Lecture Notes on Calculus 1

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Trigonometric identities

$$\boxed{a \pm b} \quad\Downarrow \qquad\qquad \boxed{2a}\quad\Downarrow$$

$$\left.\begin{aligned}
\sin(a \pm b) &= \sin a \cos b \pm \sin b \cos a \\
\cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\
\tan(a \pm b) &= \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b} \\
\cot(a \pm b) &= \frac{\cot a \cot b \mp 1}{\cot b \pm \cot a} \quad \textbf{(!!)}
\end{aligned}\right\} \Longrightarrow
\begin{aligned}
\sin(2a) &= 2 \sin a \cos a \\
\cos(2a) &= \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a \\
\tan(2a) &= \frac{2 \tan a}{1 - \tan^2 a} \\
\cot(2a) &= \frac{\cot^2 a - 1}{2 \cot a}
\end{aligned}$$

- ▶ $\sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b$
- ▶ $\cos(a+b)\cos(a-b) = \cos^2 a - \sin^2 b$

$$\boxed{3a} \Longrightarrow \quad \begin{aligned} \sin(3a) &= -4\sin^3 a + 3 \sin a \\ \cos(3a) &= +4 \cos^3 a - 3 \cos a \end{aligned} \qquad \tan(3a) = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a}$$

In terms of

$$\boxed{\cos 2a}\quad\Downarrow \qquad\qquad \boxed{\tan(a/2)}\quad\Downarrow$$

$$\sin^2 a = \frac{1 - \cos(2a)}{2} \quad \cos^2 a = \frac{1 + \cos(2a)}{2} \quad \sin a = \frac{2\tan(a/2)}{1 + \tan^2(a/2)} \quad \cos a = \frac{1 - \tan^2(a/2)}{1 + \tan^2(a/2)}$$

$$\tan^2 a = \frac{1 - \cos(2a)}{1 + \cos(2a)} \quad \cot^2 a = \frac{1 + \cos(2a)}{1 - \cos(2a)} \quad \tan a = \frac{2\tan(a/2)}{1 - \tan^2(a/2)} \quad \cot a = \frac{1 - \tan^2(a/2)}{2\tan(a/2)}$$

Transformation to

$$\boxed{\text{sum}}\quad\Downarrow \qquad\qquad \boxed{\text{product}}\quad\Downarrow$$

$$\left.\begin{aligned}
2 \sin a \cos b &= \sin(a-b) + \sin(a+b) \\
2 \cos a \cos b &= \cos(a-b) + \cos(a+b) \\
2 \sin a \sin b &= \cos(a-b) - \cos(a+b)
\end{aligned}\right\} \Longrightarrow
\begin{aligned}
\sin a \pm \sin b &= 2 \sin \frac{a \pm b}{2} \cos \frac{a \mp b}{2} \\
\cos a + \cos b &= 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2} \\
\cos a - \cos b &= 2 \sin \frac{a+b}{2} \sin \frac{b-a}{2} \quad \textbf{(!!)} \\
\tan a \pm \tan b &= \frac{\sin(a \pm b)}{\cos a \cos b} \\
\cot a \pm \cot b &= \frac{\sin(b \mp a)}{\sin a \sin b} \quad \textbf{(!!)}
\end{aligned}$$

Also note the factorizations:

- ▶ $1 \pm \sin a = \sin(\pi/2) \pm \sin a = 2 \sin \frac{(\pi/2) \pm a}{2} \cos \frac{(\pi/2) \mp a}{2}$
- ▶ $\sin a \pm \cos b = \sin a \pm \sin(\pi/2 - b) = 2 \sin \frac{a \pm (\pi/2 - b)}{2} \cos \frac{a \mp (\pi/2 - b)}{2}$
- ▶ $1 + \cos a = 2\cos^2(a/2)$
- ▶ $1 - \cos a = 2 \sin^2(a/2)$

CAL1.0: Review of inequalities

Method of sign tables

The method of sign tables relies on the following facts:

1) $f(x) = ax + b$

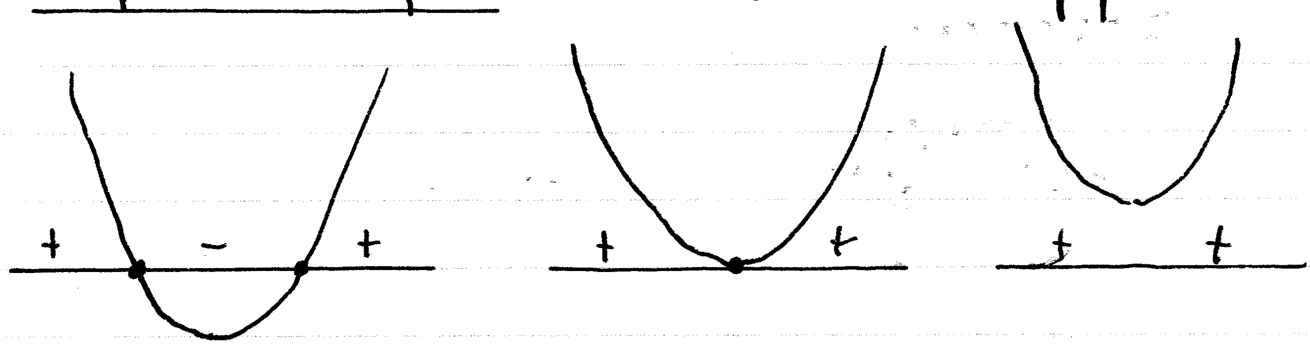
   $a > 0 \Rightarrow f$ increases $\Rightarrow$ goes from negative $\to$ positive

   $a < 0 \Rightarrow f$ decreases $\Rightarrow$ goes from positive $\to$ negative.

2) $f(x) = ax^2 + bx + c$

   Always has the same sign as coefficient "a"
   UNLESS $\Delta = b^2 - 4ac > 0$ (two zeroes) and
   x is between the two zeroes (i.e. $\rho_1 \leq x \leq \rho_2$).

Graphical examples: when $a > 0$ (convex up parabola)



$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
two zeroes	one zero	no zeroes
$\downarrow$	$\downarrow$	$\downarrow$
$\rho_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$	$\rho_1 = \rho_2 = \dfrac{-b}{2a}$	$\rho_{1,2} \notin \mathbb{R}$.

3) $f(x) = (ax+b)^{2k}$ or $f(x) = (ax^2+bx+c)^{2k}$
with $k \in \mathbb{N}$
are <u>ALWAYS POSITIVE</u>

4) $f(x) = (ax+b)^{2k+1}$ or $f(x) = (ax^2+bx+c)^{2k+1}$

$\boxed{\text{have the same sign as}}$

$f(x) = ax+b$ $\qquad\qquad$ $f(x) = ax^2+bx+c$

Applications of sign tables

1) Polynomial inequalities
2) Rational inequalities
3) Elimination of absolute values
4) Monotonicity and Convexity (Calculus).

Polynomial Inequalities

Form: $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \gtreqless 0$

Method:
- 1 Move everything to left side
- 2 Factor the left side
- 3 Identify and sort the zeroes of each factor.
- 4 Make a sign table for each factor.
- 5 Multiply the signs to get signs for inequality as a whole.
- 6 Identify carefully the intervals that correspond to the desired solution.

Examples

1) ~~xde~~ $x^3 - 4x \geqslant -3$

Solution: $x^3 - 4x \geqslant -3 \Leftrightarrow x^3 - 4x + 3 \geqslant 0 \Leftrightarrow$

$$\Leftrightarrow x^3 - x - 3x + 3 \geqslant 0 \Leftrightarrow$$
$$\Leftrightarrow x(x^2 - 1) - 3(x-1) \geqslant 0$$
$$\Leftrightarrow x(x-1)(x+1) - 3(x-1) \geqslant 0 \Leftrightarrow$$
$$\Leftrightarrow (x-1)[x(x+1) - 3] \geqslant 0$$
$$\Leftrightarrow (x-1)(x^2 + x - 3) \geqslant 0$$

