v\| = 1$.
Show that: $(u \cdot v)^2 + \|u \times v\|^2 = 1$.

(24) Let $u, v, w \in \mathbb{R}^3$ with $\|u\| = 2$, $\|v\| = 3$, and $\|w\| = 1$. If the angle φ from u to v is $\pi/4$, and the angle θ from w to $u \times v$ is $\pi/3$, then show that:

a) $u \cdot v = 3\sqrt{2}$ — c) $\|u \times v\| = 3\sqrt{2}$

b) $\|u+v\| = \sqrt{13+6\sqrt{2}}$ — d) $u \cdot (v \times w) = 3\sqrt{6}/2$

▸ Parallel vectors

Def : Let $u, v \in \mathbb{R}^3 - \{\mathbf{0}\}$ be two vectors. We say that $u \parallel v \Leftrightarrow \exists a \in \mathbb{R} : u = av$

Thm : $\forall u, v \in \mathbb{R}^3 - \{\mathbf{0}\} : (u \parallel v \Leftrightarrow u \times v = \mathbf{0})$

Proof

Let $u, v \in \mathbb{R}^3 - \{\mathbf{0}\}$ be given.

$(\Rightarrow)$: Assume that $u \parallel v$. Then

$u \parallel v \Rightarrow \exists a \in \mathbb{R} : u = av$

Choose $a \in \mathbb{R}$ such that $u = av$. It follows that

$u \times v = (av) \times v = a(v \times v) = a\mathbf{0} = \mathbf{0}$

$(\Leftarrow)$: Assume that $u \times v = \mathbf{0}$, and define

$u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$. Then :

$u \times v = \mathbf{0} \Rightarrow (u_1, u_2, u_3) \times (v_1, v_2, v_3) = (0, 0, 0) \Rightarrow$

$\Rightarrow (u_2 v_3 - u_3 v_2 , u_3 v_1 - u_1 v_3 , u_1 v_2 - u_2 v_1) = (0, 0, 0) \Rightarrow$

$$\Rightarrow \begin{cases} u_1 v_2 - u_2 v_1 = 0 \\ u_2 v_3 - u_3 v_2 = 0 \\ u_3 v_1 - u_1 v_3 = 0 \end{cases}$$

Since $v \in \mathbb{R}^3 - \{\mathbf{0}\} \Rightarrow v \neq \mathbf{0} \Rightarrow v_1 \neq 0 \lor v_2 \neq 0 \lor v_3 \neq 0$

With no loss of generality, let us assume that $v_1 \neq 0$

We choose an $a \in \mathbb{R}$ such that $u_1 = a v_1$. Then, it follows that :

$u_1 v_2 - u_2 v_1 = 0 \Rightarrow a v_1 v_2 - u_2 v_1 = 0 \Rightarrow v_1 (a v_2 - u_2) = 0$
$\Rightarrow v_1 = 0 \vee a v_2 - u_2 = 0 \Rightarrow \underline{u_2 = a v_2}$
and
$u_3 v_1 - u_1 v_3 = 0 \Rightarrow u_3 v_1 - (a v_1) v_3 = 0 \Rightarrow v_1 (u_3 - a v_3) = 0$
$\Rightarrow v_1 = 0 \vee u_3 - a v_3 = 0 \Rightarrow \underline{u_3 = a v_3}$
and therefore
$u = (u_1, u_2, u_3) = (a v_1, a v_2, a v_3) = a (v_1, v_2, v_3) = a v$
$\Rightarrow (\exists a \in \mathbb{R} : u = a v) \Rightarrow u \parallel v.$

• <u>Scalar triple product</u>

$$\boxed{\forall u, v, w \in \mathbb{R}^3 : u \cdot (v \times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}}$$

<u>Proof</u>

$u \cdot (v \times w) = (u_1, u_2, u_3) \cdot [(v_1, v_2, v_3) \times (w_1, w_2, w_3)] =$
$= (u_1, u_2, u_3) \cdot (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)$
$= u_1 (v_2 w_3 - v_3 w_2) + u_2 (v_3 w_1 - v_1 w_3) + u_3 (v_1 w_2 - v_2 w_1) =$
$= u_1 v_2 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_3 v_2 w_1 - u_2 v_1 w_3$
$- u_1 v_3 w_2 =$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

▸ Vectors perpendicular to the same vector are parallel

Thm: Let $u, v, w \in \mathbb{R}^3 - \{\mathbf{0}\}$ be given. Then
$(u \perp w \wedge v \perp w) \Leftrightarrow w \parallel u \times v$

Proof

($\Rightarrow$): Assume that $u \perp w$ and $v \perp w$. Then:

$$\begin{cases} u \perp w \\ v \perp w \end{cases} \Rightarrow \begin{cases} u \cdot w = 0 \\ v \cdot w = 0 \end{cases} \Rightarrow$$

$$\begin{aligned} \Rightarrow w \times (u \times v) &= (w \cdot v) u - (w \cdot u) v = \\ &= (v \cdot w) u - (u \cdot w) v = \\ &= 0u - 0v = \mathbf{0} - \mathbf{0} = \mathbf{0} \Rightarrow \end{aligned}$$

$\Rightarrow w \parallel u \times v$.

($\Leftarrow$): Assume that $w \parallel u \times v$. Then:

$w \parallel u \times v \Rightarrow \exists \lambda \in \mathbb{R} : w = \lambda (u \times v)$

Choose an $\lambda \in \mathbb{R}$ such that $w = \lambda (u \times v)$.

It follows that:

$$\begin{aligned} u \cdot w &= u \cdot [\lambda (u \times v)] = \lambda [u \cdot (u \times v)] = \\ &= \lambda [v \cdot (u \times u)] = \lambda (v \cdot \mathbf{0}) \\ &= \lambda 0 = 0 \Rightarrow u \perp w \end{aligned}$$

and

$$\begin{aligned} v \cdot w &= v \cdot [\lambda (u \times v)] = \lambda [v \cdot (u \times v)] = \lambda [u \cdot (v \times v)] \\ &= \lambda (u \cdot \mathbf{0}) = \lambda 0 = 0 \Rightarrow v \perp w \end{aligned}$$

and therefore: $u \perp w \wedge v \perp w$.

EXERCISES

(25) Use the cross-product to find the set of all vectors u such that $u \perp v$ and $u \perp w$, for $v = (1,2,3)$ and $w = (-2,2,-4)$.

(26) Let $u, v, w \in \mathbb{R}^3$ be given. Show that:

a) $u // v \wedge v // w \Rightarrow u // w$

b) $u // v \wedge u // w \Rightarrow u // (v+w)$

c) $u \cdot (u \times v) = 0$

d) $u \perp v \Rightarrow u \times (u \times v) = -\|u\|^2 v$

e) $(u \times v) \times (v \times w) = [(w \times u) \cdot v] v$

f) $u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = \mathbf{0}$

g) $(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (b \cdot c)(a \cdot d)$

h) $(u - v) \times (u + v) = 2(u \times v)$

Lines in $\mathbb{R}^3$

Let O be the origin of our coordinate system.

- **Def**: The parametric equation for a line (ℓ) going through the points A, B is:

$$\boxed{(\ell): \ \vec{r} = \vec{OA} + t\,\vec{AB} \ , \ \forall t \in \mathbb{R}}$$

The above statement is equivalent to:

$$\boxed{M \in (\ell) \Leftrightarrow \exists\, t \in \mathbb{R} : \vec{OM} = \vec{OA} + t\,\vec{AB}}$$

- $\vec{AB}$ = direction vector of (ℓ).

- For $\vec{r} = (x, y, z)$, $\vec{OA} = (x_0, y_0, z_0)$, and $\vec{AB} = (a, b, c)$, the parametric equation is equivalent to:

$$(\ell): \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases} , \ \forall t \in \mathbb{R}.$$

- Eliminating t from the above equations gives the symmetric equations representation of the line (ℓ):

$$\boxed{(\ell): \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$

The symemetric equations can be reduced to a system of the form

$$(\ell): \begin{cases} A_1 x + B_1 y + C_1 = 0 \\ A_2 y + B_2 z + C_2 = 0 \end{cases}$$

which essentially defines the line (ℓ) as an intersection of two planes.

Relative position of two lines

Consider the lines:

$(\ell_1): \vec{r} = \vec{a}_1 + t\vec{b}_1 \quad , \forall t \in \mathbb{R}$

$(\ell_2): \vec{r} = \vec{a}_2 + t\vec{b}_2 \quad , \forall t \in \mathbb{R}$

Then:

$$(\ell_1) // (\ell_2) \Leftrightarrow \vec{b}_1 // \vec{b}_2 \Leftrightarrow \vec{b}_1 \times \vec{b}_2 = \mathbf{0}$$

EXAMPLES

a) Write the symmetric equations for the line (AB) with $A(1,2,-1)$ and $B(5,4,1)$.

Solution

$$\left.\begin{array}{l} A(1,2,-1) \\ B(5,4,1) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \vec{OA} = (1,2,-1) \\ \vec{AB} = (5-1,\, 4-2,\, 1-(-1)) = (4,2,2) \end{array}\right.$$

therefore:

$$(\ell):\ (x,y,z) = \vec{OA} + t\,\vec{AB} = (1,2,-1) + t\,(4,2,2) =$$
$$= (1+4t,\, 2+2t,\, -1+2t) \Leftrightarrow$$

$$\Leftrightarrow \left\{\begin{array}{l} x = 1+4t \\ y = 2+2t \\ z = -1+2t \end{array}\right. \Leftrightarrow \frac{x-1}{4} = \frac{y-2}{2} = \frac{z+1}{2} \Leftrightarrow$$

$$\Leftrightarrow \left\{\begin{array}{l} 2(x-1) = 4(y-2) \\ 2(y-2) = 2(z+1) \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} 2x-2 = 4y-8 \\ 2y-4 = 2z+2 \end{array}\right. \Leftrightarrow$$

$$\Leftrightarrow \left\{\begin{array}{l} 2x-4y+6=0 \\ 2y-2z-6=0 \end{array}\right. \Leftrightarrow \boxed{\left\{\begin{array}{l} x-2y+3=0 \\ y-z-3=0 \end{array}\right.}$$

b) Write the parametric equation for the line (ℓ) defined by

$$(\ell): \begin{cases} x-3y+2=0 \\ 2y-z+5=0 \end{cases}$$

<u>Solution</u>

Since,

$$(x,y,z)\in(\ell) \Leftrightarrow \begin{cases} x-3y+2=0 \\ 2y-z+5=0 \end{cases} \Leftrightarrow \begin{cases} x=3y-2 \\ z=2y+5 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow (x,y,z)=(3y-2,\ y,\ 2y+5)=$$
$$=(3y,y,2y)+(-2,0,5)=$$
$$= y(3,1,2)+(-2,0,5)$$

$$\Leftrightarrow \exists t\in\mathbb{R}: (x,y,z)=(3,1,2)t+(-2,0,5)$$

it follows that

$$(\ell):\ (x,y,z)=(-2,0,5)+(3,1,2)t$$

EXERCISES

(27) Write and simplify the symmetric equations for the line (AB) with

a) $A(1,1,3)$ and $B(2,1,1)$

b) $A(5,3,2)$ and $B(7,2,4)$

c) $A(1,9,7)$ and $B(6,2,2)$

d) $A(2,1,6)$ and $B(5,3,2)$

↳ To confirm your results, check whether your symmetric equations are satisfied by the points A and B. Since two points define a unique line, if the answer is yes, then you have in fact proved that your answer is correct.

(28) Write the parametric equations for the following lines, defined by symmetric equations

a) $(\ell): \begin{cases} 2x+y-3=0 \\ y+2z+1=0 \end{cases}$

b) $(\ell): \begin{cases} 3x-2y+2=0 \\ y+2z-5=0 \end{cases}$

c) $(\ell): \begin{cases} 5x+3y-6=0 \\ 2y-6z+1=0 \end{cases}$

d) $(\ell): \begin{cases} 2x+5y+3=0 \\ 4y-3z-2=0 \end{cases}$

↳ To confirm your results, use the parametric equations to obtain two points A,B (e.g. try $t=0$ and $t=2$). Then confirm that the points A,B satisfy the original symmetric equations. If they do, then you have shown that your answer is correct.

Planes in $\mathbb{R}^3$

- Let A, B, C be three non-collinear points (i.e. A, B, C are not on the same line). Then, these three points define a unique plane with equation:

$$\boxed{(p):\ \vec{r} = \vec{OA} + t\,\vec{AB} + s\,\vec{AC}\ ,\ \forall t, s \in \mathbb{R}}$$

Equivalently, if we let
$\vec{OA} = (x_0, y_0, z_0)$, $\vec{AB} = (a_1, a_2, a_3)$, $\vec{AC} = (b_1, b_2, b_3)$
then:

$$\boxed{(p):\ \begin{cases} x = x_0 + a_1 t + b_1 s \\ y = y_0 + a_2 t + b_2 s \\ z = z_0 + a_3 t + b_3 s \end{cases},\ \forall t, s \in \mathbb{R}}$$

Similarly, the belonging condition for (p) is:

$$\boxed{M \in (p) \Leftrightarrow \exists t, s \in \mathbb{R} : \vec{OM} = \vec{OA} + t\,\vec{AB} + s\,\vec{AC}}$$

- Eliminating t, s gives an equivalent equation of the form:

$$\boxed{(p):\ Ax + By + Cz + D = 0}$$

which is called the <u>scalar equation</u> of (p).

Scalar equation for plane from 3 points

Let A, B, C be three points with $\vec{AB} \times \vec{AC} \neq \vec{0}$.
The plane (p) defined by A, B, C has scalar equation:

$$\boxed{(p): \ (\vec{AB} \times \vec{AC}) \cdot (\vec{r} - \vec{OA}) = 0}$$

with $r = (x, y, z)$. Equivalently:

$$\boxed{M \in (p) \iff (\vec{AB} \times \vec{AC}) \cdot (\vec{OM} - \vec{OA}) = 0}$$

Here $\vec{n} \equiv \vec{AB} \times \vec{AC}$ = normal vector of (p).
$\vec{n}$ is $\perp$ to every line of (p).

To prove this we use the following lemma:

Lemma: $\boxed{\begin{array}{l} \text{If } \vec{AB} \times \vec{AC} \neq \vec{0}, \text{ then} \\ \forall u \in \mathbb{R}^3 : \exists! x_1, x_2, x_3 \in \mathbb{R} : u = x_1 \vec{AB} + x_2 \vec{AC} + x_3 (\vec{AB} \times \vec{AC}) \end{array}}$

Proof

Define: $(a_1, a_2, a_3) = \vec{AB}$, $(b_1, b_2, b_3) = \vec{AC}$, and
$(c_1, c_2, c_3) = \vec{AB} \times \vec{AC}$.

Let $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ be given. Then:

$$u = x_1 \vec{AB} + x_2 \vec{AC} + x_3 (\vec{AB} \times \vec{AC}) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a_1 x_1 + a_2 x_2 + a_3 x_3 = u_1 \\ b_1 x_1 + b_2 x_2 + b_3 x_3 = u_2 \\ c_1 x_1 + c_2 x_2 + c_3 x_3 = u_3 \end{cases} \quad (1).$$

Since:

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$= [(a_1, a_2, a_3) \times (b_1, b_2, b_3)] \cdot (c_1, c_2, c_3) =$$

$$= (\vec{AB} \times \vec{AC}) \cdot (\vec{AB} \times \vec{AC}) = \| \vec{AB} \times \vec{AC} \|^2 > 0,$$

because $\vec{AB} \times \vec{AC} \neq \vec{0}$. $\Rightarrow$

$\Rightarrow$ equation (1) has a unique solution $(x_1, x_2, x_3) \in \mathbb{R}^3$, which proves the claim. $\square$

$\hookrightarrow$ It follows that the vectors $\vec{AB}$, $\vec{AC}$, and $\vec{AB} \times \vec{AC}$ can be used to define a non-orthogonal coordinate system in which the coordinates of the vector $\vec{u}$ are (x_1, x_2, x_3). We say therefore that $\vec{AB}, \vec{AC}, \vec{AB} \times \vec{AC}$ are linearly independent.

Now we prove the main result:

$$\boxed{M \in (p) \Leftrightarrow (\vec{AB} \times \vec{AC}) \cdot (\vec{OM} - \vec{OA}) = 0}$$

Proof

($\Rightarrow$)

Assume $M \in (p) \Rightarrow \exists\, t, s \in \mathbb{R} : \vec{OM} = \vec{OA} + t\vec{AB} + s\vec{AC} \Rightarrow$

$$\Rightarrow (\vec{AB} \times \vec{AC}) \cdot (\vec{OM} - \vec{OA}) =$$
$$= (\vec{AB} \times \vec{AC}) \cdot (\vec{OA} + t\vec{AB} + s\vec{AC} - \vec{OA}) =$$
$$= (\vec{AB} \times \vec{AC}) \cdot (t\vec{AB} + s\vec{AC}) =$$
$$= (AB \times \vec{AC}) \cdot (t\vec{AB}) + (\vec{AB} \times \vec{AC}) \cdot (s\vec{AC}) =$$
$$= t\vec{AB} \cdot (\vec{AB} \times \vec{AC}) + s\vec{AC} \cdot (\vec{AB} \times \vec{AC}) =$$
$$= t\vec{AC} \cdot (\vec{AB} \times \vec{AB}) + s\vec{AB} \cdot (\vec{AC} \times \vec{AC}) =$$
$$= t\vec{AC} \cdot \vec{0} + s\vec{AB} \cdot \vec{0} = t0 + s0 = 0$$

($\Leftarrow$)

Assume that $(\vec{AB} \times \vec{AC}) \cdot (\vec{OM} - \vec{OA}) = 0$ (1)

By the Lemma above, let $x_1, x_2, x_3 \in \mathbb{R}$ such that

$$\vec{AM} = x_1\vec{AB} + x_2\vec{AC} + x_3(\vec{AB} \times \vec{AC}).$$

Also note that from the ($\Rightarrow$) proof:

$$(\vec{AB} \times \vec{AC}) \cdot (x_1 AB + x_2\vec{AC}) = 0$$

Since:

$$(\vec{AB} \times \vec{AC}) \cdot (\vec{OM} - \vec{OA}) = (AB \times AC) \cdot (\vec{OA} + \vec{AM} - \vec{OA}) =$$
$$= (\vec{AB} \times \vec{AC}) \cdot \vec{AM} =$$
$$= (\vec{AB} \times \vec{AC}) \cdot (x_1\vec{AB} + x_2\vec{AC} + x_3(\vec{AB} \times \vec{AC})) =$$
$$= (\vec{AB} \times \vec{AC}) \cdot [x_3(\vec{AB} \times \vec{AC})] =$$
$$= x_3(\vec{AB} \times \vec{AC}) \cdot (\vec{AB} \times \vec{AC}) = x_3\|\vec{AB} \times \vec{AC}\|^2.$$

it follows from (1) that

$$\left.\begin{array}{l} x_3\|\vec{AB} \times \vec{AC}\|^2 = 0 \\ \vec{AB} \times \vec{AC} \neq \vec{0} \end{array}\right\} \Rightarrow x_3 = 0 \Rightarrow \vec{AM} = x_1\vec{AB} + x_2\vec{AC}$$

$$\Rightarrow \vec{OM} = \vec{OA} + \vec{AM} = \vec{OA} + x_1 \vec{AB} + x_2 \vec{AC} \Rightarrow$$
$$\Rightarrow M \in (p) \qquad \square$$

EXAMPLES

Find the equation of the plane (p) containing the points $A(3,1,2)$, $B(4,3,1)$, and $C(2,5,6)$

Solution

$$\left.\begin{array}{l} A(3,1,2) \\ B(4,3,1) \end{array}\right\} \Rightarrow \vec{AB} = (4-3, 3-1, 1-2) = (1,2,-1) \qquad (1)$$

$$\left.\begin{array}{l} A(3,1,2) \\ C(2,5,6) \end{array}\right\} \Rightarrow \vec{AC} = (2-3, 5-1, 6-2) = (-1,4,4) \qquad (2)$$

From Eq.(1) and Eq.(2):

$$\vec{AB} \times \vec{AC} = (1,2,-1) \times (-1,4,4) =$$

$$= \left|\begin{array}{ccc|cc} e_1 & e_2 & e_3 & e_1 & e_2 \\ 1 & 2 & -1 & 1 & 2 \\ -1 & 4 & 4 & -1 & 4 \end{array}\right| =$$

$$= (2\cdot 4)e_1 + (-1)(-1)e_2 + (1\cdot 4)e_3 - (-1)2e_3 - 4(-1)e_1 - (4\cdot 1)e_2 =$$

$$= 8e_1 + e_2 + 4e_3 + 2e_3 + 4e_1 - 4e_2 =$$

$$= (8+4)e_1 + (1-4)e_2 + (4+2)e_3 = 12e_1 - 3e_2 + 6e_3$$

$$= (12, -3, 6) \quad , \quad \text{and therefore}$$

$$(p): (\vec{AB} \times \vec{AC}) \cdot [(x,y,z) - \vec{OA}] = 0 \Leftrightarrow$$

$$\Leftrightarrow (12,-3,6) \cdot (x-3, y-1, z-2) = 0$$

$\Leftrightarrow 12(x-3)-3(y-1)+6(z-2)=0 \Leftrightarrow$
$\Leftrightarrow 4(x-3)-(y-1)+2(z-2)=0 \Leftrightarrow$
$\Leftrightarrow 4x-12-y+1+2z-4=0 \Leftrightarrow$
$\Leftrightarrow 4x-y+2z-15=0 \Leftrightarrow$
$\Leftrightarrow 4x-y+2z=15.$

Thus: $(p): 4x-y+2z=15.$

$\hookrightarrow$ Note that the plane $(p): Ax+By+Cz+D=0$ has normal vector $\vec{n}=(A,B,C)$.

▶ <u>Plane equation to parametric equation</u>

The first step is to solve the plane equation for one of the 3 variables and use the result to rewrite (x,y,z) as in the following example.

6) Write the parametric equations for the plane $(p): 2x+y+3z=7$.

<u>Solution</u>

We note that

$2x+y+3z=7 \Leftrightarrow y=7-2x-3z \Leftrightarrow$

$$\begin{aligned}\Leftrightarrow (x,y,z) &= (x, 7-2x-3z, z) = \\ &= (0,7,0)+(x,-2x,0)+(0,-3z,z) \\ &= (0,7,0)+x(1,-2,0)+z(0,-3,1)\end{aligned}$$

$\Leftrightarrow \exists t, s \in \mathbb{R} : (x, y, z) = (0, 7, 0) + t(1, -2, 0) + s(0, -3, 1)$
and therefore
$(p) : (x, y, z) = (0, 7, 0) + t(1, -2, 0) + s(0, -3, 1), \forall t, s \in \mathbb{R}.$

▸ Parametric equation to plane equations

We use the parametric equation to obtain ~~3 collinear~~ 3 non-collinear points A, B, C (e.g. use $(t, s) = (0,0), (1,0), (0,1)$). From the 3 points we then derive the plane equation.

c) Write the plane equation for the plane

$$(p): \begin{cases} x = 1 + 2t + s \\ y = 3 - t - 2s \\ z = 2 + 3t + s \end{cases}$$

Solution

We obtain 3 points:

$(t, s) = (0,0) \longrightarrow A(1,3,2)$

$(t, s) = (1,0) \longrightarrow B(3,2,5)$

$(t, s) = (0,1) \longrightarrow C(2,1,3)$

It follows that

$$\left.\begin{aligned} \vec{AB} &= (3-1, 2-3, 5-2) = (2,-1,3) \\ \vec{AC} &= (2-1, 1-3, 3-2) = (1,-2,1) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \vec{AB} \times \vec{AC} = (2,-1,3) \times (1,-2,1) =$$

$$= \left| \begin{matrix} e_1 & e_2 & e_3 \\ 2 & -1 & 3 \\ 1 & -2 & 1 \end{matrix} \right| \begin{matrix} e_1 & e_2 \\ 2 & -1 \\ 1 & -2 \end{matrix} =$$

$$= (-1)1e_1 + 3\cdot 1e_2 + 2(-2)e_3 - 1(-1)e_3 - (-2)3e_1 - 1\cdot 2e_2$$

$$= -e_1 + 3e_2 - 4e_3 + e_3 + 6e_1 - 2e_2 =$$

$$= (-1+6)e_1 + (3-2)e_2 + (-4+1)e_3 =$$
$$= 5e_1 + e_2 - 3e_3 = (5,1,-3)$$

and therefore

$$(p):\ (\vec{AB} \times \vec{AC}) \cdot [(x,y,z) - \vec{OA}] = 0 \Leftrightarrow$$
$$\Leftrightarrow (5,1,-3) \cdot [(x,y,z) - (1,3,2)] = 0 \Leftrightarrow$$
$$\Leftrightarrow (5,1,-3) \cdot (x-1, y-3, z-2) = 0 \Leftrightarrow$$
$$\Leftrightarrow 5(x-1) + (y-3) - 3(z-2) = 0 \Leftrightarrow$$
$$\Leftrightarrow 5x - 5 + y - 3 - 3z + 6 = 0 \Leftrightarrow$$
$$\Leftrightarrow 5x + y - 3z - 2 = 0 \Leftrightarrow 5x + y - 3z = 2$$

and therefore

$$\underline{(p):\ 5x + y - 3z = 2}$$

EXERCISES

29) Write the plane equation $(p): Ax + By + Cz + D = 0$ for the plane (p) defined by three non-collinear points A, B, C with coordinates:

a) $A(1,2,1)$, $B(4,1,0)$, $C(0,3,5)$
b) $A(0,0,0)$, $B(1,1,1)$, $C(1,-1,1)$
c) $A(3,0,0)$, $B(0,1,2)$, $C(1,0,2)$
d) $A(3,2,2)$, $B(3,5,3)$, $C(0,1,2)$

To confirm your answers, it is sufficient to verify that the plane equation is satisfied by the coordinates of the points A, B, C. Note that 3 collinear points define a unique plane.

30) Write the parametric equations for the planes (p) defined by the following plane equations:

a) $(p): 2x + y + 7z = 3$
b) $(p): x + 3y - 2z = 6$
c) $(p): 3x - 2y - z = 5$
d) $(p): x - y + 3z = -1$
e) $(p): 2x + 5y + 3z = 2$

↳ To confirm your work, use the parametric equations to generate 3 noncollinear points and confirm that these 3 points satisfy the original plane equation.

(31) Write the plane equations for the planes (p) defined by the following parametric equations:

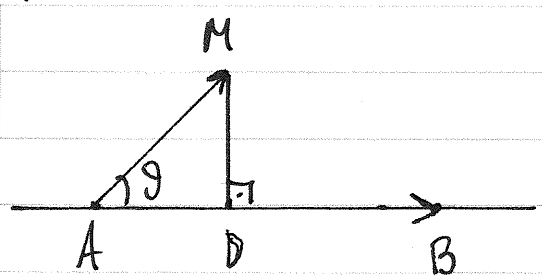
a) $(p): \begin{cases} x = 2 + 3t + 5s \\ y = 1 - t - 2s \\ z = 3 + t + s \end{cases}$ b) $(p): \begin{cases} x = 1 + 2t + s \\ y = 7 + t - 3s \\ z = 2 - t - 9s \end{cases}$

c) $(p): \begin{cases} x = t - s \\ y = t + 2s \\ z = 3t - 2s \end{cases}$

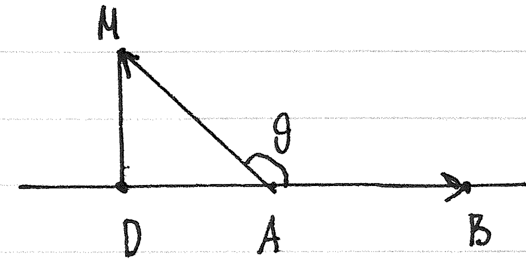
↳ To confirm your work, substitute x, y, z in terms of t, s from the parametric equations into the plane equation, and confirm that the plane equation is satisfied for all values of t and s.

Distances between points, lines, and planes

(1) → Distance of point M from line (AB)



$\vartheta \in [0, \pi/2]$



$\vartheta \in (\pi/2, \pi]$

$$\boxed{d(M, AB) = \frac{\|\vec{AB} \times \vec{AM}\|}{\|\vec{AB}\|}}$$

Proof

Let $D \in (AB)$ such that $MD \perp AB$. Define the interior angle $\vartheta = M\hat{A}B$. We distinguish between the following cases:

Case 1: Assume $\vartheta \in [0, \pi/2]$.

Then $d(M, AB) = MD = AM \sin\vartheta$.

Case 2: Assume $\vartheta \in (\pi/2, \pi]$. Then:

$$d(M, AB) = MD = AM \sin(\pi - \vartheta) = -AM \sin(-\vartheta) =$$
$$= AM \sin\vartheta$$

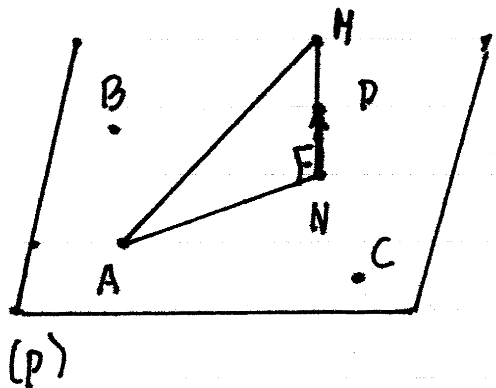
In both cases we find $d(M, AB) = AM \sin\vartheta$.

It follows that:

$$\|\vec{AB} \times \vec{AM}\| = \|\vec{AB}\| \|\vec{AM}\| \sin\vartheta = \|\vec{AB}\| [AM \sin\vartheta] =$$
$$= \|\vec{AB}\| d(M, AB) \Rightarrow d(M, AB) = \frac{\|\vec{AB} \times \vec{AM}\|}{\|\vec{AB}\|}$$

② → Distance of point from plane

Case 1 : The distance $d(M,(p))$ between the point M and the plane (p) defined by the points A, B, C is given by



$$\boxed{d(M,(p)) = \frac{|(\vec{AB} \times \vec{AC}) \cdot \vec{MA}|}{\|\vec{AB} \times \vec{AC}\|}}$$

Proof

Let $N \in (p)$ be the projection of M on (p) such that $MN \perp (p)$. Let $D \in (MN)$ such that $\vec{ND} = \vec{AB} \times \vec{AC}$. It follows that:

$$d(M,(p)) = MN = \|\vec{MN}\| = \|\mathrm{proj}_{\vec{ND}}(\vec{AM})\| =$$

$$= |\mathrm{comp}_{\vec{ND}}(\vec{AM})| = \left| \frac{\vec{AM} \cdot \vec{ND}}{\|\vec{ND}\|} \right| =$$

$$= \frac{|\vec{AM} \cdot \vec{ND}|}{\|\vec{ND}\|} = \frac{|\vec{AM} \cdot (\vec{AB} \times \vec{AC})|}{\|\vec{AB} \times \vec{AC}\|} =$$

$$= \frac{|(\vec{AB} \times \vec{AC}) \cdot \vec{MA}|}{\|\vec{AB} \times \vec{AC}\|}$$

Case 2: The distance between $M(x_0, y_0, z_0)$ and the plane $(p): Ax+By+Cz+D=0$ is given by:

$$\boxed{d(M,(p)) = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}}$$

Proof

Let $N, P, Q \in (p)$ be three non-collinear points on the plane (p) with $N(x_1, y_1, z_1)$. Since $N(x_1, y_1, z_1) \in (p) \Rightarrow Ax_1+By_1+Cz_1+D=0 \Rightarrow$
$\Rightarrow \underline{Ax_1+By_1+Cz_1=-D}.$ (1)
We also note that:

$$\left.\begin{array}{l}(p): Ax+By+Cz+D=0 \Rightarrow \vec{u}=(A,B,C)\perp(p)\\ \vec{NP}\times\vec{NQ}\perp(p)\end{array}\right\}\Rightarrow$$

$\Rightarrow \vec{u} \parallel \vec{NP}\times\vec{NQ} \Rightarrow \exists \lambda\in\mathbb{R}: \underline{\vec{NP}\times\vec{NQ}=\lambda\vec{u}}$ (2)

and

$$\left.\begin{array}{l}M(x_0, y_0, z_0)\\ N(x_1, y_1, z_1)\end{array}\right\}\Rightarrow \underline{\vec{NM}=(x_0-x_1,\ y_0-y_1,\ z_0-z_1)} \quad (3)$$

From (1),(2),(3), using the previous result, it follows that

$$d(M,(p)) = \frac{|\vec{NM}\cdot(\vec{NP}\times\vec{NQ})|}{\|\vec{NP}\times\vec{NQ}\|} = \frac{|\vec{NM}\cdot(\lambda\vec{u})|}{\|\lambda\vec{u}\|} =$$

$$= \frac{|\lambda(\vec{NM} \cdot \vec{u})|}{\|\lambda \vec{u}\|} = \frac{|\lambda| \cdot |\vec{NM} \cdot \vec{u}|}{|\lambda| \|\vec{u}\|} = \frac{|\vec{NM} \cdot \vec{u}|}{\|\vec{u}\|} =$$

$$= \frac{|(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (A, B, C)|}{\|(A, B, C)\|} =$$

$$= \frac{|A(x_0 - x_1) + B(y_0 - y_1) + C(z_0 - z_1)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|(Ax_0 + By_0 + Cz_0) - (Ax_1 + By_1 + Cz_1)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|Ax_0 + By_0 + Cz_0 - (-D)|}{\sqrt{A^2 + B^2 + C^2}} =$$

$$= \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} \qquad \square$$

EXERCISES

(32) Find the distance between

a) The point $M(3,1,5)$ and the line (AB) passing through $A(1,2,1)$ and $B(0,0,4)$.

b) The point $M(1,1,-2)$ and the line (AB) given by

$$(AB): \begin{cases} x+y-3=0 \\ y+2z+1=0 \end{cases}$$

c) The point $M(2,2,1)$ and the plane (p) defined by the points $A(1,1,0)$, $B(3,2,4)$, and $C(-1,2,0)$

d) The point $M(1,3,-1)$ and the plane (p) given by

$$(p): \begin{cases} x = 2+t+s \\ y = 1-2t+3s \\ z = 3-t-s \end{cases}$$

e) The point $M(1,2,1)$ and the plane $(p): 2x+3y-z=3$.

(33) Find all $a \in \mathbb{R}$ such that:

a) The distance of $M(1,1,a)$ from the line (AB) passing through $A(0,2,2)$ and $B(3,1,1)$ is equal to 10

b) The distance of $M(a,a,3)$ from the line

$$(AB): \begin{cases} 2x+y-1=0 \\ y+5z-2=0 \end{cases}$$

is minimized

c) The distance between the point $M(1, a, 2a+1)$ and the plane defined by the points $A(2,0,2)$, $B(3,0,0)$, $C(1,-1,3)$ is equal to 6.

d) The distance between the point $M(1,1,2)$ and the plane $(p): x-3y+2z=a$ is equal to a.

CAL3.2: Vector-valued functions

VECTOR-VALUED FUNCTIONS

Def: A vector-valued function is a mapping $f: A \to \mathbb{R}^3$ with $A \subseteq \mathbb{R}$

As such, vector-valued functions are mapping a NUMBER to a VECTOR

Limit of a vector-valued function

Def: Let $f: A \to \mathbb{R}^3$ be a vector-valued function, let $a \in A$ be a limit point of A, and let $u \in \mathbb{R}^3$. Then

$$\lim_{t \to a} f(t) = u \Leftrightarrow \lim_{t \to a} \|f(t) - u\| = 0$$

- This definition piggybacks on the Calculus 1 definition of the limit for a function $g: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$:

$$\lim_{t \to a} g(t) = l \Leftrightarrow \forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall t \in A : (0 < |t-a| < \delta \Rightarrow |g(t) - l| < \varepsilon)$$

The negation of this definition is given by

$$\lim_{t \to a} g(t) \neq l \Leftrightarrow \exists \varepsilon \in (0, +\infty) : \forall \delta \in (0, +\infty) : \exists t \in A : (0 < |t-a| < \delta \wedge |g(t) - l| \geq \varepsilon)$$

and we use it to prove the following theorem:

Thm: Let $f: A \to \mathbb{R}^3$ be a vector-valued function with $A \subseteq \mathbb{R}$ and let $a \in A$ be a limit point of A and assume that

$\forall t \in A : f(t) = (x(t), y(t), z(t))$

Let $u = (u_1, u_2, u_3) \in \mathbb{R}^3$. Then:

$$\lim_{t \to a} f(t) = u \Leftrightarrow$$
$$\Leftrightarrow \left(\lim_{t \to a} x(t) = u_1 \land \lim_{t \to a} y(t) = u_2 \land \lim_{t \to a} z(t) = u_3 \right)$$

▸ It follows that when the limits converge, we have:

$$\lim_{t \to a} f(t) = \left(\lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right)$$

Proof

($\Rightarrow$): Assume that $\lim_{t \to a} f(t) = u$. To show that

$\lim_{t \to a} x(t) = u_1$, assume that $\lim_{t \to a} x(t) \neq u_1$, in order

to derive a contradiction. Then,

$$\lim_{t \to a} x(t) \neq u_1 \Rightarrow \exists \varepsilon \in (0, +\infty) : \forall \delta \in (0, +\infty) : \exists t \in A :$$
$$: (0 < |t-a| < \delta \land |x(t) - u_1| \geq \varepsilon)$$

Choose some $\varepsilon \in (0, +\infty)$ such that

$\forall \delta \in (0, +\infty) : \exists t \in A : (0 < |t-a| < \delta \land |x(t) - u_1| \geq \varepsilon)$

Let $\delta \in (0, +\infty)$ be given and choose some $t \in A$

such that

$$\begin{cases} 0 < |t-a| < \delta \\ |x(t) - u_1| \geq \varepsilon \end{cases}$$

Then, it follows that

$$\|f(t) - u\| = \sqrt{[x(t)-u_1]^2 + [y(t)-u_2]^2 + [z(t)-u_3]^2}$$
$$\geq \sqrt{[x(t)-u_1]^2} = |x(t) - u_1| \geq \varepsilon \Rightarrow$$
$$\Rightarrow \|f(t) - u\| \geq \varepsilon$$

We have thus shown that

$$\exists \varepsilon \in (0, +\infty) : \forall \delta \in (0, +\infty) : \exists t \in A : \begin{cases} 0 < |t-a| < \delta \\ \|f(t) - u\| \geq \varepsilon \end{cases}$$

$$\Rightarrow \lim_{t \to a} \|f(t) - u\| \neq 0 \Rightarrow \lim_{t \to a} f(t) \neq u$$

which is a contradiction, since by hypothesis we have: $\lim_{t \to a} f(t) = u$. It follows that $\lim_{t \to a} x(t) = u_1$

Similarly, we can show that $\lim_{t \to a} y(t) = u_2$ and

$\lim_{t \to a} z(t) = u_3$.

<u>($\Leftarrow$)</u>: Assume that $\lim_{t \to a} x(t) = u_1 \wedge \lim_{t \to a} y(t) = u_2 \wedge \lim_{t \to a} z(t) = u_3$.

Then:

$$\lim_{t \to a} \|f(t) - u\| = \lim_{t \to a} \sqrt{[x(t)-u_1]^2 + [y(t)-u_2]^2 + [z(t)-u_3]^2}$$
$$= \sqrt{(u_1-u_1)^2 + (u_2-u_2)^2 + (u_3-u_3)^2}$$
$$= 0 \Rightarrow \lim_{t \to a} f(t) = u.$$

Continuity of vector-valued functions

<u>Def</u> : Let $f: A \to \mathbb{R}^3$ be a vector-valued function, and let $J \subseteq A$, and $a \in A$.

a) f continuous at $t=a \Leftrightarrow \lim\limits_{t \to a} f(t) = f(a)$

b) f continuous at $J \Leftrightarrow \forall a \in J : f$ continuous at $t=a$.

Derivatives of vector-valued functions

<u>Def</u> : Let $f: A \to \mathbb{R}^3$ be a vector-valued function. We say that:

a) f differentiable at $t \in A \Leftrightarrow$

$$\Leftrightarrow \exists v \in \mathbb{R}^3 : \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = v$$

b) f differentiable at $J \subseteq A \Leftrightarrow$
$\Leftrightarrow \forall a \in J : f$ differentiable at $t=a$.

<u>Def</u> : Let $f: A \to \mathbb{R}^3$ be a vector-valued function that is differentiable in $J \subseteq A$. We define the derivative function $\dot{f}: J \to \mathbb{R}^3$ as:

$$\boxed{\forall t \in J : \dot{f}(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}}$$

▸ <u>notation</u> : $\dot{f}(t) = \dfrac{df(t)}{dt}$

For higher-order derivatives:

$$\ddot{f}(t) = \frac{d\dot{f}(t)}{dt} = \frac{d^2 f(t)}{dt^2}$$

$$\dddot{f}(t) = \frac{d\ddot{f}(t)}{dt} = \frac{d^3 f(t)}{dt^3}$$

<u>Thm</u> : Let $f : A \to \mathbb{R}^3$ with $f(t) = (x(t), y(t), z(t))$, $\forall t \in A$ and assume that f differentiable on B with $B \subseteq A$. Then,
$\forall t \in B : \dot{f}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$

▸ <u>interpretation</u>

If a vector-valued function $r : [0, +\infty) \to \mathbb{R}^3$ represents the position vector of an object in motion, then:

$u(t) = \dot{r}(t)$ = the velocity of the object
$a(t) = \dot{u}(t) = \ddot{r}(t)$ = the acceleration of the object
$\|u(t)\| = \|\dot{r}(t)\|$ = the speed of the object

▸ Note the careful distinction between the terms <u>speed</u> and <u>velocity</u>.

EXAMPLE

For circular motion given by
$r(t) = (\rho\cos(\omega t), \rho\sin(\omega t), 0)$, $\forall t \in \mathbb{R}$
show that

a) $u(t) \perp r(t)$, $\forall t \in \mathbb{R}$

b) $a(t) = -\omega^2 r(t)$, $\forall t \in \mathbb{R}$

c) $\|a(t)\| = \|u(t)\|^2/\rho$, $\forall t \in \mathbb{R}$

Solution

a) $u(t) = \dot{r}(t) = (d/dt)(\rho\cos(\omega t), \rho\sin(\omega t), 0) =$
$= (-\omega\rho\sin(\omega t), \omega\rho\cos(\omega t), 0) =$
$= \omega\rho(-\sin(\omega t), \cos(\omega t), 0)$, $\forall t \in \mathbb{R} \Rightarrow$
$\Rightarrow u(t) \cdot r(t) = [\omega\rho(-\sin(\omega t), \cos(\omega t), 0)] \cdot (\rho\cos(\omega t), \rho\sin(\omega t), 0)$
$= \omega\rho^2[(-\sin(\omega t))\cos(\omega t) + \cos(\omega t)\sin(\omega t) + 0 \cdot 0] =$
$= \omega\rho^2 \cdot 0 = 0$, $\forall t \in \mathbb{R} \Rightarrow$
$\Rightarrow u(t) \perp r(t)$, $\forall t \in \mathbb{R}$.

b) $a(t) = \dot{u}(t) = (d/dt)[\omega\rho(-\sin(\omega t), \cos(\omega t), 0)] =$
$= \omega\rho(-\omega\cos(\omega t), -\omega\sin(\omega t), 0) =$
$= -\omega^2(\rho\cos(\omega t), \rho\sin(\omega t), 0)$
$= -\omega^2 r(t)$, $\forall t \in \mathbb{R}$.

c) $\|a(t)\| = \|-\omega^2 r(t)\| = |-\omega^2| \, \|r(t)\| = \omega^2\|r(t)\| =$
$= \omega^2\|(\rho\cos(\omega t), \rho\sin(\omega t), 0)\| =$
$= \omega^2\|\rho(\cos(\omega t), \sin(\omega t, 0)\| =$
$= \omega^2|\rho| \, \|(\cos(\omega t), \sin(\omega t), 0)\| =$
$= \omega^2\rho\sqrt{\cos^2(\omega t) + \sin^2(\omega t) + 0^2} = \omega^2\rho$

and

$$\|u(t)\| = \| \omega\rho\,(-\sin(\omega t),\ \cos(\omega t),\ 0)\| =$$
$$= |\omega\rho|\ \|(-\sin(\omega t),\ \cos(\omega t),\ 0)\| =$$
$$= \omega\rho\sqrt{(-\sin(\omega t))^2 + \cos^2(\omega t) + 0^2}$$
$$= \omega\rho\sqrt{\sin^2(\omega t) + \cos^2(\omega t)} = \omega\rho\ ,\ \forall t\in\mathbb{R}$$

it follows that

$$\|a(t)\| = \omega^2\rho = \frac{\omega^2\rho^2}{\rho} = \frac{(\omega\rho)^2}{\rho} = \frac{\|u(t)\|^2}{\rho}\ ,\ \forall t\in\mathbb{R}.$$

EXERCISES

① Find the default domain for the following vector-valued functions:

a) $r(t) = \left(\dfrac{2}{t-1}, \sqrt{3-t}, \ln(t+3)\right)$

b) $r(t) = (2t^2+1, \operatorname{Arcsin}(2t-3), \operatorname{Arctan}(2t+3))$

c) $r(t) = (\ln[(t-2)^2(3t-2)^3], t^2+3, t/(t-3))$

d) $r(t) = (\operatorname{Arcsin}(3t+1), \operatorname{Arcsin}(2t-1), 1/t)$

e) $r(t) = (\log_x(2x+1), 1/(x+1), 0)$

↳ The default domain is defined as the widest possible subset of $\mathbb{R}$ for which the function definition can be evaluated into real-valued vectors.

② Evaluate the following limits:

a) $\lim\limits_{t\to 3} (2(t-3)^2, -7t^3, 0)$

b) $\lim\limits_{t\to 2} \left(\dfrac{t-2}{t^2-4}, \dfrac{t^2+3t-10}{t-2}, 0\right)$

c) $\lim\limits_{t\to 0} \left(\dfrac{\sin(2t)\cos(3t)}{5t}, \dfrac{e^t-1}{3t}, \dfrac{2t}{t+2}\right)$

d) $\lim_{t\to 0}\left(\frac{t^2\cos(1/t)}{e^t-1}, \frac{\ln(t)}{t^2}, 0\right)$

e) $\lim_{t\to +\infty}\left(\log_{2x}(3x+1), \log_{x-2}(2x+5), 0\right)$

f) $\lim_{t\to 0}(t^t, t^{2t}, t^{3t})$

g) $\lim_{t\to 0^+}(t^2\ln t, \ln(t^3+1), 2t-1)$

(3) Find the derivatives $\dot{r}(t)$ and $\ddot{r}(t)$ for the following vector-valued functions.

a) $r(t) = (2t^2, (t-3)^3, (2t+1)^2)$
b) $r(t) = (t^2e^{-t}, te^{-t^2}, t^3e^{-t})$
c) $r(t) = (\ln(t^2+1), \ln(t^3+2t^2-t), \ln(3t))$
d) $r(t) = (t^t, t^{2t}, t^{3t})$

Properties of differentiation

Thm: Let $u : A \to \mathbb{R}^3$ and $v : A \to \mathbb{R}^3$ be two vector-valued functions. Then:

a) $(d/dt)[u(t) + v(t)] = \dot{u}(t) + \dot{v}(t)$

b) $(d/dt)[\lambda u(t)] = \lambda \dot{u}(t)$

c) $(d/dt)[f(t) u(t)] = f'(t) u(t) + f(t) \dot{u}(t)$

d) $(d/dt)[u(t) \cdot v(t)] = \dot{u}(t) \cdot v(t) + u(t) \cdot \dot{v}(t)$

e) $(d/dt)[u(t) \times v(t)] = \dot{u}(t) \times v(t) + u(t) \times \dot{v}(t)$

We give selected proofs for properties (d), (e) using tensor notation

Proof of (d)

$$(d/dt)(u(t) \cdot v(t)) = (d/dt)[u_a(t) v_a(t)] =$$
$$= \dot{u}_a(t) v_a(t) + u_a(t) \dot{v}_a(t) =$$
$$= \dot{u}(t) \cdot v(t) + u(t) \cdot \dot{v}(t).$$

Proof of (e)

$$(d/dt)[u(t) \times v(t)]_a = (d/dt)[\varepsilon_{abc} u_b(t) u_c(t)] =$$
$$= \varepsilon_{abc} (d/dt)[u_b(t) u_c(t)] = \varepsilon_{abc}[\dot{u}_b(t) v_c(t) + u_b(t) \dot{v}_c(t)]$$
$$= \varepsilon_{abc} \dot{u}_b(t) v_c(t) + \varepsilon_{abc} u_b(t) \dot{v}_c(t) =$$
$$= [\dot{u}(t) \times v(t)]_a + [u(t) \times \dot{v}(t)]_a =$$
$$= [\dot{u}(t) \times v(t) + u(t) \times \dot{v}(t)]_a \Rightarrow$$
$$\Rightarrow (d/dt)[u(t) \times v(t)] = \dot{u}(t) \times v(t) + u(t) \times \dot{v}(t)$$

APPLICATION

Show that

$\|u(t)\|$ constant on $\mathbb{R} \Longleftrightarrow \forall t \in \mathbb{R}: \dot{u}(t) \perp u(t)$.

Solution

We note that

$(d/dt)\|u(t)\|^2 = (d/dt)[u(t) \cdot u(t)] =$

$= \dot{u}(t) \cdot u(t) + u(t) \cdot \dot{u}(t) = \dot{u}(t) \cdot u(t) + \dot{u}(t) \cdot u(t)$

$= 2\dot{u}(t) \cdot u(t)$

($\Rightarrow$) : Assume that $\|u(t)\|$ constant on $\mathbb{R}$. Then:

$\|u(t)\|$ constant on $\mathbb{R} \Rightarrow \|u(t)\|^2$ constant on $\mathbb{R} \Rightarrow$

$\Rightarrow (d/dt)\|u(t)\|^2 = 0 \Rightarrow 2\dot{u}(t) \cdot u(t) = 0, \forall t \in \mathbb{R} \Rightarrow$

$\Rightarrow \dot{u}(t) \cdot u(t) = 0, \forall t \in \mathbb{R} \Rightarrow \dot{u}(t) \perp u(t), \forall t \in \mathbb{R}$.