$x-1 \to$ zeroes : $+1$

$x^2+x-3 \to \Delta = 1^2 - 4 \cdot 1 \cdot (-3) = 13 > 0$

$\Rightarrow$ two zeroes

$$\rho_{1,2} = \frac{-1 \pm \sqrt{13}}{2}$$

Sort zeroes: $(-1-\sqrt{13})/2$, 1 , $(-1+\sqrt{13})/2$

To compare these two:

$$\frac{-1+\sqrt{13}}{2} > \frac{-1+\sqrt{9}}{2} = \frac{-1+3}{2} = 1$$

- check the nearest perfect squares of the number under the root. This is usually sufficient for a decisive comparison.

x		$(-1-\sqrt{13})/2$		1		$(-1+\sqrt{13})/2$	
$x-1$	$-$		$-$	0	$+$		$+$
x^2+x-3	$+$	0	$-$		$-$	0	$+$
$f(x)$	$-$	0	$+$	0	$-$	0	$+$

thus $x \in \left[\frac{-1-\sqrt{13}}{2}, 1\right] \cup \left[\frac{-1+\sqrt{13}}{2}, +\infty\right)$.

2) $(2x^4 - x^2)(x^2-3)^2(2-x)^3 < 0$ (1)

▸ Note that now we do not allow equality with 0.

Solution:

(1) $\Leftrightarrow$ $x^2(2x^2-1)(x^2-\cancel{2}3)^2(2-x)^3 < 0.$

Zeroes: $0, \pm 1/\sqrt{2} = \pm \frac{\sqrt{2}}{2}, \pm\sqrt{3}, 2$

Sorted: $-\sqrt{3}, -\sqrt{2}/2, 0, \sqrt{2}/2, \sqrt{3}, 2.$

x		$-\sqrt{3}$		$-\sqrt{2}/2$		0		$\sqrt{2}/2$		$\sqrt{3}$		2	
x^2	+		+		+	0	+		+		+		+
$2x^2-1$	+		+	0	−		−	0	+		+		+
$(x^2-3)^2$	+	0	+		+		+		+	0	+		+
$(2-x)^3$	+		+		+		+		+		+	0	−
$f(x)$	+	0	+	0	−	0	−	0	+	0	+	0	−

Thus (1) $\Leftrightarrow$ $x \in (-\sqrt{2}/2, 0) \cup (0, \sqrt{2}/2) \cup (2, +\infty)$.

Rational Inequalities

Form : $\dfrac{P(x)}{Q(x)} \gtrless 0$

with P, Q polynomials.

Method : The method entails the same steps as with polynomial inequalities. However, the zeroes of numerator factors must be distinguished from the zeroes of denominator factors.

- Denominator zeroes are shown with the ‡ symbol instead of ϕ in the last entry of your sign table because at these zeroes, the expression is undefined.
- Denominator zeroes are to be excluded from the solution set.

examples

1) $\dfrac{x-5}{x-3} \geqslant \dfrac{x-2}{x-1}$ (1)

Solution:

$$(1) \Leftrightarrow \frac{x-5}{x-3} - \frac{x-2}{x-1} \geqslant 0 \Leftrightarrow \frac{(x-5)(x-1)-(x-2)(x-3)}{(x-3)(x-1)} \geqslant 0$$

$$\Leftrightarrow \frac{(x^2-6x+5)-(x^2-5x+6)}{(x-3)(x-1)} \geq 0$$

$$\Leftrightarrow \frac{(-6+5)x+(5-6)}{(x-3)(x-1)} \geq 0$$

$$\Leftrightarrow \frac{-x-1}{(x-3)(x-1)} \geq 0. \quad (2)$$

Zeroes: $-1, 3, 1$

x		-1		1		3	
$-x-1$	$+$	0	$-$		$-$		$-$
$x-3$	$-$		$-$		$-$	0	$+$
$x-1$	$-$		$-$	0	$+$		$+$
$f(x)$	$+$	0	$-$	$\Vert$	$+$	$\Vert$	$-$

$(2) \Leftrightarrow x \in (-\infty, -1] \cup (1, 3)$

↳ Note that -1 is a zero of $f(x)$ but 1 and 3 are not, so they are not included in the solution.