($\Leftarrow$): Assume that $\forall t \in \mathbb{R}: \dot{u}(t) \perp u(t)$. Then:

$\dot{u}(t) \perp u(t), \forall t \in \mathbb{R} \Rightarrow \dot{u}(t) \cdot u(t) = 0, \forall t \in \mathbb{R} \Rightarrow$

$\Rightarrow (d/dt)\|u(t)\|^2 = 2\dot{u}(t) \cdot u(t) = 2 \cdot 0 = 0 \Rightarrow$

$\Rightarrow \|u(t)\|^2$ constant on $\mathbb{R} \Rightarrow$

$\Rightarrow \exists a \in [0, +\infty) : \forall t \in \mathbb{R} : \|u(t)\|^2 = a$.

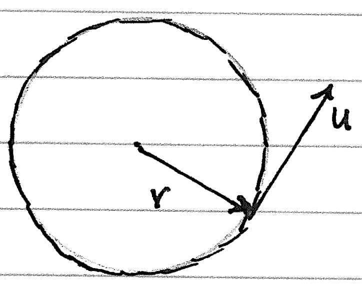
Choose $a \in [0, +\infty)$ such that $\forall t \in \mathbb{R} : \|u(t)\|^2 = a$.

Then :

$\|u(t)\| \geq 0, \forall t \in \mathbb{R} \Rightarrow$

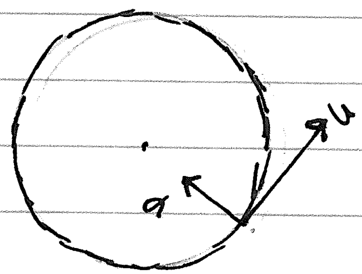
$\Rightarrow \|u(t)\| = |\,\|u(t)\|\,| = \sqrt{\|u(t)\|^2} = \sqrt{a}, \forall t \in \mathbb{R} \Rightarrow$

$\Rightarrow \|u(t)\|$ constant on $\mathbb{R}$.



This result admits two interpretations for the case of circular motion.

a) Since in circular motion the distance from the origin is constant, then the velocity vector is always tangent to the circle.



b) Since in circular motion the speed is constant, the accelaration is perpendicular to the velocity and therefore parallel to the corresponding radius. Note that this result holds for any motion where the speed is constant.

EXERCISES

(4) Prove the differentiation properties a, b, c. from the lecture notes.

(5) Let $u : A \to \mathbb{R}^3$ be a vector-valued function and $f : A \to \mathbb{R}$ a function with $A \subseteq \mathbb{R}$.
If both u and f are differentiable on A, then prove the scalar quotient rule:

$$\frac{d}{dt}\left[\frac{1}{f(t)}\, u(t)\right] = \left[\frac{1}{f(t)}\right]^2 \left(f(t)\,\dot{u}(t) - f'(t)\,u(t)\right)$$

(6) Let $r : \mathbb{R} \to \mathbb{R}^3$ be a differentiable vector-valued function representing the motion of an object through space. We define:

Angular momentum : $L = r(t) \times \dot{r}(t)$

torque : $\tau = r(t) \times \ddot{r}(t)$

a) Show that $dL/dt = \tau$.

b) Let $a, b \in \mathbb{R}^3$ and $r(t) = a\cos(\omega t) + b\sin(\omega t)$ with $\omega \in \mathbb{R}$. Show that $L(t) = \omega(a \times b)$.

(7) Let $u : \mathbb{R} \to \mathbb{R}^3$ be a differentiable vector-valued function and let $v \in \mathbb{R}^3$. Show that:

$$(\forall t \in \mathbb{R} : \|u(t) - v\| = c) \Rightarrow \dot{u}(t) \perp u(t) - v$$

(8) Let $u : \mathbb{R} \to \mathbb{R}^3$ and $v : \mathbb{R} \to \mathbb{R}^3$ be differentiable vector-valued functions. Show that:

a) $\forall t \in \mathbb{R} : \|u(t) - v(t)\| = \|v(t)\| \Rightarrow$
$\Rightarrow \forall t \in \mathbb{R} : u(t) \cdot \dot{v}(t) = [u(t) - v(t)] \cdot \dot{u}(t)$

b) $\forall t \in \mathbb{R} : \begin{cases} \|u(t)\| = c_1 \\ \|\dot{u}(t)\| = c_2 \end{cases} \Rightarrow \forall t \in \mathbb{R} : u(t) \,//\, \ddot{u}(t)$

(Hint: First show that $(d/dt)(\|u(t)\| \|\dot{u}(t)\|) =$
$= u(t) \times \ddot{u}(t)$, using $\forall t \in \mathbb{R} : \|u(t)\| = c_1$)

c) $\forall t \in \mathbb{R} : |(d/dt)\|u(t)\|| \leq \|(d/dt) u(t)\|$
(Hint: Use $|u|^2 = u \cdot u$ and $|\cos\vartheta| \leq 1$)

d) $\forall t \in \mathbb{R} : (d/dt)\|u(t) \times v(t)\| \leq \|\dot{u}(t)\| \|v(t)\| + \|u(t)\| \|\dot{v}(t)\|$
(Hint: Use the previous result and the triangle inequality $\|u + v\| \leq \|u\| + \|v\|$).

(9) Let $r : \mathbb{R} \to \mathbb{R}^3$ be a ~~twice~~ triple-differentiable vector-valued function, and let $u : \mathbb{R} \to \mathbb{R}$ be a function given by $u(t) = r(t) \cdot (\dot{r}(t) \times \ddot{r}(t))$. Show that

$$du/dt = r(t) \cdot (\dot{r}(t) \times \dddot{r}(t))$$

Arclength

- Let $(c): r(t),\ \forall t \in [a,b]$ be a finite curve. The length of the curve is given by:

$$\boxed{\ell = \int_a^b \|\dot{r}(t)\| dt}$$

- For the more general case of an infinite curve $(c): r(t),\ \forall t \in \mathbb{R}$, we define the <u>arclength function</u> $s(t)$ as:

$$\boxed{s(t) = \int_{t_0}^{t} \|\dot{r}(\tau)\| d\tau \ , \ \forall t \in \mathbb{R}}$$

Here $t_0 \in \mathbb{R}$ represents an initial time, usually chosen by default as $t_0 = 0$. The <u>arclength function</u> $s(t)$ gives the distance travelled during the interval $[t_0, t]$ for $t > t_0$.

- We note, from the fundamental theorem of calculus, that

$$\boxed{\frac{ds(t)}{dt} = \|\dot{r}(t)\|}$$

EXAMPLES

Find the arclength function from $t_0 = 0$ for the curve (c): $r(t) = (e^{2t}\cos 2t, 2, e^{2t}\sin 2t)$.

Solution

For $x(t) = e^{2t}\cos 2t$, $y(t) = 2$, and $z(t) = e^{2t}\sin 2t$, we find that:

$$\dot{x}(t) = (e^{2t})'\cos 2t + e^{2t}(\cos 2t)' =$$
$$= 2e^{2t}\cos 2t - 2e^{2t}\sin 2t =$$
$$= 2e^{2t}(\cos 2t - \sin 2t),$$
$$\dot{y}(t) = 0,$$
$$\dot{z}(t) = (e^{2t})'\sin 2t + e^{2t}(\sin 2t)' =$$
$$= 2e^{2t}\sin 2t + 2e^{2t}\cos 2t =$$
$$= 2e^{2t}(\sin 2t + \cos 2t).$$

It follows that:

$$|\dot{r}(t)|^2 = [\dot{x}(t)]^2 + [\dot{y}(t)]^2 + [\dot{z}(t)]^2 =$$
$$= [2e^{2t}(\cos 2t - \sin 2t)]^2 + [2e^{2t}(\sin 2t + \cos 2t)]^2 =$$
$$= 4e^{4t}[(\cos 2t - \sin 2t)^2 + (\cos 2t + \sin 2t)^2]$$
$$= 4e^{4t}[\cos^2 2t - 2\cos 2t\sin 2t + \sin^2 2t + \cos^2 2t +$$
$$+ 2\cos 2t\sin 2t + \sin^2 2t]$$
$$= 4e^{4t}[2\cos^2 2t + 2\sin^2 2t] = 4e^{4t}\cdot 2 = 8e^{4t} \Rightarrow$$
$$\Rightarrow |\dot{r}(t)| = 2\sqrt{2}\, e^{2t} \Rightarrow$$
$$\Rightarrow \ell = \int_0^t \|\dot{r}(t)\| dt =$$

$$\Rightarrow s(t) = \int_0^t \|\dot{r}(\tau)\| d\tau = \int_0^t 2\sqrt{2}\, e^{2\tau}\, d\tau =$$

$$= 2\sqrt{2} \int_0^t e^{2\tau}\, d\tau = 2\sqrt{2} \left[\frac{e^{2\tau}}{2} \right]_0^t =$$

$$= 2\sqrt{2}\ \frac{e^{2t} - 1}{2} = \sqrt{2}\,(e^{2t} - 1).$$

EXERCISES

(10) Find the arclength function from $t_0 = 0$ for the curves defined by

a) $r(t) = (a, bt^2, ct^3)$

b) $r(t) = (at^2, bt^3, ct^3)$

c) $r(t) = (at^3, bt^3, ct^3)$

d) $r(t) = (at, b\sin(\omega t), b\cos(\omega t))$

e) $r(t) = (t\sin(\omega t), t\cos(\omega t), ut)$

(11) Consider the decaying helix curve defined by $r(t) = (e^{-at}\cos(\omega t), e^{-at}\sin(\omega t), ut)$

a) Find the arclength function from $t_0 = 0$ for the special case $u = 0$

b) Extend the previous result to the case $u \neq 0$.

Curvature of a curve (c)

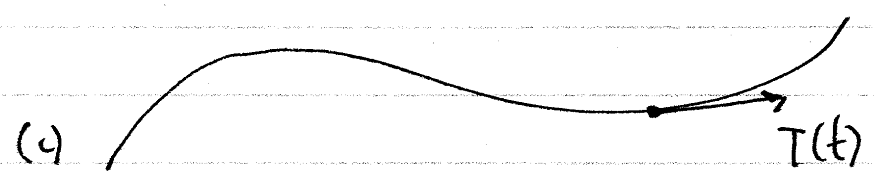
Consider a parameterized curve $(c): (x,y,z) = r(t), \forall t \in A$ representing the motion of an object, and assume that

$$\forall t \in A : \|\dot{r}(t)\| \neq 0$$

meaning that the object is always in motion.

- We define the <u>unit tangent vector</u> $T(t)$ such that

$$\boxed{\forall t \in A : T(t) = \frac{\dot{r}(t)}{\|\dot{r}(t)\|}}$$



- If $\$(t)$ is the arclength function of the curve definition $(x,y,z) = r(t)$, given by:

$$\$(t) = \int_0^t \|\dot{r}(\tau)\| \, d\tau \ , \ \forall t \in A$$

we can define a corresponding inverse function

$$\tau(\$_0) = t \Leftrightarrow \$(t) = \$_0$$

and write T as a function of $\$$: $T(\tau(\$))$

We can then define the curvature κ as the rate of change in the direction of $T(t)$ with respect to the arclength $s(t)$:

$$\kappa = \left\| \frac{dT}{ds} \right\|$$

The reason why we are not using dT/dt to define curvature is because it depends on the speed with which $r(t)$ traces the curve (c) whose curvature we wish to calculate. Multiplying the speed by a factor of 2 would increase dT/dt by the same factor, without an underlying change in the curvature of the curve traced by $r(t)$.

Calculation of $\kappa(t)$

<u>Thm</u>: $\boxed{\forall t \in A: \quad \kappa(t) = \frac{\|\dot{T}(t)\|}{\|\dot{r}(t)\|}}$

<u>Proof</u>

From the chain rule, we have

$$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} = \frac{dT}{ds}\|\dot{r}(t)\| \Rightarrow$$

$$\Rightarrow \frac{dT}{ds} = \frac{1}{\|\dot{r}(t)\|}\frac{dT}{dt} = \frac{\dot{T}(t)}{\|\dot{r}(t)\|} \Rightarrow$$

$$\Rightarrow \kappa(t) = \left\| \frac{dT}{ds} \right\| = \left\| \frac{\dot{T}(t)}{\|\dot{r}(t)\|} \right\| = \left| \frac{1}{\|\dot{r}(t)\|} \right| \|\dot{T}(t)\|$$

$$= \frac{\|\dot{T}(t)\|}{\|\dot{r}(t)\|}$$

<u>Thm</u> : $\boxed{\forall t \in A: \ \kappa(t) = \dfrac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^3}}$

<u>Proof</u>

First, we note that the arclength function $s(t)$ satisfies

$$s'(t) = \frac{d}{dt} \int_{t_0}^{t} \|\dot{r}(\tau)\| d\tau = \|\dot{r}(t)\|$$

▸ Begin with calculating $\|\dot{r}(t) \times \ddot{r}(t)\|$ in terms of $T(t)$.

Since $T(t) = \dot{r}(t) / \|\dot{r}(t)\| \Rightarrow$

$$\Rightarrow \dot{r}(t) = T(t) \|\dot{r}(t)\| = T(t) s'(t) \Rightarrow$$

$$\Rightarrow \ddot{r}(t) = (d/dt)[T(t) s'(t)] = \dot{T}(t) s'(t) + T(t) s''(t)$$

it follows that

$$\begin{aligned}\dot{r}(t) \times \ddot{r}(t) &= [T(t) s'(t)] \times [\dot{T}(t) s'(t) + T(t) s''(t)] = \\ &= [s'(t)]^2 [T(t) \times \dot{T}(t)] + s'(t) s''(t) [T(t) \times T(t)] \\ &= [s'(t)]^2 [T(t) \times \dot{T}(t)] + s'(t) s''(t) \mathbf{0} = \\ &= [s'(t)]^2 [T(t) \times \dot{T}(t)] = \|\dot{r}(t)\|^2 [T(t) \times \dot{T}(t)]\end{aligned}$$

We also note that

$$(\forall t \in A: \|T(t)\| = 1) \Rightarrow (\forall t \in A: T(t) \perp \dot{T}(t))$$

$$\Rightarrow \vartheta(T(t), \dot{T}(t)) = \pi/2$$

and therefore

$$\begin{aligned}\|\dot{r}(t) \times \ddot{r}(t)\| &= \| \|\dot{r}(t)\|^2 [T(t) \times \dot{T}(t)] \| = \\ &= \|\dot{r}(t)\|^2 \|T(t) \times \dot{T}(t)\| =\end{aligned}$$

$$= \|\dot{r}(t)\|^2 \, \|T(t)\| \, \|\dot{T}(t)\| \, \sin\theta\,(T(t), \dot{T}(t)) =$$

$$= \|\dot{r}(t)\|^2 \cdot 1 \cdot \|\dot{T}(t)\| \, \sin(\pi/2) =$$

$$= \|\dot{r}(t)\|^2 \, \|\dot{T}(t)\| \Rightarrow$$

$$\Rightarrow \|\dot{T}(t)\| = \frac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^2} \Rightarrow$$

$$\Rightarrow K(t) = \frac{\|\dot{T}(t)\|}{\|\dot{r}(t)\|} = \frac{1}{\|\dot{r}(t)\|} \, \frac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^2} =$$

$$= \frac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^3}$$

APPLICATION

Show that a circle (c) with radius R has constant curvature $\kappa(t) = 1/R$.

Proof

Consider the circle (c): $r(t) = (R\cos t, R\sin t, 0)$.

It follows that

$$\dot{r}(t) = (-R\sin t, R\cos t, 0)$$

$$\ddot{r}(t) = (-R\cos t, -R\sin t, 0)$$

and therefore

$$\dot{r}(t) \times \ddot{r}(t) = (-R\sin t, R\cos t, 0) \times (-R\cos t, -R\sin t, 0)$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ -R\sin t & R\cos t & 0 \\ -R\cos t & -R\sin t & 0 \end{vmatrix} \begin{matrix} e_1 & e_2 \\ -R\sin t & R\cos t \\ -R\cos t & -R\sin t \end{matrix} =$$

$$= e_3(-R\sin t)(-R\sin t) - e_3(R\cos t)(-R\cos t)$$

$$= e_3\left[R^2\sin^2 t + R^2\cos^2 t\right] = R^2 e_3 (\sin^2 t + \cos^2 t)$$

$$= R^2 e_3 \Rightarrow$$

$$\Rightarrow \|\dot{r}(t) \times \ddot{r}(t)\| = \|R^2 e_3\| = R^2 \|e_3\| = R^2$$

Also

$$\|\dot{r}(t)\| = \|(-R\sin t, R\cos t, 0)\| = \sqrt{(-R\sin t)^2 + (R\cos t)^2 + 0^2}$$

$$= \sqrt{R^2(\cos^2 t + \sin^2 t)} = \sqrt{R^2} = R$$

and we conclude that

$$\kappa(t) = \frac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^3} = \frac{R^2}{R^3} = \frac{1}{R}$$

APPLICATION

Show that the curvature of a two-dimensional curve (c): $y=f(x)$ at x is given by

$$\boxed{\kappa(x)=\frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}}$$

Proof

We rewrite the definition of the curve parametrically as (c): $r(x)=(x,f(x),0)$ and note that

$$\begin{cases}\dot r(x)=(1,f'(x),0)\\ \ddot r(x)=(0,f''(x),0)\end{cases}$$

It follows that

$$\dot r(x)\times\ddot r(x)=(1,f'(x),0)\times(0,f''(x),0)=$$

$$=\left|\begin{array}{ccc|cc} e_1 & e_2 & e_3 & e_1 & e_2\\ 1 & f'(x) & 0 & 1 & f'(x)\\ 0 & f''(x) & 0 & 0 & f''(x)\end{array}\right| =$$

$$=f''(x)e_3 \Rightarrow$$

$$\Rightarrow \|\dot r(x)\times\ddot r(x)\|=\|f''(x)e_3\|=|f''(x)|\,\|e_3\|=|f''(x)|$$

and

$$\|\dot r(x)\|=\|(1,f'(x),0)\|=\sqrt{1^2+[f'(x)]^2+0^2}$$

$$=\sqrt{1+[f'(x)]^2}$$

and therefore

$$\kappa(x)=\frac{\|\dot r(x)\times\ddot r(x)\|}{\|\dot r(x)\|^3}=\frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$$

EXERCISES

(12) Find the curvature $\kappa(t)$ for the curves defined by

a) $r(t) = (at^2, bt, ct^3)$

b) $r(t) = (a\cos(\omega t), a\sin(\omega t), ut)$

c) $r(t) = (\cos^3 t, 0, \sin^3 t)$ at $t = \pi/2$

d) $r(t) = (3\cosh(t/3), t, 0)$ at $t = 1$

e) $r(t) = (e^{at}\cos(\omega t), e^{at}\sin(\omega t), e^{at})$

f) $r(t) = (e^{at}, e^{-at}, ut)$

g) $r(t) = (\ln t, at, bt^2)$ at $t = 1$

(13) For the following curves, find the (x, y) coordinates of the point where the curvature is maximum

a) $(c): y = \ln x$

b) $(c): y = \cosh x$

c) $(c): y = \sin x \wedge x \in [-\pi, \pi]$

d) $(c): y = \sinh x$

e) $(c): y = e^x$

f) $(c): y = \ln(\cos x) \wedge x \in [-\pi/2, \pi/2)$

CAL3.3: Scalar fields

SCALAR FIELDS

▼ Definitions

- We define the n-dimensional space $\mathbb{R}^n$ as follows:
  $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid \forall k \in [n] : x_k \in \mathbb{R}\}$
  with $[n]$ defined as
  $[n] = \{1, 2, 3, \ldots, n\}$
- The elements $x \in \mathbb{R}^n$ are n-dimensional vectors with $x = (x_1, x_2, \ldots, x_n)$. The numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$ are the components of x.

▶ Algebra on $\mathbb{R}^n$

Let $x, y, z \in \mathbb{R}^n$ with $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, and $z = (z_1, z_2, \ldots, z_n)$. We define:

$x = y \Leftrightarrow \forall k \in [n] : x_k = y_k$

$z = x + y \Leftrightarrow \forall k \in [n] : z_k = x_k + y_k$

$z = ax \Leftrightarrow \forall k \in [n] : z_k = a x_k$ (with $a \in \mathbb{R}$)

$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{k=1}^{n} x_k y_k$ (inner product)

$\|x\| = (x \cdot x)^{1/2} = \sqrt{\sum_{k=1}^{n} x_k^2}$ (norm).

Balls on $\mathbb{R}^n$

- Let $x, y \in \mathbb{R}^n$ be two vectors. Assume that x, y represent two points in an n-dimensional space.. Then $\|x-y\|$ is the distance between x and y.
- We therefore define:

$$B(x,\rho) = \{u \in \mathbb{R}^n \mid \|x-u\| < \rho\}$$

with $x \in \mathbb{R}^n$ and $\rho \in (0, +\infty)$

- The set $B(x,\rho)$ contains all the points in $\mathbb{R}^n$ whose distance from x is less than ρ.

Open and closed sets

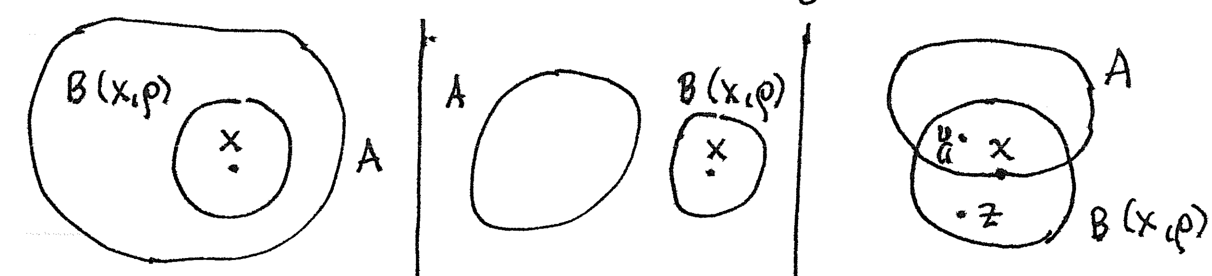
- Let $A \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. We say that

x <u>interior</u> to $A \Leftrightarrow \exists \rho \in (0,+\infty): B(x,\rho) \subseteq A$

x <u>exterior</u> to $A \Leftrightarrow \exists \rho \in (0,+\infty): B(x,\rho) \cap A = \emptyset$

x <u>boundary point</u> to $A \Leftrightarrow$

$\Leftrightarrow \forall \rho \in (0,+\infty): \exists y, z \in B(x,\rho): (y \in A \wedge z \notin A)$.



x interior to A	x exterior to A	x boundary point to A.

- We may therefore define:
  $\mathrm{int}(A) = \{x \in \mathbb{R}^n \mid x \text{ interior to } A\}$
  $\mathrm{ext}(A) = \{x \in \mathbb{R}^n \mid x \text{ exterior to } A\}$
  $\partial A = \{x \in \mathbb{R}^n \mid x \text{ boundary point to } A\}$.
- We say that:
  A is an <u>open</u> set $\Leftrightarrow A \cap \partial A = \emptyset \Leftrightarrow A = \mathrm{int}(A)$
  A is a <u>closed</u> set $\Leftrightarrow \partial A \subseteq A$
- It follows that
  a) An open set does not contain any of the points in its boundary.
  b) A closed set includes all of the points in its boundary.

▶ <u>Scalar fields</u>

- A <u>scalar field</u> is a mapping $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$.
- A is the <u>domain</u> of f, and we write $\mathrm{dom}(f) = A$.
- The range $f(A)$ of A is defined as:
  $f(A) = \{f(x) \mid x \in A\}$.

▼ Limits of scalar fields

- Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $x_0 \in \text{int}(A)$. We define

$$\boxed{\lim_{x \to x_0} f(x) = \ell \Leftrightarrow \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (x \in B(x_0, \delta) - \{x_0\} \Rightarrow |f(x) - \ell| < \varepsilon)}$$

- This definition is similar to the Weierstrass definition of $\lim_{x \to a} f(x) = \ell$ for functions of one variable.

↳ Properties of limits

Thm : Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$, and let $a \in \text{int}(A)$. Assume that

$$\lim_{x \to a} f(x) = \ell_1 \wedge \lim_{x \to a} g(x) = \ell_2. \quad \text{Then:}$$

a) $\lim_{x \to a} [f(x) + g(x)] = \ell_1 + \ell_2$

b) $\lim_{x \to a} [f(x) g(x)] = \ell_1 \ell_2$

c) $\forall \lambda \in \mathbb{R} : \lim_{x \to a} [\lambda f(x)] = \lambda \lim_{x \to a} f(x)$

d) $\ell_2 \neq 0 \Rightarrow \lim_{x \to a} \left[\dfrac{f(x)}{g(x)} \right] = \dfrac{\ell_1}{\ell_2}$

Limit of polynomials

Def: A polynomial $f \in \mathbb{R}^n[x]$ is a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ with

$$\boxed{f(x) = \sum_{(k_1,\dots,k_n) \in ([m] \cup \{0\})^n} A_{k_1 k_2 \cdots k_n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}}$$

with $x = (x_1, x_2, \dots, x_n)$ and $[m] = \{1, 2, 3, \dots, m\}$.

▶ $\mathbb{R}^n[x]$ = the set of all polynomials on $\mathbb{R}^n$.

Thm: $\boxed{f \in \mathbb{R}[x] \Rightarrow \lim_{x \to x_0} f(x) = f(x_0), \ \forall x_0 \in \mathbb{R}^n}$

EXAMPLE

Evaluate: $\displaystyle\lim_{(x,y) \to (1,3)} [3 + x^2 + y^2 + 3xy]$

Solution

$$\lim_{(x,y) \to (1,3)} [3 + x^2 + y^2 + 3xy] = 3 + 1^2 + 3^2 + 3 \cdot 1 \cdot 3 =$$
$$= 3 + 1 + 9 + 9 =$$
$$= 22.$$

Zero-bounded theorem

Def: Let $f: A \to \mathbb{R}$ be a scalar field with $A \subseteq \mathbb{R}^n$. Let $B \subseteq A$. We say that
$$f \text{ bounded in } B \Leftrightarrow \exists a \in (0, +\infty) : \forall x \in B : |f(x)| \leq a$$

Let $N(x_0, \delta)$ be an open ball with center x_0 and radius δ defined as:
$$N(x_0, \delta) = \{x \in \mathbb{R}^n \mid 0 < \|x - x_0\| < \delta\}$$
The zero-bounded theorem reads:

Thm: Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ be scalar fields with $A \subseteq \mathbb{R}^n$ and let $x_0 \in \text{int}(A)$. Then
$$\begin{cases} \exists \delta \in (0, +\infty) : f \text{ bounded at } N(x_0, \delta) \\ \lim_{x \to x_0} g(x) = 0 \end{cases} \Rightarrow \lim_{x \to x_0} [f(x) g(x)] = 0$$

EXAMPLE

Evaluate the limit $\displaystyle\lim_{(x,y) \to (0,0)} \frac{3x^2(x^2 - y^2)}{x^2 + y^2}$

Solution

Define $\displaystyle f(x,y) = \frac{3x^2}{x^2 + y^2}, \ \forall (x,y) \in \mathbb{R}^2 - \{(0,0)\}$

and $g(x,y) = x^2 - y^2, \ \forall (x,y) \in \mathbb{R}^2$.

We note that:

$$|b(x,y)| = \left| \frac{3x^2}{x^2+y^2} \right| = \frac{3|x^2|}{|x^2+y^2|} = \frac{3x^2}{x^2+y^2} \leq$$

$$\leq \frac{3(x^2+y^2)}{x^2+y^2} = 3, \ \forall (x,y) \in \mathbb{R}^2 - \{(0,0)\} \Rightarrow$$

$\Rightarrow$ b bounded at $\mathbb{R}^2 - \{(0,0)\}$. (1)

Also:

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} (x^2-y^2) = 0^2-0^2 = 0 \quad (2)$$

From (1) and (2):

$$\lim_{(x,y)\to(0,0)} \frac{3x^2(x^2-y^2)}{x^2+y^2} = 0 .$$

Limit restricted to paths

Let $x_0 \in \mathbb{R}^n$ be given. We define $\mathcal{P}(x_0|A)$ as the set of all continuous mappings $\gamma : (0,a) \to A - \{x_0\}$ with $a \in (0,+\infty)$ and $A \subseteq \mathbb{R}^n$ such that $\lim_{t \to 0^+} \|\gamma(t) - x_0\| = 0$. The corresponding belonging condition is:

$$\boxed{\gamma \in \mathcal{P}(x_0|A) \Leftrightarrow \begin{cases} \exists a \in (0,+\infty) : (\gamma : (0,a) \to A - \{x_0\}) \land \\ \qquad \land \gamma \text{ continuous on } (0,a)) \\ \lim_{t \to 0^+} \|\gamma(t) - x_0\| = 0 \end{cases}}$$

We now define path-restricted limits as follows:

Def: Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field, let $x_0 \in \text{int}(A)$, and let $\gamma \in \mathcal{P}(x_0|A)$ be a path. Then, we define the limit of f along the path γ as:

$$\lim_{x \in \gamma} f(x) = \lim_{t \to 0^+} f(\gamma(t))$$

These path limits are a generalization of the "side limits" from single-variable calculus. Then, we can show that:

$$\lim_{x \to x_0} f(x) = \ell \iff \forall \gamma \in \mathcal{P}(x_0 | A) : \lim_{x \in \gamma} f(x) = \ell$$

$$\exists \gamma_1, \gamma_2 \in \mathcal{P}(x_0 | A) : \left(\lim_{x \in \gamma_1} f(x) \neq \lim_{x \in \gamma_2} f(x) \right) \implies$$

$$\implies \lim_{x \to x_0} f(x) \text{ does not exist}$$

It follows that $\lim_{x \to x_0} f(x)$ will converge if and only if all paths limits for all paths $\gamma \in \mathcal{P}(x_0 | A)$ agree. This includes both linear and nonlinear paths. As we will see from a counterexample below, agreement between just the linear paths is not sufficient to ensure that $\lim_{x \to x_0} f(x)$ converges.

EXAMPLE

Evaluate the following limit or show it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

Solution

Let $\forall (x,y) \in \mathbb{R}^2 - \{0\}: f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ and consider the path

$$\gamma(\theta): \begin{cases} x = t\cos\theta \\ y = t\sin\theta \end{cases}, \text{ with } t \to 0^+$$

Then:

$$\lim_{(x,y)\in\gamma(\theta)} f(x,y) = \lim_{(x,y)\in\gamma(\theta)} \frac{x^2-y^2}{x^2+y^2} = \lim_{t\to 0^+} \frac{(t\cos\theta)^2-(t\sin\theta)^2}{(t\cos\theta)^2+(t\sin\theta)^2}$$

$$= \lim_{t\to 0^+} \frac{t^2(\cos^2\theta-\sin^2\theta)}{t^2(\cos^2\theta+\sin^2\theta)} = \cos^2\theta - \sin^2\theta = \cos(2\theta)$$

For $\theta = 0$: $\displaystyle\lim_{(x,y)\in\gamma(0)} f(x,y) = \cos(2\cdot 0) = \cos 0 = 1$ (1)

For $\theta = \pi/4$: $\displaystyle\lim_{(x,y)\in\gamma(\pi/4)} f(x,y) = \cos(2(\pi/4)) = \cos(\pi/2) = 0$ (2)

From Eq.(1) and Eq.(2):

$\displaystyle\lim_{(x,y)\in\gamma(0)} f(x,y) \neq \lim_{(x,y)\in\gamma(\pi/4)} f(x,y) \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

COUNTEREXAMPLE

Consider the function $f(x,y) = \dfrac{xy^2}{x^2+y^4}, \forall (x,y) \in \mathbb{R}^2 - \{0\}$

and $\gamma(\vartheta)$ be a linear path towards $(0,0)$ defined as

$$\gamma(\vartheta): \begin{cases} x = t\cos\vartheta \\ y = t\sin\vartheta \end{cases}, \text{ with } t \to 0^+$$

Show that:

a) $\forall \vartheta \in [0, 2\pi): \lim\limits_{(x,y)\in\gamma(\vartheta)} f(x,y) = 0$

b) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Solution

a) Let $\vartheta \in [0,2\pi)$ be given. Then:

$$\lim_{(x,y)\in\gamma(\vartheta)} f(x,y) = \lim_{(x,y)\in\gamma(\vartheta)} \frac{xy^2}{x^2+y^4} = \lim_{t\to 0^+} \frac{(t\cos\vartheta)(t\sin\vartheta)^2}{(t\cos\vartheta)^2+(t\sin\vartheta)^4} =$$

$$= \lim_{t\to 0^+} \frac{t^3\cos\vartheta\sin^2\vartheta}{t^2\cos^2\vartheta + t^4\sin^4\vartheta} = \lim_{t\to 0^+} \frac{t^3\cos\vartheta\sin^2\vartheta}{t^2[\cos^2\vartheta + t^2\sin^4\vartheta]}$$

$$= \lim_{t\to 0^+} \frac{t\cos\vartheta\sin^2\vartheta}{\cos^2\vartheta + t^2\sin^4\vartheta} \quad (1)$$

We distinguish between the following cases:

Case 1: Assume that $\cos\vartheta \neq 0$. Then:

$$\lim_{(x,y)\in\gamma(\vartheta)} f(x,y) = \frac{0\cos\vartheta\sin^2\vartheta}{\cos^2\vartheta + 0^2\sin^4\vartheta} = \frac{0}{\cos\vartheta} = 0$$

Case 2: Assume that $\cos\vartheta = 0$. Then

$\sin^2\vartheta = 1 - \cos^2\vartheta = 1 - 0 = 1 \Rightarrow$

and it follows that

$$\lim_{(x,y)\in\gamma(\theta)} f(x,y) = \lim_{t\to 0^+} \frac{t\cdot 0\cdot 1}{0^2 + t^2\cdot 1^2} = \lim_{t\to 0^+} \frac{0}{t^2} = 0$$

We conclude that $\forall\vartheta\in[0,2\pi): \lim\limits_{(x,y)\in\gamma(\vartheta)} f(x,y) = 0$

b) Consider the path $\gamma_0: \begin{cases} x = t^2 \\ y = t \end{cases}$ with $t\to 0^+$

Then, we have

$$\lim_{(x,y)\in\gamma_0} f(x,y) = \lim_{(x,y)\in\gamma_0} \frac{xy^2}{x^2+y^4} = \lim_{t\to 0^+} \frac{t^2(t)^2}{(t^2)^2+t^4}$$

$$= \lim_{t\to 0^+} \frac{t^4}{t^4+t^4} = \lim_{t\to 0^+} \frac{t^4}{2t^4} = \frac{1}{2} \Rightarrow$$

$$\Rightarrow \lim_{(x,y)\in\gamma_0} f(x,y) \neq \lim_{(x,y)\in\gamma(0)} f(x,y) \Rightarrow$$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

EXERCISES

① Evaluate the following limits or show that they do not exist.

a) $\lim\limits_{(x,y)\to(1,3)} \left[x^2 y(x+y)\right]$

b) $\lim\limits_{(x,y,z)\to(2,-1,4)} \left[x^3+y^3+z^3-3xyz\right]$

c) $\lim\limits_{(x,y)\to(1,0)} \dfrac{xy+x^2}{x^2+y^2+3}$

d) $\lim\limits_{(x,y)\to(3,3)} \dfrac{x-3}{\sqrt{y^2-9}}$

e) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 y}{x^2+y^2}$

f) $\lim\limits_{(x,y)\to(0,0)} \dfrac{|x|}{|x|+|y|}$

g) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$

h) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{1+y^2}$

i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2+y^2}$

j) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x(x+y)^2}{x^2+y^2}$

k) $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x}{x+y^3}$

l) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x^2-y^2}$

② Show that

a) $\forall x,y\in\mathbb{R}: \begin{cases} |x^3| \leqslant |x|(x^2+y^2) \\ |y^3| \leqslant |y|(x^2+y^2) \end{cases}$

b) Consider the function

$$\forall (x,y)\in\mathbb{R}^2-\{\mathbf{0}\}: f(x,y) = \frac{x^3+y^3}{x^2+y^2}$$

Use (a) to show that: $\forall (x,y)\in\mathbb{R}^2-\{\mathbf{0}\}: |f(x,y)| \leqslant |x|+|y|$.

c) Use (b) to show that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

③ Let $f(x,y) = \dfrac{x^a y^b}{x^2+y^2}$, $\forall (x,y) \in \mathbb{R}^2 - \{0\}$

Show that:

a) $a+b > 2 \Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

b) $a+b \leq 2 \Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Continuity of scalar fields

Def: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $x_0 \in \operatorname{int}(A)$. We say that:

$$f \text{ continuous at } x_0 \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)$$

$$f \text{ continuous on } B \Leftrightarrow \forall x_0 \in B : f \text{ continuous at } x_0$$

Composition theorem

The composition theorem for scalar fields reads:

Thm: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $g: B \to \mathbb{R}$ with $B \subseteq \mathbb{R}$ be a function and let $x_0 \in A$. Define $\forall x \in G : h(x) = f(g(x))$ with $G = \{x \in B \mid g(x) \in A\}$ and assume that $x_0 \in G$. Then

$$\left.\begin{array}{l} f \text{ continuous at } x_0 \\ g \text{ continuous at } f(x_0) \end{array}\right\} \Rightarrow h \text{ continuous at } x_0$$

- The following are immediate consequences of the composition theorem:

$$\lim_{x \to x_0} f(x) = a \Rightarrow \lim_{x \to x_0} \sin(f(x)) = \sin a$$

$$\lim_{x \to x_0} f(x) = a \Rightarrow \lim_{x \to x_0} \cos(f(x)) = \cos a$$

$$\left.\begin{array}{l}\forall k \in \mathbb{Z}: a \neq k\pi + \pi/2 \\ \lim\limits_{x \to x_0} f(x) = a\end{array}\right\} \Rightarrow \lim_{x \to x_0} \tan(f(x)) = \tan a$$

$$\left.\begin{array}{l}\forall k \in \mathbb{Z}: a \neq k\pi \\ \lim\limits_{x \to x_0} f(x) = a\end{array}\right\} \Rightarrow \lim_{x \to x_0} \cot(f(x)) = \cot a$$

▸ In certain limit calculations we can take advantage of the composition theorem, but the following composition theorem corollary can result in more economical solutions

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Prop: (Composition theorem corollary)

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $g: B \to \mathbb{R}$ with $B \subseteq \mathbb{R}$ be a function. Let $x_0 \in \mathbb{R}^n$ be a limit point of A and let l_0 be a limit point of B. Then:

$$\left.\begin{array}{l}\lim\limits_{x \to x_0} f(x) = l_0 \wedge \lim\limits_{t \to l_0} g(t) = l \\ \exists \delta \in (0, +\infty): \forall x \in N(x_0, \delta): f(x) \neq l_0\end{array}\right\} \Rightarrow \lim_{x \to x_0} g(f(x)) = l$$

EXAMPLES

a) $f(x,y) = \dfrac{x^4[\sin(xy)+\sin(x+y)]}{(x^2+y^2)^2} \leftarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$

Solution

Since

$$\lim_{(x,y)\to(0,0)} (xy) = 0\cdot 0 = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \sin(xy) = \sin 0 = 0 \qquad (1)$$

$$\lim_{(x,y)\to(0,0)} (x+y) = 0+0 = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \sin(x+y) = \sin 0 = 0 \qquad (2)$$

then from Eq.(1) and Eq.(2):

$$\lim_{(x,y)\to(0,0)} [\sin(xy)+\sin(x+y)] = 0+0 = 0 \qquad (3)$$

Define $b(x,y) = \dfrac{x^4}{(x^2+y^2)^2}$, $\forall (x,y) \in \mathbb{R}^2 - \{(0,0)\}$

and note that

$$|b(x,y)| = \left|\frac{x^4}{(x^2+y^2)^2}\right| = \frac{x^4}{(x^2+y^2)^2} =$$

$$= \frac{x^4}{x^4+2x^2y^2+y^4} \leq \frac{x^4+2x^2y^2+y^4}{x^4+2x^2y^2+y^4}$$

$$= 1, \ \forall (x,y) \in \mathbb{R}^2 - \{(0,0)\} \Rightarrow$$

$$\Rightarrow b \text{ bounded on } \mathbb{R}^2 - \{(0,0)\} \qquad (4)$$

From Eq.(3) and Eq.(4), via the zero-bounded theorem,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

b) $f(x,y) = \dfrac{\sin(x^2+y^2)}{3(x^2+y^2)} \leftarrow \lim\limits_{(x,y)\to(0,0)} f(x,y).$

Solution

1st method: (via composition theorem - not recommended!)
Define the auxiliary functions

$$g(x) = \begin{cases} \sin(x)/(3x) & , \text{ if } x \in \mathbb{R}-\{0\} \\ 1/3 & , \text{ if } x = 0 \end{cases}$$

$$h(x,y) = x^2+y^2 \quad , \quad \forall (x,y) \in \mathbb{R}^2$$

and note that

$$f(x,y) = g(h(x,y)) \quad , \quad \forall (x,y) \in \mathbb{R}^2 - \{(0,0)\}$$

We show the assumptions needed by the composition theorem:

$$\lim_{(x,y)\to(0,0)} h(x,y) = \lim_{(x,y)\to(0,0)} (x^2+y^2) = 0^2+0^2 = 0 = h(0,0)$$

$\Rightarrow h$ continuous at $(x,y)=(0,0)$ (1)

and

$$\lim_{x\to h(0,0)} g(x) = \lim_{x\to 0} \frac{\sin x}{3x} = \frac{1}{3} \lim_{x\to 0} \frac{\sin x}{x} =$$

$$= \frac{1}{3} = g(0) \Rightarrow$$

$\Rightarrow g$ continuous at $x = h(0,0)$. (2)

From Eq.(1) and Eq.(2), via the composition theorem, it follows that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} g(h(x,y)) =$$

$$= g(h(0,0)) = g(0) = 1/3 \qquad \square$$

<u>2nd method</u>: (via the composition corollary).

We note that

$$\lim_{(x,y)\to(0,0)} (x^2+y^2) = 0^2+0^2 = 0$$

and for $t = x^2+y^2$

$$\lim_{t\to 0} \frac{\sin(t)}{3t} = \frac{1}{3} \lim_{t\to 0} \frac{\sin t}{t} = \frac{1}{3}$$

and

$$x^2+y^2 \neq 0 \;,\; \forall (x,y) \in \mathbb{R}^2 - \{(0,0)\}$$

so from the composition theorem it follows that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{3(x^2+y^2)} = \frac{1}{3} \qquad \square$$

→ Using the composition theorem corollary eliminates the need to explicitly define the auxilliary functions $g(x)$ and $h(x,y)$, and substantially simplifies the writing of the solution.

EXERCISES

④ Evaluate the following limits or show that they do not exist.

a) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{\tan(x^2+y^2+z^2)}{x^2+y^2+z^2}$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4+2x^2y^2+y^4-4}{x^2+y^2-2}$

d) $\lim\limits_{(x,y)\to(0,0)} \sin(x+y)\exp(-1/(x^2+y^2))$

e) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{\sin x+\sin y+\sin z}{\ln(x^2+y^2+z^2)}$

f) $\lim\limits_{(x,y)\to(0,0)} \left[\dfrac{1}{\sin^2(x^2+y^2)} - \dfrac{1}{x^2+y^2}\right]$

g) $\lim\limits_{(x,y)\to(0,0)} \tan(x^2+y^4)\ln(x^2+y^4)$

h) $\lim\limits_{(x,y,z)\to(0,0,0)} (x^2+y^2+z^2)^{\sin(x^2+y^2+z^2)}$

⑤ a) Show that : $\forall (x,y)\in\mathbb{R}^2-\{0\}: x^2+xy+y^2>0$

b) Use the result from (a) to evaluate $\lim\limits_{(x,y)\to(0,0)} \sin(x^2+xy+y^2)\ln(x^2+xy+y^2)$.

Directional and Partial Derivatives

- Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $x \in \operatorname{int} A$ and $y \in \mathbb{R}^n$. The directional derivative $f'(x|y)$ is defined as

$$f'(x|y) = \lim_{h\to 0} \frac{f(x+hy) - f(x)}{h}$$

when the limit exists.

- $f'(x|y)$ give the rate of change of f along the line connecting x and $x+y$ at the point x. However, it is important to realize that $f'(x|y)$ depends on BOTH the direction AND the magnitude of y:

$$\boxed{f'(x|\lambda y) = \lambda f'(x|y) \quad , \forall \lambda \in \mathbb{R} - \{0\}}$$

- Method: To calculate the directional derivative $f'(x|y)$ we use the following proposition:

$$g(t) = f(x+ty) \; , \forall t \in (-\varepsilon, \varepsilon) \Rightarrow f'(x|y) = g'(0)$$

EXAMPLE

Evaluate $f'(x,y|e)$ for $f(x,y) = xy(x+y)$ and $e = (1,3)$.

Solution

Let $x,y \in \mathbb{R}$ be given. Define:

$$g(t) = f((x,y) + t(1,3)) = f(x+t, y+3t) =$$
$$= (x+t)(y+3t)(x+t+y+3t) =$$
$$= (x+t)(y+3t)(x+y+4t) \Rightarrow$$
$$\Rightarrow g'(t) = (y+3t)(x+y+4t) + (x+t)\cdot 3(x+y+4t) +$$
$$+ (x+t)(y+3t)\cdot 4 \Rightarrow$$
$$\Rightarrow f'(x,y|(1,3)) = g'(0) =$$
$$= y(x+y) + 3x(x+y) + 4xy =$$
$$= xy + y^2 + 3x^2 + 3xy + 4xy =$$
$$= 3x^2 + 8xy + y^2.$$

↳ While calculating $g'(t)$ we treat x,y as given constants.

↦ Mean Value Theorem

Thm: Let $f: A \to \mathbb{R}$ be a scalar field with $A \subseteq \mathbb{R}^n$. Then:

$$\boxed{\begin{aligned} &f'(x+ty|y) \text{ exists}, \ \forall t \in [0,1] \Rightarrow \\ &\Rightarrow \exists \xi \in (0,1): f(x+y) - f(x) = f'(x+\xi y|y) \end{aligned}}$$

EXERCISES

⑥ Use the definition to evaluate the directional derivative $f'(x,y|e)$ for

a) $f(x,y) = x^2y^3$ and $e = (1,-2)$

b) $f(x,y) = x^2(x-y)^3$ and $e = (-2,3)$

c) $f(x,y) = \dfrac{x^2}{x^3+y^3}$ and $e = (2,5)$

d) $f(x,y) = \dfrac{x(x+y)^2}{x-y}$ and $e = (-1,3)$

e) $f(x,y) = \ln(x^2+y^2)$ and $e = (2,1)$

f) $f(x,y) = \sin(xy)$ and $e = (1,1)$

g) $f(x,y) = \mathrm{Arctan}(x^2-y^2)$ and $e = (3,-2)$

Directional derivatives and continuity.

Surprise : The existence of directional derivatives in all directions does not guarantee that your scalar field is continuous!! This is different from single variable calculus.

COUNTEREXAMPLE

Consider the function:

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

• Directional derivatives

For $d = (a,b)$ with $a \neq 0$

$$f'(0,0|d) = \lim_{h\to 0} \frac{f(ha,hb) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(ha,hb)}{h} =$$

$$= \lim_{h\to 0} \left[\frac{1}{h} \frac{(ha)(hb)^2}{(ha)^2+(hb)^4} \right] = \lim_{h\to 0} \left[\frac{h^3 ab^2}{h^3(a^2+h^2b^4)} \right]$$

$$= \lim_{h\to 0} \left[\frac{ab^2}{a^2+h^2b^4} \right] = \frac{ab^2}{a^2+0} = \frac{b^2}{a}$$

For $d=(0,b)$:

$$f'(0,0|d) = \lim_{h\to 0} \frac{f(0,hb)-f(0,0)}{h} = \lim_{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

Thus $f'(0,0|d)$ exists for all directions $d \in \mathbb{R}^2 - \{(0,0)\}$.

- <u>Continuity</u>

For $(\gamma_1): \begin{cases} x=0 \\ y=t \end{cases}$ with $t\to 0^+$, we have:

$$\lim_{\gamma_1} f(x,y) = \lim_{t\to 0^+} f(0,t) = 0 \quad (\text{since } f(0,t)=0,\ \forall t\in\mathbb{R}).$$

For $(\gamma_2): \begin{cases} x=t^2 \\ y=t \end{cases}$ with $t\to 0^+$, we have:

$$\lim_{\gamma_2} f(x,y) = \lim_{t\to 0^+} f(t^2,t) = \lim_{t\to 0^+} \frac{t^2 t^2}{(t^2)^2+t^4} =$$

$$= \lim_{t\to 0^+} \frac{t^4}{t^4+t^4} = \lim_{t\to 0^+} \frac{t^4}{2t^4} = \frac{1}{2}$$

Since $\lim_{\gamma_1} f(x,y) \neq \lim_{\gamma_2} f(x,y) \Rightarrow$

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist $\Rightarrow$

$\Rightarrow f$ not continuous at $(0,0)$

Partial Derivatives

- Let $f(x) = f(x_1, x_2, \ldots, x_n)$, $\forall x \in A$ with $A \subseteq \mathbb{R}^n$ be a scalar field and consider the unit vectors:

$$e_1 = (1, 0, 0, \ldots, 0)$$
$$e_2 = (0, 1, 0, \ldots, 0)$$
$$e_3 = (0, 0, 1, \ldots, 0)$$
$$\vdots$$
$$e_n = (0, 0, 0, \ldots, 1)$$

We define the partial derivative of f with respect to x_K as:

$$\frac{\partial f}{\partial x_K} = f'(x \mid e_K)$$

- <u>Method</u>: To calculate $\partial f / \partial x_K$ we differentiate f with respect to x_K treating all other variables as constant.