CAUTION: If the fraction has cancellations then you must find the domain of the inequality before solving it:

example: $$\frac{(x+1)(x^2+4x+4)}{(x^2+5x+6)} \geqslant 0. \quad (1)$$

$$(1) \Leftrightarrow \frac{(x+1)(x+2)^2}{(x+2)(x+3)} \geqslant 0 \Leftrightarrow \frac{(x+1)(x+2)}{x+3} \geqslant 0$$

▶ Domain:

$x^2+5x+6 \neq 0 \Leftrightarrow \underline{x \in \mathbb{R} - \{-2,-3\} = A}$

x		-3		-2		-1	
$x+1$	$-$		$-$		$-$	0	$+$
$x+2$	$-$		$-$	0	$+$		$+$
$x+3$	$-$	0	$+$		$+$		$+$
$f(x)$	$-$	$\|\|$	$+$	$\|\|$	$-$	0	$+$

thus

$(1) \Leftrightarrow \underline{x \in (-3,-2) \cup [-1,+\infty)}$

-2 looks like a numerator zero but it cannot solve the original inequality because the domain
$$A = \mathbb{R} - \{-2,-3\}$$
of the inequality EXCLUDES -2 !!

Elimination of absolute values

Method : Sign tables can be used in conjuction with expressions that involve absolute values.

Example : Simplify
$$f(x) = |x^2+3x+2| + |x+5|.$$

Solution:

Zeroes : $-2, -1, -5$

x		-5		-2		-1	
x^2+3x+2	$+$		$+$	0	$-$	0	$+$
$x+5$	$-$	0	$+$		$+$		$+$

Distinguish 4 cases

a) $x \in (-\infty, -5)$
$$f(x) = (x^2+3x+2) - (x+5) = x^2+2x-3$$
b) $x \in [-5, -2)$
$$f(x) = (x^2+3x+2) + (x+5) = x^2+4x+7$$
c) $x \in [-2, -1)$
$$f(x) = -(x^2+3x+2) + (x+5) = -x^2-2x+3$$
d) $x \in [-1, +\infty)$
$$f(x) = \cdots = x^2+4x+7.$$

thus

$$f(x) = \begin{cases} x^2 + 2x - 3, & x \in (-\infty, -5) \\ x^2 + 4x + 7, & x \in [-5, -2) \cup [-1, +\infty) \\ -x^2 - 2x + 3, & x \in [-2, -1). \end{cases}$$

Similar applications are possible to equations / inequalities with absolute values where a case distinction is necessary in setting up your solution

CAL1.1: Functions and domains

PRELIMINARIES

Sets

- A set is an unordered collection of elements. An element can be a number or another set. We denote sets with upper-case letters: A, B, C, etc.
- Notation

  $x \in A$: x is an element of the set A
  x belongs to A.

  $x \notin A$: x is not an element of the set A
  x does not belong to A.
- Fundamentally, sets are defined via belonging condition of the form:

  $$x \in A \Longleftrightarrow p(x)$$

  Here, $p(x)$ is a predicate, i.e. a statement about x which is either true or false.
  "$\Longleftrightarrow$" means that $x \in A$ implies $p(x)$ and $p(x)$ implies $x \in A$.

Logic Operations

Let p, q be two statements that are either true or false. We define the following logic operations:

$p \wedge q$: p true AND q true

$p \vee q$: p true OR q true
(At least one of p or q is true, or both)

$p \underline{\vee} q$: either p is true OR q is true, but not both.

$p \Rightarrow q$: if p is true, then q is true
p implies q.

$p \Leftrightarrow q$: p is true <u>if and only if</u> q is true.
p implies q and q implies p.

Quantified statements

Let A be a set and $p(x)$ be a predicate. We define the following:

$\forall x \in A : p(x)$ ⟶ <u>For all</u> $x \in A$, $p(x)$ is true

$\exists x \in A : p(x)$ ⟶ There exists at least one $x \in A$ such that $p(x)$ is true.