- <u>Notation</u>: Other notations for partial derivatives, for example for 2 variables:

$$\frac{\partial f(x,y)}{\partial x} = f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = D_1 f(x,y)$$

$$\frac{\partial f(x,y)}{\partial y} = f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = D_2 f(x,y)$$

EXAMPLE

For $f(x,y) = xy^2(x^2+y^2)^3$, evaluate $\partial f/\partial x$ and $\partial f/\partial y$.

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[xy^2(x^2+y^2)^3\right] =$$

$$= \frac{\partial}{\partial x}\left[xy^2\right]\cdot(x^2+y^2)^3 + xy^2\left[\frac{\partial}{\partial x}(x^2+y^2)^3\right] =$$

$$= y^2(x^2+y^2)^3 + xy^2\, 3(x^2+y^2)^2\left[(\partial/\partial x)(x^2+y^2)\right]$$

$$= y^2(x^2+y^2)^3 + 3xy^2(x^2+y^2)^2\cdot(2x) =$$

$$= y^2(x^2+y^2)^2\left[(x^2+y^2) + 3x(2x)\right] =$$

$$= y^2(x^2+y^2)^2(x^2+y^2+6x^2) =$$

$$= y^2(x^2+y^2)^2(7x^2+y^2).$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[xy^2(x^2+y^2)^3\right] =$$

$$= \left[\frac{\partial}{\partial y}(xy^2)\right](x^2+y^2)^3 + xy^2\left[\frac{\partial}{\partial y}(x^2+y^2)^3\right] =$$

$$= 2xy(x^2+y^2)^3 + xy^2\cdot 3(x^2+y^2)^2\left[\frac{\partial}{\partial y}(x^2+y^2)\right] =$$

$$= 2xy(x^2+y^2)^3 + 3xy^2(x^2+y^2)^2\, 2y =$$

$$= 2xy(x^2+y^2)^2\left[(x^2+y^2) + 3y(y)\right] =$$

$$= 2xy(x^2+y^2)^2(x^2+y^2+3y^2) =$$

$$= 2xy(x^2+y^2)^2(x^2+4y^2).$$

EXERCISES

(7) Evaluate the following partial derivatives

a) $f(x,y) = x^3+y^3$; $\partial f/\partial x$, $\partial f/\partial y$

b) $f(x,y) = \sqrt{1-x^2-y^2}$; $\partial f/\partial x$, $\partial f/\partial y$

c) $f(x,y) = \sin(2x)\cos(3y)$; $\partial f/\partial x$, $\partial f/\partial y$

d) $f(x,y) = \tan(x/y)$; $\partial f/\partial x$, $\partial f/\partial y$

e) $f(x,y,z) = \dfrac{x}{(x^2+y^2+z^2)^{3/2}}$; $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$.

f) $f(x,y) = \exp(-y/x^2)$; $\partial f/\partial x$, $\partial f/\partial y$

g) $f(x,y,z) = \ln\left(\sqrt{x^2+y^2+z^2}\right)$; $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$

h) $f(x,y) = \mathrm{Arctan}\left(\sqrt{x^3+y^3}\right)$; $\partial f/\partial x$, $\partial f/\partial y$

i) $f(x,y) = \mathrm{Arcsin}\left(\sqrt{1-x^2y^2}\right)$; $\partial f/\partial x$

Mixed partial derivatives

Mixed partial derivatives are defined by successive partial differentiation, as follows:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = D_1 D_1 f$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy} = D_2 D_2 f$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy} = D_1 D_2 f$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx} = D_2 D_1 f$$

▸ Clairut's theorem (is $f_{xy} = f_{yx}$?)

Thm: Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ be a scalar field. Let $(a,b) \in A$ be a point and let $S \subseteq A$ be an open set such that $(a,b) \in \operatorname{int}(S)$.
Assume that

a) f_x, f_y, f_{xy} exist in S (i.e. for all points in S)
b) f_{xy} continuous in S

Then: $f_{xy}(a,b) = f_{yx}(a,b)$.

↳ In condition (b) we can replace f_{xy} with f_{yx}.

- **Example with $f_{xy}(a,b) \neq f_{yx}(a,b)$**

Consider the function

$$f(x,y)=\begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & , \text{if } (x,y)\in\mathbb{R}^2-\{(0,0)\} \\ 0 & , \text{if } (x,y)=(0,0) \end{cases}$$

We will show that $f_{xy}(0,0)=1$ and $f_{yx}(0,0)=-1$.

Solution

- **Calculation of $f_{xy}(0,0)$**

We begin with evaluating $f_y(x,0)$ for all $x\in\mathbb{R}$.
We note that $\forall x\in\mathbb{R}: f(x,0)=0$. It follows that

$$\forall x\in\mathbb{R}: f_y(x,0)=\lim_{h\to 0}\frac{f((x,0)+h(0,1))-f(x,0)}{h}=$$

$$=\lim_{h\to 0}\frac{f(x,h)-0}{h}\underset{h\neq 0}{=}\lim_{h\to 0}\left[\frac{1}{h}\frac{xh(x^2-h^2)}{x^2+h^2}\right]=$$

$$=\lim_{h\to 0}\frac{x(x^2-h^2)}{x^2+h^2} \qquad (1)$$

We distinguish between the following cases

Case 1: Assume that $x\neq 0$. Then, from Eq.(1)

$$f_y(x,0)=\lim_{h\to 0}\frac{x(x^2-h)}{x^2+h^2}=\frac{x(x^2-0)}{x^2+0}=\frac{x^3}{x^2}=x, \forall x\in\mathbb{R}-\{0\}$$

Case 2: Assume that $x=0$. Then, from Eq.(1)

$$f_y(0,0) = \lim_{h\to 0} \frac{0(0-h)}{0+h^2} = \lim_{h\to 0} 0 = 0$$

We conclude from both cases that

$$\forall x \in \mathbb{R} : f_y(x,0) = \begin{cases} x & , \text{if } x \in \mathbb{R}-\{0\} \\ 0 & , \text{if } x=0 \end{cases} = x$$

$$\Rightarrow \forall x \in \mathbb{R} : f_{xy}(x,0) = (\partial/\partial x)x = 1$$

$$\Rightarrow \underline{f_{xy}(0,0) = 1.}$$

• <u>Calculation of $f_{yx}(0,0)$</u>

We begin by evaluating $f_x(0,y)$ for all $y \in \mathbb{R}$.
We note that $\forall y \in \mathbb{R} : f(0,y) = 0$. It follows that

$$\forall y \in \mathbb{R} : f_x(0,y) = \lim_{h\to 0} \frac{f((0,y)+h(1,0)) - f(0,y)}{h} =$$

$$= \lim_{h\to 0} \frac{f(h,y) - f(0,y)}{h} = \lim_{h\to 0} \frac{f(h,y)}{h} \underset{h\neq 0}{=}$$

$$= \lim_{h\to 0} \left[\frac{1}{h} \frac{hy(h^2-y^2)}{h^2+y^2} \right] = \lim_{h\to 0} \frac{y(h^2-y^2)}{h^2+y^2} \qquad (2)$$

We distinguish between the following cases.

<u>Case 1</u>: Assume that $y \neq 0$. Then, from Eq. (2)

$$f_x(0,y) = \lim_{h\to 0} \frac{y(h^2-y^2)}{h^2+y^2} = \frac{y(0-y^2)}{0+y^2} =$$

$$= \frac{-y^3}{y^2} = -y \, , \; \forall y \in \mathbb{R}-\{0\}.$$

<u>Case 2</u>: Assume that $y=0$. Then, from Eq. (2)

$$f_x(0,0) = \lim_{h \to 0} \frac{0(h^2-0)}{h^2+0} = \lim_{h\to 0} 0 = 0$$

We conclude from both cases that

$$f_x(0,y) = \begin{cases} -y &, \text{ if } y \in \mathbb{R}-\{0\} \\ 0 &, \text{ if } y=0 \end{cases} = -y \,, \ \forall y \in \mathbb{R}$$

$$\Rightarrow f_{yx}(0,y) = (\partial/\partial y)(-y) = -1 \,, \ \forall y \in \mathbb{R}$$

$$\Rightarrow \underline{f_{yx}(0,0) = -1.}$$

EXERCISES

(8) Evaluate the following mixed partial derivatives

a) $f(x,y) = \sin(x^3+2y^2)$; f_{xxy}

b) $f(x,y,z) = x^3y^5z^8$; f_{xyz}

c) $f(x,y) = \dfrac{\exp(-x^2/y)}{\sqrt{y}}$; f_{xx}, f_{xy}

d) $f(x,y,z) = \mathrm{Arctan}(x^2+yz)$; f_{xy}, f_{yz}, f_{zx}

e) $f(x,y,z) = \cos(xy+yz+zx)$; f_{xxy}, f_{yyzz}

(9) Show that the scalar field

$u(x,t) = \sin(nx)\exp(-n^2 t)$, $\forall (x,t) \in \mathbb{R}^2$

satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for all $n \in \mathbb{R}$.

(10) A field $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ is harmonic if and only if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Show that the following scalar fields are harmonic:

a) $f(x,y) = x$

b) $f(x,y) = e^x \cos y$

c) $f(x,y) = \mathrm{Arctan}(y/x)$

d) $f(x,y) = \ln(x^2+y^2)$

(11) Find all $a, b, c, d \in \mathbb{R}$ such that
$f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$
is harmonic.

(12) Find all $a, b \in \mathbb{R}$ such that
$f(x,y) = \cos(ax)\exp(by)$
is harmonic.

(13) Use Clairut's theorem to show that there does not exist a scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ with $\partial f/\partial x = y^2$ and $\partial f/\partial y = x$.

Application of partial derivatives to error propagation

Suppose that a variable z is related to a set of variables $x_1, x_2, \ldots, x_n$ via the equation

$$z = f(x_1, x_2, \ldots, x_n)$$

Let us assume that the value of $x_1, x_2, \ldots, x_n$ is not known exactly, but we have error estimates $x_1 \pm \sigma_1$, $x_2 \pm \sigma_2, \ldots, x_n \pm \sigma_n$. Then, the error estimate $z \pm \sigma$ is given by the equation

$$\boxed{\sigma = \sqrt{\sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k}\right)^2 \sigma_k^2}}$$

Here, σ is the error in the variables $x_1, x_2, \ldots, x_n$ as propagated via f onto the variable z.

▸ Typically, in experiments, some quantity z is calculated in two different ways from experimental data resulting in $z_1 \pm \sigma_1$ and $z_2 \pm \sigma_2$. We consider these two results to be consistent if

$$[z_1 - \sigma_1, z_1 + \sigma_1] \cap [z_2 - \sigma_2, z_2 + \sigma_2] \neq \emptyset$$

EXAMPLE

The oscillation period of an LC circuit is given by $T = 2\pi\sqrt{LC}$. Write the error of T in terms of the errors in L and C.

Solution

Since,

$$\frac{\partial T}{\partial L} = \frac{\partial}{\partial L}\left[2\pi\sqrt{LC}\right] = 2\pi\sqrt{C}\,(\partial/\partial L)\sqrt{L} =$$

$$= 2\pi\sqrt{C}\,\frac{1}{2\sqrt{L}} = \pi\sqrt{\frac{C}{L}}$$

and

$$\frac{\partial T}{\partial C} = \frac{\partial}{\partial C}\left[2\pi\sqrt{LC}\right] = 2\pi\sqrt{L}\,(\partial/\partial C)\sqrt{C} =$$

$$= 2\pi\sqrt{L}\,\frac{1}{2\sqrt{C}} = \pi\sqrt{\frac{L}{C}}$$

it follows that

$$\sigma_T = \sqrt{(\partial T/\partial L)^2\sigma_L^2 + (\partial T/\partial C)^2\sigma_C^2} =$$

$$= \sqrt{\left[\pi\sqrt{C/L}\right]^2\sigma_L^2 + \left[\pi\sqrt{L/C}\right]^2\sigma_C^2} =$$

$$= \sqrt{\pi^2(C/L)\sigma_L^2 + \pi^2(L/C)\sigma_C^2} =$$

$$= \pi\sqrt{(C/L)\sigma_L^2 + (L/C)\sigma_C^2}$$

EXERCISES

(14) Calculate the error propagation in the following expressions

a) $T = 2\pi\sqrt{l/g}$

b) $F = G\dfrac{m_1 m_2}{r^2}$

c) $E = mc^2$

d) $m = \dfrac{m_0}{\sqrt{1-(u/c)^2}}$

e) $x = x_0\sqrt{1-(u/c)^2}$

f) $y = y_0 \exp(-at)\cos(\omega t)$

g) $p = \exp(-nx^2)$

▼ Differentiable scalar fields

- Directional derivatives account for the rate of change of the scalar field f accross linear directions. A proper definition of differentiability has to account for curved directions as well.
- <u>Def</u>: Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar field. We say that T linear $\Leftrightarrow \forall \lambda_1, \lambda_2 \in \mathbb{R} : \forall x, y \in \mathbb{R}^n : T(\lambda_1 x + \lambda_2 y) = \lambda_1 T(x) + \lambda_2 T(y)$.

↳ Definition of Total Derivative (Young-Fréchet)

<u>Def</u>: Let $f: S \longrightarrow \mathbb{R}$ with $S \subseteq \mathbb{R}^n$ be a scalar field, and let $a \in \operatorname{int}(S)$.

We say that <u>f is differentiable at a</u> if and only if there exist:

a) A <u>linear</u> scalar field $T_a : \mathbb{R}^n \longrightarrow \mathbb{R}$

b) A scalar field $E_a : \mathbb{R}^n \longrightarrow \mathbb{R}$

c) A number $\rho \in (0, +\infty)$

such that

$$\begin{cases} \forall x \in B(\mathbf{0}, \rho) : f(a+x) = f(a) + T_a(x) + \|x\| E_a(x) \\ \lim\limits_{x \to \mathbf{0}} E_a(x) = 0 \end{cases}$$

- $T_a(x)$ is the <u>total derivative</u> of f at a. Note that T_a itself is a scalar field, satisfying:
$\forall \lambda_1, \lambda_2 \in \mathbb{R} : \forall x, y \in \mathbb{R}^n : T_a(\lambda_1 x + \lambda_2 y) = \lambda_1 T_a(x) + \lambda_2 T_a(y)$

Gradient of a scalar field

- Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field. We define the gradient $\nabla f : A \to \mathbb{R}^n$ as

$$\boxed{\nabla f = \langle D_1 f, D_2 f, D_3 f, \ldots, D_n f \rangle}$$

provided that the partial derivatives exist.

- Note that ∇f is NOT a scalar field. It is a vector field.

Properties of differentiability.

Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field, and let $a \in \operatorname{int}(A)$. It can be shown that:

1) f differentiable at $a \Rightarrow \forall x \in \mathbb{R}^n : T_a(x) = f'(a|x) = \nabla f(a) \cdot x$
2) f differentiable at $a \Rightarrow f$ continuous at a.
3) $\left.\begin{array}{l} \exists \rho \in (0, +\infty) : D_1 f, D_2 f, \ldots, D_n f \text{ exist on } B(a, \rho) \\ D_1 f, D_2 f, \ldots, D_n f \text{ continuous at } a \end{array}\right\} \Rightarrow$
   $\Rightarrow f$ differentiable at a.

- Properties (1), (2) are consequences of differentiable. Property (3) is a sufficient condition for differentiability.
- Property (1) indicates a method for evaluating directional derivatives via the gradient ∇f.

EXAMPLE

For $f(x,y) = xe^y + x^2y$ evaluate $f'(2,0|(a,b))$ for all $a,b \in \mathbb{R}-\{0\}$.

Solution

Since,

$f_x(x,y) = (\partial/\partial x)(xe^y + x^2y) = e^y + 2xy \Rightarrow$

$\Rightarrow f_x(2,0) = e^0 + 2\cdot 2\cdot 0 = 1+0 = 1$

and

$f_y(x,y) = (\partial/\partial y)(xe^y + x^2y) = xe^y + x^2 \Rightarrow$

$\Rightarrow f_y(2,0) = 2\cdot e^0 + 2^2 = 2+4 = 6$

it follows that

$$\begin{aligned} f'(2,0|(a,b)) &= \nabla f(2,0)\cdot(a,b) = \\ &= (f_x(2,0), f_y(2,0))\cdot(a,b) = \\ &= af_x(2,0) + bf_y(2,0) = \\ &= a + 6b \end{aligned}$$

EXERCISE

(15) Use the gradient to evaluate the directional derivatives $f'(x,y|e)$ for the scalar fields f in the direction $e \in \mathbb{R}^3$.

a) $f(x,y) = (3x^2+y^3)^5(2x^2-y)^3$; $e=(2,4)$

b) $f(x,y) = x^2y^3\sqrt{x^4+y^4}$; $e=(1,-1)$

c) $f(x,y) = \ln(x^3+y^3-3xy)$; $e=(2,-3)$

d) $f(x,y) = xy\exp(-(x+y)^2)$; $e=(1,3)$

e) $f(x,y) = \exp(-2x^2)\sin(3y)$; $e=(-2,3)$

f) $f(x,y) = \operatorname{Arctan}\left(\dfrac{x^2}{x^2+y^2}\right)$; $e=(2,1)$

g) $f(x,y) = \operatorname{Arcsin}\left(\sqrt{x^4+y^4}\right)$; $e=(-1,1)$

▼ The chain rule

Thm: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $a: I \to \text{int}(A)$ be a vector function with $I \subseteq \mathbb{R}$. Assume that:

a) $a(t)$ differentiable at t

b) f differentiable at $a(t)$.

Then:

$$\boxed{(d/dt)\, f(a(t)) = \nabla f(a(t)) \bullet \dot{a}(t)}$$

▶ Application to scalar fields on $\mathbb{R}^2$

1) For $z = f(x,y)$ with $x = x(t)$ and $y = y(t)$

$$\Rightarrow \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

2) For $z = f(x,y)$ with $x = x(t,s)$ and $y = y(t,s)$

$$\Rightarrow \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

→ The chain rule can be used to derive rules for transforming partial derivatives to other coordinates systems.

EXAMPLE

Consider the two-dimensional polar coordinates system defined via

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}$$

Given a scalar field $f(x,y)$, write $\partial f/\partial r, \partial f/\partial \vartheta$ in terms of $\partial f/\partial x, \partial f/\partial y$.

Solution

We note that

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\vartheta) = \cos\vartheta\ , \quad \frac{\partial x}{\partial \vartheta} = \frac{\partial}{\partial \vartheta}(r\cos\vartheta) = -r\sin\vartheta$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(r\sin\vartheta) = \sin\vartheta\ , \quad \frac{\partial y}{\partial \vartheta} = \frac{\partial}{\partial \vartheta}(r\sin\vartheta) = r\cos\vartheta$$

and therefore:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} =$$

$$= (\cos\vartheta)\frac{\partial f}{\partial x} + (\sin\vartheta)\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial \vartheta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \vartheta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \vartheta} =$$

$$= (-r\sin\vartheta)\frac{\partial f}{\partial x} + (r\cos\vartheta)\frac{\partial f}{\partial y}$$

Using matrix notation:

$$\begin{bmatrix} \partial f/\partial r \\ \partial f/\partial \vartheta \end{bmatrix} = \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -r\sin\vartheta & r\cos\vartheta \end{bmatrix} \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix}$$

Let $A = \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -r\sin\vartheta & r\cos\vartheta \end{bmatrix}$. Then

$$\det A = \begin{vmatrix} \cos\vartheta & \sin\vartheta \\ -r\sin\vartheta & r\cos\vartheta \end{vmatrix} = \cos\vartheta \cdot (r\cos\vartheta) - \sin\vartheta\,(-r\sin\vartheta)$$

$$= r(\cos^2\vartheta + \sin^2\vartheta) = r.$$

and therefore:

$$A^{-1} = \frac{1}{r}\begin{bmatrix} r\cos\vartheta & -\sin\vartheta \\ +r\sin\vartheta & \cos\vartheta \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -(\sin\vartheta)/r \\ \sin\vartheta & (\cos\vartheta)/r \end{bmatrix}$$

It follows that

$$\frac{\partial f}{\partial x} = (\cos\vartheta)\frac{\partial f}{\partial r} - \frac{\sin\vartheta}{r}\frac{\partial f}{\partial \vartheta}$$

$$\frac{\partial f}{\partial y} = (\sin\vartheta)\frac{\partial f}{\partial r} + \frac{\cos\vartheta}{r}\frac{\partial f}{\partial \vartheta}$$

EXERCISES

(16) Consider the coordinate transformation

$$\begin{cases} x = \xi + t \\ y = \xi - t \end{cases}$$

Show that for a scalar field $f(x,y)$

$$\left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 = \frac{\partial f}{\partial \xi}\,\frac{\partial f}{\partial t}$$

(17) Consider the function $u(x,y)$ in polar coordinates:

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}$$

a) Show that

$$\|\nabla u\|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \vartheta}\right)^2$$

b) For $u(r,\vartheta) = r^3 \sin^3\vartheta$ in polar coordinates evaluate $\|\nabla u\|^2$.

(18) Consider the spherical coordinate transformation

$$\begin{cases} x = \rho \sin\varphi \cos\vartheta \\ y = \rho \sin\varphi \sin\vartheta \\ z = \rho \cos\varphi \end{cases}$$

Given a function $f(x,y,z)$, write the derivatives $\partial f/\partial\rho$, $\partial f/\partial\vartheta$, $\partial f/\partial\varphi$ in terms of $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$.

(19) Let $r=(x,y,z)$ be a vector and consider a scalar field $f(x,y,z) = F(\|r\|)$, $\forall (x,y,z) \in \mathbb{R}^3 - \{0\}$

a) Show that

$$\nabla f(r) = \frac{F'(\|r\|)}{\|r\|} r \quad , \text{ and}$$

$$\|\nabla f(r)\| = |F'(r)|$$

b) Use part (a) to evaluate ∇f and $\|\nabla f\|$ for the following scalar fields:

1) $f(x,y,z) = \exp(-(x^2+y^2+z^2))$

2) $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

3) $f(x,y,z) = \ln(\sqrt{x^2+y^2+z^2})$

(20) Let $f(x,y,z)$ be a scalar functions such that $f(x,y,z) = u(x-y, y-z, z-x)$, $\forall (x,y,z) \in \mathbb{R}^3$

Show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$$

(21) Let $f(x,y,z)$ be a scalar field such that $\forall t \in (0,+\infty) : \forall (x,y,z) \in \mathbb{R}^3 : f(tx,ty,tz) = t^n f(x,y,z)$

Show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

(22) Consider a scalar field $f(x,y)$ rewritten in terms of polar coordinates

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}$$

Show that:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \vartheta^2}$$

Chain Rule and Implicit Differentiation

Implicit Function Theorem

The Implicit Function Theorem, given below, shows that given a constraint

$$F(x_1, x_2, \ldots, x_n) = 0$$

if the conditions of the theorem are satisfied, then it implicitly defines a new function

$$x_n = f_n(x_1, x_2, \ldots, x_{n-1})$$

such that

$$F(x_1, x_2, \ldots, x_{n-1}, f(x_1, x_2, \ldots, x_{n-1})) = 0.$$

Then, the derivative of x_n with respect to x_a with $a \in [n-1]$ is written:

$$\left(\frac{\partial x_n}{\partial x_a}\right)_{x_1, \ldots, x_{a-1}, x_{a+1}, \ldots, x_n}$$

<u>Thm</u>: Let $F: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field, let $a \in \operatorname{int}(A)$, and let $\rho \in (0, +\infty)$ be given. Assume that

$$\begin{cases} F(a) = 0 \\ D_1F, D_2F, \ldots, D_nF \text{ continuous on } B(a,\rho) \\ \forall x \in B(a,\rho): D_nF \neq 0 \end{cases}$$

Then there is a $f_n: B \to \mathbb{R}$ with $B \subseteq \mathbb{R}^{n-1}$ such that $\forall (x_1, \ldots, x_{n-1}) \in B: F(x_1, \ldots, x_{n-1}, f_n(x_1, \ldots, x_{n-1})) = 0$

• The case $F(x,y)=0$

Consider the function $y=f(x)$ defined implicitly by the equation

$$F(x,y)=0$$

It follows that $F(x,f(x))=0$, and we use the chain rule to differentiate with respect to x:

$$(d/dx)F(x,y)=0 \Rightarrow$$

$$\Rightarrow \frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0 \Rightarrow$$

$$\Rightarrow \frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0 \Rightarrow \frac{\partial F}{\partial y}\frac{dy}{dx}=-\frac{\partial F}{\partial x}$$

$$\Rightarrow \boxed{\frac{dy}{dx}=\frac{-\partial F/\partial x}{\partial F/\partial y}}$$

• The general case

Consider the constraint

$$F(x_1,x_2,\dots,x_n)=0$$

and let it define implicitly a function

$$x_n=f_n(x_1,x_2,\dots,x_{n-1})$$

Then we can show that

$$\boxed{\left(\frac{\partial x_n}{\partial x_a}\right)_{x_1,\dots,x_{a-1},x_{a+1},\dots,x_n}=\frac{\partial f_n}{\partial x_a}=\frac{-\partial F/\partial x_a}{\partial F/\partial x_n}}$$

for all $a\in[n-1]$.

Proof

We note that $\forall a, b \in [n-1] : \partial x_b / \partial x_a = \delta_{ab}$ with

$$\delta_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}$$

and therefore

$$\frac{dF}{dx_a} = \sum_{b=1}^{n-1} \frac{\partial F}{\partial x_b} \frac{\partial x_b}{\partial x_a} + \frac{\partial F}{\partial x_n} \left(\frac{\partial x_n}{\partial x_a} \right)_{x_1, \dots, x_{a-1}, x_{a+1}, \dots, x_n}$$

$$= \sum_{b=1}^{n-1} \frac{\partial F}{\partial x_b} \delta_{ab} + \frac{\partial F}{\partial x_n} \left(\frac{\partial x_n}{\partial x_a} \right)_{x_1, \dots, x_{a-1}, x_{a+1}, \dots, x_n}$$

$$= \frac{\partial F}{\partial x_a} + \frac{\partial F}{\partial x_n} \left(\frac{\partial x_n}{\partial x_a} \right)_{x_1, \dots, x_{a-1}, x_{a+1}, \dots, x_n}$$

It follows that

$$dF/dx_a = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial F}{\partial x_a} + \frac{\partial F}{\partial x_n} \left(\frac{\partial x_n}{\partial x_a} \right)_{x_1, \dots, x_{a-1}, x_{a+1}, \dots, x_n} = 0$$

$$\Rightarrow \left(\frac{\partial x_n}{\partial x_a} \right)_{x_1, \dots, x_{a-1}, x_{a+1}, \dots, x_n} = \frac{-\partial F / \partial x_a}{\partial F / \partial x_n}$$

EXAMPLES

a) If x,y,z are constrained via
$x^3+y^3+z^3+6xyz=1$
evaluate $(\partial z/\partial x)_y$, $(\partial z/\partial y)_x$, $(\partial x/\partial y)_z$
in terms of x,y,z.

Solution

Define $F(x,y,z) = x^3+y^3+z^3+6xyz-1$
We note that

$$\partial F/\partial x = (\partial/\partial x)(x^3+y^3+z^3+6xyz-1) = 3x^2+6yz$$
$$= 3(x^2+2yz)$$
$$\partial F/\partial y = (\partial/\partial y)(x^3+y^3+z^3+6xyz-1) =$$
$$= 3y^2+6xz = 3(y^2+2xz)$$
$$\partial F/\partial z = (\partial/\partial z)(x^3+y^3+z^3+6xyz-1) =$$
$$= 3z^2+6xy = 3(z^2+2xy)$$

and therefore

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{-\partial F/\partial x}{\partial F/\partial z} = \frac{-3(x^2+2yz)}{3(z^2+2xy)} = \frac{-(x^2+2yz)}{z^2+2xy}$$
$$\left(\frac{\partial z}{\partial y}\right)_x = \frac{-\partial F/\partial y}{\partial F/\partial z} = \frac{-3(y^2+2xz)}{3(z^2+2xy)} = \frac{-(y^2+2xz)}{z^2+2xy}$$
$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{-\partial F/\partial y}{\partial F/\partial x} = \frac{-3(y^2+2xz)}{3(x^2+2yz)} = \frac{-(y^2+2xz)}{x^2+2yz}$$

b) Consider the van der Waals equations

$$[P + a(n/V)^2][(V/n) - b] = RT$$

where a, b, R are constants and

P : gas pressure

V : gas volume

T : gas temperature

n : amount of gas molecules.

Use implicit differentiation to evaluate

$(\partial P/\partial n)_{V,T}$, $(\partial T/\partial n)_{P,V}$, $(\partial V/\partial T)_{P,n}$

<u>Solution</u>

Define

$$F(P,V,T,n) = [P + a(n/V)^2][(V/n) - b] - RT$$
$$= P(V/n) - bP + a(V/n)^{-1} - ab(V/n)^{-2} - RT$$

and note that

$$\partial F/\partial P = (V/n) - b$$

$$\partial F/\partial V = (\partial/\partial V)[P(V/n) + a(V/n)^{-1} - ab(V/n)^{-2}] =$$
$$= P/n + (-1)(V/n)^{-2}(1/n) - ab(-2)(V/n)^{-3}(1/n)$$
$$= (1/n)[P - (V/n)^{-2} + 2ab(V/n)^{-3}]$$

$$\partial F/\partial T = -R$$

$$\partial F/\partial n = (\partial/\partial n)[P(n/V)^{-1} + a(n/V) - ab(n/V)^2] =$$
$$= P(-1)(n/V)^{-2}(1/V) + a(1/V) - ab(n/V)2(1/V)$$
$$= (1/V)[-P(n/V)^{-2} + a - 2ab(n/V)]$$

It follows that:

$$\left(\frac{\partial P}{\partial n}\right)_{V,T} = \frac{-\partial F/\partial n}{\partial F/\partial P} =$$

$$= \frac{-(1/V)\left[-P(n/V)^{-2} + a - 2ab(n/V)\right]}{V/n - b}$$

$$\left(\frac{\partial T}{\partial n}\right)_{P,V} = \frac{-\partial F/\partial n}{\partial F/\partial T} =$$

$$= \frac{-(1/V)\left[-P(n/V)^{-2} + a - 2ab(n/V)\right]}{-R}$$

$$= \frac{1}{RV}\left[-P(V/n)^{2} + a - 2ab(n/V)\right]$$

$$\left(\frac{\partial V}{\partial T}\right)_{P,n} = \frac{-\partial F/\partial T}{\partial F/\partial V} =$$

$$= \frac{-(-R)}{(1/n)\left[P - (V/n)^{-2} + 2ab(V/n)^{-3}\right]}$$

$$= \frac{nR}{P - (n/V)^{2} + 2ab(n/V)^{3}}$$

EXERCISES

(23) Evaluate the following partial derivatives using implicit differentiation based on partial derivatives of the implicit definition

a) $x^3y + y^3z + z^3x = (xyz)^2$; $\partial z/\partial x$, $\partial z/\partial y$, $\partial x/\partial y$

b) $x^3 + y^3 + z^3 + (x+y)(y+z)(z+x)$; $\partial y/\partial x$, $\partial y/\partial z$, $\partial z/\partial x$

c) $\exp(x^2y) + \cos(y^2z) = xz$; $\partial z/\partial x$, $\partial x/\partial z$, $\partial x/\partial y$

d) $\dfrac{1}{x^2+y^2} + \dfrac{1}{y^2+z^2} + \dfrac{1}{z^2+x^2} = 1$; $\partial y/\partial z$, $\partial x/\partial z$, $\partial x/\partial y$

e) $\dfrac{xy}{(x+y)^2} + \dfrac{yz}{(y+z)^2} + \dfrac{zx}{(z+x)^2} = xyz$; $\partial y/\partial x$, $\partial z/\partial y$, $\partial x/\partial z$

(24) According to the law of cosines, a triangle $\hat{ABC}$ with $a = BC$ and $b = CA$ and $c = AB$ satisfies

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

Use implicit differentiation based on partial derivatives of the implicit definition to evaluate $\partial a/\partial A$, $\partial b/\partial A$, $\partial c/\partial A$.

CAL3.4: Optimization of scalar fields

OPTIMIZATION ON SCALAR FIELDS

Maximum and minimum values

Def: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field. Let $p \in A$ be a point in A. We say that

a) p maximum of $f \iff \forall x \in A : f(x) \leq f(p)$

b) p minimum of $f \iff \forall x \in A : f(x) \geq f(p)$

c) p local maximum of $f \iff$
$\iff \exists a \in (0, +\infty) : \forall x \in B(p, a) \cap A : f(x) \leq f(p)$

d) p local minimum of $f \iff$
$\iff \exists a \in (0, +\infty) : \forall x \in B(p, a) \cap A : f(x) \geq f(p)$

▸ If p is a local minimum or local maximum, we say that p is a local extremum.

Generalized Fermat theorem

The following result generalizes the Fermat theorem from Calculus 1.

Thm: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field. Then

$$\left.\begin{array}{l} p \in \operatorname{int}(A) \\ f \text{ differentiable on } p \\ p \text{ local extremum of } f \end{array}\right\} \Rightarrow \nabla f(p) = \mathbf{0}$$

Proof

With no loss of generality, assume that ~~$p \in \mathrm{int}(A)$~~
p is a local maximum of f. Then,

$p \in \mathrm{int}(A) \Rightarrow \exists a \in (0,+\infty): B(p,a) \subseteq A$

Choose $a \in (0,+\infty)$ such that $B(p,a) \subseteq A$.

Furthermore:

p local maximum of $\Rightarrow$

$\Rightarrow \exists b \in (0,+\infty): \forall x \in B(p,b) \cap A: f(x) \leq f(p)$.

Choose $b \in (0,+\infty)$ such that $\forall x \in B(p,b) \cap A: f(x) \leq f(p)$.

Let $\rho = \min\{a,b\}$ and define:

$\forall k \in [n]: \forall t \in (-\rho,\rho): g_k(t) = f(p + te_k)$

with $e_1, e_2, \ldots, e_n$ the unit vectors of the coordinate system.

It follows that

$g_k(t) = f(p+te_k) \leq f(p) = g(0), \ \forall t \in (-\rho,\rho) \Rightarrow$

$\Rightarrow t=0$ local max of g_k (1)

We also have:

f differentiable on $p \Rightarrow \forall k \in [n]: g_k$ differentiable on 0. (2)

$0 \in (-\rho,\rho)$ (3)

From Eq. (1), (2), (3) via the Fermat theorem:

$\forall k \in [n]: g_k'(0) = 0 \Rightarrow$

$\Rightarrow \forall k \in [n]: \partial f(p)/\partial x_k = f'(p \mid e_k) = g_k'(0) = 0$

$\Rightarrow \nabla f(p) = \mathbf{0}$ $\square$

• It follows from the generalized Fermat theorem that local extrema $p \in A$ can occur only at points where at least one of the following conditions is satisfied:

a) $\nabla f(p) = 0$

b) f not differentiable at p

c) $p \in \partial A$ (p is on the boundary of A)

• If $p \in A$ satisfies one of these conditions, we say that p is a <u>stationary point</u> (or <u>critical point</u>) of f.

• To find the local min/max of a function; we

a) Find all stationary points

b) Use classification theorems to see if the point is a min or max.

↳ Failure of 1st derivative test

We will now show that the 1st derivative test cannot be generalized to functions with 2 or more variables. This is bad news, since the 1st derivative test is the best method for functions of 1 variable.

Consider the following conjecture:

<u>Conjecture:</u> Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ and let $p \in \text{int} A$ with $\nabla f(p) = \mathbf{0}$. Then:

a) p local max $\Leftrightarrow \forall u \in \mathbb{R}^n : \exists a > 0 : g(t) = f(p + tu) \searrow (0, a)$

b) p local min $\Leftrightarrow \forall u \in \mathbb{R}^n : \exists a > 0 : g(t) = f(p + tu) \nearrow (0, a)$

Intuitively we expect this conjecture to be true. However, we will now exhibit a <u>counter example:</u> disproving the conjecture.

COUNTEREXAMPLE

Let $f(x,y) = 3x^4 - 4x^2y + y^2$, $\forall (x,y) \in \mathbb{R}^2$. Then $(0,0)$ is a stationary point and satisfies the conditions of the conjecture but it is NOT a local max or min of f.

Proof

▶ We show that $(0,0)$ is a stationary point.

$$\left.\begin{aligned} \partial f/\partial x &= (\partial/\partial x)(3x^4 - 4x^2y + y^2) = 12x^3 - 8xy \\ \partial f/\partial y &= (\partial/\partial y)(3x^4 - 4x^2y + y^2) = -4x^2 + 2y \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0 \wedge \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0 \Rightarrow \nabla f(0,0) = \mathbf{0}$$

$\Rightarrow (0,0)$ is a stationary point.

▶ We show that the conditions of conjecture are satisfied.

Let $u = (a,b) \in \mathbb{R}^2 - \{\mathbf{0}\}$ with $a \neq 0 \vee b \neq 0$ be given. We define:

$$\begin{aligned} g(t) &= f((0,0) + tu) = f((0,0) + t(a,b)) = f(ta, tb) = \\ &= 3(at)^4 - 4(at)^2(bt) + (bt)^2 = \\ &= 3a^4t^4 - 4a^2bt^3 + b^2t^2, \ \forall t \in [0, +\infty) \Rightarrow \end{aligned}$$

$$\Rightarrow g'(t) = 12a^4t^3 - 12a^2bt^2 + 2b^2t =$$

$$= 2t(6a^4t^2 - 6a^2bt + b^2)$$
$$= 2t\varphi(t|a,b), \forall t \in [0,+\infty)$$
with $\varphi(t|a,b) = 6a^4t^2 - 6a^2bt + b^2, \forall t \in [0,+\infty)$.

We distinguish between the following cases.

<u>Case 1:</u> Assume that $b \neq 0$. Then since $\varphi(0|a,b) = b^2 > 0$, it follows that there is some interval $(0,t_0)$ such that

$$(\forall t \in (0,t_0) : \varphi(t|a,b) > 0) \Rightarrow$$
$$\Rightarrow \forall t \in (0,t_0) : g'(t) = 2t\varphi(t|a,b) > 0$$
$$\Rightarrow g(t) = f((0,0)+tu) \nearrow (0,t_0).$$

<u>Case 2:</u> Assume that $b=0$. Then

$$g'(t) = 2t\varphi(t|a,0) = 2t(6a^4t^2 - 0 + 0) =$$
$$= 12a^4t^3 \geq 0, \forall t \in (0,+\infty) \Rightarrow$$
$$\Rightarrow g(t) = f((0,0)+tu) \nearrow (0,+\infty)$$

We conclude that in both cases the conditions of the conjecture are satisfied, and the conjecture would imply that $(0,0)$ is a local minimum.

▶ We show that $(0,0)$ is not a local minimum. First, we note that $f(0,0)=0$. Then we consider the values of f along the curve $(c): y = 2x^2$. We see that

$$f(x,2x^2) = 3x^4 - 4x^2(2x^2) + (2x^2)^2 =$$
$$= 3x^4 - 8x^4 + 4x^4 = -x^4 < 0, \forall x \in (0,+\infty)$$
$$\Rightarrow \forall r > 0 : \exists x \in B((0,0),r) : f(x) < 0 = f(\mathbf{0})$$
$$\Rightarrow \mathbf{0} \text{ not local min of } f \text{ !}$$

↳ Second derivative test

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ and let $p \in \text{int}(A)$ with $\nabla f(p) = \mathbf{0}$. Define

$$D = f_{xx}(p) f_{yy}(p) - [f_{xy}(p)]^2$$

Then:

a) $D > 0 \wedge f_{xx}(p) > 0 \Rightarrow p$ local minimum

b) $D > 0 \wedge f_{xx}(p) < 0 \Rightarrow p$ local maximum

c) $D < 0 \Rightarrow p$ saddle point

d) $D = 0 \Rightarrow$ inconclusive.

EXAMPLES

a) Classify all stationary points of $f(x,y) = x^3 + y^3 - 3xy$.

<u>Solution</u>

$$f_x(x,y) = (\partial/\partial x)(x^3 + y^3 - 3xy) = 3x^2 - 3y$$
$$f_y(x,y) = (\partial/\partial y)(x^3 + y^3 - 3xy) = 3y^2 - 3x$$

(x,y) stationary point $\Leftrightarrow \nabla f(x,y) = \mathbf{0} \Leftrightarrow \begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases}$

$$\Leftrightarrow \begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 - y = 0 \\ y^2 - x = 0 \end{cases} \Leftrightarrow \begin{cases} y = x^2 \\ x^4 - x = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = x^2 \\ x(x-1)(x^2+x+1) = 0 \end{cases} \Leftrightarrow \begin{cases} y = x^2 \\ x = 0 \end{cases} \vee \begin{cases} y = x^2 \\ x = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases} \vee \begin{cases} x=1 \\ y=1 \end{cases}$$

Thus $(0,0)$ and $(1,1)$ are stationary points.

Evaluate:

$f_{xx}(x,y) = (\partial/\partial x)(3x^2-3y) = 6x$

$f_{xy}(x,y) = (\partial/\partial x)(3y^2-3x) = -3$

$f_{yy}(x,y) = (\partial/\partial y)(3y^2-3x) = 6y$

$D = f_{xx}(x,y) f_{yy}(x,y) - [f_{xy}(x,y)]^2 =$

$= (6x)(6y) - (-3)^2 = 36xy - 9 = 9(4xy-1)$

For $(x,y) = (0,0)$:

$f_{xx}(0,0) = 0$

$D(0,0) = 9(4\cdot 0 - 1) = -9 < 0 \Rightarrow (0,0)$ is a saddle point.

For $(x,y) = (1,1)$:

$$\left. \begin{array}{l} D(1,1) = 9(4\cdot 1\cdot 1 - 1) = 9\cdot 3 = 27 > 0 \\ f_{xx}(1,1) = 6\cdot 1 = 6 > 0 \end{array} \right\} \Rightarrow (1,1) \text{ is a local minimum.}$$

↳ If $D=0$, then the 2nd derivative test is inconclusive. In that case, if the stationary point is a saddle point, we could be able to identify it as such using the following 1st derivative test. Note however that for local min or max, the 1st derivative test does not work as we have shown previously.

1st derivative test for saddle points

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ and let $p \in \text{int}(A)$ with $\nabla f(p) = 0$. Then:

$$\left[\exists a, b \in \mathbb{R}^2 : \exists t_1, t_2 \in (0, +\infty) : \begin{cases} g_1(t) = f(p + at) \nearrow (0, t_1) \\ g_2(t) = f(p + bt) \searrow (0, t_2) \end{cases}\right] \Rightarrow$$

$\Rightarrow$ p saddle point of f.

EXAMPLE

Show that $f(x,y) = x^4 y^7$ has a saddle point at $(0,0)$.

Solution

$$\left.\begin{aligned} f_x(x,y) &= (\partial/\partial x)(x^4 y^7) = 4x^3 y^7 \\ f_y(x,y) &= (\partial/\partial y)(x^4 y^7) = 7x^4 y^6 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \nabla f(0,0) = (f_x(0,0), f_y(0,0)) =$$

$$= (4 \cdot 0 \cdot 0, 7 \cdot 0 0) = (0,0) = \mathbf{0} \Rightarrow$$

$\Rightarrow (0,0)$ is a stationary point.

- Let $a, b \in \mathbb{R} - \{0\}$ be given. Define:

$$g(t) = f((0,0) + t(a,b)) = f(at, bt) = (at)^4 (bt)^7 =$$

$$= a^4 b^7 t^{11}, \forall t \in (0, +\infty) \Rightarrow$$

$$\Rightarrow g'(t) = 11 a^4 b^7 t^{10}, \forall t \in (0, +\infty).$$

For $a = b = 1$: $g'(t) = 11 t^{10}, \forall t \in (0, +\infty) \Rightarrow$

$$\Rightarrow g'(t) > 0, \forall t \in (0, +\infty) \Rightarrow g \nearrow (0, +\infty).$$

For $a = 1$ and $b = -1$: $g'(t) = 11 \cdot 1^4 \cdot (-1)^7 t^{10} = -11t^{10}, \forall t \in (0, +\infty)$
$\Rightarrow g'(t) < 0, \forall t \in (0, +\infty) \Rightarrow g \downarrow (0, +\infty)$
It follows that $(0,0)$ is a saddle point.

$\mapsto$ To see why the second derivative test fails, we note that:

$f_{xx}(x,y) = (\partial/\partial x)(4x^3y^7) = 12x^2y^7$
$f_{xy}(x,y) = (\partial/\partial x)(7x^4y^6) = 28x^3y^6$
$f_{yy}(x,y) = (\partial/\partial y)(7x^4y^6) = 42x^4y^5$
and therefore:
$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - [f_{xy}(x,y)]^2 =$
$= (12x^2y^7)(42x^4y^5) - (28x^3y^6)^2 =$
$= 504x^6y^{12} - 784x^6y^{12} = -280x^6y^{12} \Rightarrow$
$\Rightarrow D(0,0) = -280 \cdot 0^6 \cdot 0^{12} = 0 \Rightarrow$
$\Rightarrow$ 2nd derivative test inconclusive with respect to $(0,0)$.

EXERCISES

(1) Find all the local min and max for the following functions

a) $f(x,y) = x^2 + 3y^2$

b) $f(x,y) = 2x^2 - 5y^2$

c) $f(x,y) = x^2 + 2x + y^2 - 4y$

d) $f(x,y) = x^2 - y^2 + 8(x+y)$

e) $f(x,y) = x^2 + 2y^2 + xy - 2x - 4y$

f) $f(x,y) = 3x^2 + 2y^2 - 2xy + x - 3y$

g) $f(x,y) = x^3 - 3xy^2 + y^3$

h) $f(x,y) = x^3 + y^3 - 3a^3xy$

i) $f(x,y) = \sin x \sin y \sin(x+y)$

(2) Similarly for the functions

a) $f(x,y) = (x-1)^3(y-2)^4$

b) $f(x,y) = x^2y^2(x-1)^5$

(3) Let $f(x,y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f$ with $a > 0$ and $b^2 < ac$. Show that

a) f has a local minimum at some point (x_1, y_1)

b) $f(x_1, y_1) = dx_1 + ey_1 + f$.

▼ Constrained Optimization

• **The problem:** Let $f: A \to \mathbb{R}$ and $g_k: A \to \mathbb{R}$ for $k \in [m]$ with $A \subseteq \mathbb{R}^n$. The problem is to identify the local min/max of

$$z = f(x),\ \forall x \in A$$

under the constraints

$$\forall k \in [m]: g_k(x) = 0$$

▶ Lagrange multiplier theorem

Def: Let $u_1, u_2, \ldots, u_m \in \mathbb{R}^n$ be m vectors. We say that

$u_1, u_2, \ldots, u_m$ are linearly independent $\iff \forall \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$: $(\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_m u_m = \mathbf{0} \Rightarrow$
$\Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_m = 0)$

Thm: Consider the constrained optimization problem

$$\begin{cases} z = f(x),\ \forall x \in A \\ \forall k \in [m]: g_k(x) = 0 \end{cases}$$

with $f: A \to \mathbb{R}$ and $g_k: A \to \mathbb{R}$.

Assume that:

a) $f, g_1, g_2, \ldots, g_m$ are differentiable at $p \in A$.

b) $\forall k \in [m]: g_k(p) = 0$

c) $\nabla g_1(p), \nabla g_2(p), \ldots, \nabla g_m(p)$ are linearly independent

d) p is a local min or max of the constrained problem.

Then:

$$\exists \lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R} : \nabla f(p) = \sum_{k=1}^{m} \lambda_k \nabla g_k(p)$$

↦ The Lagrange multipliers method gives the stationary points of the constrained optimization problem but does not establish whether these points are local min or max. Doing so is very difficult, but there are occasional tricks we can try.

▶ <u>When constraints yield a finite curve.</u>

- Recall from single variable calculus that for any function $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$; and $a, b \in A$:

  f continuous in $[a, b] \Rightarrow f$ bounded in $[a, b]$.
- Let S be the set defined by the problem constraints:

  $S = \{ x \in \mathbb{R}^n \mid \forall k \in [m] : g_k(x) = 0 \}$

<u>Def</u>: We say that S is a finite curve if we can define a parameterization $\gamma : [0,1] \to S$ such that $\gamma([0,1]) = S$ and γ continuous in $[0,1]$.

Now we can parameterize the function f on S by defining $g(t) = f(\gamma(t))$. Then, it follows that:

$$\left.\begin{array}{l} f \text{ continuous in } A \Rightarrow f \text{ continuous in } S \\ \gamma \text{ continuous in } [0,1] \end{array}\right\} \Rightarrow$$

$\Rightarrow g$ continuous in $[0,1] \Rightarrow$

$\Rightarrow \exists\, t_1, t_2 \in [0,1] : \forall t \in [0,1] : g(t_1) \leq g(t) \leq g(t_2)$.

The points $\gamma(t_1), \gamma(t_2)$ will show up as stationary points of the Lagrange multiplier method solutions and are correspondingly the minimum and maximum of f under the given constraints.

- <u>To conclude:</u> If S is a finite curve, then the constrained optimization has a maximum and a minimum among the existing stationary points. To identify the minimum and maximum, we simply evaluate the function for all stationary points, and choose the stationary points that give the minimum and maximum values.

EXAMPLE

Find the minimum and maximum value of
$f(x,y,z) = x+2y+3z$
under the constraints

$$\begin{cases} x^2+y^2=2 \\ y+z=1 \end{cases}$$

Solution

Define $g_1(x,y,z) = 2-x^2-y^2$
$g_2(x,y,z) = y+z-1$

Note that:
$\nabla f(x,y,z) = (1,2,3)$
$\nabla g_1(x,y,z) = (-2x,-2y,0)$
$\nabla g_2(x,y,z) = (0,1,1)$
▶ Linear independence

$$\nabla g_1(x,y,z) \times \nabla g_2(x,y,z) = (-2x,-2y,0) \times (0,1,1) =$$

$$= \begin{vmatrix} i & j & k \\ -2x & -2y & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} i & j & -j+k \\ -2x & -2y & 2y \\ 0 & 1 & 0 \end{vmatrix} =$$

(-1)

$$= -1 \cdot \begin{vmatrix} i & -j+k \\ -2x & 2y \end{vmatrix} = -\left(2yi-(-2x)(-j+k)\right) =$$

$= -2x(-j+k) - 2yi$

therefore

$\nabla g_1(x,y,z), \nabla g_2(x,y,z)$ linearly independent $\Leftrightarrow$

$\Leftrightarrow \nabla g_1(x,y,z) \times \nabla g_2(x,y,z) \neq \mathbf{0} \Leftrightarrow -2x(-j+k) - 2yi \neq 0 \Leftrightarrow$

$\Leftrightarrow \underline{\underline{x \neq 0 \lor y \neq 0}}$

↳ Recall that $\nabla g_1, \nabla g_2$ need to be linearly independent to apply the Lagrange multiplier theorem, when evaluated at the stationary points. For two vectors we use the criterio:

$$\boxed{\begin{array}{l}\text{For } u,v \in \mathbb{R}^3: \\ u,v \text{ linearly independent} \Leftrightarrow u \times v \neq \mathbf{0}\end{array}}$$

▶ Stationary points

(x,y,z) is a stationary point $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} \nabla f(x,y,z) = \lambda_1 \nabla g_1(x,y,z) + \lambda_2 \nabla g_2(x,y,z) \\ x^2+y^2=2 \\ y+z=1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (1,2,3) = \lambda_1(-2x,-2y,0) + \lambda_2(0,1,1) \\ x^2+y^2=2 \\ y+z=1 \end{cases} \Leftrightarrow$$

$$\begin{cases} 1=-2\lambda_1 x \\ 2=-2\lambda_1 y+\lambda_2 \\ 3=\lambda_2 \\ x^2+y^2=2 \\ y+z=1 \end{cases} \Leftrightarrow \begin{cases} 1=-2\lambda_1 x \\ 2=-2\lambda_1 y+3 \\ x^2+y^2=2 \\ y+z=1 \end{cases} \Leftrightarrow \begin{cases} 2\lambda_1 x=-1 & (1) \\ 2\lambda_1 y=1 & (2) \\ x^2+y^2=2 & (3) \\ y+z=1 & (4) \end{cases}$$

If $\lambda_1=0$, then equations (1) and (2) are inconsistent.
So, assume $\lambda_1\neq 0$. Then:

$(1)\Leftrightarrow x=\dfrac{-1}{2\lambda_1}$ and $(2)\Leftrightarrow y=\dfrac{1}{2\lambda_1}$ and $(4)\Leftrightarrow z=1-y$

It follows that

$$(3)\Leftrightarrow x^2+y^2=2\Leftrightarrow\left(\frac{-1}{2\lambda_1}\right)^2+\left(\frac{1}{2\lambda_1}\right)^2=2\Leftrightarrow$$

$$\Leftrightarrow 2\,\frac{1}{4\lambda_1^2}=2\Leftrightarrow 4\lambda_1^2=1\Leftrightarrow 2\lambda_1=1 \vee 2\lambda_1=-1\Leftrightarrow$$

$$\Leftrightarrow \lambda_1=\frac{1}{2} \vee \lambda_1=\frac{-1}{2}.$$

Thus the system of equations (1), (2), (3), (4) is equivalent to:

$$\begin{cases} x=-1/(2\lambda_1) \\ y=1/(2\lambda_1) \\ \lambda_1=1/2 \vee \lambda_1=-1/2 \\ z=1-y \end{cases} \Leftrightarrow$$

$$\begin{cases} x = (-1/2)\cdot 2 = -1 \\ y = (1/2)\cdot 2 = 1 \\ z = 1-1 = 0 \end{cases} \vee \begin{cases} x = (-1/2)(-2) = 1 \\ y = (+1/2)(-2) = -1 \\ z = 1-(-1) = 2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow (x,y,z) = (-1,1,0) \vee (x,y,z) = (1,-1,2)$$

Note that both stationary points satisfy the linear independence condition.

- The set
  $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=2 \wedge y+z=1\}$
  is the intersection of a cylindrical surface and a plane, therefore it is a finite curve. Consequently:
  $f(-1,1,0) = (-1)+2\cdot 1+3\cdot 0 = -1+2 = 1$
  $f(1,-1,2) = (1)+2(-1)+3\cdot 2 = 1-2+6 = 5$
  thus $(-1,1,0)$ is the minimum and
  $(1,-1,2)$ is the maximum.

EXAMPLES

a) Find the stationary points of
$f(x,y,z) = xyz$ with $x,y,z \in (0,+\infty)$
under the constraint
$xy + yz + zx = 1.$

Solution

Define $g(x,y,z) = xy + yz + zx - 1$. Note that

$f_x(x,y,z) = (\partial/\partial x)(xyz) = yz \quad (1)$

$f_y(x,y,z) = (\partial/\partial y)(xyz) = zx \quad (2)$

$f_z(x,y,z) = (\partial/\partial z)(xyz) = xy \quad (3)$

From (1),(2),(3): $\nabla f(x,y,z) = (yz, zx, xy)$.

$g_x(x,y,z) = (\partial/\partial x)(xy+yz+zx-1) = y+z \quad (4)$

$g_y(x,y,z) = (\partial/\partial y)(xy+yz+zx-1) = z+x \quad (5)$

$g_z(x,y,z) = (\partial/\partial z)(xy+yz+zx-1) = x+y. \quad (6)$

From (4),(5),(6): $\nabla g(x,y,z) = (y+z, z+x, x+y)$.

It follows that:

(x,y,z) is a stationary point $\Leftrightarrow \begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 0 \end{cases}$

$\Leftrightarrow \begin{cases} (yz, zx, xy) = \lambda (y+z, z+x, x+y) \\ xy + yz + zx = 1 \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} xy = \lambda(x+y) \\ yz = \lambda(y+z) \\ zx = \lambda(z+x) \\ xy+yz+zx = 1 \end{cases} \Leftrightarrow \begin{cases} xyz = \lambda z(x+y) & (7) \\ xyz = \lambda x(y+z) & (8) \\ xyz = \lambda y(z+x) & (9) \\ xy+yz+zx = 1 & (10). \end{cases}$$

From (7) and (8):

$$\lambda z(x+y) = \lambda x(y+z) \Leftrightarrow \lambda[x(y+z) - z(x+y)] = 0 \Leftrightarrow$$
$$\Leftrightarrow \lambda(xy+xz-zx-zy) = 0 \Leftrightarrow \lambda(xy-zy) = 0 \Leftrightarrow$$
$$\Leftrightarrow \lambda y(x-z) = 0 \Leftrightarrow \lambda = 0 \vee y = 0 \vee x-z = 0 \Leftrightarrow$$
$$\Leftrightarrow \lambda = 0 \vee y = 0 \vee x = z. \quad (11)$$

We note that for $\lambda = 0$:

$$\begin{cases} xy = 0 \\ yz = 0 \\ zx = 0 \\ xy+yz+zx = 1 \end{cases} \Leftrightarrow \begin{cases} xy = 0 \\ yz = 0 \\ zx = 0 \\ 0+0+0 = 1 \end{cases} \longleftarrow \text{inconsistent.}$$

therefore $\lambda \neq 0$.

We have also assumed $y > 0 \Rightarrow y \neq 0$.

It follows that $(11) \Leftrightarrow x = z$.

Since the system remains invariant under the transformation $x \to y \to z \to x$, we can similarly show that $y = x$ and $z = y$. Thus, the system defined by (7), (8), (9), (10) is equivalent to

$$\begin{cases} x=y \\ y=z \\ z=x \\ xy+yz+zx=1 \end{cases} \Leftrightarrow \begin{cases} x=y=z \\ x^2+x^2+x^2=1 \end{cases} \Leftrightarrow \begin{cases} x=y=z \\ 3x^2=1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=y=z \\ x^2=1/3 \end{cases} \Leftrightarrow x=y=z=\frac{1}{\sqrt{3}} \quad \text{(because } x>0\text{)}.$$

Therefore : the stationary point is:

$$(x,y,z)=\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

EXERCISES

(4) Use the method of Lagrange multipliers to find the minimum or maximum point of the following functions subject to the given constants.

a) $f(x,y) = 3x + 4y$ with $x^2 + y^2 = 9$

b) $f(x,y) = 2xy$ with $2x^2 + y^2 = 16$

c) $f(x,y) = x^2 + y^2$ with $3x + 2y = 10$

d) $f(x,y) = x^2y + x + y$ with $xy = 9$

e) $f(x,y) = x^2 + y^2$ with $x^4 + y^4 = 1$

f) $f(x,y) = x^2y^2$ with $x^2 + y^2 = 4$

(5) Similarly for the following functions (two constraints)

a) $f(x,y,z) = x + y + z$

with $\begin{cases} x^2 + y^2 + z^2 = 9 \\ x^2 + y^2 + 16z^2 = 36 \end{cases}$

b) $f(x,y,z) = x + 2y + z$

with $\begin{cases} 2x + z = 4 \\ x^2 + y^2 = 1 \end{cases}$

c) $f(x,y,z) = x^2 + y^2 + z^2$

with $\begin{cases} x + 2y + z = 3 \\ x - y = 4 \end{cases}$

⑥ Show that the point on the line $(\ell): ax+by=c$ closest to the origin has coordinates (x_0, y_0) with

$$x_0 = \frac{ac}{a^2+b^2} \wedge y_0 = \frac{bc}{a^2+b^2}$$

⑦ Find the maximum point of
$f(x,y) = x^a y^b$, $\forall (x,y) \in [0,+\infty) \times [0,+\infty)$
on the unit circle
$(c): x^2+y^2=1$
with $a, b \in (0,+\infty)$

⑧ Find the maximum point of
$f(x,y,z) = x^a y^b z^c$, $\forall (x,y,z) \in [0,+\infty) \times [0,+\infty) \times [0,+\infty)$
on the unit ~~circle~~ sphere
$(c): x^2+y^2+z^2=1$
with $a, b, c \in (0,+\infty)$.

⑨ Consider scalar fields $f: \mathbb{R}^2 \to \mathbb{R}$ and $g: \mathbb{R}^2 \to \mathbb{R}$ and assume that $f(x,y)$ has a maximum under the constraint $g(x,y)=a$ at the point $(x(a), y(a))$ which is a function of the <u>budget variable</u> $a \in \mathbb{R}$.
Show that:

a) $\forall a \in \mathbb{R}: \nabla g(x(a), y(a)) \cdot (x'(a), y'(a)) = 1$

b) The Lagrange multiplier $\lambda(a)$ corresponding to the point $(x(a), y(a))$ is given by:

$$\lambda(a) = \frac{d}{da} f(x(a), y(a))$$

↳ The problem gives a physical interpretation of the Lagrange multiplier $\lambda(a)$. It shows how fast the maximum value $f(x(a), y(a))$ of the scalar field f increases as we increase the budget value a. A similar result can be obtained for the most general constrained optimization problem.

⑩ Boltzmann distribution

Consider the entropy $S(x_1, x_2, \ldots, x_n)$ defined as

$$S(x_1, x_2, \ldots, x_n) = \sum_{a=1}^{n} x_a \ln x_a =$$

$$= x_1 \ln x_1 + x_2 \ln x_2 + \cdots + x_n \ln x_n$$

with $x_1, x_2, \ldots, x_n \in (0, +\infty)$ subject to the constraints

$$\begin{cases} \displaystyle\sum_{a=1}^{n} x_a = N \\ \displaystyle\sum_{a=1}^{n} E_a x_a = E \end{cases}$$

with $E_1, E_2, \ldots, E_n, E, A \in (0, +\infty)$. Show that the entropy is maximized when

$$\forall a \in [n] : x_a = \frac{\exp(\mu E_a)}{A}$$

with $A = (1/N) \sum_{a=1}^{n} \exp(\mu E_a)$

and find the value of the constant μ.

▼ Optimization problems on a bounded set

Def: Let $A \subseteq \mathbb{R}^n$. We say that

$$A \text{ bounded} \iff \exists p \in \mathbb{R}^n : \exists a \in (0, +\infty) : A \subseteq B(p, a)$$

Thm: Let $f : A \longrightarrow \mathbb{R}$ with $A \subseteq \mathbb{R}^n$. Assume that:

a) f continuous at A

b) A is closed (i.e. $\partial A \subseteq A$)

c) A is bounded.

Then:

$$\exists p_1, p_2 \in A : \forall x \in A : f(p_1) \leq f(x) \leq f(p_2).$$

↳ This theorem is called the extremum value theorem and it generalizes the extremum value theorem of single-variable calculus.

▶ The problem: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and we want to optimize it under the constraint $g(x, y) \leq 0$. We assume that the set

$$S = \{(x, y) \in \mathbb{R}^2 \mid g(x, y) \leq 0\}$$

is closed and bounded.

EXAMPLE

Find the minimum and maximum of the scalar field
$f(x,y) = x^2+y^2+2$
under the constraint $x^2+y^2/4 \leq 1$.

Solution

Define $S = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2/4 \leq 1\}$.

▶ Interior stationary points

$\partial f/\partial x = (\partial/\partial x)(x^2+y^2+2) = 2x$

$\partial f/\partial y = (\partial/\partial y)(x^2+y^2+2) = 2y$

(x,y) stationary point on $\mathrm{int}(S) \Leftrightarrow \nabla f(x,y) = 0 \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \partial f/\partial x = 0 \\ \partial f/\partial y = 0 \end{cases} \Leftrightarrow (2x,2y) = (0,0) \Leftrightarrow (x,y) = (0,0) \in \mathrm{int}(S)$$

accepted.

▶ Boundary stationary points

Note that $\partial S = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2/4 = 1\}$.

Define $g(x,y) = x^2+y^2/4-1$. Then

$$\begin{cases} \partial g/\partial x = 2x \\ \partial g/\partial y = 2y/4 = y/2 \end{cases} \Rightarrow \nabla g(x,y) = (2x, y/2)$$

Let $(x,y) \in \partial S$ be given. Then:

$$(x,y) \in \partial S \Rightarrow x^2+y^2/4 = 1 \Rightarrow (x,y) \neq (0,0) \Rightarrow$$
$$\Rightarrow \nabla g(x,y) = (2x, y/2) \neq (0,0) \Rightarrow$$
$$\Rightarrow \nabla g(x,y) \text{ linearly independent.}$$

and thus: $\forall (x,y) \in \partial S : \nabla g(x,y)$ linearly independent.

It follows that

$$(x,y) \text{ stationary point} \Leftrightarrow \begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (2x, 2y) = \lambda (2x, y/2) \\ x^2 + y^2/4 = 1 \end{cases} \Leftrightarrow \begin{cases} 2x = 2\lambda x \\ 2y = \lambda y/2 \\ 4x^2 + y^2 = 4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda x - x = 0 \\ \lambda y - 4y = 0 \\ 4x^2 + y^2 = 4 \end{cases} \Leftrightarrow \begin{cases} (\lambda - 1)x = 0 \\ (\lambda - 4)y = 0 \\ 4x^2 + y^2 = 4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda = 1 \\ (1-4)y = 0 \\ 4x^2 + y^2 = 4 \end{cases} \vee \begin{cases} x = 0 \\ (\lambda - 4)y = 0 \\ 0 + y^2 = 4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda = 1 \\ y = 0 \\ 4x^2 = 4 \end{cases} \vee \begin{cases} x = 0 \\ \lambda = 4 \\ y^2 = 4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda = 1 \\ y = 0 \\ x = 1 \end{cases} \vee \begin{cases} \lambda = 1 \\ y = 0 \\ x = -1 \end{cases} \vee \begin{cases} x = 0 \\ \lambda = 4 \\ y = 2 \end{cases} \vee \begin{cases} x = 0 \\ \lambda = 4 \\ y = -2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow (x,y) \in \{(1,0), (-1,0), (0,2), (0,-2)\}.$$

▸ <u>Point classification</u>

$f(0,0) = 2 + 0^2 + 0^2 = 2$

$f(1,0) = 2 + 1^2 + 0^2 = 3$

$f(-1,0) = 2 + (-1)^2 + 0^2 = 3$

$f(0,2) = 2 + 0^2 + 2^2 = 6$

$f(0,-2) = 2 + 0^2 + (-2)^2 = 6$

Since S is closed and bounded it follows that f has minimum at $(0,0)$ and maximum at $(0,2)$ and $(0,-2)$.

$\mapsto$ The boundary stationary points can also be found by parameterizing the boundary ∂S as follows.

<u>2nd method</u>: We parameterize ∂S with

$$\begin{cases} x(t) = \cos t \\ y(t) = 2\sin t \end{cases}, \quad \forall t \in [0, 2\pi)$$

It follows that:

$$g(t) \equiv f(x(t), y(t)) = f(\cos t, 2\sin t) =$$
$$= 2 + \cos^2 t + 4\sin^2 t = 2 + (\cos^2 t + \sin^2 t) + 3\sin^2 t$$
$$= 2 + 1 + 3\sin^2 t = 3 + 3\sin^2 t \Rightarrow$$
$$\Rightarrow g'(t) = 3(\sin^2 t)' = 3 \cdot 2\sin t \, (\sin t)' = 6\sin t \cos t =$$
$$= 3(2\sin t \cos t) = 3\sin(2t)$$

Solve:

$$g'(t) = 0 \Leftrightarrow 3\sin(2t) = 0 \Leftrightarrow \sin(2t) = 0 \Leftrightarrow 2t = k\pi \Leftrightarrow$$
$$\Leftrightarrow t = \frac{k\pi}{2}$$

Since $t \in [0, 2\pi) \Leftrightarrow 0 \leq t < 2\pi \Leftrightarrow 0 \leq \frac{k\pi}{2} < 2\pi \Leftrightarrow$
$$\Leftrightarrow 0 \leq k/2 < 2 \Leftrightarrow 0 \leq k < 4 \Leftrightarrow$$
$$\Leftrightarrow k \in \{0, 1, 2, 3\}$$

Thus we find 4 stationary points:

$t = 0$: $(x, y) = (\cos 0, 2\sin 0) = (1, 0)$

$t = \pi/2$: $(x, y) = (\cos(\pi/2), 2\sin(\pi/2)) = (0, 2)$

$t = \pi$: $(x, y) = (\cos\pi, 2\sin\pi) = (-1, 0)$

$t = 3\pi/2$: $(x, y) = (\cos(3\pi/2), 2\sin(3\pi/2)) = (0, -2)$

EXERCISES

(11) Find the minimum and maximum of the following functions constrained on the set S

a) $f(x,y) = 1+x+y$
on $S = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$

b) $f(x,y) = x+y-2xy$
on $S = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4\}$

c) $f(x,y) = x^2+y^2+3x-xy$
on $S = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 9\}$

d) $f(x,y) = \dfrac{x}{y^2+1}$ on $S = \{(x,y) \in \mathbb{R}^2 \mid x^2/4+y^2/9 \leq 1\}$

e) $f(x,y) = (1+x+y)^2$ on $S = \{(x,y) \in \mathbb{R}^2 \mid x^2/4+y^2/16 \leq 1\}$

CAL3.5: Multiple integrals

MULTIPLE INTEGRALS

Definition of Multiple integrals

We are mainly interested in defining the double and triple integral. However, in order to eliminate repetitiveness, we will just give one definition for the general n-dimensional integral. The definition is made in two steps:

a) We define the multiple integral over a box region
b) We then extend the definition over a general bounded region.

Finally we introduce the Fubini theorems that reduce the multiple integral to simpler integrals.

① → Multiple integral over a box region

Let us consider a scalar field $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ and also consider a box region

$$S = \prod_{k=1}^{n} [a_k, b_k] =$$

$$= \{(x_1, x_2, \dots, x_n) \mid \forall k \in [n] : x_k \in [a_k, b_k]\}$$

with $\forall k \in [n] : (a_k, b_k \in \mathbb{R} \land a_k < b_k)$.

We define the integral

$$I = \int_{\$} dx\, f(x)$$

using Riemann sums as follows:

1) We divide each interval $[a_k, b_k]$ with $k \in [n]$ involved in forming the box $\$$ onto N subintervals $[x_{km}, x_{k,m+1}]$ with
$\forall m \in \{0\} \cup [N] : x_{km} = a_k + (k/N)(b_k - a_k)$
This results in dividing the box $\$$ into N^n smaller boxes that we can index via a set $\mathrm{Map}([n],[N])$ that contains all possible mappings $\sigma : [n] \to [N]$. Then, for each choice $\sigma \in \mathrm{Map}([n],[N])$ the corresponding small box is given by

$$\forall \sigma \in \mathrm{Map}([n],[N]) : \$(\sigma) = \prod_{k=1}^{n} [x_{k,\sigma(k)-1}, x_{k,\sigma(k)}]$$

2) For each small box, we define the minimum and maximum value of the scalar field f as

$$\forall \sigma \in \mathrm{Map}([n],[N]) : m_\sigma(f \mid \$, N) = \min_{x \in \$(\sigma)} f(x)$$

$$\forall \sigma \in \mathrm{Map}([n],[N]) : M_\sigma(f \mid \$, N) = \max_{x \in \$(\sigma)} f(x)$$

3) We form the Riemann sums $L_N(f|\$)$ and $U_N(f|\$)$ given by:

$$L_N(f|\$) = \sum_{\sigma \in Map([n],[N])} m_\sigma(f|\$,N) \left[\prod_{k=1}^{n} (x_{k,\sigma(k)} - x_{k,\sigma(k)-1}) \right]$$

$$U_N(f|\$) = \sum_{\sigma \in Map([n],[N])} M_\sigma(f|\$,N) \left[\prod_{k=1}^{n} (x_{k,\sigma(k)} - x_{k,\sigma(k)-1}) \right]$$

Based on the above, we define integrability and the n-dimensional integral as follows:

Def: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ and let $\$ \subseteq A$ be a box domain. Then, we say that

f integrable on $\$ \Leftrightarrow \exists l \in \mathbb{R}: \lim_{m \in \mathbb{N}^*} L_m(f|\$) = \lim_{m \in \mathbb{N}^*} U_m(f|\$) = l$

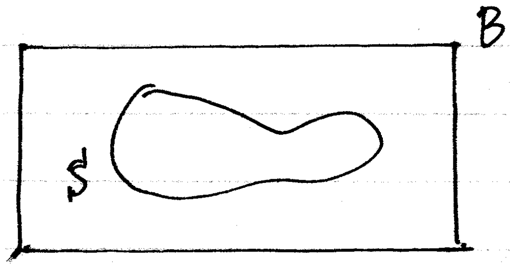
If f is indeed integrable on $\$$, then we write

$$l = \int_{\$} dx\, f(x)$$

and say that l is the integral of f over $\$$.

② → Multiple integral over a closed bounded set

- Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $S \subseteq A$ be a closed and bounded set. It follows that we can define a box domain $B \subseteq \mathbb{R}^n$ such that $S \subseteq B$.



Given such a box domain, we define a new scalar field $F: B \to \mathbb{R}$ such that

$$F(x) = \begin{cases} f(x), & \text{if } x \in S \\ 0, & \text{if } x \in B - S \end{cases}$$

and we use F to define the integral of f over S as:

$$\int_S dx\, f(x) = \int_B dx\, F(x)$$

- We also extend the definition further to the cases $S = \emptyset$ and $S = \mathbb{R}^n$ such that

$$\int_\emptyset dx\, f(x) = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} dx\, f(x) = \lim_{a \to +\infty} \int_{B(\mathbf{0}, a)} dx\, f(x)$$

<u>notation</u>: In the previous notation $x \in \mathbb{R}^n$ is a vector. For the cases $n=2$ and $n=3$ we use the following notation and terminology:

a) For $n=2$, we write:

$$I = \iint_S dx\,dy\, f(x,y) = \iint_S f(x,y)\,dx\,dy$$

with $x,y \in \mathbb{R}$ and $S \subseteq \mathbb{R}^2$ and designate I as a <u>double integral</u>.

b) For $n=3$, we write

$$I = \iiint_S dx\,dy\,dz\, f(x,y,z) = \iiint_S f(x,y,z)\,dx\,dy\,dz$$

with $x,y,z \in \mathbb{R}$ and $S \subseteq \mathbb{R}^3$ and designate I as a <u>triple integral</u>.

→ <u>Properties of multiple integrals</u>

Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be scalar fields. Let $S \subseteq A$ be a closed and bounded domain.

1) <u>Linearity</u>

$$\forall \lambda_1, \lambda_2 \in \mathbb{R}: \int_S dx\,[\lambda_1 f(x) + \lambda_2 g(x)] =$$

$$= \lambda_1 \int_S f(x)\,dx + \lambda_2 \int_S g(x)\,dx$$

2) <u>Comparison theorem</u>

$$\left[\forall x \in \mathcal{S} : f(x) \leq g(x)\right] \Rightarrow \int_{\mathcal{S}} dx\, f(x) \leqslant \int_{\mathcal{S}} dx\, g(x)$$

3) <u>Inclusion - Exclusion principle</u>

Let $\mathcal{S}_1 \subseteq A$ and $\mathcal{S}_2 \subseteq A$ be closed and bounded domains. Then:

$$\int_{\mathcal{S}_1 \cup \mathcal{S}_2} f(x)\, dx = \int_{\mathcal{S}_1} f(x)\, dx + \int_{\mathcal{S}_2} f(x)\, dx - \int_{\mathcal{S}_1 \cap \mathcal{S}_2} f(x)\, dx$$

▶ Special case:

$$\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset \Rightarrow \int_{\mathcal{S}_1 \cup \mathcal{S}_2} f(x)\, dx = \int_{\mathcal{S}_1} f(x)\, dx + \int_{\mathcal{S}_2} f(x)\, dx$$

▼ Evaluate double integrals in boxed domains

Double integrals in box domains can be evaluated using Fubini's theorem. Below, we give a simplified corollary of Fubini's theorem that is specific to double integrals

Thm: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ and let $S = [a_1, a_2] \times [b_1, b_2] \subseteq A$ be a boxed domain. Then:
f continuous on $S \Rightarrow$

$$\Rightarrow \iint_S f(x,y)\, dx\, dy = \int_{a_1}^{a_2} dx \int_{b_1}^{b_2} dy\, f(x,y) =$$
$$= \int_{b_1}^{b_2} dy \int_{a_1}^{a_2} dx\, f(x,y)$$

notation: The notation

$$\int_{a_1}^{a_2} dx \int_{b_1}^{b_2} dy\, f(x,y)$$

is an iterated integral, and it is equivalent to

$$\int_{a_1}^{a_2} dx \left[\int_{b_1}^{b_2} dy\, f(x,y) \right]$$

where we first do the y integral and then do the x integral. There is an equivalent notation that reads

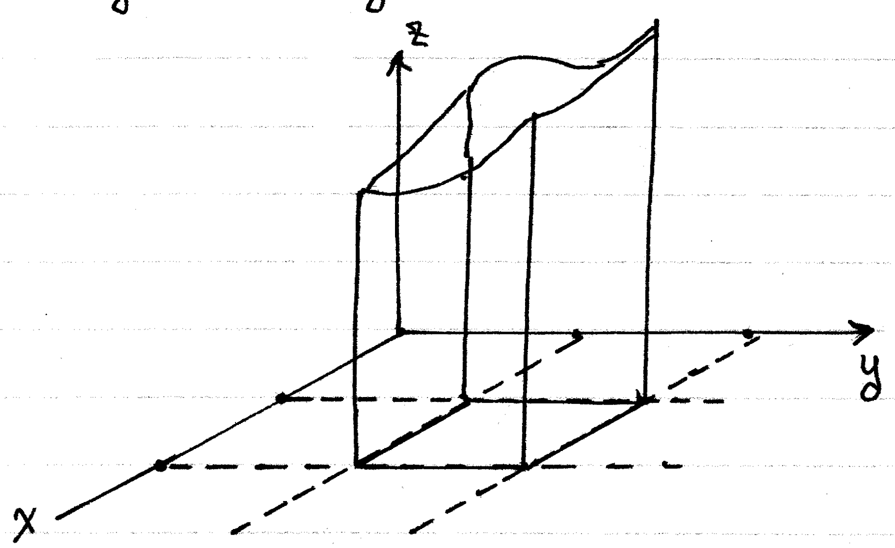
$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(x,y)\,dy\,dx$$

in common use, but it should be avoided because it is confusing, stupid, idiotic, pathetic, annoying, and absolutely horrible.

● Geometric interpretations of the double integral.

① → Volume of solid under a surface

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ be a scalar field such that $\forall (x,y) \in A : f(x,y) \geq 0$



Now consider a solid V under the surface $(S): z = f(x,y)$, shown above, defined by

$$V = \{(x,y,z) \mid (x,y) \in A \land 0 \leq z \leq f(x,y)\}$$

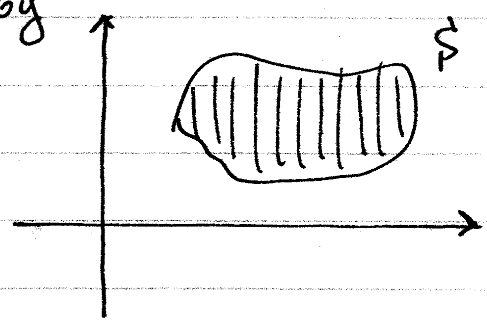
Then, the volume of V is given by

$$\boxed{\mathrm{vol}(V) = \iint_A f(x,y)\,dx\,dy}$$

② → <u>Area of a surface $\mathcal{S}$</u>

Let $\mathcal{S} \subseteq \mathbb{R}^2$ be a closed and bounded set. Then, the area of $\mathcal{S}$ is given by

$$\boxed{\mathrm{area}(\mathcal{S}) = \iint_{\mathcal{S}} dx\,dy}$$



③ → <u>Center of mass</u>

If the closed and bounded set $\mathcal{S} \subseteq \mathbb{R}^2$ represents a two-dimensional solid object with density $\rho(x,y)$ then it has <u>mass</u>

$$\boxed{m = \iint_{\mathcal{S}} dx\,dy\,\rho(x,y)}$$

and <u>center of mass</u> $M(\bar{x},\bar{y})$ with $\bar{x},\bar{y}$ given by

$$\boxed{\begin{aligned} \bar{x} &= \frac{1}{m} \iint_S x\rho(x,y)\,dx\,dy \\ \bar{y} &= \frac{1}{m} \iint_S y\rho(x,y)\,dx\,dy \end{aligned}}$$

For the special case $\forall (x,y) \in S : \rho(x,y) = 1$ the calculation above gives the geometric center of mass, also known as <u>Barycenter.</u>

EXAMPLES

a) Evaluate the integral:

$$I = \iint_A \frac{1+x^2}{1+y^2}\, dx\, dy \quad \text{for } A = [0,1]\times[0,1]$$

Solution

$$I = \iint_A \frac{1+x^2}{1+y^2}\, dx\, dy = \int_0^1 dx \int_0^1 dy\, \frac{1+x^2}{1+y^2} =$$

$$= \int_0^1 dx\, (1+x^2) \int_0^1 \frac{dy}{1+y^2} = \int_0^1 dx\, (1+x^2) \Big[\operatorname{Arctan} y \Big]_0^1 =$$

$$= \int_0^1 dx\, (1+x^2)(\operatorname{Arctan} 1 - \operatorname{Arctan} 0) =$$

$$= \left(\frac{\pi}{4} - 0 \right) \int_0^1 dx\, (1+x^2) = \frac{\pi}{4} \left[x + \frac{x^3}{3} \right]_0^1 =$$

$$= \frac{\pi}{4} \left[(1-0) + \frac{1^3 - 0^3}{3} \right] = \frac{\pi}{4}\, \frac{4}{3} =$$

$$= \frac{\pi}{3}$$

6) Find the volume of the solid under the surface (s): $z = x\sqrt{x^2+y}$ and above the rectangle $A = [0,1]\times[0,1]$ on the xy-plane.

Solution

$$V = \iint_A dxdy\ x\sqrt{x^2+y} = \int_0^1 dx \int_0^1 dy\ x\sqrt{x^2+y} =$$

$$= \int_0^1 dx\ x\left[\int_0^1 dy \sqrt{x^2+y}\right]$$

Let $t = g(y) = \sqrt{x^2+y} \Leftrightarrow t^2 = x^2+y \Leftrightarrow y = t^2 - x^2$.
Thus: $dy = 2t\,dt$ and furthermore:
for $y=0$: $g(0) = \sqrt{x^2+0} = \sqrt{x^2}$
for $y=1$: $g(1) = \sqrt{x^2+1}$ -
It follows that:

$$V = \int_0^1 dx\ x\left[\int_{\sqrt{x^2}}^{\sqrt{1+x^2}} dt\ (2t)\,t\right] = 2\int_0^1 dx\ x\left[\int_x^{\sqrt{1+x^2}} t^2 dt\right] =$$

$$= 2\int_0^1 dx\ x\left[\frac{t^3}{3}\right]_x^{\sqrt{1+x^2}} = 2\int_0^1 dx\ x\left[\frac{(\sqrt{1+x^2})^3 - x^3}{3}\right] =$$

$$= \frac{2}{3}\int_0^1 dx\ x\left[(1+x^2)\sqrt{1+x^2} - x^3\right] =$$

$$= \frac{2}{3}\int_0^1 dx\ x(1+x^2)\sqrt{1+x^2} - \frac{2}{3}\int_0^1 dx\ x^4 =$$

$$= \frac{2}{3}(I_1 - I_2) \quad \text{with}$$

$$I_1 = \int_0^1 dx\ x(1+x^2)\sqrt{1+x^2} \quad \text{and} \quad I_2 = \int_0^1 x^4 dx$$

- For I_2:

$$I_2 = \int_0^1 x^4 dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1^5 - 0^5}{5} = \frac{1}{5}$$

- For I_1:

Let $x = \tan\vartheta \Rightarrow \begin{cases} dx = d\vartheta/\cos^2\vartheta \\ \text{For } x=0 : \vartheta = 0 \\ \text{For } x=1 : \vartheta = \pi/4 \end{cases} \Rightarrow$

$$\Rightarrow I_1 = \int_0^{\pi/4} \frac{d\vartheta}{\cos^2\vartheta} \tan\vartheta\ (1+\tan^2\vartheta) \cdot \sqrt{1+\tan^2\vartheta} =$$

$$= \int_0^{\pi/4} \frac{d\vartheta}{\cos^2\vartheta} \tan\vartheta \frac{1}{\cos^2\vartheta} \sqrt{\frac{1}{\cos^2\vartheta}} =$$

$$= \int_0^{\pi/4} d\vartheta \frac{\tan\vartheta}{\cos^4\vartheta\,|\cos\vartheta|} = \int_0^{\pi/4} d\vartheta \frac{\tan\vartheta}{\cos^5\vartheta} =$$

$$= \int_0^{\pi/4} d\vartheta \frac{\sin\vartheta}{\cos^5\vartheta\cos\vartheta} = \int_0^{\pi/4} d\vartheta \frac{\sin\vartheta}{\cos^6\vartheta}$$

Let $s = \cos\vartheta = g(\vartheta) \Rightarrow \begin{cases} ds = -\sin\vartheta\, d\vartheta \Rightarrow \sin\vartheta\, d\vartheta = -ds \\ g(0) = \cos 0 = 1 \\ g(\pi/4) = \cos(\pi/4) = \sqrt{2}/2 \end{cases} \Rightarrow$

$$\Rightarrow I_1 = \int_1^{\sqrt{2}/2} d\varsigma \cdot (-1) \frac{1}{\varsigma^6} = \int_1^{\sqrt{2}/2} d\varsigma \, (-\varsigma^{-6}) =$$

$$= \left[\frac{-\varsigma^{-5}}{-5} \right]_1^{\sqrt{2}/2} = \left[\frac{1}{5\varsigma^5} \right]_1^{\sqrt{2}/2} =$$

$$= \frac{1}{5} \left[\frac{1}{(1/\sqrt{2})^5} - \frac{1}{1^5} \right] =$$

$$= \frac{1}{5} \left[(\sqrt{2})^5 - 1 \right] = \frac{4\sqrt{2} - 1}{5}$$

It follows that

$$V = \frac{2}{3} (I_1 - I_2) = \frac{2}{3} \left[\frac{4\sqrt{2} - 1}{5} - \frac{1}{5} \right] =$$

$$= \frac{2}{3} \, \frac{4\sqrt{2} - 1 - 1}{5} = \frac{2(4\sqrt{2} - 2)}{15} = \frac{4(2\sqrt{2} - 1)}{15}$$

EXERCISES

① Evaluate the following double integrals

a) $I = \iint_A xy(x+y)^2 \, dx\,dy$ with $A = [1,2] \times [1,3]$

b) $I = \iint_A (x^3+y^3-3xy) \, dx\,dy$ with $A = [0,a] \times [0,a]$

c) $I = \iint_A x^2 \sin y \, dx\,dy$ with $A = [1,b] \times [0,\pi/4]$

d) $I = \iint_A \sqrt{x+y} \, dx\,dy$ with $A = [0,a] \times [0,b]$

e) $I = \iint_A \frac{dx\,dy}{x+y}$ with $A = [1,a] \times [0,b]$

f) $I = \iint_A \frac{dx\,dy}{\sqrt{x+y}}$ with $A = [0,a] \times [0,b]$

g) $I = \iint_A \frac{\ln(xy)}{y} \, dx\,dy$ with $A = [1,a] \times [1,b]$

h) $I = \iint_A \frac{y}{1+xy} \, dx\,dy$ with $A = [0,1] \times [0,1]$

i) $I = \iint_A \exp(ax+by) \, dx\,dy$ with $A = [0,1] \times [0,2]$

(9) Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ be a continuous scalar field. Show that if there is a field $F: A \to \mathbb{R}$ such that

$$\forall (x,y) \in A, \quad f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

Then, it follows that over the region

$$A = [a_1, a_2] \times [b_1, b_2]$$

we have

$$\iint_A f(x,y)\,dx\,dy = F(a_2, b_2) - F(a_1, b_2) - F(a_2, b_1) + F(a_1, b_1)$$

Double integrals over simple regions

① x-simple regions

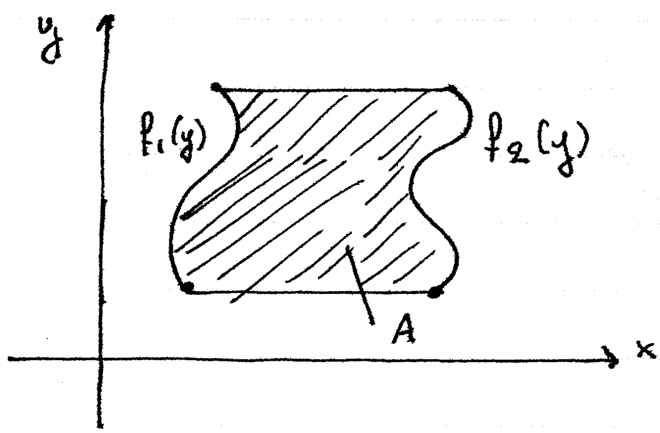
- Let $A \subseteq \mathbb{R}^2$. We say that A is <u>x-simple</u> if and only if it can be written us:

$$A = \{(x,y) \in \mathbb{R}^2 \mid f_1(y) \le x \le f_2(y) \wedge y \in [a,b]\}$$

with f_1, f_2 functions with $f_1 : [a,b] \to \mathbb{R}$ and $f_2 : [a,b] \to \mathbb{R}$.

- If A is x-simple then

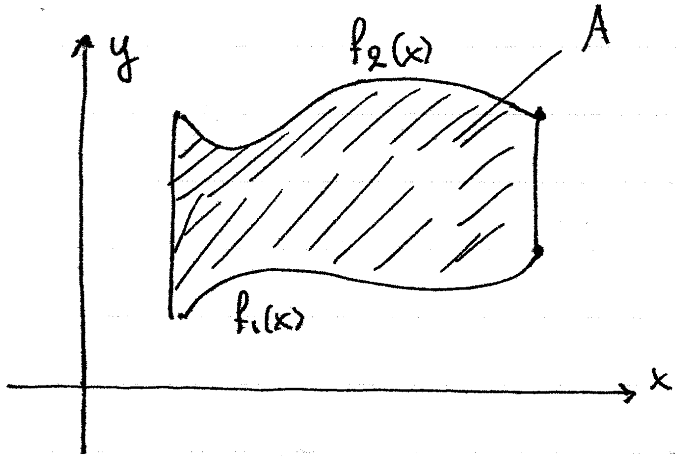
$$\boxed{\iint_A f(x,y)\,dx\,dy = \int_a^b dy \int_{f_1(y)}^{f_2(y)} dx\; f(x,y)}$$



② y-simple regions

- Let $A \subseteq \mathbb{R}^2$. We say that A is y-simple if and only if it can be written as:

$$A = \{ (x,y) \in \mathbb{R}^2 \mid x \in [a,b] \wedge f_1(x) \leq y \leq f_2(x) \}$$
with f_1, f_2 functions with $f_1 : [a,b] \to \mathbb{R}$ and $f_2 : [a,b] \to \mathbb{R}$.



- If A is y-simple then:

$$\boxed{\iint_A f(x,y)\,dx\,dy = \int_a^b dx \int_{f_1(x)}^{f_2(x)} dy\; f(x,y)}$$

EXAMPLES

a) Evaluate the integral $I = \iint_A 2ye^x \, dxdy$ with

$A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 \wedge 0 \leq x \leq y^2\}$

Solution

$$I = \iint_A 2y e^x \, dxdy = \int_0^1 dy \int_0^{y^2} dx \, 2ye^x = \int_0^1 dy \, 2y \left[\int_0^{y^2} e^x \, dx \right] =$$

$$= \int_0^1 dy \, 2y \left[e^x \right]_0^{y^2} = \int_0^1 dy \, 2y \, (e^{y^2} - e^0) =$$

$$= \int_0^1 2y e^{y^2} dy - \int_0^1 2y \, dy = I_1 - I_2 \qquad (1)$$

For $I_1 = \int_0^1 2y e^{y^2} dy$

Let $t = e^{y^2} = g(y) \Rightarrow \begin{cases} dt = 2ye^{y^2} dy \\ g(0) = e^{0^2} = 1 \\ g(1) = e^{1^2} = e \end{cases} \Rightarrow$

$$\Rightarrow I_1 = \int_1^e dt = [t]_1^e = e - 1.$$

For $I_2 = \int_0^1 2y dy = [y^2]_0^1 = 1^2 - 0^2 = 1$

It follows that $I = I_1 - I_2 = (e-1) - 1 = e - 2$.

b) Evaluate the following integral:

$$I = \iint_A (4x + 10y)\, dx\, dy$$

over the region
$A = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1] \wedge -x \leq y \leq x^2\}$

<u>Solution</u>

$$I = \iint_A (4x + 10y)\, dx\, dy = 4 \iint_A x\, dx\, dy + 10 \iint_A y\, dx\, dy =$$

$$= 4 I_1 + 10 I_2 \qquad \text{with}$$

$$I_1 = \iint_A x\, dx\, dy = \int_0^1 dx \int_{-x}^{x^2} dy\; x = \int_0^1 dx\; x \left[\int_{-x}^{x^2} dy \right] =$$

$$= \int_0^1 dx\; x \,[y]_{-x}^{x^2} = \int_0^1 dx\; x\,(x^2 - (-x)) =$$

$$= \int_0^1 (x^3 + x^2)\, dx = \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_0^1 =$$

$$= \frac{1^4 - 0^4}{4} + \frac{1^3 - 0^3}{3} = \frac{1}{4} + \frac{1}{3} = \frac{4+3}{12} = \frac{7}{12}$$

and

$$I_2 = \iint_A y\,dx\,dy = \int_0^1 dx \int_{-x}^{x^2} dy\, y = \int_0^1 dx \left[\frac{y^2}{2}\right]_{-x}^{x^2} =$$

$$= \int_0^1 dx\ \frac{(x^2)^2 - (-x)^2}{2} = \int_0^1 dx\ \frac{x^4 - x^2}{2} =$$

$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2}\left[\frac{1^5 - 0^5}{5} - \frac{1^3 - 0^3}{3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{5} - \frac{1}{3}\right] = \frac{1}{2}\ \frac{3-5}{15} = \frac{1}{2}\ \frac{-2}{15} =$$

$$= \frac{-1}{15}.$$

It follows that

$$I = 4I_1 + 10I_2 = 4\cdot\left(\frac{7}{12}\right) + 10\cdot\left(\frac{-1}{15}\right) = \frac{7}{3} - \frac{2}{3} = \frac{5}{3}.$$

▶ Application to volumes

- Before setting up a volume integral, you should:

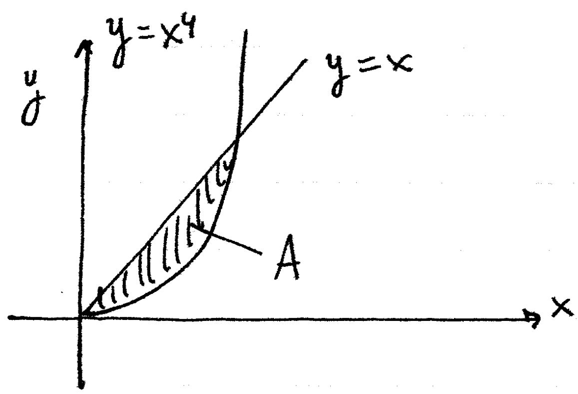
a) Define the region $A \subseteq \mathbb{R}^2$ over which we integrate using set-theoretic notation.

b) Define the solid S in terms of A, again using set theoretic notation.

EXAMPLE

Find the volume of the solid under the plane $(p): x+2y-z=0$ and bounded by the surfaces $(p_1): y=x$ and $(p_2): y=x^4$.

Solution



We note that

$$x=x^4 \Leftrightarrow x^4-x=0 \Leftrightarrow$$
$$\Leftrightarrow x(x^3-1)=0$$
$$\Leftrightarrow x=0 \vee x^3-1=0$$
$$\Leftrightarrow x=0 \vee x^3=1$$
$$\Leftrightarrow x=0 \vee x=1$$

Thus, the region on the xy-plane bounded by (p_1) and (p_2) is given by:

$$A=\{(x,y)\in\mathbb{R}^2 \mid x\in[0,1] \wedge x^4 \leq y \leq x\}$$

It follows that the solid we are interested in is given by:

$$S = \{(x,y,z) \in \mathbb{R}^2 \mid (x,y) \in A \wedge 0 \leq z \leq x+2y\}$$

It follows that the volume of S is given by:

$$V = \iint_A (x+2y)\,dx\,dy = \iint_A x\,dx\,dy + 2\iint_A y\,dx\,dy = I_1 + 2I_2$$

with

$$I_1 = \iint_A x\,dx\,dy = \int_0^1 dx \int_{x^4}^{x} dy\; x = \int_0^1 dx\; x\left[\int_{x^4}^{x} dy\right] =$$

$$= \int_0^1 dx\; x\,[y]_{x^4}^{x} = \int_0^1 dx\; x(x-x^4) = \int_0^1 (x^2-x^5)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 = \frac{1^3-0^3}{3} - \frac{1^6-0^6}{6} = \frac{1}{3} - \frac{1}{6} =$$

$$= \frac{2-1}{6} = \frac{1}{6}$$

and

$$I_2 = \iint_A y\,dx\,dy = \int_0^1 dx \int_{x^4}^{x} dy\; y = \int_0^1 dx \left[\frac{y^2}{2}\right]_{x^4}^{x} =$$

$$= \int_0^1 dx\; \frac{x^2-(x^4)^2}{2} = \frac{1}{2}\int_0^1 (x^2-x^8)\,dx =$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^9}{9}\right]_0^1 = \frac{1}{2}\left[\frac{1^3-0^3}{3} - \frac{1^9-0^9}{9}\right] =$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{9}\right) = \frac{1}{2}\,\frac{3-1}{9} = \frac{1}{2}\,\frac{2}{9} = \frac{1}{9}$$

It follows that:

$$V = I_1 + 2I_2 = \frac{1}{6} + 2\cdot\frac{1}{9} = \frac{3+2\cdot 2}{18} =$$

$$= \frac{3+4}{18} = \frac{7}{18}\,.$$

EXERCISES

③ Evaluate the following double integrals

a) $I = \iint_A x^2 y \, dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq a \wedge x \leq y \leq 3x+1\}$

b) $I = \iint_A dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq 4 \wedge 0 \leq y \leq \ln x\}$

c) $I = \iint_A x \, dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq a \wedge 1 \leq y \leq \exp(x^2)\}$

d) $I = \iint_A \frac{dx dy}{x^2+y^2}$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq a \wedge 0 \leq y \leq x\}$

e) $I = \iint_A e^x \cos y \, dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq \pi/4 \wedge 0 \leq x \leq \sin y\}$

f) $I = \iint_A (x+y)\,dx\,dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant a \wedge 0 \leq y \leq \sqrt{a^2 - x^2}\}$

Changing the order of iterated integrals

In some double integrals, defined over an x-simple or y-simple domain, we may find that in the corresponding iterated integral, the first integral may not have an antiderivative that can be defined using elementary functions, while the second integral evaluates to an outcome that can be expressed with elementary functions. If the domain of integration is both x-simple and y-simple, then we can try reversing the order of the corresponding iterated integral, and see whether this results in iterated integrals that can be evaluated one at a time.

Note that in order to apply this technique, it is important that the solution include a mathematical proof that the x-simple and y-simple representations of the domain of integration are equal.