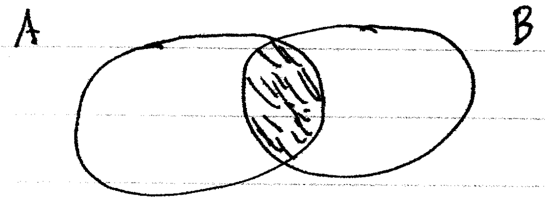
Set algebra

- Sets can be defined by listing elements.
  e.g.: $A = \{2, 3, 6, 8\}$
  Note that elements cannot be repeated and the order of listing is not important. The corresponding belonging condition is:
  $x \in A \Leftrightarrow x = 2 \vee x = 3 \vee x = 6 \vee x = 8$

- Let A, B be two sets. We define the following <u>set operations</u>; in terms of belonging conditions:

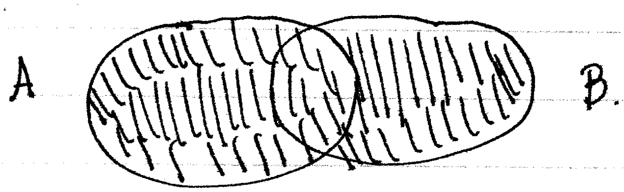
a) <u>Intersection</u> : $A \cap B$

$x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B)$



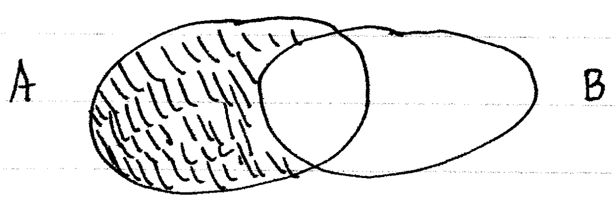
b) <u>Union</u> : $A \cup B$

$x \in A \cup B \Leftrightarrow (x \in A \vee x \in B)$.



c) <u>Difference</u> : $A - B$

$x \in A - B \Leftrightarrow (x \in A \wedge x \notin B)$.



- Let A, B be two sets. We say that
  a) $A = B$ if and only if the sets A, B have the same elements.
  b) $A \subseteq B$ if and only if all the elements of the set A also belong to the set B.

These definitions more formally are written as follows:

$$A \subseteq B \Leftrightarrow \forall x \in A : x \in B$$
$$A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$$

- We define $\emptyset = \{\}$ as the empty set, i.e. a set with no elements.

EXAMPLE

Let $A = \{1,2,3,4,5,6\}$ and $B = \{2,3,7,8\}$. Evaluate the sets $G = (A \cup B) - (A \cap B)$. and $D = A - (A \cup B)$

Solution

$$\begin{aligned} G &= (A \cup B) - (A \cap B) = \\ &= (\{1,2,3,4,5,6\} \cup \{2,3,7,8\}) - (\{1,2,3,4,5,6\} \cap \{2,3,7,8\}) = \\ &= \{1,2,3,4,5,6,7,8\} - \{2,3\} = \\ &= \{1,4,5,6,7,8\} \end{aligned}$$

$$\begin{aligned} D &= A - (A \cup B) \\ &= \{1,2,3,4,5,6\} - (\{1,2,3,4,5,6\} \cup \{2,3,7,8\}) \\ &= \{1,2,3,4,5,6\} - \{1,2,3,4,5,6,7,8\} \\ &= \emptyset. \end{aligned}$$

Number sets

- We define the following number sets:
  1) Natural numbers : $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
  2) Integers: $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$
  3) Rationals numbers: $\mathbb{Q}$.
     A number $x \in \mathbb{Q}$ if and only if there are two integers $a \in \mathbb{Z}$ and $b \in \mathbb{N} - \{0\}$ such that $x = a/b$.
     i.e.: $x \in \mathbb{Q} \Leftrightarrow \exists a \in \mathbb{Z} : \exists b \in \mathbb{N} - \{0\} : x = a/b$.
  4) Real numbers : $\mathbb{R}$
     The set $\mathbb{R}$ of real numbers is informally defined as the set of all numbers that can be approximated by a convergent sequence of rational numbers.
- Let A be a set. and $p(x)$ a predicate. We define
  $$S = \{x \in A \mid p(x)\}$$
  via the following belonging condition:
  $$x \in S \Leftrightarrow (x \in A \wedge p(x)).$$
- We use this to define intervals:

$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$	$[a, +\infty) = \{x \in \mathbb{R} \mid a \leq x\}$
$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$	$(a, +\infty) = \{x \in \mathbb{R} \mid a < x\}$
$(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$	$(-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$
$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$	$(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$

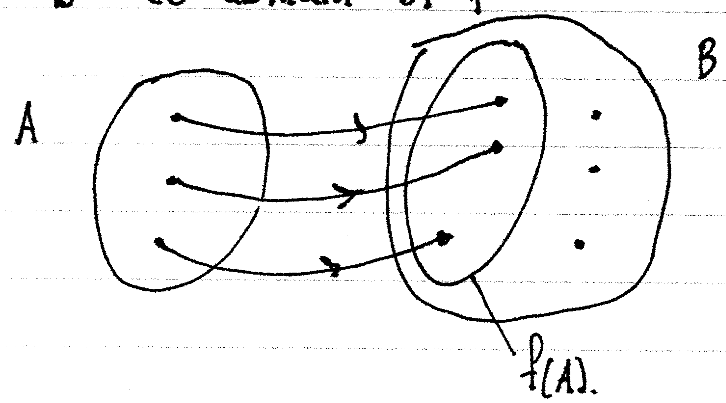
- In general, for a subset $S \subseteq \mathbb{R}$:
  $$S \text{ interval} \Leftrightarrow \forall a, b \in S : (a < b \Rightarrow [a,b] \subseteq S)$$

Mappings and Functions

- A <u>mapping</u> $f: A \to B$ is a rule that maps every element $x \in A$ to a unique element $f(x) \in B$.

  $\mathrm{dom}(f) = A \longrightarrow$ domain of f.

  $B =$ co-domain of f



- A function $f: A \to \mathbb{R}$ is a mapping with domain $A \subseteq \mathbb{R}$ and co-domain $\mathbb{R}$.

$\mapsto$ Range of a mapping/function

- Given a mapping $f: A \to B$, the range $f(A)$ of f is the set of all elements of B for which there is at least one $x \in A$ that maps to these elements. Formally, we define $f(A)$ via the following belonging condition:

  $$y \in f(A) \Leftrightarrow \exists x \in A : f(x) = y.$$

("$y \in f(A)$ if and only if there is at least one $x \in A$ such that $f(x) = y$").

- **notation**: Given an expression $\varphi(x)$ and a predicate $p(x)$ and a set A, the set

$$S = \{\varphi(x) \mid x \in A \wedge p(x)\}$$

is defined via the following belonging condition:

$$y \in S \Leftrightarrow \exists x \in A : (p(x) \wedge y = f(x))$$

Using this notation, we may rewrite the range definition as:

$$f(A) = \{f(x) \mid x \in A\}.$$

- In general, for $S \subseteq A$, we define

$$f(S) = \{f(x) \mid x \in S\}.$$

EXAMPLE

Given the function $f(x) = x^2+3$, $\forall x \in \mathbb{R}$ and $S = \{-1, 0, 1\}$ evaluate $f(S)$.

Solution

Since:

$$\left.\begin{aligned} f(-1) &= (-1)^2+3 = 1+3 = 4 \\ f(0) &= 0^2+3 = 3 \\ f(1) &= 1^2+3 = 1+3 = 4 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow f(S) = f(\{-1, 0, 1\}) = \{3, 4\}.$$

Defining a function / implied domain

- To define a function f, it is necessary to give both
  a) The algorithm for $f(x)$
  b) The domain A of f.

This can be done succinctly using quantifier notation or verbally.

e.g: Let f be the function $f(x) = x^3 + x^2 + 2x + 1$ with domain $A = \mathbb{R}$.

OR more succinctly:

Define $f(x) = x^3 + x^2 + 2x + 1$, $\forall x \in \mathbb{R}$.

- If the domain is not given, then we assume the <u>default domain</u> to be the <u>largest possible subset of $\mathbb{R}$.</u> This is also called, the <u>implied domain</u>.