Recall that in order to show that the two sets A, B satisfy $A = B$, we have to show that

$$\begin{cases} x \in A \Rightarrow x \in B \\ x \in B \Rightarrow x \in A \end{cases}$$

EXAMPLE

Evaluate the integral

$$I = \iint_A \exp(y^2)\, dx\, dy$$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 4 \wedge x/2 \leq y \leq 2\}$

Solution

1st method: (without exchanging the order of integration)

$$I = \iint_A \exp(y^2)\, dx\, dy = \int_0^4 dx \int_{x/2}^2 dy\, \exp(y^2) =$$

$$= \int_0^4 dx \left[[(d/dx)x] \int_{x/2}^2 dy\, \exp(y^2) \right] =$$

$$= \left[x \int_{x/2}^2 dy\, \exp(y^2) \right]_{x=0}^{x=4} - \int_0^4 dx \left[x \frac{d}{dx} \int_{x/2}^2 dy\, \exp(y^2) \right] =$$

$$= \left[4 \int_2^2 dy\, \exp(y^2) - 0 \int_0^2 dy\, \exp(y^2) \right]$$

$$- \int_0^4 dx\, x \left[-(x/2)' \exp((x/2)^2) \right] =$$

$$= (4 \cdot 0 - 0) + \int_0^4 (1/2)\, x \exp((x/2)^2)\, dx$$

Let $t = \exp((x/2)^2)$. Then:

$$dt = [\exp((x/2)^2)]'\, dx = [(x/2)^2]' \exp((x/2)^2)\, dx$$

$$= 2(x/2)(x/2)' \exp((x/2)^2)\, dx =$$

$$= (1/2)\, x \exp((x/2)^2)\, dx$$

For $x = 0 \Rightarrow t = \exp(0) = 1$

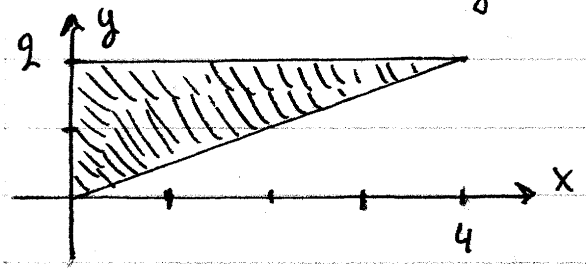
$x = 4 \Rightarrow t = \exp((4/2)^2) = e^4$

It follows that

$$I = \int_1^{e^4} dt = e^4 - 1$$

<u>2nd method</u>: (with exchanging order of integration).

Recall that $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 4 \wedge x/2 \leq y \leq 2\}$



To exchange the integral order, we define

$B = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 2 \wedge 0 \leq x \leq 2y\}$

and claim that $A = B$.

Proof of claim:

$(\Rightarrow)$: Assume that $(x,y) \in A$. Then:

$(x,y) \in A \Rightarrow 0 \leq x \leq 4 \wedge x/2 \leq y \leq 2$

and it follows that

$y \geq x/2 \geq 0/2 = 0 \Rightarrow y \geq 0$

$y \leq 2$, by hypothesis

$0 \leq x$, by hypothesis

$x = 2(x/2) \leq 2y \Rightarrow x \leq 2y$

From the above, it follows that

$0 \leq y \leq 2 \wedge 0 \leq x \leq 2y \Rightarrow (x,y) \in B$

($\Leftarrow$): Assume that $(x,y) \in B$. Then

$(x,y) \in B \Rightarrow 0 \leq y \leq 2 \wedge 0 \leq x \leq 2y$

It follows that

$0 \leq x$, by hypothesis

$x \leq 2y \leq 2 \cdot 2 = 4 \Rightarrow x \leq 4$

$x/2 \leq (2y)/2 = y \Rightarrow x/2 \leq y$

$y \leq 2$, by hypothesis

From the above, it follows that

$0 \leq x \leq 4 \wedge x/2 \leq y \leq 2 \Rightarrow (x,y) \in A$

This proves the claim: $A = B$.

It follows that

$$I = \iint_A \exp(y^2)\, dx dy = \iint_B \exp(y^2)\, dx dy =$$

$$= \int_0^2 dy \int_0^{2y} dx \exp(y^2) = \int_0^2 dy \left[\exp(y^2) \int_0^{2y} dx \right] =$$

$$= \int_0^2 dy \exp(y^2)(2y) = \int_0^2 dy \exp(y^2)(y^2)' =$$

$$= \int_0^2 dy\, [\exp(y^2)]' = [\exp(y^2)]_0^2 =$$

$$= \exp(2^2) - \exp(0^2) = e^4 - 1.$$

EXERCISES

(4) Evaluate the following double integrals by changing the order of integration

a) $I = \iint_A \sqrt{x^3+1}\, dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid \sqrt{y} \leqslant x \leqslant 2 \wedge 0 \leqslant y \leqslant 4\}$

b) $I = \iint_A x \exp(y^3)\, dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1 \wedge x \leqslant y \leqslant 1\}$

c) $I = \iint_A \frac{x\, dx dy}{\sqrt{3x^2+y}}$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant \sqrt{y} \wedge 0 \leqslant y \leqslant 9\}$

d) $I = \iint_A \sqrt{ax^2+by}\, dx dy$

with $a, b \in (0, +\infty)$ and

$A = \{(x,y) \in \mathbb{R}^2 \mid \sqrt{y} \leqslant x \leqslant 2 \wedge 0 \leqslant y \leqslant 4\}$

⑤ Let $f: \mathbb{R} \to \mathbb{R}$ be a function, continuous on $\mathbb{R}$, and define

$$\forall t \in \mathbb{R}: \quad g(t) = \iint_{A(t)} f(y)\, dx\, dy$$

with

$$\forall t \in (0, +\infty): \quad A(t) = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq t \wedge 0 \leq y \leq x\}$$

a) Use the fundamental theorem of calculus to show that

$$\forall t \in (0, +\infty): \quad g''(t) = f(t)$$

b) Exchange the order of the integrals to show that

$$\forall t \in (0, +\infty): \quad g(t) = \int_0^t (t - y) f(y)\, dy.$$

▼ Change of variables in multiple integrals

Consider a change of variables defined by the equations

$$\begin{cases} x_1 = g_1(u_1, u_2, \dots, u_n) \\ x_2 = g_2(u_1, u_2, \dots, u_n) \\ \vdots \\ x_n = g_n(u_1, u_2, \dots, u_n) \end{cases}, \forall (u_1, u_2, \dots, u_n) \in B$$

and define $g: B \to \mathbb{R}^n$ such that

$\forall u \in B: g(u) = (g_1(u), g_2(u), \dots, g_n(u))$

with $u = (u_1, u_2, \dots, u_n)$.

We begin with the following definitions

Smooth change of variables

1) We define the <u>derivative matrix</u> of g as:

$$Dg(u) = \begin{bmatrix} \partial g_1/\partial u_1 & \partial g_1/\partial u_2 & \cdots & \partial g_1/\partial u_n \\ \partial g_2/\partial u_1 & \partial g_2/\partial u_2 & \cdots & \partial g_2/\partial u_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial g_n/\partial u_1 & \partial g_n/\partial u_2 & \cdots & \partial g_n/\partial u_n \end{bmatrix}, \forall u \in B$$

and define the Jacobian of g as the determinant of the derivative matrix $Dg(u)$:

$$\frac{\partial(g_1, g_2, \dots, g_n)}{\partial(u_1, u_2, \dots, u_n)} = \det(Dg(u)), \quad \forall u \in B$$

2) We say that, given $\mathcal{S} \subseteq B$,

g one-to-one on $\mathcal{S} \iff \forall u, v \in \mathcal{S} : (g(u) = g(v) \Rightarrow u = v)$

<u>Def</u> : A transformation $g : B \to \mathbb{R}^n$ with $B \subseteq \mathbb{R}^n$ and components:

$\forall u \in B : g(u) = (g_1(u), g_2(u), \dots, g_n(u))$

is a <u>smooth change of variables</u> if and only if

$$\begin{cases} g_1, g_2, \dots, g_n \text{ differentiable on } B \\ Dg \text{ continuous on } B \\ g \text{ one-to-one on } B - \partial B \\ \dfrac{\partial(g_1, g_2, \dots, g_n)}{\partial(u_1, u_2, \dots, u_n)} \neq 0, \ \forall (u_1, \dots, u_n) \in B - \partial B \end{cases}$$

Note that in the above definition, the last two requirements can be violated on the boundary ∂B of the domain B. The corresponding points $u \in \partial B$ are then considered <u>singular points</u> of the otherwise smooth change of variables.

● → <u>Changing variables on multiple integrals</u>

Let us consider the integral of a scalar field $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ over a region $\mathcal{S} \subseteq A$ defined as

$$\mathcal{S} = \{ g(u) \mid u \in \mathcal{S}_0 \} = g(\mathcal{S}_0)$$

with $x = g(u)$ representing a smooth change of variables. Then, it follows that

$$\boxed{\int_{S} f(x)\,dx = \int_{S_0} f(g(u)) \left| \frac{\partial g}{\partial u} \right| du}$$

with $dx = dx_1 dx_2 \cdots dx_n$

$du = du_1 du_2 \cdots du_n$

and $\dfrac{\partial g}{\partial u} = \dfrac{\partial(g_1, g_2, \dots, g_n)}{\partial(u_1, u_2, \dots, u_n)}$

Formally, the corresponding transformation of the integral differentials reads:

$$\boxed{dx_1 dx_2 \cdots dx_n = \frac{\partial(g_1, g_2, \dots, g_n)}{\partial(u_1, u_2, \dots, u_n)}\, du_1 du_2 \cdots du_n}$$

- This change of variables can be beneficial when one of the following happens:
  a) The domain S_0 is substantially simpler than S.
  b) The resulting integrand is simpler.
- Note that a change of variables from x to u requires a mapping g from u back to x (i.e. in the reverse direction). This is similar to the backsubstitution method in Calculus 1.

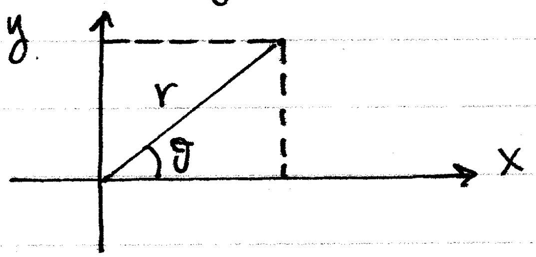
Change of variables to polar coordinates

Consider the change of variables

$$\begin{cases} x = g_1(r,\vartheta) = r\cos\vartheta , \quad \forall r \in [0,+\infty), \forall \vartheta \in [0,2\pi] \\ y = g_2(r,\vartheta) = r\sin\vartheta \end{cases}$$

or equivalently

$$(x,y) = g(r,\vartheta) = (r\cos\vartheta, r\sin\vartheta) , \quad \forall (r,\vartheta) \in [0,+\infty) \times [0,2\pi]$$



The corresponding Jacobian is given by

$$\frac{\partial(g_1,g_2)}{\partial(r,\vartheta)} = \begin{vmatrix} \partial g_1/\partial r & \partial g_1/\partial \vartheta \\ \partial g_2/\partial r & \partial g_2/\partial\vartheta \end{vmatrix} = \begin{vmatrix} \cos\vartheta & -r\sin\vartheta \\ \sin\vartheta & r\cos\vartheta \end{vmatrix}$$

$$= (\cos\vartheta)(r\cos\vartheta) - (\sin\vartheta)(-r\sin\vartheta) =$$

$$= r\cos^2\vartheta + r\sin^2\vartheta = r(\cos^2\vartheta + \sin^2\vartheta) = r$$

and therefore

$$dx\,dy = \left|\frac{\partial(g_1,g_2)}{\partial(r,\vartheta)}\right| dr\,d\vartheta = |r|\,dr\,d\vartheta = r\,dr\,d\vartheta$$

It follows that the integral

$$I = \iint_A f(x,y)\,dx\,dy$$

over a set A given by

$$A = \{(r\cos\vartheta, r\sin\vartheta) \mid (r,\vartheta) \in B\}$$

with $B \subseteq [0, +\infty) \times [0, 2\pi]$ can be evaluated with the change of variables

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}$$

as follows:

$$\boxed{\iint_A f(x,y)\,dx\,dy = \iint_B f(r\cos\vartheta, r\sin\vartheta)\, r\,dr\,d\vartheta}$$

- The resulting integral in terms of $dr\,d\vartheta$ can be evaluated if B is a boxed domain or an r-simple or ϑ-simple domain.
- It is also worth noting that:

$$\boxed{x^2 + y^2 = r^2.}$$

- The corresponding formal relation between the cartesian and polar differentials is given by:

$$\boxed{dx\,dy = r\,dr\,d\vartheta}$$

EXAMPLES

Evaluate the integral $I = \iint_A \cos(\pi x^2 + \pi y^2)\, dx dy$ with A given by

$A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4\}$

Solution

We note that

$$A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leqslant x^2 + y^2 \leqslant 4\} =$$

$$= \{(r\cos\vartheta, r\sin\vartheta) \mid r \in [1,2] \wedge \vartheta \in [0, 2\pi]\}$$

and it follows that:

$$I = \iint_A \cos(\pi x^2 + \pi y^2)\, dx dy = \int_1^2 dr \int_0^{2\pi} d\vartheta\; r\cos(\pi r^2) =$$

$$= \int_1^2 dr\; r\cos(\pi r^2) \left[\int_0^{2\pi} d\vartheta \right] = \int_1^2 dr\; r\cos(\pi r^2)\, 2\pi =$$

$$= 2\pi \int_1^2 r\cos(\pi r^2)\, dr$$

Let $t = \pi r^2 = g(r) \Rightarrow \begin{cases} dt = 2\pi r\, dr \\ g(1) = \pi \cdot 1^2 = \pi \\ g(2) = \pi \cdot 2^2 = 4\pi \end{cases} \Rightarrow$

$$\Rightarrow I = \int_{\pi}^{4\pi} \cos(t)\,dt = [\sin(t)]_{\pi}^{4\pi} = \sin(4\pi) - \sin(\pi)$$

$$= \sin 0 - \sin \pi = 0 - 0 = 0.$$

▶ Application: Gaussian integral

$$\boxed{\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}.}$$

Proof

Define $I = \int_{-\infty}^{+\infty} e^{-x^2}\,dx$ and note that:

$$J = \iint_{\mathbb{R}^2} \exp(-x^2-y^2)\,dx\,dy = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, \exp(-x^2-y^2) =$$

$$= \int_{-\infty}^{+\infty} dx \, e^{-x^2} \left[\int_{-\infty}^{+\infty} dy \, e^{-y^2} \right] = \int_{-\infty}^{+\infty} dx \, e^{-x^2} I =$$

$$= I \int_{-\infty}^{+\infty} dx \, e^{-x^2} = I \cdot I = I^2.$$

We now calculate J:

$$J = \iint_{\mathbb{R}^2} \exp(-x^2-y^2)\,dx\,dy = \int_0^{+\infty} dr \int_0^{2\pi} d\vartheta \; r e^{-r^2} =$$

$$= \int_0^{+\infty} dr\; r e^{-r^2}\left[\int_0^{2\pi} d\vartheta\right] = \int_0^{+\infty} 2\pi r e^{-r^2}\,dr =$$

$$= 2\pi \int_0^{+\infty} r e^{-r^2}\,dr .$$

Let $\underset{= g(r)}{\rho = -r^2} \Rightarrow \begin{cases} d\rho = -2r\,dr \Rightarrow r\,dr = (-1/2)\,d\rho \\ g(0) = 0 \\ g(+\infty) = \lim_{r\to+\infty}(-r^2) = -\infty \end{cases} \Rightarrow$

$$\Rightarrow J = 2\pi \int_0^{-\infty} e^{\rho}(-1/2)\,d\rho = -\pi \int_0^{-\infty} e^{\rho}\,d\rho = -\pi\left[e^{\rho}\right]_0^{-\infty} =$$

$$= -\pi\left[\lim_{\rho\to-\infty} e^{\rho} - e^0\right] = -\pi[0 - 1] = \pi \Rightarrow$$

$$\Rightarrow I^2 = \pi \Rightarrow I = \sqrt{\pi} \lor I = -\sqrt{\pi}.$$

Since $\forall x \in \mathbb{R}: e^{-x^2} > 0 \Rightarrow \int_{-\infty}^{+\infty} e^{-x^2}\,dx > 0 \Rightarrow I > 0$

and therefore $I = \sqrt{\pi}$
(the solution $I = -\sqrt{\pi}$ is rejected).

EXERCISES

⑥ Evaluate the following integrals using change of variables to polar coordinates

a) $I = \iint_A \sqrt{x^2+y^2}\, dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x^2+y^2 \leq a^2\}$

b) $I = \iint_A xy\, dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4 \wedge x \geq 0 \wedge y \geq 0\}$

c) $I = \iint_A y(x^2+y^2)^3\, dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0 \wedge x^2+y^2 \leq a^2\}$

d) $I = \iint_A \frac{y\, dx dy}{x^2+y^2}$

with $A = \{(x,y) \in \mathbb{R}^2 \mid y \geq 1/2 \wedge x^2+y^2 \leq 1\}$

e) $I = \iint_A \operatorname{Arctan}\left(\frac{y}{x}\right) dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq a \wedge 0 \leq y \leq \sqrt{a^2-x^2}\}$

f) $I=\iint_A x\,dx\,dy$

with $A=\{(x,y)\in\mathbb{R}^2 \mid 0\leq x\leq 1/2 \wedge x\sqrt{3}\leq y\leq\sqrt{1-x^2}\}$

g) $I=\iint_A |xy|\,dx\,dy$

with $A=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq a^2\}$

h) $I=\iint_A \dfrac{dx\,dy}{(x^2+y^2)\sqrt{x^2+y^2}}$

with $A=\{(x,y)\in\mathbb{R}^2 \mid x+y\geq 1 \wedge x^2+y^2\leq 1\}$

i) $I=\iint_A (x-y)\,dx\,dy$

with $A=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 1 \wedge x+y\geq 1\}$

j) $I=\iint_A y\,dx\,dy$

with $A=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 1 \wedge (x-1)^2+y^2\leq 1\}$.

Evaluation of triple integrals over boxed domains

Let $A \subseteq \mathbb{R}^3$ be a boxed domain given by

$$A = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid x \in [a_1, a_2] \wedge y \in [b_1, b_2] \wedge z \in [c_1, c_2]\}$$

and consider a scalar field $f : A \to \mathbb{R}$ such that f is continuous on A. The triple integral of f over A can be evaluated, according to Fubini's theorem by:

$$\boxed{\iiint_A f(x,y,z)\, dx\, dy\, dz = \int_{a_1}^{a_2} dx \int_{b_1}^{b_2} dy \int_{c_1}^{c_2} dz\, f(x,y,z)}$$

It can also be shown that permuting the order of the iterated integrals gives the same result.

Separable scalar fields

We say that the scalar field $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^3$ is <u>separable</u> if and only if it satisfies
$\forall (x,y,z) \in A : f(x,y,z) = f_1(x) f_2(y) f_3(z)$
Then, we can show that:

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \left[\int_{a_1}^{a_2} f_1(x)\,dx\right]\left[\int_{b_1}^{b_2} f_2(y)\,dy\right]\left[\int_{c_1}^{c_2} f_3(z)\,dz\right]$$

Proof

$$I = \iiint_A f(x,y,z)\,dx\,dy\,dz = \iiint_A f_1(x) f_2(y) f_3(z)\,dx\,dy\,dz$$

$$= \int_{a_1}^{a_2} dx \int_{b_1}^{b_2} dy \int_{c_1}^{c_2} dz\; f_1(x) f_2(y) f_3(z) =$$

$$= \int_{a_1}^{a_2} dx\, f_1(x) \left[\int_{b_1}^{b_2} dy \int_{c_1}^{c_2} dz\, f_2(y) f_3(z)\right] =$$

$$= \left[\int_{b_1}^{b_2} dy \int_{c_1}^{c_2} dz\, f_2(y) f_3(z)\right]\left[\int_{a_1}^{a_2} f_1(x)\,dx\right] =$$

$$= \left[\int_{b_1}^{b_2} dy\, f_2(y) \left(\int_{c_1}^{c_2} dz\, f_3(z)\right)\right]\left[\int_{a_1}^{a_2} f_1(x)\,dx\right] =$$

$$= \left[\int_{a_1}^{a_2} f_1(x)\,dx\right]\left[\int_{b_1}^{b_2} f_2(y)\,dy\right]\left[\int_{c_1}^{c_2} f_3(z)\,dz\right]$$

EXAMPLE

Evaluate $I = \iiint_A x^2yz\,dxdydz$

with $A = [1,2]\times[0,1]\times[0,2]$

Solution

$$I = \iiint_A x^2yz\,dxdydz = \int_1^2 dx \int_0^1 dy \int_0^2 dz\; x^2yz =$$

$$= \left[\int_1^2 x^2\,dx\right]\left[\int_0^1 y\,dy\right]\left[\int_0^2 z\,dz\right] = I_1 I_2 I_3$$

with

$$I_1 = \int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3-1^3}{3} = \frac{8-1}{3} = \frac{7}{3}$$

$$I_2 = \int_0^1 y\,dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1^2-0^2}{2} = \frac{1}{2}$$

$$I_3 = \int_0^2 z\,dz = \left[\frac{z^2}{2}\right]_0^2 = \frac{2^2-0^2}{2} = \frac{4}{2} = 2$$

and therefore

$$I = I_1 I_2 I_3 = \frac{7}{3}\cdot\frac{1}{2}\cdot 2 = \frac{7}{3}$$

EXERCISES

(7) Evaluate the following triple integrals

a) $I = \iiint_A z^3 \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [1,3] \wedge y \in [-1,2] \wedge z \in [1,2]\}$

b) $I = \iiint_A x \exp(ay + bz) \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,1] \wedge y \in [0,2] \wedge z \in [0,2]\}$

c) $I = \iiint_A \frac{x \, dx\,dy\,dz}{(y+z)^2}$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,1] \wedge y \in [1,a] \wedge z \in [1,a]\}$
and $a \in (1,+\infty)$.

d) $I = \iiint_A (x-y)(y-z)(z-x) \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,a] \wedge y \in [0,b] \wedge z \in [0,c]\}$
and $a,b,c \in (0,+\infty)$.

e) $I = \iiint_A (x+y)^3 \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,a] \wedge y \in [0,a] \wedge z \in [0,a]\}$
and $a \in (0,+\infty)$

f) $I = \iiint_A (x+y+z)^2 \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [1,a] \wedge y \in [1,a] \wedge z \in [1,a]\}$

with $a \in (1,+\infty)$.

Evaluation of triple integrals on simple domains

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^3$ be a scalar field continuous at A. We distinguish between the following cases:

1. z-simple regions

For $A = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in \mathcal{S}_{xy} \wedge g_1(x,y) \leq z \leq g_2(x,y)\} \Rightarrow$

$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \iint_{\mathcal{S}_{xy}} dx\,dy \int_{g_1(x,y)}^{g_2(x,y)} dz\; f(x,y,z)$$

2. x-simple regions

For $A = \{(x,y,z) \in \mathbb{R}^3 \mid (y,z) \in \mathcal{S}_{yz} \wedge g_1(y,z) \leq x \leq g_2(y,z)\} \Rightarrow$

$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \iint_{\mathcal{S}_{xy}} dy\,dz \int_{g_1(y,z)}^{g_2(y,z)} dx\; f(x,y,z)$$

3. y-simple regions

For $A = \{(x,y,z) \in \mathbb{R}^3 \mid (x,z) \in \mathcal{S}_{xz} \wedge g_1(x,z) \leq y \leq g_2(x,z)\}$

$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \iint_{\mathcal{S}_{xz}} dx\,dz \int_{g_1(x,z)}^{g_2(x,z)} dy\; f(x,y,z).$$

④ <u>xy-simple regions</u>

For $A = \{(x,y,z) \in \mathbb{R}^3 \mid z \in [a,b] \wedge (x,y) \in S_{xy}(z)\} \Rightarrow$

$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \int_a^b dz \iint_{S_{xy}(z)} dx\,dy\; f(x,y,z)$$

⑤ <u>yz-simple regions</u>

For $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [a,b] \wedge (y,z) \in S_{yz}(x)\} \Rightarrow$

$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \int_a^b dx \iint_{S_{yz}(x)} dy\,dz\; f(x,y,z)$$

⑥ <u>xz-simple regions</u>

For $A = \{(x,y,z) \in \mathbb{R}^3 \mid y \in [a,b] \wedge (x,z) \in S_{xz}(y)\} \Rightarrow$

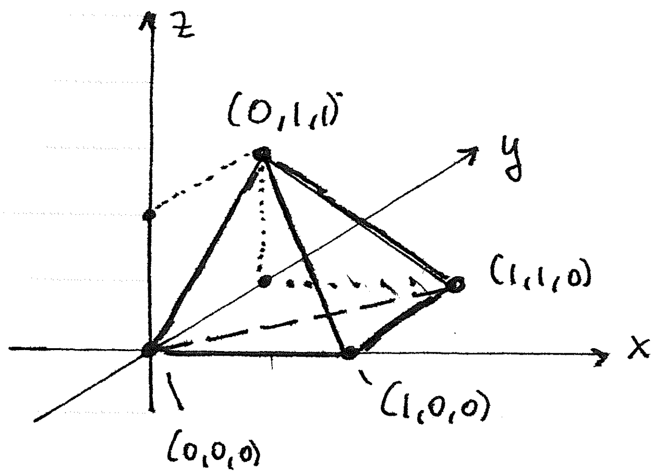
$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \int_a^b dy \iint_{S_{xz}(y)} dx\,dz\; f(x,y,z).$$

EXAMPLES

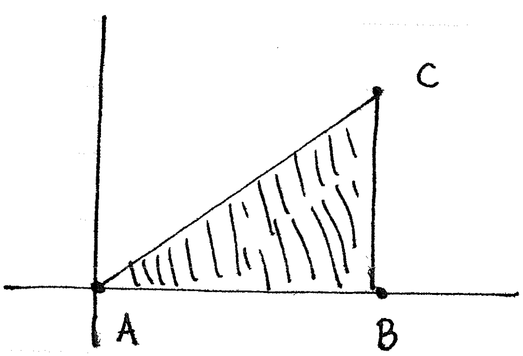
Evaluate the integral $I = \iiint_T xy \, dx\,dy\,dz$

if T is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(0,1,1)$.

Solution



Consider the xy-cross-section of the trapezoid at $z=0$:



$A(0,0,0)$
$B(1,0,0)$
$C(1,1,0)$

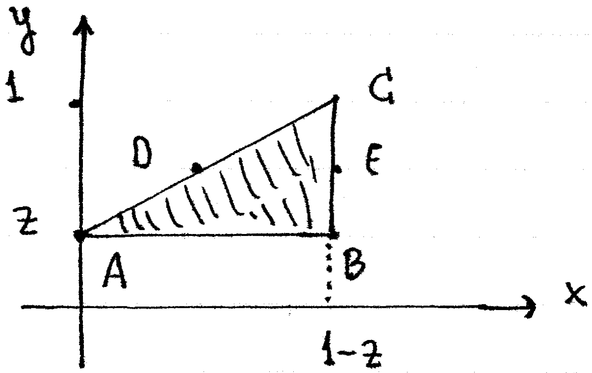
Note that for $0<z<1$, the point A, B, C come together linearly at the point $(0,1,1)$ at the top of the tetrahedron. It follows that for a given z, the coordinates of the vertices A, B, C are:

For $(0,0,0) \longrightarrow (0,1,1)$: $A(0, z, z)$

For $(1,0,0) \longrightarrow (0,1,1)$: $B(1-z, z, z)$

For $(1,1,0) \longrightarrow (0,1,1)$: $C(1-z, 1, z)$

with $0 \leq z \leq 1$. We sketch A, B, C:



It follows that the cross-section of the tetrahedron at z is given by:

$$S_{xy}(z) = \{(x,y) \in \mathbb{R}^2 \mid z \leq y \leq 1 \wedge f_1(y,z) \leq x \leq f_2(y,z)\}$$

To determine $f_1(y,z)$ and $f_2(y,z)$, let $D \in AC$ and $E \in BC$ such that $y_D = y_E = y$. Then we see that $f_1(y,z) = x_D$ and $f_2(y,z) = x_E$. We note that:

$$D \in AC \Leftrightarrow \frac{y_D - y_A}{x_D - x_A} = \frac{y_C - y_A}{x_C - x_A} \Leftrightarrow$$

$$\Leftrightarrow \frac{y-z}{f_1(y,z) - 0} = \frac{1-z}{(1-z)-0} \Leftrightarrow$$

$$\Leftrightarrow \frac{y-z}{f_1(y,z)} = 1 \Leftrightarrow f_1(y,z) = y - z.$$

Since BC is vertical (i.e. $x_B = x_C = 1-z$) it follows that

$f_2(y,z) = x_E = x_B = 1-z$

and therefore

$S_{xy}(z) = \{(x,y) \in \mathbb{R}^2 \mid z \leq y \leq 1 \wedge y-z \leq x \leq 1-z\}$

The trapezoid itself is represented by:

$T = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in S_{xy}(z) \wedge z \in [0,1]\}$

We may now evaluate the integral:

$$I = \iiint_T xy \, dx dy dz = \int_0^1 dz \iint_{S_{xy}(z)} dx dy \; xy =$$

$$= \int_0^1 dz \int_z^1 dy \int_{y-z}^{1-z} dx \; xy = \int_0^1 dz \int_z^1 dy \; y \left[\int_{y-z}^{1-z} x \, dx \right] =$$

$$= \int_0^1 dz \int_z^1 dy \; y \left[\frac{x^2}{2} \right]_{y-z}^{1-z} = \int_0^1 dz \int_z^1 dy \; y \left[\frac{(1-z)^2 - (y-z)^2}{2} \right]$$

$$= \frac{1}{2} \int_0^1 dz \int_z^1 dy \; y \left[(1-2z+\cancel{z^2}) - (y^2 - 2yz + \cancel{z^2}) \right] =$$

$$= \frac{1}{2} \int_0^1 dz \int_z^1 dy \; y(1-2z-y^2+2yz) =$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^1 dz \int_z^1 dy \ \left(y - 2yz - y^3 + 2y^2 z\right) = \\
&= \frac{1}{2}\int_0^1 dz \ \left[\frac{y^2}{2} - y^2 z - \frac{y^4}{4} + \frac{2y^3 z}{3}\right]_z^1 = \\
&= \frac{1}{2}\,\frac{1}{12}\int_0^1 dz \ \left[6y^2 - 12y^2 z - 3y^4 + 8y^3 z\right]_z^1 = \\
&= \frac{1}{24}\int_0^1 dz \left[6(1-z^2) - 12(1-z^2)z - 3(1-z^4) + 8(1-z^3)z\right] \\
&= \frac{1}{24}\int_0^1 dz \left[6 - 6z^2 - 12z + 12z^3 - 3 + 3z^4 + 8z - 8z^4\right] = \\
&= \frac{1}{24}\int_0^1 dz \left[(3-8)z^4 + 12z^3 - 6z^2 + (-12+8)z + (6-3)\right] = \\
&= \frac{1}{24}\int_0^1 dz \left(-5z^4 + 12z^3 - 6z^2 - 4z + 3\right) = \\
&= \frac{1}{24}\left[-\frac{5z^5}{5} + 12\,\frac{z^4}{4} - 6\,\frac{z^3}{3} - 4\,\frac{z^2}{2} + 3z\right]_0^1 = \\
&= \frac{1}{24}\left[-z^5 + 3z^4 - 2z^3 - 2z^2 + 3z\right]_0^1 = \\
&= \frac{1}{24}\left[-1 + 3 - 2 - 2 + 3\right] = \frac{1}{24}
\end{aligned}
$$

EXERCISES

⑧ Evaluate the following triple integrals

a) $I = \iiint_A \exp(x+y+z)\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid y \leq z \leq x \wedge 0 \leq y \leq x \wedge 0 \leq x \leq a\}$
and $a \in (0,+\infty)$

b) $I = \iiint_A xyz\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq z \leq a \wedge 0 \leq y \leq \sqrt{a^2-x^2}$
$\wedge\, 0 \leq x \leq a\}$
and $a \in (0,+\infty)$

c) $I = \iiint_A x\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq z \leq a\}$
and $a \in (0,+\infty)$

d) $I = \iiint_A z^2\,dx\,dy\,dz$

with $A = \{(x,y,z) \in [0,+\infty)^3 \mid x+y+z \leq a\}$
and $a \in (0,+\infty)$.

e) $I = \iiint_A z\,dx\,dy\,dz$

with $A = \{(x,y,z) \in [0,+\infty)^3 \mid x^2 \leq y \leq 2a \wedge 0 \leq x \leq a$
$\wedge\, x-y \leq z \leq x+y\}$
and $a \in (0,+\infty)$

f) $I = \iiint_A \dfrac{y+z}{x}\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid 1 \leq x \leq a \wedge 0 \leq y \leq a-x \wedge 0 \leq z \leq a-x-y\}$
and $a \in (1,+\infty)$

g) $I = \iiint_A \sin\left(\dfrac{x}{y}\right)\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq z \leq a \wedge 0 \leq y \leq \sin(2z) \wedge 0 \leq x \leq 2yz\}$
and $a \in (0,\pi/2)$.

h) $I = \iiint_A xy\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq x \leq a \wedge 0 \leq y \leq \sqrt{a^2-x^2} \wedge$
$\wedge\, 0 \leq z \leq \sqrt{a^2-x^2-y^2}\}$
and $a \in (0,+\infty)$

i) $I = \iiint_A e^z\,dx\,dy\,dz$ with $A \subseteq \mathbb{R}^3$ the tetrahedron with vertices $(0,0,0)$, $(a,0,0)$, $(0,a,0)$, $(0,0,b)$ and $a,b \in (0,+\infty)$.

Change of variables in $\mathbb{R}^3$

Consider a change of variables

$$\begin{cases} x = g_1(u,v,w) \\ y = g_2(u,v,w) \\ z = g_3(u,v,w) \end{cases}, \quad \forall (u,v,w) \in B$$

that is a smooth change of variables from cartesian coordinates (x,y,z) to a new coordinate system (u,v,w).

Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^3$ be a scalar field and let

$$S = \{(g_1(u,v,w), g_2(u,v,w), g_3(u,v,w)) \mid (u,v,w) \in S_0\}$$

be the domain of integration, with $S_0 \subseteq B$. Then, it follows that

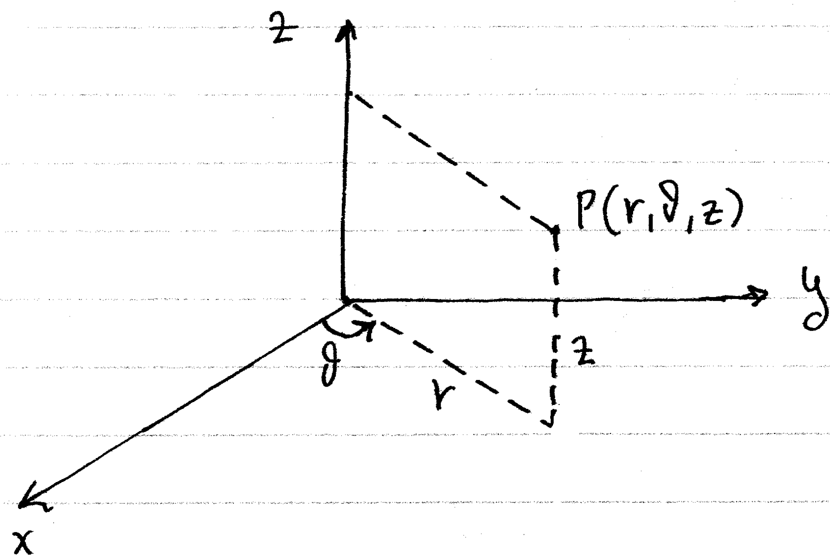
$$\boxed{\begin{aligned} &\iiint_S f(x,y,z)\,dx\,dy\,dz = \\ &= \iiint_{S_0} f(g_1(u,v,w), g_2(u,v,w), g_3(u,v,w)) \left| \frac{\partial(g_1,g_2,g_3)}{\partial(u,v,w)} \right| du\,dv\,dw \end{aligned}}$$

The corresponding transformation of the differential reads:

$$\boxed{dx\,dy\,dz = \left| \frac{\partial(g_1,g_2,g_3)}{\partial(u,v,w)} \right| du\,dv\,dw}$$

(1) → Cylindrical coordinates

$$\boxed{\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \quad , \forall (r,\vartheta,z) \in [0,+\infty) \times [0,2\pi] \times \mathbb{R} \\ z = z \end{cases}}$$



▸ $$\boxed{dx\,dy\,dz = r\,dr\,d\vartheta\,dz}$$

▸ Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^3$ and consider the domain of integration

$S = \{(r\cos\vartheta, r\sin\vartheta, z) \mid (r,\vartheta,z) \in S_0\}$

Then:

$$I = \iiint_S f(x,y,z)\,dx\,dy\,dz =$$

$$= \iiint_{S_0} f(r\cos\vartheta, r\sin\vartheta, z)\, r\,dr\,d\vartheta\,dz$$

Proof

$x = r\cos\vartheta \land y = r\sin\vartheta \land z = z \Rightarrow$

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(r,\vartheta,z)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\vartheta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial\vartheta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial\vartheta & \partial z/\partial z \end{vmatrix} =$$

$$= \left|\begin{matrix} \cos\vartheta & -r\sin\vartheta & 0 \\ \sin\vartheta & r\cos\vartheta & 0 \\ 0 & 0 & 1 \end{matrix}\right| \begin{matrix} \cos\vartheta & -r\sin\vartheta \\ \sin\vartheta & r\cos\vartheta \\ 0 & 0 \end{matrix} =$$

$$= (\cos\vartheta)(r\cos\vartheta)1 + 0 + 0 - 0 - 0 - 1(\sin\vartheta)(-r\sin\vartheta)$$

$$= r\cos^2\vartheta + r\sin^2\vartheta = r(\cos^2\vartheta + \sin^2\vartheta) = r \Rightarrow$$

$$\Rightarrow dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(r,\vartheta,z)}\right| dr\,d\vartheta\,dz =$$

$$= |r|\,dr\,d\vartheta\,dz = r\,dr\,d\vartheta\,dz \Rightarrow$$

$$\Rightarrow I = \iiint_S f(x,y,z)\,dx\,dy\,dz =$$

$$= \iiint_{S_0} f(r\cos\vartheta, r\sin\vartheta, z)\,r\,dr\,d\vartheta\,dz.$$

EXAMPLE

Evaluate the integral $I = \iiint_A (x^2+y^2)\,dx\,dy\,dz$

with A given by:

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [-2,2] \wedge y \in [-\sqrt{4-x^2}, \sqrt{4-x^2}] \wedge$$
$$\wedge\, z \in [\sqrt{x^2+y^2}, 2]\}$$

by converting to cylindrical coordinates.

Solution

- First we determine the domain B of the cylindrical integral.

Let $x = r\cos\theta \wedge y = r\sin\theta \wedge z = z$. Then $x^2+y^2 = r^2$.

We note that

$$y \in [-\sqrt{4-x^2}, \sqrt{4-x^2}] \Leftrightarrow -\sqrt{4-x^2} \leqslant y \leqslant \sqrt{4-x^2} \Leftrightarrow$$
$$\Leftrightarrow y^2 \leqslant (\sqrt{4-x^2})^2 \Leftrightarrow y^2 \leqslant 4-x^2 \Leftrightarrow x^2+y^2 \leqslant 4 \Leftrightarrow 0 \leqslant r^2 \leqslant 4$$
$$\Leftrightarrow 0 \leqslant r \leqslant 2. \text{ (since } x^2+y^2 \geqslant 0, \forall x,y \in \mathbb{R}).$$

and

$$z \in [\sqrt{x^2+y^2}, 2] \Leftrightarrow \sqrt{x^2+y^2} \leqslant z \leqslant 2$$
$$\Leftrightarrow r \leqslant z \leqslant 2$$

The condition $x \in [-2,2]$ is implied by the condition on y via the domain restriction $4-x^2 \geqslant 0$, so it is redundant and does not introduce further restrictions.

Indeed: $|x| = |r\cos\theta| = |r||\cos\theta| \leqslant |r| = r \leqslant 2 \Rightarrow$

$$\Rightarrow |x| \leqslant 2 \Rightarrow x \in [-2,2].$$

Since there are no restrictions on the angle ϑ, we have $\vartheta \in [0, 2\pi]$.

It follows that

$A = \{ (r\cos\vartheta, r\sin\vartheta, z) \mid \vartheta \in [0, 2\pi] \wedge r \in [0, 2] \wedge z \in [r, 2] \}$

Define:

$B = \{ (r, \vartheta, z) \mid \vartheta \in [0, 2\pi] \wedge r \in [0, 2] \wedge z \in [r, 2] \}$

• 2 Now we change variables and evaluate the integral.

$$I = \iiint_A (x^2 + y^2)\, dx\, dy\, dz = \iiint_B r^2\, r\, dr\, d\vartheta\, dz =$$

$$= \iiint_B r^3\, dr\, d\vartheta\, dz = \int_0^{2\pi} d\vartheta \int_0^2 dr \int_r^2 dz\; r^3 =$$

$$= \int_0^{2\pi} d\vartheta \int_0^2 dr\; r^3 \left[\int_r^2 dz \right] = \int_0^{2\pi} d\vartheta \int_0^2 dr\; r^3 (2 - r) =$$

$$= \int_0^{2\pi} d\vartheta \int_0^2 dr\; (2r^3 - r^4) = \int_0^{2\pi} d\vartheta \left[\frac{2r^4}{4} - \frac{r^5}{5} \right]_0^2 =$$

$$= \left[\frac{r^4}{2} - \frac{r^5}{5} \right]_0^2 \int_0^{2\pi} d\vartheta = 2\pi \left[\frac{2^4 - 0^4}{2} - \frac{2^5 - 0}{5} \right]$$

$$= 2\pi \left[8 - \frac{32}{5} \right] = 2\pi \cdot \frac{40 - 32}{5} = 2\pi \cdot \frac{8}{5} = \frac{16\pi}{5}$$

EXERCISES

(9) Evaluate the following integrals using change of variables to cylindrical coordinates

a) $I = \iiint_A (x^2+y^2)\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq a^2 \wedge 0 \leq z \leq b\}$
and $a, b \in (0,+\infty)$.

b) $I = \iiint_A xz\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq a^2 \wedge x \geq 0 \wedge 0 \leq z \leq b\}$
and $a, b \in (0,+\infty)$

c) $I = \iiint_A xy\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid a^2 \leq x^2+y^2 \leq b^2 \wedge x \geq 0 \wedge y \geq 0 \wedge 0 \leq z \leq b\}$
and $a, b \in (0,+\infty)$ with $a \leq b$.

d) $I = \iiint_A z(x+y)^2\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq a^2 \wedge x \geq y \geq 0 \wedge 0 \leq z \leq b\}$
and $a, b \in (0,+\infty)$

e) $I = \iiint_A z\sqrt{x^2+y^2}\, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq z \leq a^2-x^2-y^2\}$
and $a \in (0,+\infty)$

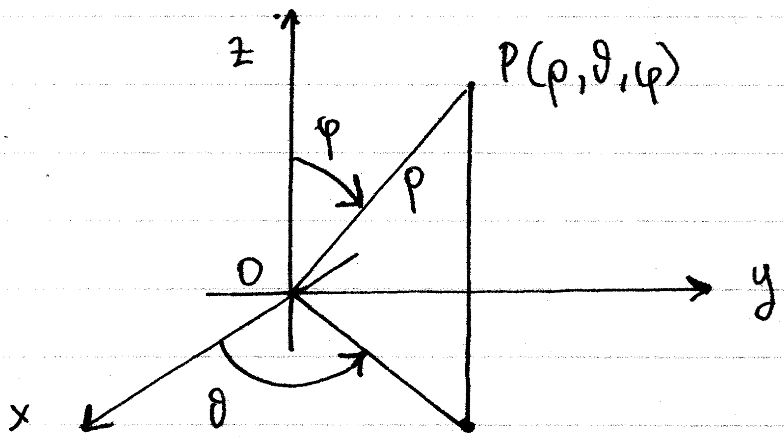
f) $I = \iiint_A z\, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq z \leq a\}$
and $a \in (0,+\infty)$.

g) $I = \iiint_A (x-y)^2\, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq z \leq x^2+y^2 \leq a^2\}$
and $a \in (0,+\infty)$.

2 → Spherical coordinates



ρ = distance from O to P

ϑ = angle from x-axis to projection of OP onto the xy plane

φ = angle from z-axis to OP.

$$\boxed{\begin{cases} x = \rho \sin\varphi \cos\vartheta \\ y = \rho \sin\varphi \sin\vartheta \quad , \forall (\rho, \vartheta, \varphi) \in [0, +\infty) \times [0, 2\pi] \times [0, \pi] \\ z = \rho \cos\varphi \end{cases}}$$

Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^3$ and let

$S = \{ (\rho \sin\varphi \cos\vartheta, \rho \sin\varphi \sin\vartheta, \rho \cos\varphi) \mid (\rho, \vartheta, \varphi) \in S_0 \}$

Then:

$$I = \iiint_S f(x, y, z)\, dx\, dy\, dz =$$

$$= \iiint_{S_0} f(\rho \sin\varphi \cos\vartheta, \rho \sin\varphi \sin\vartheta, \rho \cos\varphi)\, \rho^2 \sin\varphi \, d\rho\, d\vartheta\, d\varphi$$

The formal transformation for the differential is given by:

$$\boxed{dx\,dy\,dz = \rho^2 \sin\varphi \, d\rho\, d\vartheta\, d\varphi}$$

Proof

$$\begin{cases} x = \rho \sin\varphi \cos\vartheta \\ y = \rho \sin\varphi \sin\vartheta \\ z = \rho\cos\varphi \end{cases} \Rightarrow$$

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(\rho,\vartheta,\varphi)} = \begin{vmatrix} \partial x/\partial\rho & \partial x/\partial\vartheta & \partial x/\partial\varphi \\ \partial y/\partial\rho & \partial y/\partial\vartheta & \partial y/\partial\varphi \\ \partial z/\partial\rho & \partial z/\partial\vartheta & \partial z/\partial\varphi \end{vmatrix} =$$

$$= \begin{vmatrix} \sin\varphi\cos\vartheta & -\rho\sin\varphi\sin\vartheta & \rho\cos\varphi\cos\vartheta \\ \sin\varphi\sin\vartheta & \rho\sin\varphi\cos\vartheta & \rho\cos\varphi\sin\vartheta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix}$$

$$= (\sin\varphi\cos\vartheta)(\rho\sin\varphi\cos\vartheta)(-\rho\sin\varphi) +$$
$$+ (-\rho\sin\varphi\sin\vartheta)(\rho\cos\varphi\sin\vartheta)\cos\varphi + 0$$
$$- (\cos\varphi)(\rho\sin\varphi\cos\vartheta)(\rho\cos\varphi\cos\vartheta) - 0$$

$$- (-\rho\sin\varphi)(\sin\varphi\sin\vartheta)(-\rho\sin\varphi\sin\vartheta)$$

$$= -\rho^2\sin^3\varphi\cos^2\vartheta - \rho^2\sin\varphi\cos^2\varphi\sin^2\vartheta$$
$$-\rho^2\sin\varphi\cos^2\varphi\cos^2\vartheta - \rho^2\sin^3\varphi\sin^2\vartheta =$$

$$= -\rho^2 \sin\varphi \left(\sin^2\varphi \cos^2\vartheta + \cos^2\varphi \sin^2\vartheta + \cos^2\varphi \cos^2\vartheta + \sin^2\varphi \sin^2\vartheta \right) =$$

$$= -\rho^2 \sin\varphi \left[\sin^2\varphi \left(\cos^2\vartheta + \sin^2\vartheta \right) + \cos^2\varphi \left(\cos^2\vartheta + \sin^2\vartheta \right) \right]$$

$$= -\rho^2 \sin\varphi \left[\sin^2\varphi + \cos^2\varphi \right] =$$

$$= -\rho^2 \sin\varphi \Rightarrow$$

$$\Rightarrow dx\,dy\,dz = \left| \frac{\partial(x,y,z)}{\partial(\rho,\vartheta,\varphi)} \right| d\rho\, d\vartheta\, d\varphi =$$

$$= \left| -\rho^2 \sin\varphi \right| d\rho\, d\vartheta\, d\varphi =$$

$$= \rho^2 \left| \sin\varphi \right| d\rho\, d\vartheta\, d\varphi =$$

$$= \rho^2 \sin\varphi\, d\rho\, d\vartheta\, d\varphi . \Rightarrow$$

$$\Rightarrow I = \iiint_S f(x,y,z)\, dx\,dy\,dz =$$

$$= \iiint_{S_0} f(\rho \sin\varphi \cos\vartheta, \rho \sin\varphi \cos\vartheta, \rho \cos\varphi) \rho^2 \sin\varphi\, d\rho\, d\vartheta\, d\varphi \quad \square$$

Note that $\varphi \in [0, \pi] \Rightarrow \sin\varphi \geq 0 \Rightarrow |\sin\varphi| = \sin\varphi$.

EXAMPLE

Use spherical coordinates to evaluate the integral

$$I = \iiint_A \exp\left((x^2+y^2+z^2)^{3/2}\right) dx\,dy\,dz$$

over the region $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\}$

Solution

Let

$$\begin{cases} x = \rho \sin\varphi \cos\vartheta \\ y = \rho \sin\varphi \sin\vartheta \\ z = \rho \cos\varphi \end{cases}$$

with $\rho \in [0, +\infty)$ and $\vartheta \in [0, 2\pi)$ and $\varphi \in [0, \pi]$.

Then $dx\,dy\,dz = \rho^2 \sin\varphi \, d\rho\, d\vartheta\, d\varphi$

and we note that

$x^2+y^2+z^2 \leq 1 \Leftrightarrow \rho^2 \leq 1 \Leftrightarrow 0 \leq \rho \leq 1 \Leftrightarrow \rho \in [0,1]$

There are no constraints on ϑ and φ. It follows that

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\} =$$
$$= \{(\rho \sin\varphi \cos\vartheta, \rho \sin\varphi \sin\vartheta, \rho\cos\varphi) \mid (\rho,\vartheta,\varphi) \in B\}$$

with

$$B = \{(\rho,\vartheta,\varphi) \mid \rho \in [0,1] \land \vartheta \in [0,2\pi) \land \varphi \in [0,\pi]\} =$$
$$= [0,1] \times [0,2\pi) \times [0,\pi]$$

and therefore:

$$I = \iiint_A \exp\left((x^2+y^2+z^2)^{3/2}\right) dx\,dy\,dz =$$

$$= \iiint_B \exp\left((\rho^2)^{3/2}\right) \rho^2 \sin\varphi \, d\rho\, d\vartheta\, d\varphi =$$

$$= \iiint_B \rho^2 e^{\rho^3} \sin\varphi \, d\rho\, d\vartheta\, d\varphi =$$

$$= \int_0^1 d\rho \int_0^{2\pi} d\vartheta \int_0^{\pi} d\varphi \; \rho^2 e^{\rho^3} \sin\varphi =$$

$$= \left[\int_0^1 d\rho\, \rho^2 e^{\rho^3}\right]\left[\int_0^{\pi} d\varphi \sin\varphi\right]\left[\int_0^{2\pi} d\vartheta\right] = 2\pi I_1 I_2$$

with

$$I_1 = \int_0^1 d\rho\, \rho^2 \exp(\rho^3) = \frac{1}{3}\int_0^1 3\rho^2 \exp(\rho^3)\, d\rho$$

$$= \frac{1}{3}\int_0^1 \left[\exp(\rho^3)\right]' d\rho = \frac{1}{3}\left[\exp(\rho^3)\right]_0^1 =$$

$$= \frac{e^1 - e^0}{3} = \frac{e-1}{3}$$

and

$$I_2 = \int_0^{\pi} \sin\varphi\, d\varphi = \left[-\cos\varphi\right]_0^{\pi} = -\left[\cos\varphi\right]_0^{\pi} =$$

$$= -\left[\cos\pi - \cos 0\right] = -\left[-\cos 0 - \cos 0\right] =$$

$$= -(-1-1) = 2$$

It follows that

$$I = 2\pi I_1 I_2 = 2\pi \frac{e-1}{3} \cdot 2 = \frac{4\pi(e-1)}{3}$$

EXERCISES

(10) Evaluate the following triple integrals using change of variables to spherical coordinates.

a) $I = \iiint_A y\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0 \wedge y \geq 0 \wedge z \geq 0 \wedge x^2+y^2+z^2 \leq a^2\}$ and $a \in (0,+\infty)$

b) $I = \iiint_A \dfrac{dx\,dy\,dz}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid a^2 \leq x^2+y^2+z^2 \leq b^2\}$ and $a, b \in (0,+\infty)$ with $a < b$.

c) $I = \iiint_A (x^2+y^2)\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq a^2\}$ and $a \in (0,+\infty)$.

d) $I = \iiint_A \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 2z\}$

e) $I = \iiint_A \frac{z\,dx\,dy\,dz}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 4a^2 \wedge z \geq a\}$

and $a \in (0,+\infty)$

f) $I = \iiint_A \exp\left(\frac{1}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}\right) dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid -a \leq x \leq a \wedge$

$\wedge -\sqrt{a^2-x^2} \leq y \leq \sqrt{a^2-x^2} \wedge$

$\wedge\, 0 \leq z \leq \sqrt{a^2-x^2-y^2}\,\}$

and $a \in (0,+\infty)$.

CAL3.6: Vector fields

VECTOR FIELDS

Definition

- A <u>three-dimensional vector field</u> f is a mapping $f: A \longrightarrow \mathbb{R}^3$ with $A \subseteq \mathbb{R}^3$.
- If f is a vector field, we write:
  $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$
- The scalar fields f_1, f_2, f_3 are the <u>components</u> of the vector field f.

Derivatives of a vector field

- Let $f: A \to \mathbb{R}^3$ be a vector field with components f_1, f_2, f_3 that are assumed to be partially differentiable. Then we define:

a) The divergence of f

$$\boxed{\nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}}$$

b) The curl of f

$$\boxed{\nabla \times f = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}}$$

- Note that
  $\nabla \cdot f$ is a scalar field
  $\nabla \times f$ is a vector field.
- Let $\varphi : A \to \mathbb{R}$ be a scalar field with $A \subseteq \mathbb{R}^3$.
  Then we define:

a) The gradient of φ

$$\boxed{\nabla \varphi = (\partial \varphi / \partial x, \partial \varphi / \partial y, \partial \varphi / \partial z)}$$

b) The Laplacian of φ

$$\boxed{\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}}$$

Tensor notation

Tensor notation makes it easier to work with the derivatives of vector fields. We follow the following guidelines:

a) We write the vector field "f" as "f_a", representing the "a" component of f. With tensor notation we always work in terms of the components of the involved vector fields.

b) Repeating indices are automatically summed over all components, when associated with a product

<u>example:</u> $f_a g_a = f_1 g_1 + f_2 g_2 + f_3 g_3 = f \cdot g$
represents the dot product
However, for the vector sum
$(f+g)_a = f_a + g_a$
no summation is implied.

c) We abbreviate the partial derivative $\partial / \partial x_a$ as ∂_a.
We may thus write:
$$\nabla \cdot f = \partial_a f_a \; ; \quad (\nabla \varphi)_a = \partial_a \varphi \; ; \quad \nabla^2 \varphi = \partial_a \partial_a \varphi$$

d) To define the curl, we introduce the Levi-Civita tensor ε_{abc} as:

$$\boxed{\begin{aligned} \varepsilon_{abc} &= \frac{(a-b)(b-c)(c-a)}{2} = \\ &= \begin{cases} +1, & \text{if } (a,b,c) \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1, & \text{if } (a,b,c) \in \{(3,2,1),(1,3,2),(2,1,3)\} \\ 0, & \text{if } a=b \vee b=c \vee c=a \end{cases} \end{aligned}}$$

Then, the curl reads:
$$(\nabla \times f)_a = \varepsilon_{abc} \partial_b f_c$$
Likewise, for two vector fields f, g, the cross-product reads:
$$(f \times g)_a = \varepsilon_{abc} f_b g_c$$

- To summarize: Given the vector fields f, g and the scalar field φ:

$(f+g)_a = f_a + g_a$ $f \cdot g = f_a g_a$ $f \times g = \varepsilon_{abc} f_b g_c$	$\nabla\varphi = \partial_a \varphi$ $\nabla^2 \varphi = \partial_a \partial_a \varphi$	$\nabla \cdot f = \partial_a f_a$ $\nabla \times f = \varepsilon_{abc} \partial_b f_c$

▶ Kronecker delta

We define:

$$\delta_{ab} = \begin{cases} 1 \ , \text{ if } a = b \\ 0 \ , \text{ if } a \neq b \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that $\delta_{ab} f_b = f_a$

(because in the implied summation $b = 1, 2, 3$ the only non-zero contribution occurs when $a = b$).

Similar contractions are possible. For example:

$\delta_{ab} \varepsilon_{acd} = \varepsilon_{bcd}$

(because in the implied summation $a = 1, 2, 3$ the only non-zero contribution occurs when $a = b$).

▶ Properties of Levi-Civita tensor

We can show that ε_{abc} satisfy the following properties:

1) $$\boxed{\begin{aligned} \varepsilon_{abc} f_b f_c &= \mathbf{0}_a \\ \varepsilon_{abc} \partial_b \partial_c &= \mathbf{0}_a \end{aligned}}$$ ← $\mathbf{0}_a$ is the "a" component of the field $\mathbf{0} = (0,0,0)$.

2) $$\boxed{\begin{aligned} \varepsilon_{abc} &= \varepsilon_{bca} = \varepsilon_{cab} \\ \varepsilon_{abc} &= -\varepsilon_{cba} \end{aligned}}$$

3) Relation to the Kronecker delta

$$\boxed{\varepsilon_{abc}\, \varepsilon_{pqr} = \begin{vmatrix} \delta_{ap} & \delta_{aq} & \delta_{ar} \\ \delta_{bp} & \delta_{bq} & \delta_{br} \\ \delta_{cp} & \delta_{cq} & \delta_{cr} \end{vmatrix}}$$

4) Contracted epsilon identities

$$\boxed{\begin{aligned} \varepsilon_{abc}\, \varepsilon_{apq} &= \delta_{bp}\delta_{cq} - \delta_{bq}\delta_{cp} \\ \varepsilon_{abp}\, \varepsilon_{abq} &= 2\delta_{pq} \\ \varepsilon_{abc}\, \varepsilon_{abc} &= 6 \end{aligned}}$$

Note that (4) is a consequence of (3). All properties of the curl and cross-product are encapsulated in properties (1), (2), (3).

Second derivatives

Tensor notation can be used to establish the following properties about 2nd derivatives:

1) $\boxed{\nabla \times (\nabla \varphi) = \mathbf{0}}$

Proof

$$[\nabla \times (\nabla\varphi)]_a = \varepsilon_{abc}\,\partial_b\,(\nabla\varphi)_c = \varepsilon_{abc}\,\partial_b\,(\partial_c\,\varphi) =$$
$$= (\varepsilon_{abc}\,\partial_b\partial_c)\,\varphi = \mathbf{0}_a\,\varphi = \mathbf{0}_a \Rightarrow$$
$$\Rightarrow \nabla \times \nabla\varphi = \mathbf{0}. \qquad \square$$

2) $\boxed{\nabla \cdot (\nabla \times f) = 0}$

Proof

$$\nabla \cdot (\nabla \times f) = \partial_a (\nabla \times f)_a = \partial_a(\varepsilon_{abc}\,\partial_b f_c) =$$
$$= \varepsilon_{abc}\,\partial_a\partial_b f_c = (\varepsilon_{cab}\,\partial_a\partial_b) f_c$$
$$= \mathbf{0}_c f_c = 0$$

3) $\boxed{\nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \nabla^2 f}$

Proof

$$[\nabla\times(\nabla\times f)]_a = \varepsilon_{abc}\,\partial_b(\nabla\times f)_c = \varepsilon_{abc}\partial_b(\varepsilon_{cpq}\partial_p f_q)$$
$$= \varepsilon_{abc}\varepsilon_{cpq}\partial_b\partial_p f_q = \varepsilon_{cab}\,\varepsilon_{cpq}\,\partial_b\partial_p f_q =$$
$$= (\delta_{ap}\delta_{bq} - \delta_{aq}\delta_{bp})\,\partial_b\partial_p f_q =$$
$$= \delta_{ap}\,\delta_{bq}\,\partial_b\,\partial_p f_q - \delta_{aq}\,\delta_{bp}\,\partial_b\partial_p f_q =$$
$$= \delta_{ap}\,\partial_q\,\partial_p f_q - \delta_{aq}\,\partial_p\partial_p f_q =$$
$$= \partial_q\partial_a f_q - \partial_p\partial_p f_a = \partial_a(\partial_q f_q) - \partial_p\partial_p f_a =$$
$$= \partial_a(\nabla\cdot f) - \nabla^2 f_a = [\nabla(\nabla\cdot f)]_a - \nabla^2 f_a \Rightarrow$$
$$\Rightarrow \nabla\times(\nabla\times f) = \nabla(\nabla\cdot f) - \nabla^2 f. \qquad \square$$

Note that δ_{ab} and ε_{abc} are constants so they freely commute with the operator ∂_a. Be careful, however. Sometimes it is necessary to employ the product rule. For example:

EXAMPLE

Show that: $\nabla\cdot(\varphi f) = f\cdot\nabla\varphi + \varphi\nabla\cdot f$ with φ scalar field and f vector field.

Solution

$$\nabla\cdot(\varphi f) = \partial_a(\varphi f)_a = \partial_a(\varphi f_a) = f_a\partial_a\varphi + \varphi\,\partial_a f_a$$
$$= (f\cdot\nabla)\varphi + \varphi(\nabla\cdot f) =$$
$$= f\cdot\nabla\varphi + \varphi(\nabla\cdot f).$$

EXERCISES

① Evaluate $\nabla \cdot F$ and $\nabla \times F$ for the following vector fields:

a) $F(x,y,z) = (x^3, 2xy, yz^3)$

b) $F(x,y,z) = (x^a, y^a, z^a)$ with $a \in \mathbb{R}$

c) $F(x,y,z) = (yz, zx, xy)$

d) $F(x,y,z) = (\cos(ax), \sin(by), c)$ with $a,b,c \in \mathbb{R}$

e) $F(x,y,z) = (e^x \cos(ay), e^x \sin(by), z)$

f) $F(x,y,z) = (x+y, y+z, z+x)$

② Evaluate $\nabla^2 f$ for the following scalar fields

a) $f(x,y,z) = x^3+y^3+z^3-3xyz$

b) $f(x,y,z) = \cos(xyz)$

c) $f(x,y,z) = \ln|xyz|$

d) $f(x,y,z) = x^a+y^a+z^a$

e) $f(x,y,z) = xe^y \sin(z)$

f) $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

③ Use tensor notation to show the following identities. Here F, G are vector fields and φ, ψ are scalar fields

a) $\nabla \cdot (F+G) = \nabla \cdot F + \nabla \cdot G$

b) $\nabla \times (F+G) = \nabla \times F + \nabla \times G$

c) $\nabla \times (\nabla\varphi) = 0$

d) $\nabla \cdot (\nabla \times F) = 0$

e) $\nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - \nabla^2 F$

f) $\nabla \cdot (\varphi F) = \varphi(\nabla \cdot F) + F \cdot (\nabla\varphi)$

g) $\nabla\|F\|^2 = 2[(F\cdot\nabla)F + F\times(\nabla\times F)]$

h) $\nabla(F\cdot G) = (F\cdot\nabla)G + (G\cdot\nabla)F + F\times(\nabla\times G) + G\times(\nabla\times F)$

i) $\nabla\cdot(F\times G) = (\nabla\times F)\cdot G - F\cdot(\nabla\times G)$

j) $\nabla\times(F\times G) = F(\nabla\cdot G) - G(\nabla\cdot F) - (F\cdot\nabla)G + (G\cdot\nabla)F$

k) $\nabla\cdot(\varphi\nabla\psi) = \varphi\nabla^2\psi + (\nabla\varphi)\cdot(\nabla\psi)$

l) $\psi\nabla^2\varphi - \varphi\nabla^2\psi = \nabla\cdot(\psi\nabla\varphi - \varphi\nabla\psi)$

m) $\nabla^2(\varphi\psi) = \varphi\nabla^2\psi + 2(\nabla\varphi)\cdot(\nabla\psi) + \psi\nabla^2\varphi$

n) $\nabla^2(\varphi F) = F\nabla^2\varphi + 2(\nabla\varphi\cdot\nabla)F + \varphi\nabla^2 F$

o) $\nabla^2(F\cdot G) = F\cdot(\nabla^2 G) - G\cdot(\nabla^2 F) + 2\nabla\cdot[(G\cdot\nabla)F + G\times(\nabla\times F)]$

p) $\nabla^2(\nabla\varphi) = \nabla(\nabla^2\varphi)$

r) $\nabla^2(\nabla\times F) = \nabla\times(\nabla^2 F)$.

4) Let $x = (x_1, x_2, x_3)$ be a vector.

(a) Explain why $\partial_a x_b = \delta_{ab}$ and $\partial_a\|x\| = x_a/\|x\|$.

(b) If f is differentiable scalar function, show that

$$\partial_a f(\|x\|) = f'(\|x\|)\, x_a/\|x\|$$

(c) Use the previous results to show that

$$\partial_a\left[\frac{x_a}{\|x\|} f(\|x\|)\right] = \frac{2f(\|x\|)}{\|x\|} + f'(\|x\|)$$

(5) Consider the vector field $F(x) = x/\|x\|^m$ with $x = (x_1, x_2, x_3)$ a vector. Show that for all $x \in \mathbb{R}^3 - \{0\}$ we have:

a) $m = 3 \Rightarrow \nabla \cdot F = 0 \wedge \nabla \times F = 0$

b) $m \neq 3 \Rightarrow \nabla \cdot F \neq 0 \wedge \nabla \times F = 0$

(6) The Navier-Stokes equations govern the velocity field $u(x_1, x_2, x_3, t)$ of an incompressible fluid, and written using tensor notation, they read

$$\begin{cases} \partial u_a/\partial t + u_b \partial_b u_a = -\partial_a p + \nu \nabla^2 u_a \\ \partial_a u_a = 0 \leftarrow \text{(incompressibility condition)} \end{cases}$$

with $\nu > 0$ the fluid viscosity and p the pressure field. Show, using tensor notation, that

a) $u_b \partial_b u_a = \partial_b (u_a u_b)$

b) $\nabla^2 p = -\partial_a \partial_b (u_a u_b)$

c) The vorticity field defined as $w = \nabla \times u$ satisfies the equation

$$\frac{\partial w_a}{\partial t} + u_b \partial_b w_a = w_b \partial_b u_a + \nu \nabla^2 w_a$$

▼ Line Integrals

- Line integrals are integrals of vector fields defined over a path.

↳ Definitions concerning paths

- Let $a : I \to \mathbb{R}^n$ be a vector-valued function. We say that

  $$a \text{ is a } \underline{\text{path}} \Leftrightarrow \begin{cases} \exists t_1, t_2 \in \mathbb{R} : I = [t_1, t_2] \\ a \text{ continuous at } [t_1, t_2]. \end{cases}$$

  Furthermore, if a is a path, we say that

  a is a <u>closed path</u> $\Leftrightarrow a(t_1) = a(t_2)$

  a is an <u>open path</u> $\Leftrightarrow a(t_1) \neq a(t_2)$.

  We say that: $a(t_1) =$ <u>initial point</u>

  $a(t_2) =$ <u>final point</u>

- Consider a path $a : [t_1, t_2] \longrightarrow \mathbb{R}^n$. We say that

  $$a \ \underline{\text{smooth}} \text{ path} \Leftrightarrow \begin{cases} a \text{ continuous at } [t_1, t_2] \\ a \text{ differentiable at } (t_1, t_2) \\ \dot{a} \text{ continuous at } (t_1, t_2). \end{cases}$$

- Let $a : [t_1, t_2] \longrightarrow \mathbb{R}^n$ be a path, and let $B \subseteq [t_1, t_2]$. The <u>restriction</u> of "a" to B is denoted as:

  $$b = a \restriction B \Leftrightarrow \begin{cases} b : B \longrightarrow \mathbb{R}^n \\ \forall t \in B : b(t) = a(t) \end{cases}$$

• Let $a : [t_1, t_2] \to \mathbb{R}^n$ be a path. We say that

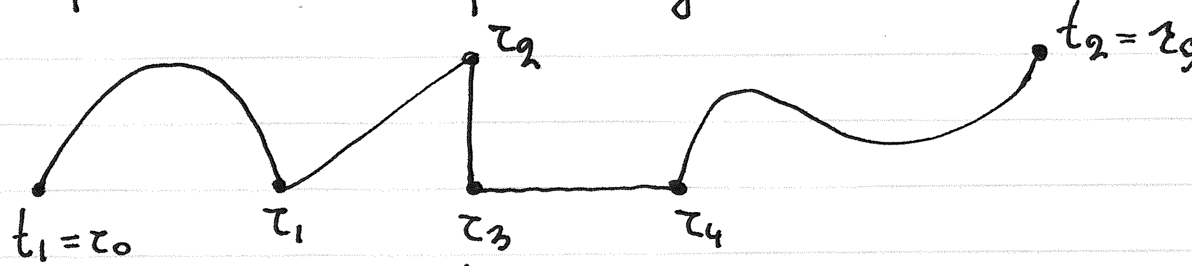
a piecewise smooth path $\Leftrightarrow$

$\Leftrightarrow \exists \tau_0, \tau_1, \ldots, \tau_n \in [t_1, t_2] :$

$$: \begin{cases} t_1 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n = t_2 \\ \forall k \in [n] : a \restriction [\tau_{k-1}, \tau_k] \text{ smooth path} \end{cases}$$

EXAMPLE

A piecewise smooth path may look like this:



▸ Connected sets

• Let $x, y \in \mathbb{R}^n$ be given points. We define:

$\mathcal{P}_A(x, y)$ = the set of all piecewise smooth paths in A with initial point x and final point y.

• Let $A \subseteq \mathbb{R}^n$ be a given region. We say that

$$\boxed{\begin{array}{l} A \text{ path-connected} \Leftrightarrow \forall x, y \in A : \mathcal{P}_A(x, y) \neq \emptyset \\ A \text{ path-disconnected} \Leftrightarrow \exists x, y \in A : \mathcal{P}_A(x, y) = \emptyset \end{array}}$$

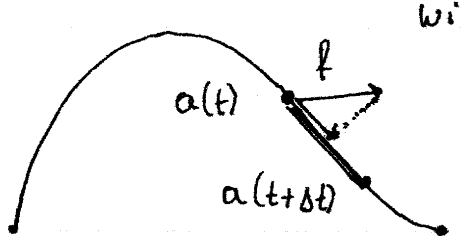
▸ interpretation: In a path-connected set A, every two points $x, y \in A$ are connected by at least one piecewise smooth path.

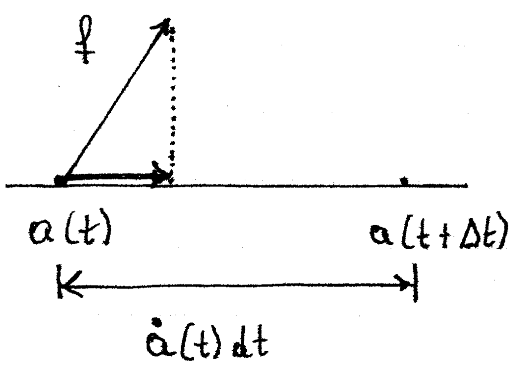
Definition of line integral

- Let $a : [t_1, t_2] \to \mathbb{R}^n$ be a piecewise smooth path and let $f : A \to \mathbb{R}^n$ be a vector field with $A \subseteq \mathbb{R}^n$ such that $a([t_1, t_2]) \subseteq A$. We define the line integral

$$\boxed{\int f \cdot da = \int_{t_1}^{t_2} f(a(t)) \cdot \dot{a}(t)\, dt}$$

▶ <u>Interpretation</u> : Consider two points $a(t)$ and $a(t+dt)$ with $dt \to 0$. Then $\dot{a}(t)dt$ represents the distance between the points $a(t)$ and $a(t+\Delta t)$. The dot product $f(a(t)) \cdot \dot{a}(t)dt$ gives the product of this point distance with the directed magnitude of the projection of the vector f along the direction defined by the points $a(t)$ and $a(t+dt)$. Consequently, only the tangential component of f contributes to the integral. The normal component of f does not contribute at all.





▶ Notation

a) When we write a line integral in the indefinite form as

$$I = \int f \cdot da$$

then the differential da points to the path $a : [t_1, t_2] \longrightarrow \mathbb{R}^n$ over which f is integrated.

b) Alternatively, if we define the path as:

$$C : a(t) = (a_1(t), \ldots, a_n(t)) \ , \ t \in [t_1, t_2]$$

with C representing the path, we may write the line integral in definite form as:

$$I = \int f \cdot da = \int_C f \cdot dl$$

Here, the l in dl is just a dummy variable that doesn't really mean anything. It is standard to use the letter "l" as the dummy variable in line integrals when written in the definite form.

c) Consider a vector field f with components:

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

and the path $a : [t_1, t_2] \longrightarrow \mathbb{R}^n$ with components:

$\dot{a}(t) = (a_1(t), a_2(t), \dots, a_n(t))$, $\forall t \in [t_1, t_2]$

The line integral can be written in indefinite component form as:

$$I = \int f \cdot da = \int f_1 da_1 + f_2 da_2 + \dots + f_n da_n$$

and in definite component form as:

$$I = \int_C f \cdot dl = \int_C f_1 dx_1 + f_2 dx_2 + \dots + f_n dx_n$$

d) For a two-dimensional vector field
$f(x,y) = (f_1(x,y), f_2(x,y))$
the component forms of a line integral of f can be written as:

$$I = \int f \cdot da = \int f_1(x,y)\, da_1 + f_2(x,y)\, da_2 =$$

$$= \int_C f_1(x,y)\, dx + f_2(x,y)\, dy$$

e) Similarly, for a three-dimensional vector field
$f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)$
the component forms of a line integral of f can be written as:

$$I = \int f \bullet da =$$

$$= \int f_1(x,y,z)\, da_1 + f_2(x,y,z)\, da_2 + f_3(x,y,z)\, da_3 =$$

$$= \int_C f_1(x,y,z)\, dx + f_2(x,y,z)\, dy + f_3(x,y,z)\, dz.$$

EXAMPLE

Evaluate the integral $I = \int_C (x^2 - y^2)\,dx + 2xy\,dy$ over the curve:

$C : a(t) = (t^2, t^3), \ \forall t \in [0,1].$

Solution

We note that $\dot{a}(t) = (2t, 3t^2), \ \forall t \in [0,1]$.
It follows that

$$I = \int_C (x^2 - y^2)\,dx + 2xy\,dy =$$

$$= \int_0^1 \left((t^2)^2 - (t^3)^2, \ 2(t^2)(t^3) \right) \bullet (2t, 3t^2)\, dt =$$

$$= \int_0^1 (t^4 - t^6, \ 2t^5) \bullet (2t, 3t^2)\, dt =$$

$$= \int_0^1 \left[2t(t^4 - t^6) + (2t^5)(3t^2) \right] dt =$$

$$= \int_0^1 (2t^5 - 2t^7 + 6t^7)\,dt = \int_0^1 (2t^5 + 4t^7)\,dt =$$

$$= \left[\frac{2t^6}{6} + \frac{4t^8}{8} \right]_0^1 = \left[\frac{t^6}{3} + \frac{t^8}{2} \right]_0^1 =$$

$$= \frac{1^6 - 0^6}{3} + \frac{1^8 - 0^8}{2} = \frac{1}{3} + \frac{1}{2} =$$

$$= \frac{2+3}{6} = \frac{5}{6}$$

Basic Properties of line integrals

Linearity

Let f, g be two vector fields, $\lambda_1, \lambda_2 \in \mathbb{R}$, and let C be a path. Then:

$$\boxed{\int_C (\lambda_1 f + \lambda_2 g) \cdot dl = \lambda_1 \int_C f \cdot dl + \lambda_2 \int_C g \cdot dl}$$

Path Equivalence

Consider two paths $a : [a_1, a_2] \longrightarrow \mathbb{R}^n$ and $b : [b_1, b_2] \longrightarrow \mathbb{R}^n$. It is possible for a and b to sketch the same curve but with different velocities. In that case we want to be able to say that a and b are <u>equivalent</u> (notation: $a \equiv b$). Furthermore, we would like equivalent paths over the same function to give equal line integrals. We give a formal definition as follows:

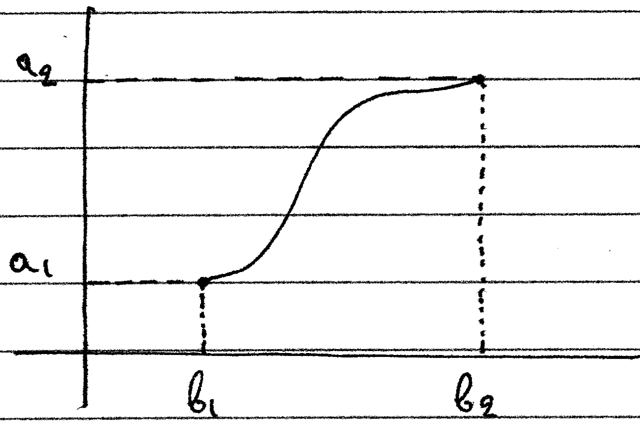
<u>Def</u>: We say that $a \equiv b$ (a is equivalent to b) if and only if there is a function $u : [b_1, b_2] \to [a_1, a_2]$ such that

a) $u([b_1, b_2]) = [a_1, a_2]$

b) u differentiable on $[b_1, b_2]$

c) $\forall t \in [b_1, b_2]: u'(t) > 0$

d) $\forall t \in [b_1, b_2]: b(t) = a(u(t))$



▶ Note that an immediate consequence of the above conditions is that

$$\begin{cases} u(b_1) = a_1 \\ u(b_2) = a_2 \end{cases}$$

<u>Thm</u>: Let $a: [a_1, a_2] \longrightarrow \mathbb{R}^n$ and $b: [b_1, b_2] \longrightarrow \mathbb{R}^n$ be two paths. Then:

$$\boxed{a \equiv b \Rightarrow \int f \cdot da = \int f \cdot db}$$

<u>Proof</u>

Since $a \equiv b$, there is a function $u: [b_1, b_2] \to [a_1, a_2]$ such that $u(b_1) = a_1 \wedge u(b_2) = a_2$ and

$\forall t \in [b_1, b_2]: b(t) = a(u(t)) \Rightarrow$

$\Rightarrow \forall t \in [b_1, b_2]: \dot{b}(t) = \dot{a}(u(t))\, u'(t)$

It follows that:

$$\int f \cdot db = \int_{b_1}^{b_2} f(b(t)) \cdot \dot{b}(t)\, dt =$$

$$= \int_{b_1}^{b_2} f(a(u(t))) \cdot \dot{a}(u(t)) u'(t)\, dt.$$

Let $\tau = u(t) \Rightarrow \begin{cases} d\tau = u'(t)\, dt \\ u(b_1) = a_1 \\ u(b_2) = a_2 \end{cases} \Rightarrow$

$$\Rightarrow \int f \cdot db = \int_{a_1}^{a_2} f(a(\tau)) \cdot \dot{a}(\tau)\, d\tau = \int f \cdot da \qquad \square$$

Path Merging

- Consider 3 path defined as:

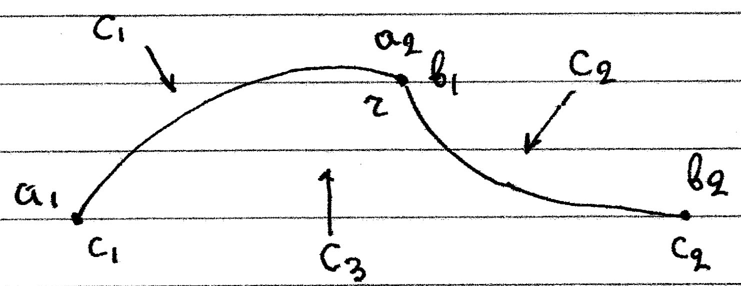
$C_1 : a(t) \quad , \forall t \in [a_1, a_2]$

$C_2 : b(t) \quad , \forall t \in [b_1, b_2]$

$C_3 : c(t) \quad , \forall t \in [c_1, c_2]$

We say that

$$\boxed{C_3 = C_1 \cup C_2 \Leftrightarrow \exists \tau \in [c_1, c_2] : \begin{cases} a \equiv c \restriction [c_1, \tau] \\ b \equiv c \restriction [\tau, c_2] \end{cases}}$$



- Let f be a vector field. Then it can be shown that:

$$\boxed{\int_{C_1 \cup C_2} f \cdot dl = \int_{C_1} f \cdot dl + \int_{C_2} f \cdot dl.}$$

Path Reversal

- Let $C : a(t)$, $\forall t \in [t_1, t_2]$ be a path. We define the reverse path $-C$ as follows:

$$-C : b(t) = a(t_1 + t_2 - t), \ \forall t \in [t_1, t_2]$$

- The reverse path $-C$ traverses the same points in space as C, but in the reverse direction.
- Given a vector field f, we can show that:

$$\boxed{\int_{-C} f \cdot dl = -\int_C f \cdot dl}$$

EXAMPLES

⑦ Evaluate the following line integrals over the given paths

a) $I = \int_{C_1} e^x dx + e^{-y} dy$

with $C_1 : (x,y) = (\ln t, \ln(bt))$, $\forall t \in [1, a+1]$
and $a, b \in (0, +\infty)$.

b) $I = \int_{C_1} (x+y)\,dx + (x-y)\,dy$

with $C_1 : (x,y) = (a\cos t, b\sin t)$, $\forall t \in [0, \pi/2]$
and $a, b \in (0, +\infty)$.

c) $I = \int_{C_1} (x+3y)\,dx + (2y-z)\,dy + (z+x)\,dz$

with $C_1 : (x,y,z) = (0,0,0) + t(a,a,b)$, $\forall t \in [0,1]$
and $a, b \in (0, +\infty)$.

d) $I = \int_{C_1} z\,dx + x^2 dy + y\,dz$

with $C_1 : (x,y,z) = (\cos t, \tan t, t)$, $\forall t \in [0, \pi/4]$

e) $I = \int_{C} \frac{-y\,dx + x\,dy}{x^2+y^2}$

with $C: (x,y) = (1,0) + t(-1,1)$, $\forall t \in [0,1]$

f) $I = \int_{C} y^2 dx + z^2 dy + (1-x^2) dz$

with $C: (x,y,z) = (a\cos t, 1, a\sin t)$, $\forall t \in [0, \pi/4]$
and $a \in (0, +\infty)$

g) $I = \int_{C} \frac{-y\,dx + x\,dy}{(x^2+y^2)^2}$

with $C: (x,y) = (a\cos t, a\sin t)$, $\forall t \in [0, \pi/6]$
and $a \in (0, +\infty)$

h) $I = \int_{C} \frac{(x+y)\,dx - (x-y)\,dy}{x^2+y^2}$

with $C: (x,y) = (a\cos t, a\sin t)$, $\forall t \in [0, 2\pi]$
and $a \in (0, +\infty)$

i) $I = \int_{C} \frac{dx + dy}{|x| + |y|}$

with C the square with vertices $(1,0), (0,1), (-1,0), (0,-1)$
traversed once in the counterclockwise direction.

j) $I = \int_{C} (x^2+y^2)\,dx + (x^2-y^2)\,dy$

with $C: (x,y) = (t, 1-|1-t|)$, $\forall t \in [0,2]$

▼ Conservative fields and potential functions

<u>Thm</u>: Let $\varphi: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field.
Let $x, y \in A$ be given. Assume that

a) A is open and path-connected.
b) φ differentiable in A
c) $\nabla\varphi$ continuous in A

Then:
$$\boxed{\forall C \in \mathcal{P}_A(x,y): \int_C \nabla\varphi \cdot d\ell = \varphi(y) - \varphi(x)}$$

<u>Proof</u>

Let $C \in \mathcal{P}_A(x,y)$ with $C: a(t), \forall t \in [a_1, a_2]$ be given such that $a(a_1) = x$ and $a(a_2) = y$. Define:
$$\forall t \in [a_1, a_2]: g(t) = \varphi(a(t))$$
Then: $\forall t \in [a_1, a_2]: g'(t) = \nabla\varphi(a(t)) \cdot \dot{a}(t)$

Since $a(t)$ is piecewise smooth, let $a_1 = t_0 < t_1 < t_2 < \cdots < t_n = a_2$ be a partition of the interval $[a_1, a_2]$ such that
$\forall k \in [n]: a \restriction [t_{k-1}, t_k]$ is a smooth path
It follows that:

$$\int_C \nabla\varphi \cdot d\ell = \int_{a_1}^{a_2} \nabla\varphi(a(t)) \cdot \dot{a}(t)\, dt = \int_{a_1}^{a_2} g'(t)\, dt =$$
$$= \sum_{k \in [n]} \int_{t_{k-1}}^{t_k} g'(t)\, dt = \sum_{k \in [n]} [g(t_k) - g(t_{k-1})] =$$

$$= g(t_n) - g(t_0) = g(a_2) - g(a_1) =$$
$$= \varphi(a(a_2)) - \varphi(a(a_1)) = \varphi(y) - \varphi(x) \quad \square$$

- We see that the line integral depends only on the initial and final points, and is independent of the path connecting the two points.

Potential Functions

Def : Let $f : A \to \mathbb{R}^n$ be a vector field with $A \subseteq \mathbb{R}^n$. We say that

$$\boxed{f \text{ conservative} \Leftrightarrow \exists \varphi : \begin{cases} \varphi : A \to \mathbb{R} \\ \forall x \in A : f(x) = \nabla \varphi(x) \end{cases}}$$

- If f is a conservative vector field and $f = \nabla \varphi$, then φ = potential function of f. and furthermore:

$$\boxed{C \in \mathcal{P}_A(a,b) \Rightarrow \int_C f \cdot dl = \varphi(b) - \varphi(a)}$$

If a vector field is conservative, the fastest way to find its potential function is as follows:

▶ Method: How to find the potential.

Consider, for example, a three-dimensional ~~function~~ vector field $f = (f_1, f_2, f_3)$. If $f = \nabla\varphi$, then:

$$\frac{\partial\varphi}{\partial x} = f_1 \quad \text{and} \quad \frac{\partial\varphi}{\partial y} = f_2 \quad \text{and} \quad \frac{\partial\varphi}{\partial z} = f_3$$

Integrating with respect to x, y, z gives:

$$\varphi(x,y,z) = \int f_1(x,y,z)\,dx + A(y,z) \qquad (1)$$

$$\varphi(x,y,z) = \int f_2(x,y,z)\,dy + B(z,x) \qquad (2)$$

$$\varphi(x,y,z) = \int f_3(x,y,z)\,dz + C(x,y) \qquad (3)$$

Here $A(y,z), B(z,x), C(x,y)$ are integration constants. To find φ, it is sufficient to define A, B, C such that equations (1), (2), (3) agree with each other. Then any one of the equations (1), (2), (3) yields the potential function $\varphi(x,y,z)$.

EXAMPLE

Show that the function

$f(x,y,z) = (2xyz + z^2 - 2y^2 + 1,\ x^2z - 4xy,\ x^2y + 2xz - 2)$

Find a potential function φ such that $f = \nabla\varphi$.

Solution

Given $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$
with $f_1(x,y,z) = 2xyz + z^2 - 2y^2 + 1$
$f_2(x,y,z) = x^2 z - 4xy$
$f_3(x,y,z) = x^2 y + 2xz - 2$

We note that

$$\varphi(x,y,z) = \int f_1(x,y,z)\,dx + A(y,z) =$$

$$= \int (2xyz + z^2 - 2y^2 + 1)\,dx + A(y,z) =$$

$$= x^2yz + xz^2 - 2xy^2 + x + A(y,z).$$

$$\varphi(x,y,z) = \int f_2(x,y,z)\,dy + B(z,x) =$$

$$= \int (x^2 z - 4xy)\,dy + B(z,x) =$$

$$= x^2yz - 2xy^2 + B(z,x)$$

$$\varphi(x,y,z) = \int f_3(x,y,z)\,dz + C(x,y) =$$

$$= \int (x^2 y + 2xz - 2)\,dz + C(x,y) =$$

$$= x^2yz + xz^2 - 2z + C(x,y).$$

•1 We guess the potential function by merging all the terms from the above 3 equations, noting that they share some but not all terms.

$$\varphi(x,y,z) = x^2yz + xz^2 - 2xy^2 + x - 2z$$

•2 Now we define the functions $A(y,z)$, $B(z,x)$, $C(x,y)$ by identifying the missing terms:

$$A(y,z) = -2z$$
$$B(z,x) = xz^2 + x - 2z$$
$$C(x,y) = -2xy + x$$

•3 In order for the proposed potential function φ to be a legitimate potential, it is necessary that in the proposed definitions of A, B, C that A be independent of x, B be independent of y, C be independent of z. If this is not possible, then the original vector field is probably not conservative.

Potential as a line integral

Def : Let $f : A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^n$ be a vector field. We say that:

$$\boxed{\begin{array}{l} f \text{ path-independent in } A \iff \\ \iff \forall x, y \in A : \exists I \in \mathbb{R} : \forall C \in \mathcal{P}_A(x,y) : \int_C f \cdot dl = I \end{array}}$$

Notation : If f is a path-independent vector field in A, then for each given $x, y \in A$, the corresponding I is denoted as

$$I = \int_x^y f \cdot dl$$

Thm : Let $f : A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^n$ be a vector field and let $a \in A$ be given. Assume that:

a) f is path-independent in A

b) A is open and path-connected

c) $\forall x \in A : \varphi(x) = \int_a^x f \cdot dl$

Then: $\forall x \in A : \nabla \varphi(x) = f(x)$

- This theorem is the generalization of the first fundamental theorem of calculus to line integrals.

▶ 2nd method : How to find the potential φ.

- 1. For each $x \in A$, choose a convenient path from some $a \in A$ to x and calculate the line integral:

$$\varphi(x) = \int_a^x f \cdot d\ell$$

- 2 Check whether $\nabla\varphi = f$. If yes, then f is conservative with potential function φ. If no, then f is not conservative

EXAMPLE

Show that the function $f(x,y) = (3x^2y, x^2y)$ is not conservative

Solution

Define the path $C(x,y): a(t) = (xt, yt)$, $\forall t \in [0,1]$ from the point $(0,0)$ to (x,y). Then

$\dot{a}(t) = (x,y)$, $\forall t \in [0,1]$

Define

$$\varphi(x,y) = \int_{C(x,y)} f \cdot dl = \int_{C(x,y)} 3x^2y\,dx + x^2y\,dy =$$

$$= \int_0^1 (3(xt)^2(yt), (xt)^2(yt)) \cdot (x,y)\,dt =$$

$$= \int_0^1 t^3 (3x^2y, x^2y) \cdot (x,y)\,dt =$$

$$= \int_0^1 t^3 (3x^2yx + x^2yxy)\,dt =$$

$$= (3x^3y + x^2y^2) \int_0^1 t^3\,dt =$$

$$= (3x^3y + x^2y^2) \left[\frac{t^4}{4} \right]_0^1 = (3x^3y + x^2y^2) \frac{1^4 - 0^4}{4}$$

$$= (3/4)x^3y + (1/4)x^2y^2$$

We note that

$$\frac{\partial \varphi}{\partial x} = (\partial/\partial x)[(3/4)x^3y + (1/4)x^2y^2] =$$
$$= (3/4)3x^2y + (1/4)2xy^2 =$$
$$= (9/4)x^2y + (1/2)xy^2 \neq 3x^2y \Rightarrow$$

$\Rightarrow \nabla\varphi \neq \mathbf{f} \Rightarrow \mathbf{f}$ not conservative.

Loop line integrals of conservative vector fields

Let $A \subseteq \mathbb{R}^n$ be an open path-connected set. Recall that for any given $x, y \in A$, $\mathcal{P}_A(x,y)$ is the set of all piecewise smooth paths from x to y that lie inside the set A. Then the set $\mathrm{Loop}(A)$ of all closed piecewise smooth paths lying inside A can be defined as:

$$\boxed{\mathrm{Loop}(A) = \bigcup_{x \in A} \mathcal{P}_A(x,x)}$$

Notation: Let $C \in \mathrm{Loop}(A)$ be a given closed path. For closed paths, in general, we use the following notation to denote a line integral, to highlight the fact that the path is closed:

$$I = \oint_C \mathbf{f} \cdot d\mathbf{l}$$

We will now show that:

Thm: Let $\mathbf{f}: A \to \mathbb{R}^n$ be a vector field. Assume that A is open and path-connected. Then, the following statements are equivalent:

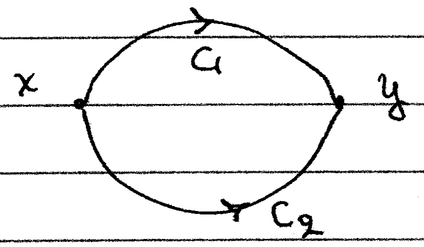
a) $\mathbf{f}$ conservative in A

b) $\mathbf{f}$ path independent in A

c) $\forall C \in \mathrm{Loop}(A): \oint_C \mathbf{f} \cdot d\mathbf{l} = 0$

Proof

- $(c) \Rightarrow (b)$: Assume that $\forall C \in \text{Loop}(A) : \oint_C f \cdot dl = 0$
We will show that f is path independent.
Let $x, y \in A$ be given. Let $C_1, C_2 \in \mathcal{P}_A(x, y)$ be given.
We define the closed path $C = C_1 \cup (-C_2)$.



It follows that

$$\oint_C f \cdot dl = \oint_{C_1 \cup (-C_2)} f \cdot dl = \int_{C_1} f \cdot dl + \int_{-C_2} f \cdot dl =$$

$$= \int_{C_1} f \cdot dl - \int_{C_2} f \cdot dl \quad (1)$$

Since : $\oint_C f \cdot dl = 0 \underset{(1)}{\Rightarrow} \int_{C_1} f \cdot dl - \int_{C_2} f \cdot dl = 0 \Rightarrow$

$$\Rightarrow \int_{C_1} f \cdot dl = \int_{C_2} f \cdot dl$$

Thus:

$\forall x, y \in A : \forall C_1, C_2 \in \mathcal{P}_A(x, y) : \int_{C_1} f \cdot dl = \int_{C_2} f \cdot dl \Rightarrow$

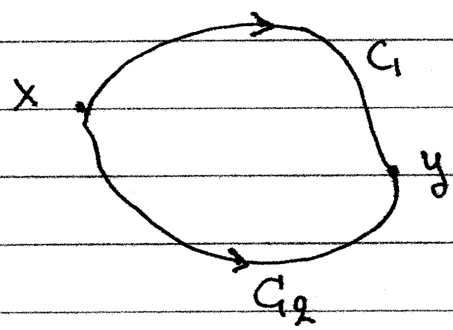
$\Rightarrow f$ path independent.

• (b) ⇒ (a) : Assume that f is path-independent. We will show that f is conservative.
Choose some $a \in A$ and define the scalar field

$$\varphi(x) = \int_a^x f \cdot dl \quad , \quad \forall x \in A$$

Since f is path-independent $\Rightarrow \forall x \in A : \nabla \varphi(x) = f(x)$
$\Rightarrow f$ conservative.

• (a) ⇒ (c) : Assume that f is conservative.
Let $C \in \mathrm{Loop}(A)$ be given. Choose $x, y \in C$ and write $C = C_1 \cup (-C_2)$ with $C_1, C_2 \in \mathcal{P}_A(x, y)$.



Since f is conservative, there is a scalar field $\varphi : A \to \mathbb{R}$ such that $\forall x \in A : \nabla \varphi(x) = f(x)$.
It follows that:

$$\int_{C_1} f \cdot dl = \int_{C_1} \nabla \varphi \cdot dl = \varphi(y) - \varphi(x) \qquad (1)$$

$$\int_{C_2} f \cdot dl = \int_{C_2} \nabla \varphi \cdot dl = \varphi(y) - \varphi(x). \qquad (2)$$

and therefore:

$$\oint_C f \cdot dl = \int_{C_1} f \cdot dl - \int_{C_2} f \cdot dl = (\varphi(y) - \varphi(x)) - (\varphi(y) - \varphi(x))$$

$$= 0 \, , \quad \forall C \in \mathrm{Loop}(A). \qquad \square$$

EXERCISES

(8) Use indefinite integrals to find a potential function for the following conservative vector fields

a) $F(x,y) = (x^2+2xy,\ y^2+2xy)$
b) $F(x,y) = (e^x\cos y,\ e^x\sin y)$
c) $F(x,y) = \left(\dfrac{x^3}{(x^4+y^4)^2},\ \dfrac{y^3}{(x^4+y^4)^2}\right)$
d) $F(x,y) = (y-1/x^2,\ x-1/y^2)$
e) $F(x,y,z) = (6xy^3+2z^2,\ 9x^2y^2,\ 4zx+1)$
f) $F(x,y,z) = (yz+1,\ zx+1,\ xy+1)$
g) $F(x,y,z) = (y+z,\ z+x,\ x+y)$
h) $F(x,y,z) = (\cos x+2yz,\ \sin y+2zx,\ z+2xy)$

(9) Use line integrals to determine and show whether or not the following vector fields are conservative

a) $F(x,y) = (2xe^y+y,\ x^2e^y+x-2y)$
b) $F(x,y) = (\sin y-y\sin x+x,\ \cos x+x\cos y+y)$
c) $F(x,y) = (\sin(xy)+xy\cos(xy),\ x^2\cos(xy))$
d) $F(x,y,z) = (2xy^3,\ x^2z^3,\ 3x^2yz^2)$
e) $F(x,y,z) = (3y^4z^2,\ 4x^3z^2,\ -3x^2y^2)$
f) $F(x,y,z) = (2x^2+8xy^2,\ 3x^3y-3xy,\ -4y^2z^2-2x^3z)$
g) $F(x,y,z) = (y^2\cos x+z^3,\ 2y\sin x-4,\ 3xz^2+2)$
h) $F(x,y,z) = (4xy-3x^2z^2+1,\ 2x^2+2,\ -2x^3z-3z^2)$

▼ Green's theorem

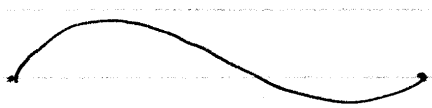
For the following discussion, we restrict ourselves to $\mathbb{R}^2$ (i.e. to two dimension).

Jordan curves

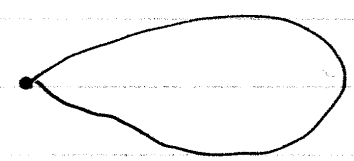
Def: Let $C : a(t),\ \forall t \in [a_1, a_2]$ be a path. We say that

$$\boxed{C \text{ simple} \iff \forall t_1, t_2 \in [a_1, a_2) : (t_1 \neq t_2 \Rightarrow a(t_1) \neq a(t_2))}$$

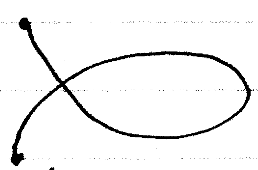
▸ interpretation: A simple path is one that does not intersect with itself short of closing upon itself to possibly form a loop. This is why in the definition above we use the set $[a_1, a_2)$ instead of $[a_1, a_2]$. For example:



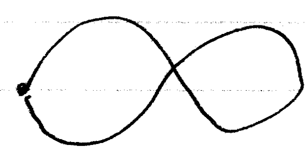
simple



simple



not simple

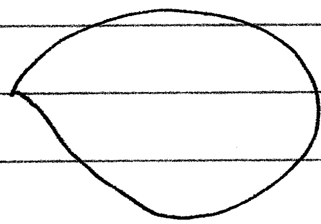


not simple

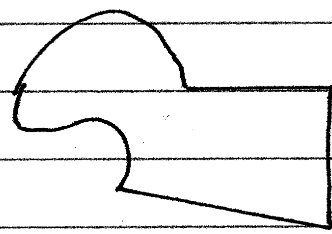
<u>Def</u> : Let G be a path. We say that

$$\boxed{G \text{ Jordan curve} \Leftrightarrow \begin{cases} G \in \text{Loop}(\mathbb{R}^2) \\ G \text{ simple} \end{cases}}$$

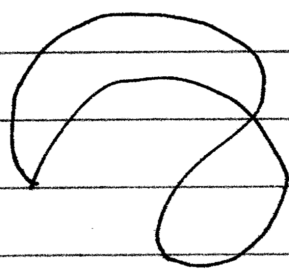
▶ <u>interpretation</u> : A Jordan curve is a simple, closed, and piecewise smooth path.



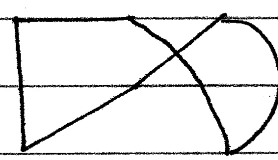
Jordan curve



Jordan curve



NOT a Jordan curve



NOT a Jordan curve.

▶ <u>Notation</u> : The set of all Jordan curves that lie in A is denoted as $\text{Jord}(A)$ and the definition of $\text{Jord}(A)$ can be written as:

$$\boxed{\text{Jord}(A) = \{ G \in \text{Loop}(A) \mid G \text{ simple} \}}$$

Jordan's theorem

Thm: For every Jordan curve G, there are two sets $I \subseteq \mathbb{R}^2$ and $E \subseteq \mathbb{R}^2$ such that all of the following statements are true:

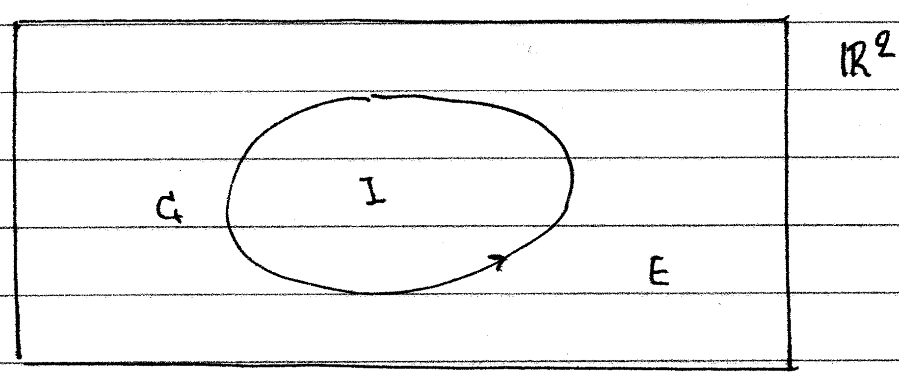
a) $I \cup G \cup E = \mathbb{R}^2$

b) I, E are open sets

c) I bounded $\wedge$ E NOT bounded

d) $\partial I = \partial E = G$ (i.e. G is the boundary set for both I and E).

▸ interpretation: As shown in the figure below, a Jordan curve G divides $\mathbb{R}^2$ into an interior set I which is open and bounded with $\partial I = G$ and an exterior set E which is open but unbounded with $\partial E = G$.



▸ notation: We denote the interior set of G as $I = \mathrm{int}(G)$ and the exterior set of G as $E = \mathrm{ext}(G)$.

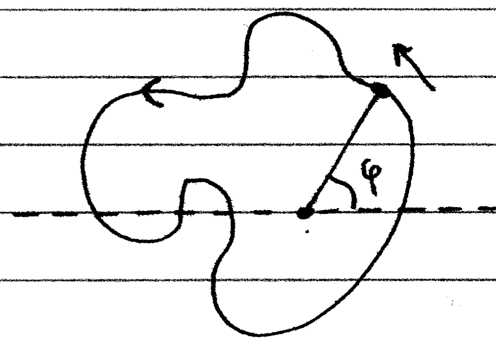
Orientation of Jordan curves

<u>Def</u>: Let $C \in \text{Jord}(\mathbb{R}^2)$ be a Jordan curve and let $(x_0, y_0) \in \text{Int}(C)$ be a point interior to the curve C. Consider the following polar coordinates representation of C:

$$C : (x, y) = (x_0, y_0) + a(t)(\cos(\varphi(t)), \sin(\varphi(t))), \forall t \in [t_1, t_2]$$

We define the winding number $w(C)$ of C as:

$$\boxed{w(C) = \frac{1}{2\pi} \int_{t_1}^{t_2} \varphi'(t)\, dt}$$



$w(C) = +1$
positive oriented
counterclockwise direction

- We can show that
  $\forall C \in \text{Jord}(\mathbb{R}^2) : w(C) = +1 \vee w(C) = -1$
  and may therefore give the following definition:

<u>Def</u>: Let $C \in \text{Jord}(\mathbb{R}^2)$ be a Jordan curve. We say that:

a) C is positive oriented $\Leftrightarrow w(C) = +1$
b) C is negative oriented $\Leftrightarrow w(C) = -1$.

- Let $C \in \mathrm{Jord}(\mathbb{R}^2)$ be a Jordan curve and let $a = (x_0, y_0) \in \mathrm{int}(C)$ be an interior point. Consider the following line integral:

$$W(C \mid a) = \oint_C \frac{-(y-y_0)dx + (x-x_0)dy}{(x-x_0)^2 + (y-y_0)^2}$$

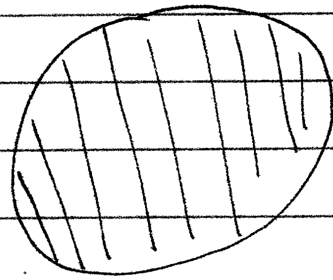
It can be shown that
$\forall a \in \mathrm{int}(C) : W(C \mid a) = \omega(C)$.
This line integral is a fairly convenient method for calculating the winding number.

Simply connected sets.

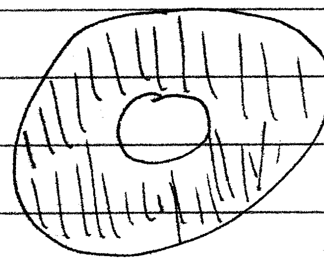
Def: Let $A \subseteq \mathbb{R}^2$. We say that

$$\boxed{A \text{ simply connected} \Leftrightarrow \begin{cases} A \text{ path-connected} \\ \forall C \in \mathrm{Jord}(A) : \mathrm{int}(C) \subseteq A \end{cases}}$$

▸ interpretation: A path-connected set can have "holes" because it is always possible to connect any two points by going around the holes. A simply connected set is not allowed to have any holes. If there is a hole, then a Jordan curve around the hole will violate the definition above.



path-connected
simply connected



path-connected but
NOT simply connected

Green's theorem

Thm: Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ be two scalar fields. Let $C \in \text{Jord}(A)$ be a Jordan curve on A. Assume that:

a) A open $\wedge$ A simply connected
b) f, g differentiable in $C \cup \text{int}(C)$
c) $\nabla f, \nabla g$ continuous in $C \cup \text{int}(C)$
d) C positive oriented (i.e. $w(C) = +1$).

Then:

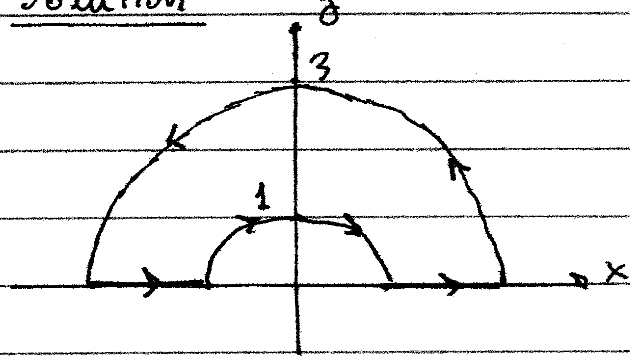
$$\boxed{\oint_C f\,dx + g\,dy = \iint_{\text{int}(C)} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy}$$

EXAMPLE

Let $A = \{(r\cos\vartheta, r\sin\vartheta) \mid r \in [1,3] \wedge \vartheta \in [0,\pi]\}$ be a region in $\mathbb{R}^2$ and let $C = \partial A$ be a Jordan curve delineating the boundary of A with C positive oriented. Evaluate the integral

$$I = \oint_C y^2 dx + x^2 dy$$

Solution



$$I = \oint_C y^2 dx + x^2 dy = \iint_{int(C)} \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\right] dxdy$$

$$= \iint_A (2x - 2y)\, dx\, dy = 2\iint_A (x-y)\, dxdy$$

Let $B = \{(r,\vartheta) \mid r \in [1,3] \wedge \vartheta \in [0,\pi]\}$. We change variables:

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases} \Rightarrow dxdy = r dr d\vartheta$$

It follows that:

$$I = 2\iint_B (r\cos\vartheta - r\sin\vartheta)\, r\, dr\, d\vartheta = 2\int_1^3 dr \int_0^\pi d\vartheta\, r^2(\cos\vartheta - \sin\vartheta)$$

$$= 2\left[\int_1^3 r^2 dr\right]\left[\int_0^\pi (\cos\vartheta - \sin\vartheta)\, d\theta\right] =$$

$$= 2\left[\frac{r^3}{3}\right]_1^3 \left[\sin\vartheta + \cos\vartheta\right]_0^\pi =$$

$$= 2\ \frac{3^3 - 1^3}{3}\left[(\sin\pi + \cos\pi) - (\sin 0 + \cos 0)\right] =$$

$$= \frac{2(27-1)}{3}\left[(0-1) - (0+1)\right] = \frac{2 \cdot 26 \cdot (-2)}{3}$$

$$= \frac{-104}{3}.$$

EXERCISES

(10) Use Green's theorem to evaluate the following line integrals by converting them to double integrals.

a) $I = \oint_{C} y^2 dx + x^2 dy$

with C the boundary $C = \partial S$ of the rectangle $S = [0,a] \times [0,b]$ traversed counterclockwise with $a, b \in (0, +\infty)$.

b) $I = \oint_{C} e^{x+y} dx + e^{x-y} dy$

with C the triangle with vertices $A(0,0)$, $B(a,0)$, $C(a,b)$ traversed counterclockwise with $a, b \in (0, +\infty)$.

c) $I = \oint_{C} x^2 y\, dx + x y^2 dy$

with C a circle with center $O(0,0)$ and radius $R > 0$, traversed counterclockwise.

d) $I = \oint_{C} (x+y) dx + (x^2 - y) dy$

with C the boundary of the region enclosed by

$(c_1): y = x^2$ and $(c_2): y = \sqrt{x}$, traversed counterclockwise.

e) $I = \oint_{C} (\ln x + y)\,dx - x^3\,dy$

with C the rectangle with vertices $A_1(1,1)$, $A_2(1,b)$, $A_3(a,1)$, $A_4(a,b)$, traversed counterclockwise with $a, b \in (1, +\infty)$.

f) $I = \oint_{C} xy\,dx + (x^2 + x)\,dy$

with C the triangle with vertices $A_1(-a,0)$, $A_2(a,0)$, $A_3(0,a)$ traversed counterclockwise, with $a \in (0, +\infty)$.

g) $I = \oint_{C} xe^y\,dx + (x + x^2 e^y)\,dy$

with C the boundary of a half-disk given by
$S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le a^2 \wedge y \ge 0\}$
traversed counterclockwise with $a \in (0, +\infty)$.

h) $I = \oint_{C} (\sin x + y)\,dx + (3x + y)\,dy$

with C a polygon with vertices $A_1(0,0)$, $A_2(a,a)$, $A_3(a, a+b)$, $A_4(0, 2a+b)$ traversed counterclockwise with $a, b \in (0, +\infty)$.

i) $I = \oint_{C} (x^2+y^2)\,dx - 2xy\,dy$

with C a circle with center $K(0,0)$ and radius $a \in (0,+\infty)$, traversed counterclockwise

▼ Applications of Green's theorem

① Area calculation

Prop: Let $C \in \mathrm{Jord}(\mathbb{R}^2)$ be a positively-oriented Jordan's curve. The area $A(C)$ of the interior $\mathrm{int}(C)$ of C is given by

$$\boxed{A(C) = \frac{1}{2}\oint_{C} (x\,dy - y\,dx)}$$

Proof

Note that the area $A(C)$ is given by $A(C) = \iint_{\mathrm{int}(C)} dx\,dy$
The line integral reads:

$$\oint_C (x\,dy - y\,dx) = \oint_C (-y\,dx + x\,dy) =$$

$$= \iint_{\mathrm{int}(C)} \left[\frac{\partial}{\partial x} x - \frac{\partial}{\partial y}(-y)\right] dx\,dy$$

$$= \iint_{\mathrm{int}(C)} (1-(-1))\,dx\,dy = \iint_{\mathrm{int}(C)} 2\,dx\,dy =$$

$$= 2\iint_{\mathrm{int}(C)} dx\,dy = 2A(C) \Rightarrow$$

$$\Rightarrow A(C) = \frac{1}{2}\oint_C (x\,dy - y\,dx) \qquad \square$$

EXAMPLE

Find the area of the ellipse

$$S = \{(x,y) \in \mathbb{R}^2 \mid x^2/a^2 + y^2/b^2 \leq 1\}$$

Solution

The boundary of the ellipse can be written as:

$$\partial S : (x,y) = (a\cos\vartheta, b\sin\vartheta) \,,\ \forall \vartheta \in [0, 2\pi]$$

Differentiating with respect to ϑ gives:

$$(\dot{x}, \dot{y}) = (-a\sin\vartheta, b\cos\vartheta) \,,\ \forall \vartheta \in [0, 2\pi]$$

It follows that the area is given by:

$$A = \iint_S dx\,dy = \frac{1}{2}\oint_{\partial S}(-y\,dx + x\,dy) =$$

$$= \frac{1}{2}\int_0^{2\pi}(-b\sin\vartheta, a\cos\vartheta)\cdot(-a\sin\vartheta, b\cos\vartheta)\,d\vartheta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[(-b\sin\vartheta)(-a\sin\vartheta) + (a\cos\vartheta)(b\cos\vartheta)\right]d\vartheta$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\sin^2\vartheta + \cos^2\vartheta)\,d\vartheta = \frac{1}{2}\int_0^{2\pi} ab\,d\vartheta =$$

$$= \frac{ab}{2}\int_0^{2\pi} d\vartheta = \frac{ab}{2}\cdot(2\pi) = \pi ab$$

Thus $\underline{A = \pi ab}$.

② Conservative vector fields in $\mathbb{R}^2$

- Green's theorem can be used to derive a simple test for determing whether a two-dimensional vector field is conservative.

<u>Thm</u> : Let $f(x,y) = (f_1(x,y), f_2(x,y))$, $\forall (x,y) \in A$ be a vector field. We assume that:

a) A open set $\land$ A simply connected

b) f_1, f_2 differentiable in A

c) $\nabla f_1, \nabla f_2$ continuous in A

Then:

$$\boxed{f \text{ conservative} \Leftrightarrow \forall (x,y) \in A : \frac{\partial f_2(x,y)}{\partial x} = \frac{\partial f_1(x,y)}{\partial y}}$$

<u>Proof</u>

($\Rightarrow$): Assume that f is conservative.
Let $\varphi : A \to \mathbb{R}$ be the corresponding potential function such that

$$\forall (x,y) \in A : f(x,y) = \nabla \varphi(x,y) \Rightarrow$$
$$\Rightarrow \forall (x,y) \in A : f_1(x,y) = \frac{\partial \varphi(x,y)}{\partial x} \land f_2(x,y) = \frac{\partial \varphi(x,y)}{\partial y}$$

It follows that:

$$\frac{\partial f_2(x,y)}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\partial \varphi(x,y)}{\partial y}\right] = \frac{\partial^2 \varphi(x,y)}{\partial x \partial y} =$$

$$= \frac{\partial^2 \varphi(x,y)}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial \varphi(x,y)}{\partial x}\right] =$$

$$= \frac{\partial f_1(x,y)}{\partial y}, \forall (x,y) \in A$$

($\Leftarrow$): Assume that $\dfrac{\partial f_2(x,y)}{\partial x} = \dfrac{\partial f_1(x,y)}{\partial y}$

▸ We will show that $\forall C \in \text{Loop}(A) : \oint_C f \cdot dl = 0$

Let $C \in \text{Loop}(A)$ be given.

▸ We note that C can be rewritten as the union of Jordan curves, as explained after the proof:

$\exists C_1, C_2, \ldots, C_n \in \text{Jord}(A) : C = C_1 \cup C_2 \cup \cdots \cup C_n$

It follows that:

$$\oint_C f \cdot dl = \sum_{a=1}^{n} \oint_{C_a} f \cdot dl = \sum_{a=1}^{n} \oint_{C_a} f_1 dx + f_2 dy =$$

$$= \sum_{a=1}^{n} w(C_a) \iint_{\text{int}(C_a)} \left[\frac{\partial f_2(x,y)}{\partial x} - \frac{\partial f_1(x,y)}{\partial y}\right] dxdy$$

$$= \sum_{a=1}^{n} w(C_a) \iint_{\text{int}(C_a)} 0 \, dxdy = 0, \forall C \in \text{Loop}(A)$$

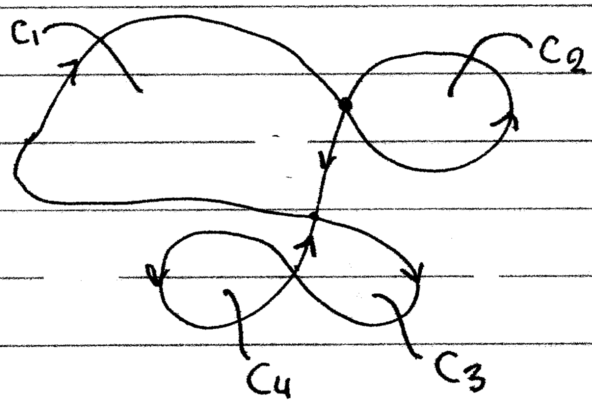
$\Rightarrow f$ conservative □

↳ In the proof above we claimed that any loop $C \in \text{Loop}(A)$ can be written as a union of Jordan curves:

$$C = C_1 \cup C_2 \cup \dots \cup C_n$$

with $C_1, C_2, \dots, C_n \in \text{Jord}(A)$.

This is not easy to prove but we can illustrate it with a graphical example:



In this figure we decompose $C = C_1 \cup C_2 \cup C_3 \cup C_4$ with the Jordan curves C_1, C_2, C_3, C_4 identified by pointing towards their interior sets: $\text{int}(C_1)$, $\text{int}(C_2)$, $\text{int}(C_3)$, and $\text{int}(C_4)$.

↳ The contrapositive statement of the theorem reads:

$$\boxed{f \text{ NOT conservative} \iff \exists (x,y) \in A : \frac{\partial f_2(x,y)}{\partial x} \neq \frac{\partial f_1(x,y)}{\partial y}}$$

EXAMPLES

a) Examine whether the field $f(x,y)=(x-y,x-2), \forall (x,y)\in\mathbb{R}^2$ is conservative

Solution

Define $\begin{cases} f_1(x,y) = x-y, \forall (x,y)\in\mathbb{R}^2 \\ f_2(x,y) = x-2, \forall (x,y)\in\mathbb{R}^2 \end{cases}$

Then:

$$\frac{\partial f_2(x,y)}{\partial x} = \frac{\partial}{\partial x}(x-2) = 1 , \forall (x,y)\in\mathbb{R}^2$$

$$\frac{\partial f_1(x,y)}{\partial y} = \frac{\partial}{\partial y}(x-y) = -1 , \forall (x,y)\in\mathbb{R}^2$$

It follows that:

$$\exists (x,y)\in\mathbb{R}^2 : \frac{\partial f_2(x,y)}{\partial x} \neq \frac{\partial f_1(x,y)}{\partial y} \Rightarrow$$

$\Rightarrow f$ NOT conservative.

b) Examine whether the field $f(x,y) = (3+2xy, x^2-3y^2), \forall (x,y)\in\mathbb{R}^2$ is conservative.

Solution

Define $\begin{cases} f_1(x,y) = 3+2xy , \forall (x,y)\in\mathbb{R}^2 \\ f_2(x,y) = x^2-3y^2 , \forall (x,y)\in\mathbb{R}^2. \end{cases}$

Then:

$$\frac{\partial f_2(x,y)}{\partial x} = \frac{\partial}{\partial x}(x^2-3y^2) = 2x \quad , \quad \forall (x,y) \in \mathbb{R}^2$$

$$\frac{\partial f_1(x,y)}{\partial y} = \frac{\partial}{\partial y}(3+2xy) = 2x \quad , \quad \forall (x,y) \in \mathbb{R}^2$$

It follows that

$$\forall (x,y) \in \mathbb{R}^2 : \frac{\partial f_2(x,y)}{\partial x} = \frac{\partial f_1(x,y)}{\partial y} \Rightarrow f \text{ conservative.}$$

EXERCISES

(11) Use line integrals to find the area of the "asteroid" S given by
$S = \{(x,y) \in \mathbb{R}^2 \mid x^{2/3} + y^{2/3} \leqslant a^{2/3}\}$
with $a \in (0, +\infty)$.
(Hint: The boundary of S can be parameterized as:
$(\partial S): (x,y) = (a\cos^3 t, a\sin^3 t), \forall t \in [0, 2\pi]$)

(12) Similarly to the previous problem, find the area of the "elliptical asteroid" S given by
$S = \{(x,y) \in \mathbb{R}^2 \mid (x/a)^{2/3} + (y/b)^{2/3} \leqslant 1\}$
with $a, b \in (0, +\infty)$

(13) Use line integrals to find the area between the cycloid (c) given by
$$(c): \begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}, \forall t \in [0, 2\pi]$$
and the x-axis

↳ Note that a different technique for calculating the area under a cycloid is given in my Calculus 2 Online Lecture Notes.

(14) An epicycloid is traced by rolling a circle with radius 1 around and outside another circle with radius $a-1$. The epicycloid (c) is given by

$$(c):\begin{cases} x = a\cos t - \cos(at) \\ y = a\sin t - \sin(at) \end{cases}, \quad \forall t \in [0, 2\pi]$$

Use line integrals to find the area enclosed by the epicycloid, with $a \in \mathbb{N} - \{0,1\}$.

(15) Area of a polygon

a) Consider the line segment (ℓ) from the point $A_1(x_1, y_1)$ to the point $A_2(x_2, y_2)$, given by

$$(\ell):\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \end{cases}, \quad \forall t \in [0,1]$$

Show that

$$\int_{(\ell)} x\,dy - y\,dx = x_1 y_2 - x_2 y_1$$

b) Now consider a polygon traced by the vertices $A_1(x_1, y_1), A_2(x_2, y_2), \ldots, A_n(x_n, y_n), A_1(x_1, y_1)$. Show, using part (a), that the area of the polygon is given by:

$$A = \left| \sum_{a=1}^{n-1} (x_a y_{a+1} - x_{a+1} y_a) + (x_n y_1 - x_1 y_n) \right|$$

▼ Parametric surfaces

Def: Let $x: A \to \mathbb{R}$, $y: A \to \mathbb{R}$, $z: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ be three scalar fields and consider the set
$$S = \{(x(t,s), y(t,s), z(t,s)) \mid (t,s) \in A\}.$$
We say that

$$\boxed{S \text{ surface} \iff \begin{cases} A \text{ simply connected} \land A \text{ closed} \\ x, y, z \text{ continuous in } A \end{cases}}$$

$$\boxed{S \text{ differentiable surface} \iff \begin{cases} S \text{ surface} \\ x, y, z \text{ differentiable in } A \end{cases}}$$

notation: Alternatively, we write the definition of S as:
$$(S): \; a(t,s) = (x(t,s), y(t,s), z(t,s)), \; \forall (t,s) \in A$$
with $a: A \to \mathbb{R}^3$.

remark: The parameters (t,s) act as a local curvilinear coordinate system on the surface. Since a surface should be a two-dimensional object, we require 2 coordinates. However, it is possible under the definitions above for a surface to collapse into a one-dimensional structure. For example:
$$(S): \; a(t,s) = (t+s, t+s, t+s), \; \forall (t,s) \in A$$
is in fact defining a line. To rule out this possibility we propose the following stronger definition.

<u>Def</u> : Let $(S): a(t,s) = (x(t,s), y(t,s), z(t,s))$, $\forall (t,s) \in A$ be a differentiable surface.

a) We define the <u>fundamental product</u> $R(t,s|a)$ of the surface (S) as:

$$\boxed{R(t,s|a) = \frac{\partial a}{\partial t} \times \frac{\partial a}{\partial s}, \quad \forall (t,s) \in A}$$

b) We say that:

$$\boxed{S \text{ smooth surface} \Leftrightarrow \begin{cases} S \text{ differentiable surface} \\ R(t,s|a) \text{ continuous in } A \\ \forall (t,s) \in \operatorname{int}(A): R(t,s|a) \neq 0 \end{cases}}$$

▸ <u>interpretation</u> : Given a point (t,s) of the surface, a small change in the parameter t should move us in a different direction than a small change in the parameter s. To prevent a surface from being degenerate, it is therefore essential that $\partial a/\partial t$ should NOT be parallel to $\partial a/\partial s$. This requirement is equivalent to the condition

$$R(t,s|a) \neq 0, \ \forall (t,s) \in \operatorname{int}(A)$$

in the above definition. We allow the condition to be violated at points on the boundary ∂A, and if such points exist, we call them <u>singular points</u>

<u>Remark:</u> The definition of the fundamental product can be rewritten as follows:

$$\boxed{\mathcal{Q}(t,s|a) = \left(\frac{\partial(y,z)}{\partial(t,s)}, \frac{\partial(z,x)}{\partial(t,s)}, \frac{\partial(x,y)}{\partial(t,s)}\right)}$$

with:

$$\boxed{\begin{aligned} \frac{\partial(y,z)}{\partial(t,s)} &= \frac{\partial y}{\partial t}\frac{\partial z}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial z}{\partial t} \\ \frac{\partial(z,x)}{\partial(t,s)} &= \frac{\partial z}{\partial t}\frac{\partial x}{\partial s} - \frac{\partial z}{\partial s}\frac{\partial x}{\partial t} \\ \frac{\partial(x,y)}{\partial(t,s)} &= \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} \end{aligned}}$$

<u>Simple surfaces</u>

Consider a surface

$$S = \{a(t,s) \mid (t,s) \in A\}$$

with $A \subseteq \mathbb{R}^2$.

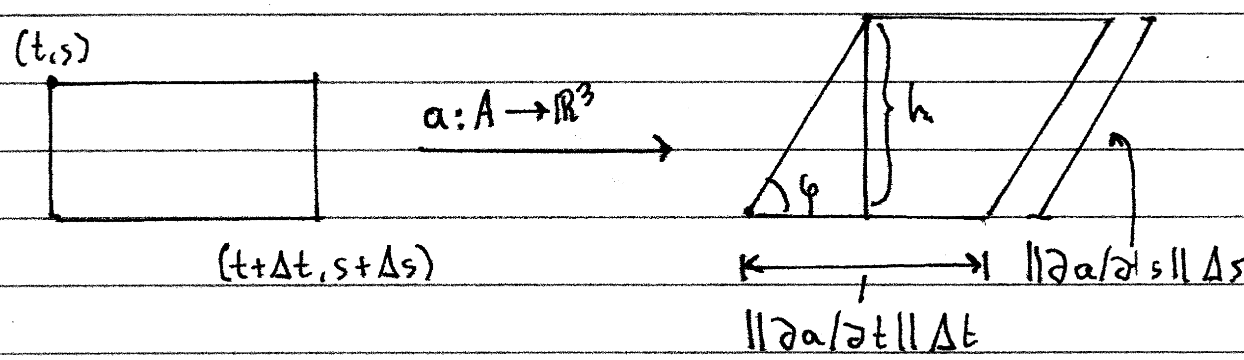
<u>Def</u>: S is <u>simple</u> $\iff a: A \to \mathbb{R}^3$ one-to-one $\iff \forall (t_1,s_1),(t_2,s_2) \in A: ((t_1,s_1) \neq (t_2,s_2) \Rightarrow a(t_1,s_1) \neq a(t_2,s_2))$

An immediate consequence of the definition is that every loop $C \in \mathrm{Loop}(A)$ in the (t,s) plane is mapped by "a" to a loop in the surface $\mathcal{S}$:

<u>Prop</u> : $\boxed{\mathcal{S} \text{ simple} \Rightarrow \forall \mathcal{C} \in \mathrm{Loop}(A) : a(\mathcal{C}) \in \mathrm{Loop}(\mathcal{S})}$

Surface Area

Consider a smooth surface $\mathcal{S} = \{ a(t) \mid ($
$$\mathcal{S} = \{ a(t,s) \mid (t,s) \in A \}$$
with $a(t,s) = (x(t,s), y(t,s), z(t,s))$, $\forall (t,s) \in A$.
Consider a "patch" on the surface $\mathcal{S}$ defined by a rectangle in the (t,s)-plane which is in turn defined by the vertices (t,s) and $(t+\Delta t, s+\Delta s)$ as shown in the following figure:



We see that:
Bottom side of patch length $\ell \approx \|\partial a/\partial t\| \Delta t$
Height $h \simeq \|\partial a/\partial s\| \Delta s \cdot \sin\varphi$
and therefore, the area of the patch is:
$$\Delta A = \ell h \cong (\|\partial a/\partial t\| \Delta t)(\|\partial a/\partial s\| \Delta s \cdot \sin\varphi) =$$
$$= (\|\partial a/\partial t\| \cdot \|\partial a/\partial s\| \cdot \sin\varphi) \Delta t \, \Delta s =$$

$= \| (\partial a/\partial t) \times (\partial a/\partial s) \| \Delta t \Delta s =$
$= \| R(t,s|a) \| \Delta t \Delta s$

with $\Delta t \to 0$ and $\Delta s \to 0$. It follows that the total area of the surface is given by:

$$\boxed{\mathrm{Area}(S) = \iint_A \| R(t,s|a) \| \, dt\, ds}$$

Expanding the norm of the fundamental product, the above equation can be rewritten as follows:

$$\boxed{\mathrm{Area}(S) = \iint_A \sqrt{\left(\frac{\partial(x,y)}{\partial(t,s)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(t,s)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(t,s)}\right)^2} \, dt\, ds}$$

Surface integrals

This result motivates the following definition of the surface integral:

Def : Let $S = \{ a(t,s) \mid (t,s) \in A \}$ with $a : A \to \mathbb{R}^3$ be a smooth surface. Let $f : B \to \mathbb{R}$ with $B \subseteq \mathbb{R}^3$ be a scalar field such that f is continuous at B and $a(A) \subseteq B$. We define the surface integral as:

$$\boxed{\iint_S f \, dS = \iint_A f(a(t,s)) \, \| R(t,s|a) \| \, dt\, ds}$$

- Note that in the above definition, both f and $d\mathcal{S}$ are scalars. The condition $a(A) \subseteq B$ ensures that the surface $\mathcal{S}$ lies in the domain of the scalar field f.
- A similar definition is possible for vector fields:

<u>Def</u> : Let $\mathcal{S} = \{a(t,s) \mid (t,s) \in A\}$ with $a: A \to \mathbb{R}^3$ be a smooth surface. Let $f: B \to \mathbb{R}^3$ with $B \subseteq \mathbb{R}^3$ be a vector field such that f is continuous at B and $a(A) \subseteq B$. We define the following surface integrals:

$$\boxed{\begin{aligned} \iint_{\mathcal{S}} f \cdot d\mathcal{S} &= \iint_A [f(a(t,s)) \cdot R(t,s \mid a)]\, dt\, ds \\ \iint_{\mathcal{S}} f \times d\mathcal{S} &= \iint_A [f(a(t,s)) \times R(t,s \mid a)]\, dt\, ds \end{aligned}}$$

→ <u>Normal vector and surface integrals</u>

Let $\mathcal{S} = \{a(t,s) \mid (t,s) \in A\}$ be a smooth curve. The vectors $\partial a/\partial t$ and $\partial a/\partial s$ define a plane tangent to the surface $\mathcal{S}$ at the given point (t,s). Since the fundamental product reads:

$$R(t,s \mid a) = \frac{\partial a(t,s)}{\partial t} \times \frac{\partial a(t,s)}{\partial s}$$

it follows that:

$$\forall (t,s) \in A : \begin{cases} \mathcal{R}(t,s \mid a) \perp \partial a(t,s)/\partial t \\ \mathcal{R}(t,s \mid a) \perp \partial a(t,s)/\partial s \end{cases}$$

Consequently, the fundamental product $\mathcal{R}(t,s|a)$ essential is <u>normal</u> to the surface $\mathcal{S}$, and we may formally define a unit-normal vector as follows:

<u>Def</u>: The <u>unit-normal</u> vector $n(t,s|a)$ of the smooth surface $\mathcal{S}$ given by
$\mathcal{S} = \{a(t,s) \mid (t,s) \in A\}$
is defined as:

$$\boxed{n(t,s|a) = \frac{\mathcal{R}(t,s|a)}{\|\mathcal{R}(t,s|a)\|}, \quad \forall (t,s) \in A}$$

It follows that:

$$\boxed{\iint_{\mathcal{S}} f \cdot d\mathcal{S} = \iint_{\mathcal{S}} (f \cdot n)\, d\mathcal{S}}$$

$$\boxed{\iint_{\mathcal{S}} f \times d\mathcal{S} = \iint_{\mathcal{S}} (f \times n)\, d\mathcal{S}}$$

- Note that the integrals on the right-hand side of the above equations are scalar surface integrals.

- Using the unit normal vector n, we can define the normal derivative of a scalar field on a surface as the directional derivative of f in the direction defined by the unit vector n

<u>Def</u> : Let $f: B \to \mathbb{R}$ be a scalar field with $a(A) \subseteq B$. Assume that f has partial derivatives in B. We define the <u>normal derivative</u>

$$\boxed{\frac{\partial f(t,s)}{\partial n} = \nabla f(x(t,s), y(t,s), z(t,s)) \cdot n(t,s|a), \forall (t,s) \in A}$$

EXERCISES

(16) Evaluate and simplify the fundamental product $\mathcal{R}(a|t,s)$ and the norm of the fundamental product $\|\mathcal{R}(a|t,s)\|$ for the following surfaces:

a) Elliptic paraboloid

$$a(t,s) = (as\cos t,\ bs\sin t,\ s^2),\ \forall (s,t)\in[0,1]\times[0,2\pi].$$

b) Ellipsoid

$$a(\varphi,\vartheta) = (a\sin\varphi\cos\vartheta,\ b\sin\varphi\sin\vartheta,\ c\cos\varphi),\ \forall(\varphi,\vartheta)\in[0,\pi]\times[0,2\pi].$$

c) Cylinder

$$a(x,\vartheta) = (x,\ r\cos\vartheta,\ r\sin\vartheta),\ \forall\ (x,\vartheta)\in[0,\ b]\times[0,2\pi]$$

d) Torus

$$a(\vartheta,\varphi) = ((a+b\cos\vartheta)\sin\varphi,\ (a+b\cos\vartheta)\cos\varphi,\ b\sin\varphi),$$
$$\forall(\vartheta,\varphi)\in[0,2\pi]\times[0,2\pi]$$

with $a,b\in(0,+\infty)$ and $0<b<a$.

(17) Evaluate the area of the surfaces defined in the previous exercise by evaluating the corresponding surface integral

$$\text{area}(S) = \iint_S dS$$

(18) Use a surface integral to evaluate the surface area of a sphere

$$S = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 = R^2\}$$

with radius R, using spherical coordinates.

▼ Fundamental product for special surfaces

We now derive the fundamental product for the special cases

① → Surface defined as $(S): z = f(x,y)$

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ and consider the surface given by

$$\begin{aligned} S &= \{(x,y,z) \mid z = f(x,y) \wedge (x,y) \in A\} \\ &= \{(x,y,f(x,y)) \mid (x,y) \in A\} \end{aligned}$$

Then:

$$\boxed{\begin{aligned} \mathcal{R}(x,y|a) &= \left(\frac{-\partial f}{\partial x}, \frac{-\partial f}{\partial y}, 1\right) \\ \|\mathcal{R}(x,y|a)\| &= \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \end{aligned}}$$

Proof

Define $a(x,y) = (x,y,f(x,y)), \; \forall (x,y) \in A$.

It follows that

$$\begin{cases} \partial a/\partial x = (1,0,\partial f/\partial x) \\ \partial a/\partial y = (0,1,\partial f/\partial y) \end{cases} \Longrightarrow$$

$$\begin{aligned} \Rightarrow \mathcal{R}(x,y|a) &= (\partial a/\partial x) \times (\partial a/\partial y) = \\ &= (1,0,\partial f/\partial x) \times (0,1,\partial f/\partial y) = \end{aligned}$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & \partial f/\partial x \\ 0 & 1 & \partial f/\partial y \end{vmatrix} \begin{matrix} e_1 & e_2 \\ 1 & 0 \\ 0 & 1 \end{matrix} =$$

$$= 0e_1 + 0e_2 + e_3 - 0e_3 - (\partial f/\partial x)e_1 - (\partial f/\partial y)e_2 =$$
$$= -(\partial f/\partial x)e_1 - (\partial f/\partial y)e_2 + e_3$$
$$= (-\partial f/\partial x, -\partial f/\partial y, 1)$$

and therefore

$$\|\mathcal{R}(x,y|a)\| = \|(-\partial f/\partial x, -\partial f/\partial y, 1)\| =$$
$$= \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

② → Surface of revolution of $(c): y = f(x)$ around x-axis

Let $f: A \to \mathbb{R}$ with $A = [a,b]$ be a function with $\forall x \in [a,b]: f(x) > 0$. Define a surface S by rotating the curve $(c): y = f(x)$ around the x-axis. It follows that

$$S = \{(x, f(x)\cos\theta, f(x)\sin\theta) \mid x \in [a,b] \wedge \theta \in [0, 2\pi]\}$$

Then

$$\boxed{\begin{aligned} &\mathcal{R}(x,\theta|a) = (f(x)f'(x), -f(x)\cos\theta, -f(x)\sin\theta) \\ &\|\mathcal{R}(x,\theta|a)\| = f(x)\sqrt{1 + [f'(x)]^2} \end{aligned}}$$

Proof

Define $a(x,\vartheta) = (x, f(x)\cos\vartheta, f(x)\sin\vartheta)$, $\forall (x,\vartheta) \in [a,b] \times [0, 2\pi]$

Then

$$\left.\begin{aligned} \partial a/\partial x &= (1, f'(x)\cos\vartheta, f'(x)\sin\vartheta) \\ \partial a/\partial\vartheta &= (0, -f(x)\sin\vartheta, f(x)\cos\vartheta) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \mathcal{N}(x,\vartheta|a) = (\partial a/\partial x) \times (\partial a/\partial\vartheta) =$$

$$= (1, f'(x)\cos\vartheta, f'(x)\sin\vartheta) \times (0, -f(x)\sin\vartheta, f(x)\cos\vartheta)$$

$$= \left|\begin{array}{ccc|cc} e_1 & e_2 & e_3 & e_1 & e_2 \\ 1 & f'(x)\cos\vartheta & f'(x)\sin\vartheta & 1 & f'(x)\cos\vartheta \\ 0 & -f(x)\sin\vartheta & f(x)\cos\vartheta & 0 & -f(x)\sin\vartheta \end{array}\right| =$$

$$= e_1 (f'(x)\cos\vartheta)(f(x)\cos\vartheta) + 0e_2 + e_3(-f'(x)\sin\vartheta)$$

$$- 0e_3 - e_1(f'(x)\sin\vartheta)(-f(x)\sin\vartheta) - e_2(f(x)\cos\vartheta)$$

$$= e_1 f(x) f'(x)(\cos^2\vartheta + \sin^2\vartheta) - (f(x)\cos\vartheta)e_2 - (f'(x)\sin\vartheta)e_3$$

$$= f(x)f'(x)e_1 - (f(x)\cos\vartheta)e_2 - (f'(x)\sin\vartheta)e_3 =$$

$$= (f(x)f'(x), -f(x)\cos\vartheta, -f(x)\sin\vartheta)$$

$$\Rightarrow \|\mathcal{N}(x,\vartheta|a)\|^2 = \|(f(x)f'(x), -f(x)\cos\vartheta, -f(x)\sin\vartheta)\|^2 =$$

$$= [f(x)f'(x)]^2 + [-f(x)\cos\vartheta]^2 + [-f(x)\sin\vartheta]^2$$

$$= [f(x)f'(x)]^2 + [f(x)]^2(\cos^2\vartheta + \sin^2\vartheta) =$$

$$= [f(x)f'(x)]^2 + [f(x)]^2$$

$$= [f(x)]^2(1 + [f'(x)]^2) \Rightarrow$$

$$\Rightarrow \|\mathcal{N}(x,\vartheta|a)\| = \sqrt{[f(x)]^2(1 + [f'(x)]^2)}$$

$$= |f(x)|\sqrt{1 + [f'(x)]^2}$$

$$= f(x)\sqrt{1 + [f'(x)]^2}$$

③ → <u>Sphere $(S): x^2+y^2+z^2=\rho^2$</u>

The surface of a sphere $(S): x^2+y^2+z^2=\rho^2$ with radius ρ is given by

$$S=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=\rho^2\}$$
$$=\{(\rho\sin\varphi\cos\vartheta,\ \rho\sin\varphi\sin\vartheta,\ \rho\cos\varphi) \mid \varphi\in[0,\pi]\wedge\vartheta\in[0,2\pi]\}$$

Then:

$$\boxed{\begin{aligned}&\mathcal{R}(\varphi,\vartheta|a)=\rho^2(\sin^2\varphi\cos\vartheta,\ \sin^2\varphi\sin\vartheta,\ \cos\varphi\sin\varphi)\\ &\|\mathcal{R}(\varphi,\vartheta|a)\|=\rho^2\sin\varphi\end{aligned}}$$

<u>Proof</u>

Let $a(\varphi,\vartheta)=(\rho\sin\varphi\cos\vartheta,\ \rho\sin\varphi\sin\vartheta,\ \rho\cos\varphi),\ \forall(\varphi,\vartheta)\in[0,\pi]\times[0,2\pi]$

It follows that

$$\left.\begin{aligned}\partial a/\partial\varphi&=(\rho\cos\varphi\cos\vartheta,\ \rho\cos\varphi\sin\vartheta,\ -\rho\sin\varphi)\\ \partial a/\partial\vartheta&=(-\rho\sin\varphi\sin\vartheta,\ \rho\sin\varphi\cos\vartheta,\ 0\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow\mathcal{R}(\varphi,\vartheta|a)=(\partial a/\partial\varphi)\times(\partial a/\partial\vartheta)=$$
$$=(\rho\cos\varphi\cos\vartheta,\ \rho\cos\varphi\sin\vartheta,\ -\rho\sin\varphi)\times(-\rho\sin\varphi\sin\vartheta,\ \rho\sin\varphi\cos\vartheta,0)$$
$$=\begin{vmatrix}e_1 & e_2 & e_3\\ \rho\cos\varphi\cos\vartheta & \rho\cos\varphi\sin\vartheta & -\rho\sin\varphi\\ -\rho\sin\varphi\sin\vartheta & \rho\sin\varphi\cos\vartheta & 0\end{vmatrix}$$

$$=0e_1+e_2(-\rho\sin\varphi)(-\rho\sin\varphi\sin\vartheta)+e_3(\rho\cos\varphi\cos\vartheta)(\rho\sin\varphi\cos\vartheta)$$
$$-e_3(\rho\cos\varphi\sin\vartheta)(-\rho\sin\varphi\sin\vartheta)-e_1(-\rho\sin\varphi)(\rho\sin\varphi\cos\vartheta)-0e_2$$

$$= e_1 \rho^2 \sin^2\varphi \cos\vartheta + e_2 \rho^2 \sin^2\varphi \sin\vartheta +$$
$$+ e_3 \rho^2 \cos\varphi \sin\varphi (\cos^2\vartheta + \sin^2\vartheta)$$
$$= e_1 \rho^2 \sin^2\varphi \cos\vartheta + e_2 \rho^2 \sin^2\varphi \sin\vartheta + e_3 \rho^2 \cos\varphi \sin\varphi$$
$$= \rho^2 (\sin^2\varphi \cos\vartheta, \sin^2\varphi \sin\vartheta, \cos\varphi \sin\varphi)$$

It follows that

$$\|Q(\varphi,\vartheta|a)\|^2 = \|\rho^2 (\sin^2\varphi \cos\vartheta, \sin^2\varphi \sin\vartheta, \cos\varphi \sin\varphi)\|^2 =$$
$$= \rho^4 [(\sin^2\varphi \cos\vartheta)^2 + (\sin^2\varphi \sin\vartheta)^2 + (\cos\varphi \sin\varphi)^2] =$$
$$= \rho^4 [\sin^4\varphi \cos^2\vartheta + \sin^4\varphi \sin^2\vartheta + \cos^2\varphi \sin^2\varphi] =$$
$$= \rho^4 [\sin^4\varphi (\cos^2\vartheta + \sin^2\vartheta) + \cos^2\varphi \sin^2\varphi] =$$
$$= \rho^4 [\sin^4\varphi + \cos^2\varphi \sin^2\varphi] =$$
$$= \rho^4 \sin^2\varphi [\sin^2\varphi + \cos^2\varphi] = \rho^4 \sin^2\varphi \Rightarrow$$
$$\Rightarrow \|Q(\varphi,\vartheta|a)\| = \sqrt{\rho^4 \sin^2\varphi} = \rho^2 \sqrt{\sin^2\varphi} =$$
$$= \rho^2 |\sin\varphi| = \rho^2 \sin\varphi$$

noting that $\varphi \in [0,\pi] \Rightarrow \sin\varphi \geq 0 \Rightarrow |\sin\varphi| = \sin\varphi$. $\square$

EXAMPLES

a) Evaluate $I = \iint_{\$} x^2 z \, d\$$ with $\$$ given by

$(\$): (x,y,z) = (\rho\cos\vartheta, \rho\sin\vartheta, \rho), \ \forall(\rho,\vartheta) \in [0,a] \times [0,2\pi]$

Solution

Define

$\$(\rho,\vartheta) = (\rho\cos\vartheta, \rho\sin\vartheta, \rho), \ \forall(\rho,\vartheta) \in [0,a] \times [0,2\pi]$

It follows that

$\partial\$/\partial\rho = (\cos\vartheta, \sin\vartheta, 1)$

$\partial\$/\partial\vartheta = (-\rho\sin\vartheta, \rho\cos\vartheta, 0)$

and therefore the fundamental product is given by

$$\mathcal{R}(\rho,\vartheta|\$) = (\partial\$/\partial\rho) \times (\partial\$/\partial\vartheta) =$$

$$= (\cos\vartheta, \sin\vartheta, 1) \times (-\rho\sin\vartheta, \rho\cos\vartheta, 0) =$$

$$= \left| \begin{matrix} e_1 & e_2 & e_3 \\ \cos\vartheta & \sin\vartheta & 1 \\ -\rho\sin\vartheta & \rho\cos\vartheta & 0 \end{matrix} \right| \begin{matrix} e_1 & e_2 \\ \cos\vartheta & \sin\vartheta \\ -\rho\sin\vartheta & \rho\cos\vartheta \end{matrix} =$$

$$= 0e_1 + 1(-\rho\sin\vartheta)e_2 + (\rho\cos\vartheta)(\cos\vartheta)e_3 - (-\rho\sin\vartheta)(\sin\vartheta)e_3$$

$$- (\rho\cos\vartheta)1e_1 - 0e_2 =$$

$$= (-\rho\cos\vartheta)e_1 + (-\rho\sin\vartheta)e_2 + \rho(\cos^2\vartheta + \sin^2\vartheta)e_3 =$$

$$= -\rho\cos\vartheta\, e_1 - \rho\sin\vartheta\, e_2 + \rho e_3 \Longrightarrow$$

$$\Rightarrow \|\mathcal{R}(\rho,\vartheta|\$)\|^2 = \|(-\rho\cos\vartheta, -\rho\sin\vartheta, \rho)\|^2 =$$

$$= (-\rho\cos\vartheta)^2 + (-\rho\sin\vartheta)^2 + \rho^2 =$$

$$= \rho^2\cos^2\vartheta + \rho^2\sin^2\vartheta + \rho^2 =$$

$$= \rho^2(\cos^2\vartheta + \sin^2\vartheta) + \rho^2 = \rho^2 + \rho^2 = 2\rho^2 \Rightarrow$$

$$\Rightarrow \|\mathcal{R}(\rho,\vartheta|S)\| = \sqrt{2\rho^2} = |\sqrt{2}\rho| = \rho\sqrt{2}$$

Define $A = \{(\rho,\vartheta) \mid \rho \in [0,a] \wedge \vartheta \in [0,2\pi]\}$.

We have

$$I = \iint_S x^2 z \, dS = \iint_A d\rho\, d\vartheta\, (\rho\cos\vartheta)^2 \rho \, \|\mathcal{R}(\rho,\vartheta|S)\| =$$

$$= \iint_A d\rho\, d\vartheta\, (\rho\cos\vartheta)^2 \rho\, (\rho\sqrt{2}) = \iint_A d\rho\, d\vartheta\, \rho^4 \sqrt{2}\cos^2\vartheta =$$

$$= \sqrt{2}\int_0^a d\rho \int_0^{2\pi} d\vartheta\, \rho^4 \cos^2\vartheta =$$

$$= \sqrt{2}\left[\int_0^a d\rho\, \rho^4\right]\left[\int_0^{2\pi} d\vartheta \cos^2\vartheta\right] = \sqrt{2} I_1 I_2$$

with

$$I_1 = \int_0^a d\rho\, \rho^4 = \left[\frac{\rho^5}{5}\right]_0^a = \frac{a^5 - 0^5}{5} = \frac{a^5}{5}$$

$$I_2 = \int_0^{2\pi} \cos^2\vartheta\, d\vartheta = \int_0^{2\pi} \frac{1+\cos(2\vartheta)}{2}\, d\vartheta =$$

$$= \left[\frac{\vartheta}{2} + \frac{\sin(2\vartheta)}{4}\right]_0^{2\pi} =$$

$$= \left[\frac{2\pi}{2} + \frac{\sin(4\pi)}{4}\right] - \left[0 + \frac{\sin 0}{4}\right] =$$

$$= \pi + (1/4)\sin 0 = \pi$$

and therefore

$$I = \sqrt{2} I_1 I_2 = \sqrt{2}\,(a^5/5)\,\pi = \frac{\pi a^5 \sqrt{2}}{5}$$

6) Evaluate the surface integral $I = \iint_S (z-x)\,dS$
with
$S = \{(x,y,z) \in \mathbb{R}^3 \mid z = x+y^2 \land x \in [0,y] \land y \in [0,1]\}$

Solution

Define $g(x,y) = x+y^2$
and $A = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,y] \land y \in [0,1]\}$
and note that
$\partial g/\partial x = (\partial/\partial x)(x+y^2) = 1$
$\partial g/\partial y = (\partial/\partial y)(x+y^2) = 2y$
It follows that

$$dS = \|R(x,y|a)\|\,dx\,dy = \sqrt{1+(\partial g/\partial x)^2+(\partial g/\partial y)^2}\,dx\,dy$$
$$= \sqrt{1+1^2+(2y)^2}\,dx\,dy = \sqrt{2+4y^2}\,dx\,dy$$

and therefore

$$I = \iint_S (z-x)\,dS = \iint_A dx\,dy\,[(x+y^2)-x]\,\|R(x,y|a)\| =$$
$$= \iint_A dx\,dy\; y^2\sqrt{2+4y^2} = \int_0^1 dy \int_0^y dx\; y^2\sqrt{2+4y^2} =$$
$$= \int_0^1 dy\left[y^2\sqrt{2+4y^2}\int_0^y dx\right] = \int_0^1 dy\; y^2(\sqrt{2+4y^2})\,y =$$
$$= \int_0^1 dy\; y^3\sqrt{2+4y^2}$$

Let $p = 2 + 4y^2 \Leftrightarrow 4y^2 = p - 2 \Leftrightarrow y^2 = \frac{p-2}{4}$

Then

$dp = (2+4y^2)' dy = 8y\,dy \Rightarrow y\,dy = (1/8)\,dp$

For $y = 0 \Rightarrow p = 2 + 4 \cdot 0^2 = 2$

For $y = 1 \Rightarrow p = 2 + 4 \cdot 1^2 = 2 + 4 = 6$

It follows that

$$I = \int_2^6 dp\, \frac{p-2}{4} \sqrt{p}\, (1/8) = \frac{1}{32} \int_2^6 dp\, \sqrt{p}\, (p-2)$$

$$= \frac{1}{32} \int_2^6 dp\, (p^{3/2} - 2p^{1/2}) = \frac{1}{32} \left[\frac{p^{5/2}}{5/2} - \frac{2p^{3/2}}{3/2} \right]_2^6$$

$$= \frac{1}{16} \left[\frac{p^2\sqrt{p}}{5} - \frac{2p\sqrt{p}}{3} \right]_2^6 =$$

$$= \frac{1}{16} \left[\left(\frac{6^2\sqrt{6}}{5} - \frac{2 \cdot 6\sqrt{6}}{3} \right) - \left(\frac{2^2\sqrt{2}}{5} - \frac{2 \cdot 2\sqrt{2}}{3} \right) \right]$$

$$= \frac{1}{16} \left[\left(\frac{36}{5} - 4 \right)\sqrt{6} - \left(\frac{4}{5} - \frac{4}{3} \right)\sqrt{2} \right]$$

$$= \frac{1}{16} \left[\frac{36 - 5 \cdot 4}{5} \sqrt{6} - \frac{4 \cdot 3 - 4 \cdot 5}{5 \cdot 3} \sqrt{2} \right] =$$

$$= \frac{1}{16} \left[\frac{36 - 20}{5} \sqrt{6} - \frac{-8}{15} \sqrt{2} \right]$$

$$= \frac{1}{16} \left(\frac{16}{5}\sqrt{6} + \frac{8}{15}\sqrt{2} \right) = \frac{\sqrt{6}}{5} + \frac{\sqrt{2}}{30}$$

$$= \frac{6\sqrt{6} + \sqrt{2}}{30}$$

c) Evaluate $I = \iint_S (z, x, 1) \cdot dS$

with $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \wedge z \geq 0\}$ oriented away from the origin.

Solution

Define $F(x, y, z) = (z, x, 1)$, $\forall (x, y, z) \in \mathbb{R}^3$.

Since

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \wedge z \geq 0\} =$$
$$= \{(\sin\varphi \cos\vartheta, \sin\varphi \sin\vartheta, \cos\varphi) \mid \varphi \in [0, \pi/2] \wedge \vartheta \in [0, 2\pi]\}$$

we define

$$A = \{(\varphi, \vartheta) \mid \varphi \in [0, \pi/2] \wedge \vartheta \in [0, 2\pi]\}.$$

and note that the fundamental product is given by:

$$R(\varphi, \vartheta) = \rho^2 (\sin^2\varphi \cos\vartheta, \sin^2\varphi \sin\vartheta, \sin\varphi \cos\varphi)$$

It follows that

$$I = \iint_S (z, x, 1) \cdot dS =$$
$$= \iint_A d\varphi\, d\vartheta\, F(\sin\varphi \cos\vartheta, \sin\varphi \sin\vartheta, \cos\varphi) \cdot R(\varphi, \vartheta \mid S) =$$
$$= \iint_A d\varphi\, d\vartheta\, (\cos\varphi, \sin\varphi \cos\vartheta, 1) \cdot (\sin^2\varphi \cos\vartheta, \sin^2\varphi \sin\vartheta, \sin\varphi \cos\varphi)$$
$$= \iint_A d\varphi\, d\vartheta \left[\cos\varphi \sin^2\varphi \cos\vartheta + \sin\varphi \cos\vartheta \sin^2\varphi \sin\vartheta + \sin\varphi \cos\varphi\right]$$
$$= \iint_A d\varphi\, d\vartheta \left[(\cos\varphi \sin^2\varphi) \cos\vartheta + (1/2) \sin^3\varphi \sin(2\vartheta) + \sin\varphi \cos\varphi\right]$$

$$= I_1 + (1/2)I_2 + I_3$$

with

$$I_1 = \iint_A d\varphi d\vartheta\ (\cos\varphi \sin^2\varphi)\cos\vartheta =$$

$$= \int_0^{\pi/2} d\varphi \int_0^{2\pi} d\vartheta\ (\cos\varphi \sin^2\varphi)\cos\vartheta =$$

$$= \left[\int_0^{2\pi} d\vartheta\ \cos\vartheta\right]\left[\int_0^{\pi/2} d\varphi\ \cos\varphi \sin^2\varphi\right] =$$

$$= \left[\sin\vartheta\right]_0^{2\pi}\left[\int_0^{\pi/2} d\varphi\ \cos\varphi \sin^2\varphi\right] =$$

$$= (\sin(2\pi) - \sin 0)\left[\int_0^{\pi/2} d\varphi\ \cos\varphi \sin^2\varphi\right] =$$

$$= 0\int_0^{\pi/2} d\varphi\ \cos\varphi \sin^2\varphi = 0$$

and

$$I_2 = \iint_A d\varphi d\vartheta\ \sin^3\varphi \sin(2\vartheta) = \int_0^{\pi/2} d\varphi \int_0^{2\pi} d\vartheta\ \sin^3\varphi \sin(2\vartheta)$$

$$= \left[\int_0^{2\pi} d\vartheta \sin(2\vartheta)\right]\left[\int_0^{\pi/2} d\varphi\ \sin^3\varphi\right] =$$

$$= \left[\frac{-\cos(2\vartheta)}{2}\right]_0^{2\pi} \int_0^{\pi/2} d\varphi\ \sin^3\varphi =$$

$$= \left[\frac{(-\cos(2\pi)) - (-\cos 0)}{2}\right]\int_0^{\pi/2} d\varphi\ \sin^3\varphi =$$

$$= 0$$

and

$$I_3 = \iint_A d\varphi d\vartheta \sin\varphi \cdot \cos\varphi = (1/2) \iint_A d\varphi d\vartheta \sin(2\varphi) =$$

$$= \int_0^{\pi/2} d\varphi \int_0^{2\pi} d\vartheta \sin(2\varphi) = \int_0^{\pi/2} d\varphi \left[\sin(2\varphi) \int_0^{2\pi} d\vartheta \right] =$$

$$= \int_0^{\pi/2} d\varphi \, 2\pi \sin(2\varphi) = 2\pi \left[\frac{-\cos(2\varphi)}{2} \right]_0^{\pi/2} =$$

$$= \pi \left[-\cos(2\varphi) \right]_0^{\pi/2} = \pi \left[(-\cos(\pi)) - (-\cos 0) \right] =$$

$$= \pi \left[(-(-1)) - (-1) \right] = \pi \left[1 + 1 \right] = 2\pi$$

and therefore

$$I = I_1 + (1/2) I_2 + I_3 = 0 + (1/2) 0 + 2\pi = 2\pi$$

EXERCISES

(19) Evaluate the following surface integrals using the definition of the surface integral

a) $I = \iint_S z \, dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x,z \in [0,a] \wedge y = a^2 - z^2\}$
and $a \in (0,+\infty)$

b) $I = \iint_S x^2 \, dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = R^2 \wedge x,y,z \in [0,+\infty)\}$
and $R \in (0,+\infty)$.

c) $I = \iint_S \dfrac{x^2 \, dS}{a-z}$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid z \in [0,b] \wedge z = a^2 - x^2 - y^2\}$
and $a \in (0,+\infty)$ and $b \in (0,a)$

d) $I = \iint_S e^{-z} \, dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = a \wedge z \in [0,a]\}$
and $a \in [0,+\infty)$

e) $I = \iint_S \dfrac{z \, dS}{x^2+y^2+z^2}$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid z \in [0,a] \wedge x^2+y^2+z^2 = a^2\}$
and $a \in (0,+\infty)$

f) $I=\iint_S y\,dS$

with $S=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=a^2 \wedge z\in[0,b]\}$
and $a\in(0,+\infty)$ and $b\in(0,a)$.

g) $I=\iint_S z\,dS$

with $S=\{(x,y,z)\in\mathbb{R}^3 \mid x,y\in[0,a] \wedge z=x^3\}$
and $a\in[0,+\infty)$

h) $I=\iint_S \frac{dS}{z}$

with $S=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=a^2 \wedge 1\leq x^2+y^2\leq b^2$
$\wedge z\geq 0\}$
and $a\in(0,+\infty)$ and $b\in(0,a)$

(10) Evaluate the following surface integrals using the surface integral definition

a) $I=\iint_S (e^z, z, x)\cdot d\vec{S}$

with (S): $(x,y,z)=(st, s+t, s)$, $\forall (s,t)\in[0,a]\times[0,a]$
and $a\in(0,+\infty)$.

b) $I=\iint_S (0,0,z^2)\cdot d\vec{S}$

with (S): $(x,y,z)=(r\cos\varphi, r\sin\varphi, r)$, $\forall(r,\varphi)\in[0,a]\times[0,2\pi]$
and $a\in(0,+\infty)$

c) $I = \iint_S (y, z, 0) \cdot dS$

with $(S): (x, y, z) = (s^3 - t, s + t, t^2), \forall (s, t) \in [0, a] \times [0, b]$
and $a, b \in (0, +\infty)$

d) $I = \iint_S (z, 0, z^2) \cdot dS$

with $(S): (x, y, z) = (s \cosh(t), s \sinh(t), s)$,
$\forall (s, t) \in [0, a] \times [0, b]$
and $a, b \in (0, +\infty)$

e) $I = \iint_S (0, 3, x) \cdot dS$

with $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = a^2 \wedge x, y, z \in [0, +\infty)\}$
and $a \in (0, +\infty)$, oriented away from the origin.

f) $I = \iint_S (x, y, z) \cdot dS$

with $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = a^2 \wedge z \in [b, c]\}$
and $a \in (0, +\infty)$ and $b, c \in (-a, a)$, oriented away from the origin.

g) $I = \iint_S (y^2, -2, x) \cdot dS$

with $S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = a \wedge x, y, z \in [0, +\infty)\}$
and $a \in (0, +\infty)$, oriented away from the origin.

▼ Stokes and Gauss theorems

Jordan-Bounded surface

<u>Def</u>: Let $\mathcal{S} = \{a(t,s) \mid (t,s) \in A\}$ be a surface with $A \subseteq \mathbb{R}^2$ and $a : A \longrightarrow \mathbb{R}^3$. Let ∂A be the boundary of A.

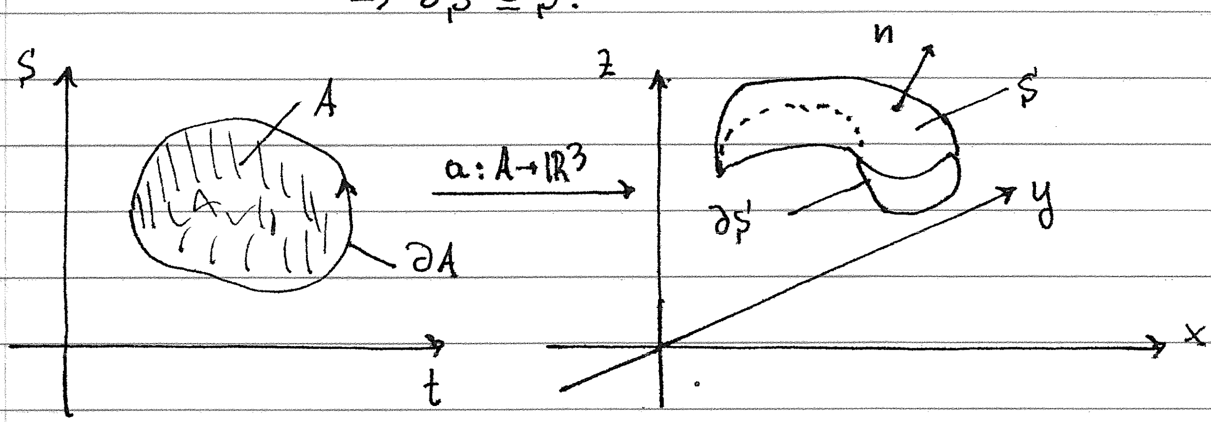
a) We define the boundary $\partial\mathcal{S}$ of the surface $\mathcal{S}$ as $\partial \mathcal{S} = \{a(t,s) \mid (t,s) \in \partial A\}$, <u>if</u> $\mathcal{S}$ is a <u>simple surface</u>

b) We say that

$$\boxed{\mathcal{S} \text{ Jordan-Bounded} \Leftrightarrow \begin{cases} \mathcal{S} \text{ simple} \wedge \mathcal{S} \text{ smooth} \\ \partial A \in \mathrm{Jord}(\mathbb{R}^2) \end{cases}}$$

▶ Note that by definition:

$\mathcal{S}$ surface $\Rightarrow A$ closed set $\Rightarrow \partial A \subseteq A \Rightarrow$
$\Rightarrow \partial\mathcal{S} \subseteq \mathcal{S}$.



▶ By convention, the direction of ∂A is counterclockwise, so that $\omega(\partial A) = +1$.

Stokes theorem

Thm: Let $S = \{a(t,s) \mid (t,s) \in A\}$ be a surface with $A \subseteq \mathbb{R}^2$, and let $f: B \longrightarrow \mathbb{R}^3$ be a vector field with $B \subseteq \mathbb{R}^3$ and $S \subseteq B$. Assume that

a) S is a Jordan-bounded surface.
b) $a(t,s)$ has continuous 2nd partial derivatives on A
c) f differentiable in B
d) ∇f continuous in B

Then:

$$\boxed{\iint_S (\nabla \times f) \cdot dS = \oint_{\partial S} f \cdot dl}$$

- The direction of the path ∂S is determined by the restriction that $\omega(\partial A) = +1$ and the mapping $a: A \longrightarrow \mathbb{R}^3$. Since $\partial A \in \mathrm{Jord}(\mathbb{R}^2)$, it follows that $\partial S \in \mathrm{Loop}(\mathbb{R}^3)$, which means that the line integral above is circular, hence the $\oint$ notation.

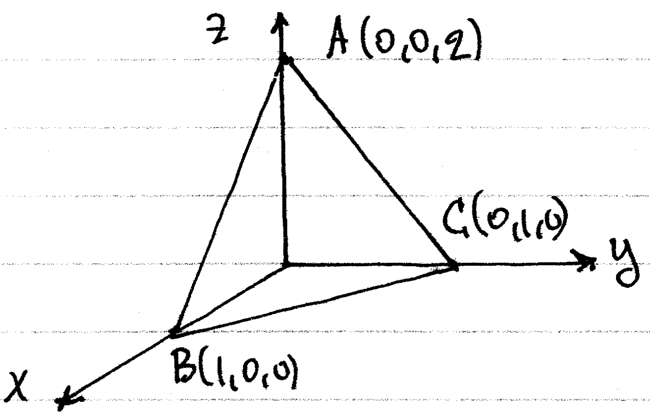
To formulate the Gauss divergence theorem we have to define first what we mean by:

a) Orientable surface
b) Closed surface.

EXAMPLES

a) Evaluate $I = \oint_C F \cdot dl$ with C a triangle with vertices $A(0,0,2)$, $B(1,0,0)$, $C(0,1,0)$ whose projection to the xy-plane is traversed counterclockwise and $f(x,y,z) = (2z, 8x-3y, 3x+y)$, $\forall (x,y,z) \in \mathbb{R}^3$.

Solution



- We need the cartesian equation for the plane (p) defined by the points A, B, C.

$\begin{cases} A(0,0,2) \\ B(1,0,0) \end{cases} \Rightarrow \vec{AB} = (1-0, 0-0, 0-2) = (1,0,-2)$

$\begin{cases} A(0,0,2) \\ C(0,1,0) \end{cases} \Rightarrow \vec{AC} = (0-0, 1-0, 0-2) = (0,1,-2)$

and therefore

$\vec{AB} \times \vec{AC} = (1,0,-2) \times (0,1,-2) =$

$$= \left|\begin{matrix} e_1 & e_2 & e_3 \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{matrix}\right| \begin{matrix} e_1 & e_2 \\ 1 & 0 \\ 0 & 1 \end{matrix} =$$

$= 0e_1 + 0e_2 + (1\cdot 1)e_3 - 0e_3 - 1(-2)e_1 - 1(-2)e_2$

$= e_3 + 2e_1 + 2e_2 = (2,2,1)$

It follows that

$(x,y,z) \in (p) \Leftrightarrow (\vec{AB} \times \vec{AC}) \cdot [(x,y,z) - (0,0,2)] = 0$

$\Leftrightarrow (2,2,1) \cdot (x,y,z-2) = 0 \Leftrightarrow$

$\Leftrightarrow 2x + 2y + (z-2) = 0 \Leftrightarrow z = 2 - 2x - 2y$

and therefore

$(p): z = 2 - 2x - 2y.$

Define $f(x,y) = 2 - 2x - 2y$, $\forall (x,y) \in \mathbb{R}^2$

The corresponding fundamental product is given by

$\mathcal{R}(x,y|S) = (-\partial f/\partial x, -\partial f/\partial y, 1) = (2,2,1)$

The projection of the triangle $\overset{\Delta}{ABC}$ onto the xy plane is given by:

$A = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1] \wedge y \in [0,1-x]\}$

The triangle $\overset{\Delta}{ABC}$ is given by

$S = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in A \wedge z = 2 - 2x - 2y\}$.

To apply the Stokes theorem, we calculate $\nabla \times F$:

$\nabla \times F = \nabla \times (2z, 8x-3y, 3x+y) =$

$$= \left|\begin{matrix} e_1 & e_2 & e_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z & 8x-3y & 3x+y \end{matrix}\right| \begin{matrix} e_1 & e_2 \\ \partial/\partial x & \partial/\partial y \\ 2z & 8x-3y \end{matrix}$$

$$= e_1(\partial/\partial y)(3x+y) + e_2(\partial/\partial z)(2z) + e_3(\partial/\partial x)(8x-3y)$$
$$- e_3(\partial/\partial y)(2z) - e_1(\partial/\partial z)(8x-3y) - e_2(\partial/\partial x)(3x+y) =$$
$$= e_1 + 2e_2 + 8e_3 - 0e_3 - 0e_1 - 3e_2 = e_1 - e_2 + 8e_3$$
$$= (1,-1,8).$$

From the Stokes theorem:

$$I = \oint_C F \cdot dl = \iint_S (\nabla \times F) \cdot dS = \iint_A dx\,dy\, (\nabla \times F) \cdot R(x,y|S)$$
$$= \iint_A dx\,dy\, (1,-1,8) \cdot (2,2,1) =$$
$$= [1 \cdot 2 + (-1) \cdot 2 + 8 \cdot 1] \iint_A dx\,dy =$$
$$= (2-2+8) \int_0^1 dx \int_0^{1-x} dy = 8 \int_0^1 dx\, (1-x) =$$
$$= 8\left[x - x^2/2\right]_0^1 = 8\left[1 - 1^2/2\right] = 8 \cdot (1/2) = 4$$

8) Evaluate the integral $I = \oint_{C} F \cdot dl$ with

$$F(x,y,z) = (-y^2, x, z^2), \quad \forall (x,y,z) \in \mathbb{R}^3$$

and C the curve defined by the intersection of the plane $(p): y+z=2$ with the cylinder $(c): x^2+y^2=1$.

Solution

The curve C is given by

$$C = \{(x,y,z) \in \mathbb{R}^3 \mid y+z=2 \land x^2+y^2=1\} =$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid z=2-y \land x^2+y^2=1\} =$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid z=2-y \land (x,y) \in \partial A\}$$

with ∂A the projection of C onto the xy-plane given by

$$\partial A = \{(\cos\vartheta, \sin\vartheta) \mid \vartheta \in [0, 2\pi]\}$$

which is the boundary of the disk A given by

$$A = \{(r\cos\vartheta, r\sin\vartheta) \mid r \in [0,1] \land \vartheta \in [0, 2\pi]\}$$

Define the surface S so that

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid z = 2-y \land (x,y) \in A\}.$$

and note that $C = \partial S$.

To evaluate I via the Stokes theorem, we need $\nabla \times F$ and $\mathbf{R}(x,y \mid S)$.

- Fundamental product $\mathbf{R}(x,y \mid S)$

Define $z = f(x,y) = 2-y$ and note that

$$\mathbf{R}(x,y \mid S) = (-\partial f/\partial x, -\partial f/\partial y, 1) =$$
$$= (-(\partial/\partial x)(2-y), -(\partial/\partial y)(2-y), 1) =$$
$$= (0, 1, 1)$$

▶ Curl $\nabla \times F$

$$\nabla \times F = \nabla \times (-y^2, x, z^2) =$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix} \begin{matrix} e_1 & e_2 \\ \partial/\partial x & \partial/\partial y \\ -y^2 & x \end{matrix} =$$

$$= e_1(\partial/\partial y)(z^2) + e_2(\partial/\partial z)(-y^2) + e_3(\partial/\partial x)\,x -$$
$$- e_3(\partial/\partial y)(-y^2) - e_1(\partial/\partial z)\,x - e_2(\partial/\partial x)\,z^2 =$$
$$= 0e_1 + 0e_2 + e_3 + 2y\,e_3 - 0e_1 - 0e_2 =$$
$$= (1+2y)\,e_3 = (0, 0, 1+2y)$$

▶ The integral

From the Stokes theorem we have:

$$I = \oint_C F \cdot d\ell = \iint_S (\nabla \times F) \cdot dS = \iint_A dxdy\, (\nabla \times F) \cdot R(x,y|S)$$
$$= \iint_A dxdy\, (0, 0, 1+2y) \cdot (0, 1, 1) = \iint_A dxdy\, (1+2y)$$

Change variables to polar coordinates:
$(x,y) = (r\cos\vartheta, r\sin\vartheta) \Rightarrow dxdy = r\,dr\,d\vartheta$
Define $B = \{(r,\vartheta) \mid r \in [0,1] \wedge \vartheta \in [0, 2\pi]\}$.
It follows that:

$$I = \iint_A dxdy\, (1+2y) = \iint_B dr\,d\vartheta\, r\,(1 + 2r\sin\vartheta)$$
$$= \iint_B dr\,d\vartheta\, (r + 2r^2 \sin\vartheta) = \int_0^{2\pi} d\vartheta \int_0^1 dr\, (r + 2r^2 \sin\vartheta) =$$

$$= \int_0^{2\pi} d\vartheta \left[\frac{r^2}{2} + \frac{2r^3}{3} \sin\vartheta \right]_0^1 =$$

$$= \int_0^{2\pi} d\vartheta \left[\frac{1^2 - 0^2}{2} + \frac{2(1^3 - 0^3)}{3} \sin\vartheta \right] =$$

$$= \int_0^{2\pi} d\vartheta \left(\frac{1}{2} + \frac{2}{3} \sin\vartheta \right) =$$

$$= \left[\frac{\vartheta}{2} - \frac{2}{3} \cos\vartheta \right]_0^{2\pi} =$$

$$= \left[\frac{2\pi}{2} - \frac{2}{3} \cos(2\pi) \right] - \left[0 - \frac{2}{3} \cos 0 \right] =$$

$$= \pi - 2/3 + 2/3 = \pi$$

EXERCISES

(21) Evaluate the following line integrals using the Stokes theorem

a) $I = \oint_{C} 2z\,dx + x\,dy + 3y\,dz$

with C the curve that is the intersection of the plane $(p): z = x$ and the cylinder $(S): x^2 + y^2 = a^2$, oriented counterclockwise as viewed from above, with $a \in (0, +\infty)$

b) $I = \oint_{C} y\,dx + z\,dy + x\,dz$

with C the triangle with vertices $A_1(0,0,0)$, $A_2(a,0,0)$, $A_3(0,a,a)$, oriented counterclockwise as viewed from above, with $a \in (0, +\infty)$

c) $I = \oint_{C} (y - x)\,dx + (x - z)\,dy + (y - z)\,dz$

with C the boundary of the surface
$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + 3y + 2z = 1 \wedge x, y, z \in [0, +\infty)\}$
oriented counterclockwise as seen from above.

d) $I = \oint_{C} (z - y)\,dx + y\,dy + x\,dz$

with C the intersection of the cylinder $(S_1): x^2 + y^2 = a^2$ with the sphere $(S_2): x^2 + y^2 + z^2 = b^2$, oriented counterclockwise as seen from above, with $b \in (0, +\infty)$ and $a \in (0, b)$.

e) $I = \oint_{C} (y-z)dx + (z-x)dy + (x-y)dz$

with C the intersection of the plane $(P): x+z=a$ with the cylinder $(S): x^2+y^2=b^2$, oriented counterclockwise as seen from above, with $a, b \in (0, +\infty)$.

f) $I = \oint_{C} x^2 dx - 2xy\,dy + yz^2 dz$

with C the boundary of the surface S given by
$S = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,a] \wedge y \in [0,b] \wedge z = xy\}$
oriented counterclockwise as seen from above, with $a, b \in (0, +\infty)$.

g) $I = \oint_{C} 2dx + xz\,dy + z^3 dz$

with C the boundary of the surface S given by
$S = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,a] \wedge y \in [0,b] \wedge z = xy^2\}$
oriented counterclockwise as seen from above, with $a, b \in (0, +\infty)$.

h) $I = \oint_{C} 2\,dx + xz\,dy + z^3 dz$

with C the boundary of the surface S given by
$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq a^2 \wedge z = x^2y^2\}$
oriented counterclockwise as seen from above, with $a \in (0, +\infty)$.

(22) Let $F(x,y,z) = f(\|(x,y,z)\|)\,(x,y,z)$ with f continuously differentiable on $\mathbb{R}-\{0\}$. Let C_1 be a Jordan curve. Show that $\oint_{C_1} F \cdot dl = 0$

(23) Let S be a Jordan-bounded surface and let f, g be continuously differentiable scalar fields. Show that

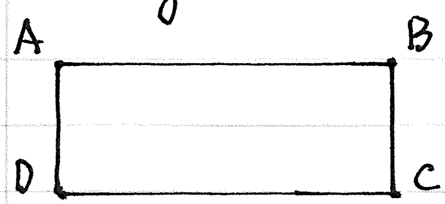
a) $\oint_{\partial S} f(\nabla g) \cdot dl = \iint_S [(\nabla f)\times(\nabla g)] \cdot dS$

b) $\oint_{\partial S} f(\nabla f) \cdot dl = 0$

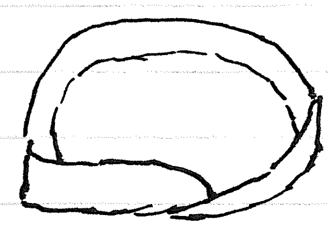
c) $\oint_{\partial S} (f\nabla g + g\nabla f) \cdot dl = 0$

Orientable surfaces

Intuitively, we say that a surface is orientable if and only if it is possible to consistently define the notion of "above the surface" and "below the surface" for every point on the surface. An example of a non-orientable surface is the Möbius strip. A Möbius strip can be constructed by folding an elongated rectangle ABCD by connecting AD with BC and



and twisting it to connect A with C and B with D. A possible parameteric representation of the Möbius strip is given via the following definition:



Def: We define the Möbius strip M as the surface
$$M = \{(x(a,\vartheta), y(a,\vartheta), z(a,\vartheta)) \mid a \in [-1,1] \wedge \vartheta \in [0,2\pi]\}$$
with
$$x(a,\vartheta) = (1 + (a/2)\cos(\vartheta/2))\cos\vartheta$$
$$y(a,\vartheta) = (1 + (a/2)\cos(\vartheta/2))\sin\vartheta$$
$$z(a,\vartheta) = (a/2)\sin(\vartheta/2)$$

We now give the following definitions to capture the concept of an orientable surface:

Def: Let $\mathcal{S} \subseteq \mathbb{R}^3$ be a smooth surface. We define the set $\mathrm{Sub}(\mathcal{S})$ of all <u>subsurfaces</u> of $\mathcal{S}$ as follows:

$$\boxed{\mathcal{S}_0 \in \mathrm{Sub}(\mathcal{S}) \Leftrightarrow \mathcal{S}_0 \subseteq \mathcal{S} \wedge \mathcal{S}_0 \text{ smooth surface}}$$

Def: Let $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{R}^3$ be two surfaces. We say that <u>$\mathcal{S}_1$ and $\mathcal{S}_2$ are homeomorphic</u> (notation: $\mathcal{S}_1 \cong \mathcal{S}_2$) if and only if there is a mapping $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ such that, the following conditions are satisfied:
a) φ one-to-one $\wedge\ \varphi(\mathcal{S}_1) = \mathcal{S}_2$
b) φ continuous at $\mathcal{S}_1$
c) φ^{-1} continuous at $\mathcal{S}_2$

▸ <u>interpretation</u>: If $\mathcal{S}_1 \cong \mathcal{S}_2$, then $\mathcal{S}_1$ can be deformed into $\mathcal{S}_2$ via "motion" and "stretching" but not "tearing". If that is the case, then the surfaces are considered topologically equivalent.

Def: Let $\mathcal{S}$ be a smooth surface. We say that:

$$\boxed{\begin{array}{l} \mathcal{S} \text{ orientable} \Leftrightarrow \forall \mathcal{S}_0 \in \mathrm{Sub}(\mathcal{S}) : \mathcal{S}_0 \not\cong \mathcal{M} \\ \mathcal{S} \text{ non-orientable} \Leftrightarrow \exists \mathcal{S}_0 \in \mathrm{Sub}(\mathcal{S}) : \mathcal{S}_0 \cong \mathcal{M} \end{array}}$$

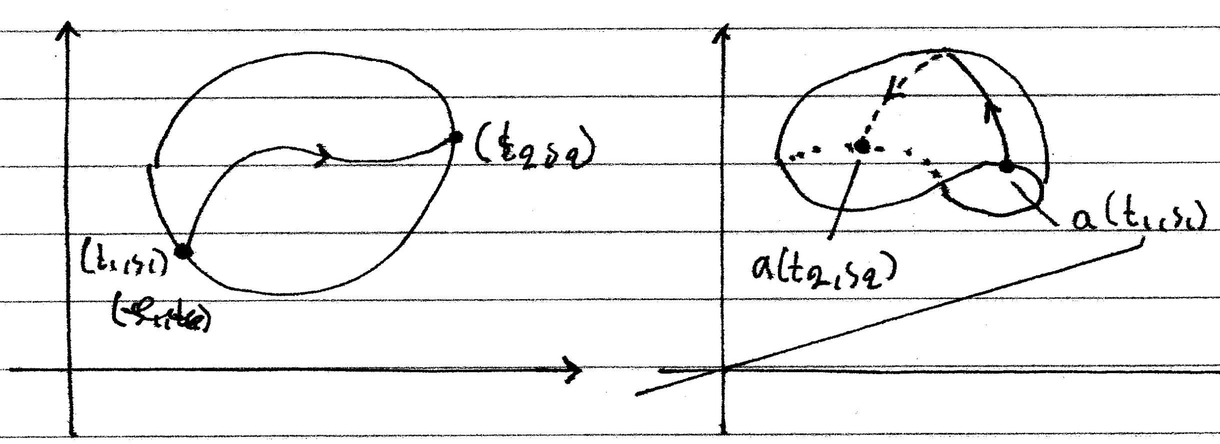
▸ Interpretation : In an orientable surface, none of its subsurfaces is homeomorphic to the Möbius strip $\mathcal{M}$. On the other hand, in a non-orientable surface, there is at least one subsurface that is homeomorphic to the Möbius strip $\mathcal{M}$.

Closed surfaces

▸ Motivation : Consider a smooth surface

$$\mathcal{S} = \{ a(t,s) \mid (t,s) \in A \}$$

By definition, A is closed and simply-connected (i.e. no holes), therefore it has a boundary $\partial A \subseteq A$. For Jordan-bounded surfaces, the image of ∂A, given by $a(\partial A)$, delineates the boundary $\partial \mathcal{S}$ of the surface $\mathcal{S}$. To see that, note that any curve C in A from ~~x~~ $(t_1, s_1) \in \partial A$ to $(t_2, s_2) \in \partial A$ with $(t_1, s_1) \neq (t_2, s_2)$ is NOT mapped into a closed curve $a(C)$ because



$a(t,s)$ is one-to-one (from the hypothesis that $\mathcal{S}$ is simple).

In more general surfaces, a may not necessarily be one-to-one, so it is possible to have $a(t_1,s_1)=a(t_2,s_2)$. Then, the curve C becomes a closed curve, and therefore the point $(x_0,y_0,z_0)=a(t_1,s_1)=a(t_2,s_2)$ should be excluded from the boundary ∂S of the surface S. The remaining points form the boundary ∂S of the surface S. Now we can define a closed surface as one that does not have a boundary.

Def: Let $S=\{a(t,s) \mid (t,s)\in A\}$ be a smooth surface and let the set of excluded points be:

$$P=\{(t,s)\in\partial A \mid \exists (t_0,s_0)\in \partial A-\{(t,s)\}: a(t,s)=a(t_0,s_0)\}$$

a) We define the boundary ∂S of S as:

$$\partial S = a(\partial A - P) = \{a(t,s) \mid (t,s)\in \partial A - P\}$$

b) We say that:

$$S \text{ is closed} \Longleftrightarrow \partial S=\emptyset$$

▶ Motivation: Closed surfaces have an "inside" and an "outside" (e.g. consider a sphere or a torus). Let $x\in\mathbb{R}^3-S$ be a point not on the surface. If x is "inside" S, then all rays emanating from x towards any direction will intersect S at an odd number of points. If x is "outside" S, then all such rays will instead intersect S at an even number of points. This observation motivates the following argument:

<u>Def</u>: Let $x \in \mathbb{R}^3$ be given. We define the set $\mathrm{Ray}(x)$ of all rays emanating from x via

$$L \in \mathrm{Ray}(x) \iff \exists a \in \mathbb{R}^3 - \{0\} : L = \{x + ta \mid t \in [0, +\infty)\}$$

<u>Prop</u>: Let $\mathcal{S} \subseteq \mathbb{R}^3$ be a smooth surface. Then:

a) $\forall x \in \mathbb{R}^3 - \{0\} : \forall L \in \mathrm{Ray}(x) : L \cap \mathcal{S}$ is a finite set.

b) Let $L_0 \in \mathrm{Ray}(x)$ be given. Then

$$|L_0 \cap \mathcal{S}| \text{ even} \Rightarrow \forall L \in \mathrm{Ray}(x) : |L \cap \mathcal{S}| \text{ even}$$
$$|L_0 \cap \mathcal{S}| \text{ odd} \Rightarrow \forall L \in \mathrm{Ray}(x) : |L \cap \mathcal{S}| \text{ odd}$$

(Here $|A|$ represents the cardinality of the set A)

<u>Def</u>: Let $\mathcal{S}$ be a closed smooth surface. Then:

$$x \in \mathrm{int}(\mathcal{S}) \iff x \notin \mathcal{S} \wedge (\exists L \in \mathrm{Ray}(x) : |L \cap \mathcal{S}| \text{ odd})$$
$$x \in \mathrm{ext}(\mathcal{S}) \iff x \notin \mathcal{S} \wedge (\exists L \in \mathrm{Ray}(x) : |L \cap \mathcal{S}| \text{ even}).$$

▸ <u>Closed surface orientation</u>

By convention, we prefer to define closed surfaces via mappings $a : A \to \mathbb{R}^3$ that yield a normal vector that always points towards the exterior $\mathrm{ext}(\mathcal{S})$ of the surface for every point on the surface. Such surfaces are called <u>positively oriented</u>. We give, therefore the following definition:

Def : Let $S = \{a(t,s) \mid (t,s) \in A\}$ be a closed and smooth surface and let $n(t,s|a)$ be the normal vector corresponding to the given representation $a(t,s)$. We say that:

$a(t,s)$ is positively oriented $\Longleftrightarrow$
$$\Longleftrightarrow \forall (t,s) \in A : \exists \tau_0 \in (0,+\infty) : \{a(t,s) + \tau n(t,s|a) \mid \tau \in (0,\tau_0)\} \subseteq$$
$$\subseteq \mathrm{ext}(S)$$

→ Gauss divergence theorem

Thm : Let $S = \{a(t,s) \mid (t,s) \in A\}$ be a surface and let $f : B \to \mathbb{R}^3$ be a vector field with $B \subseteq \mathbb{R}^3$.

We assume that:

a) S smooth $\wedge$ S closed $\wedge$ $a(t,s)$ positively oriented

b) $S \cup \mathrm{int}(S) \subseteq B$

c) f differentiable at $S \cup \mathrm{int}(S)$

d) ∇f continuous at $S \cup \mathrm{int}(S)$

Then:

$$\boxed{\iint_S f \cdot dS = \iiint_{\mathrm{int}(S)} (\nabla \cdot f)\, dx\, dy\, dz}$$

EXAMPLES

a) Evaluate $I = \iint_{\$} F \bullet d\$$ with

$F(x,y,z) = (x^3+y^3, y^3+z^3, z^3+x^3), \forall (x,y,z) \in \mathbb{R}^3$ and $\$$ a hemisphere with center $O(0,0,0)$ and radius 1 above the xy-plane

Solution

Define

$A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1 \wedge z \geq 0\} =$

$= \{(\rho\sin\varphi\cos\vartheta, \rho\sin\varphi\sin\vartheta, \rho\cos\varphi) \mid \rho \in [0,1] \wedge \varphi \in [0,\pi/2] \wedge \vartheta \in [0,2\pi]\}$

and note that $\partial A = \$$. Then:

$$I = \iint_{\$} F \bullet d\$ = \iiint_A dx\,dy\,dz (\nabla \bullet F) =$$

$$= \iiint_A dx\,dy\,dz [\partial F/\partial x + \partial F/\partial y + \partial F/\partial z] =$$

$$= \iiint_A dx\,dy\,dz [(\partial/\partial x)(x^3+y^3) + (\partial/\partial y)(y^3+z^3) + (\partial/\partial z)(z^3+x^3)]$$

$$= \iiint_A dx\,dy\,dz [3x^2+3y^2+3z^2]$$

$$= \iiint_A dx\,dy\,dz\, 3(x^2+y^2+z^2)$$

Define

$B = \{(\rho,\varphi,\vartheta) \mid \rho \in [0,1] \wedge \varphi \in [0,\pi/2] \wedge \vartheta \in [0,2\pi]\}$

and do change of variables:

$$(x,y,z) = (\rho \sin\varphi \cos\vartheta, \rho \sin\varphi \sin\vartheta, \rho \cos\varphi) \Rightarrow$$

$$\Rightarrow dx\,dy\,dz = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\vartheta$$

It follows that

$$I = \iiint_A dx\,dy\,dz \; 3(x^2+y^2+z^2) = \iiint_B 3\rho^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\vartheta =$$

$$= \int_0^1 d\rho \int_0^{\pi/2} d\varphi \int_0^{2\pi} d\vartheta \; 3\rho^4 \sin\varphi =$$

$$= \left[\int_0^1 d\rho \, 3\rho^4\right]\left[\int_0^{\pi/2} \sin\varphi \, d\varphi\right]\left[\int_0^{2\pi} d\vartheta\right] =$$

$$= \left[\frac{3\rho^5}{5}\right]_0^1 \left[-\cos\varphi\right]_0^{\pi/2} 2\pi =$$

$$= 2\pi \, \frac{3(1^5 - 0^5)}{5} \left[(-\cos(\pi/2)) - (-\cos 0)\right] =$$

$$= \frac{6\pi}{5} (0 - (-1)) = 6\pi/5 .$$

6) Evaluate the integral $I = \iint_S F \cdot dS$ where

$S = \partial A$ is the boundary of the solid
$A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,1] \wedge y \in [0,2] \wedge z \in [0,3]\}$
and $F(x,y,z) = (x^2y, xz, yz^3)$.

Solution

$$I = \iint_S F \cdot dS = \iiint_A dxdydz\, \nabla \cdot F =$$

$$= \iiint_A dxdydz\, [\partial F/\partial x + \partial F/\partial y + \partial F/\partial z] =$$

$$= \iiint_A dxdydz \left[(\partial/\partial x)(x^2y) + (\partial/\partial y)(xz) + (\partial/\partial z)(yz^3) \right] =$$

$$= \iiint_A dxdydz\, (2xy + 0 + 3yz^2) =$$

$$= \int_0^1 dx \int_0^2 dy \int_0^3 dz\, (2xy + 3yz^2) =$$

$$= \int_0^1 dx \int_0^2 dy \left[2xyz + yz^3 \right]_{z=0}^{z=3} =$$

$$= \int_0^1 dx \int_0^2 dy\, (6xy + 27y) =$$

$$= \int_0^1 dx \left[3xy^2 + \frac{27y^2}{2} \right]_0^2 = \int_0^1 dx \left[3x \cdot 2^2 + \frac{27 \cdot 2^2}{2} \right]$$

$$= \int_0^1 dx\, (12x + 54) = \left[6x^2 + 54x \right]_0^1 = 6 + 54 = 60.$$

c) Let $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 4 \wedge z \in [0,3]\}$
be a solid cylinder and let
$F(x,y,z) = (x^3 + \tan(yz), y^3 - \exp(xz), 3z + x^3)$
Evaluate the integral
$$I = \iint_{\partial A} F \cdot dS$$

Solution

We note that
$$A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 4 \wedge z \in [0,3]\}$$
$$= \{(r\cos\theta, r\sin\theta, z) \mid r \in [0,2] \wedge \theta \in [0,2\pi] \wedge z \in [0,3]\}$$
and therefore
$$I = \iint_{\partial A} F \cdot dS = \iiint_A dx\,dy\,dz\, \nabla \cdot F =$$
$$= \iiint_A dx\,dy\,dz\, [\partial F/\partial x + \partial F/\partial y + \partial F/\partial z] =$$
$$= \iiint_A dx\,dy\,dz \Big[(\partial/\partial x)(x^3 + \tan(yz)) + (\partial/\partial y)(y^3 - \exp(xz)) + (\partial/\partial z)(3z + x^3) \Big] =$$
$$= \iiint_A dx\,dy\,dz\, (3x^2 + 3y^2 + 3)$$
Change of variables:
$$(x,y,z) = (r\cos\theta, r\sin\theta, z) \Rightarrow dx\,dy\,dz = r\,dr\,d\theta\,dz$$
Define
$$B = \{(r,\theta,z) \mid r \in [0,2] \wedge \theta \in [0,2\pi) \wedge z \in [0,3]\}$$
It follows that:

$$I = \iiint_A dx\,dy\,dz\,[3(x^2+y^2)+3] =$$

$$= \iiint_B [3r^2+3]\,r\,dr\,d\vartheta\,dz = \iiint_B (3r^3+3r)\,dr\,d\vartheta\,dz =$$

$$= \int_0^2 dr \int_0^{2\pi} d\vartheta \int_0^3 dz\,(3r^3+3r) =$$

$$= \left[\int_0^2 dr\,(3r^3+3r)\right]\left[\int_0^{2\pi} d\vartheta\right]\left[\int_0^3 dz\right] =$$

$$= \left[\frac{3r^4}{4}+\frac{3r^2}{2}\right]_0^2 2\pi\cdot 3 =$$

$$= 6\pi\left[\frac{3\cdot 2^4}{4}+\frac{3\cdot 2^2}{2}\right] = 6\pi(12+6) =$$

$$= 6\pi\cdot 18 = 108\pi.$$

EXERCISES

(24) Use the Gauss divergence theorem to evaluate the following surface integrals

a) $I = \iint_{\partial S} (z, x, y) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leqslant a^2 \wedge z \geqslant 0\}$ and $a \in (0, +\infty)$.

b) $I = \iint_{\partial S} (x^3, y^3, z^3) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,a] \wedge y \in [0,b] \wedge z \in [0,c]\}$ and $a, b, c \in (0, +\infty)$.

c) $I = \iint_{\partial S} (\cos(z^2), y, \sin(x^2)) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x, y, z \in [-a, a]\}$ and $a \in (0, +\infty)$.

d) $I = \iint_{\partial S} (x^3, y^3, z^3) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leqslant a^2 \wedge z \geqslant 0\}$ and $a \in (0, +\infty)$.

e) $I = \iint_{\partial S} (x^3 yz, xy^3 z, xyz^3) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,a] \wedge y \in [0,b] \wedge z \in [0,c]\}$ and $a, b, c \in (0, +\infty)$

f) $I = \iint_{\partial S} (ax, by, cz) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leqslant \rho^2\}$ and $a, b, c \in \mathbb{R}$ and $\rho \in (0, +\infty)$.

g) $I = \iint_{\partial S} (x^2, y^2, z^2) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq z \leq a^2 - x^2 - y^2\}$

and $a \in (0, +\infty)$

h) $I = \iint_{\partial S} (x^2+y^2, y^2+z^2, xy) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq y^2+z^2 \leq a^2 \land 0 \leq x \leq b\}$

and $a, b \in (0, +\infty)$

i) $I = \iint_{\partial S} (x^2, y^2, z^2) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z \leq a \land x,y,z \in [0,+\infty)\}$

and $a \in (0, +\infty)$.

j) $I = \iint_{\partial S} (x^3+y, y^3+z, z^3+x) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid a^2 \leq x^2+y^2+z^2 \leq b^2\}$

and $a, b \in (0, +\infty)$ with $a < b$.

(25) Let f, g be scalar fields that are continuously twice differentiable on $\mathbb{R}^3$. Let S be a solid $S \subseteq \mathbb{R}^3$ with boundary ∂S a smooth, closed, and positively oriented surface. Show that:

a) $\displaystyle \iint_{\partial S} \frac{\partial f}{\partial n} \, dS = \iiint_S (\nabla^2 f) \, dx\,dy\,dz$

b) $\nabla^2 f(x,y,z) = 0, \ \forall (x,y,z) \in S \Rightarrow$

$$\Rightarrow \iint_{\partial S} f \frac{\partial f}{\partial n} \, dS = \iiint_S \|\nabla f\|^2 \, dx\,dy\,dz$$

c) $$\iint_{\partial \$} f\,\frac{\partial g}{\partial n}\,d\$ = \iiint_{\$} \left[f\nabla^2 g + (\nabla f)\cdot(\nabla g)\right] dx\,dy\,dz$$

d) $$\iint_{\partial \$} \left(f\,\frac{\partial g}{\partial n} - g\,\frac{\partial f}{\partial n}\right) d\$ = \iiint_{\$} \left[f\nabla^2 g - g\nabla^2 f\right] dx\,dy\,dz$$

(26) Let F be a vector field that is continuously twice differentiable on $\mathbb{R}^3$ and let $\$$ be a solid $\$ \subseteq \mathbb{R}^3$ with boundary $\partial \$$ a closed, smooth, and positively-oriented surface. Use tensor notation and the Gauss theorem to show that

$$\iint_{\partial \$} (\nabla \times F)\cdot d\$ = 0$$

References

The following references were consulted during the preparation of these lecture notes.

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