EXAMPLES

a) Find the implied domain of
$f(x) = (x^2+3x+1)(x+3)$

Solution

No restrictions, therefore $A = \mathbb{R}$.

↳ The default domain of any polynomial is always $A = \mathbb{R}$.

b) Find the implied domain of
$$f(x) = \frac{3x+1}{x^3-4x}$$

Solution

Require $x^3-4x \neq 0$.

Solve $x^3-4x=0 \Leftrightarrow x(x^2-4)=0 \Leftrightarrow x(x-2)(x+2)=0$
$\Leftrightarrow x=0 \lor x-2=0 \lor x+2=0 \Leftrightarrow x=0 \lor x=2 \lor x=-2$
$\Leftrightarrow x \in \{0,-2,2\}$.

It follows that $A = \mathbb{R} - \{-2,0,2\}$.

↳ For functions of the form $f(x) = P(x)/Q(x)$ with P, Q polynomials, we require $Q(x) \neq 0$.

c) Find the implied domain of

$$f(x) = \sqrt{4-x} + \sqrt{x^2+3x+2}$$

Solution

Require: $\begin{cases} 4-x \geq 0 \\ x^2+3x+2 \geq 0 \end{cases}$

Solve: $4-x \geq 0 \Leftrightarrow 4 \geq x \Leftrightarrow x \in (-\infty, 4] = S_1$

For $x^2+3x+2 \geq 0$ we have:

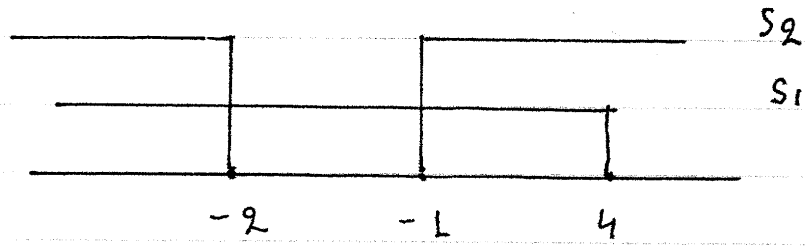
$\Delta = b^2 - 4ac = 3^2 - 4 \cdot 1 \cdot 2 = 9 - 8 = 1 \Rightarrow$

$$\Rightarrow x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-3 \pm \sqrt{1}}{2 \cdot 1} = \frac{-3 \pm 1}{2} = \begin{cases} -2 \\ -1 \end{cases}$$

x		-2		-1	
x^2+3x+2	$+$	0	$-$	0	$+$

It follows that

$x^2+3x+2 \geq 0 \Leftrightarrow x \in (-\infty, -2] \cup [-1, +\infty) = S_2.$



The domain is $A = S_1 \cap S_2 = (-\infty, -2] \cup [-1, 4]$

↳ For functions containing terms of the form $\sqrt{\varphi(x)}$ we require $\varphi(x) \geq 0$.

d) Find the implied domain of $f(x) = \sin x + \tan(3x)$.

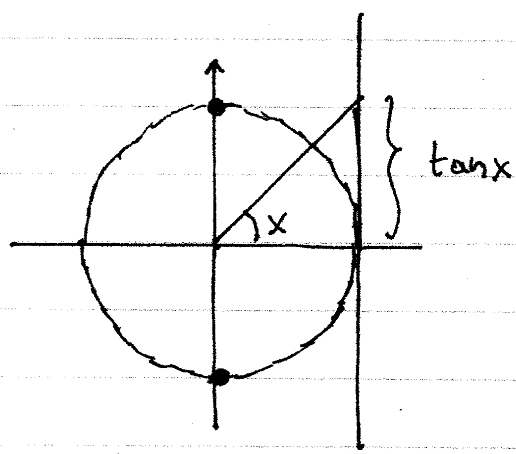
Solution

Require that $\forall k \in \mathbb{Z} : 3x \neq k\pi + \pi/2$

Solve $3x = k\pi + \pi/2 \Leftrightarrow x = k\pi/3 + \pi/6$

It follows that $A = \mathbb{R} - \left\{ \frac{k\pi}{3} + \frac{\pi}{6} \,\middle|\, k \in \mathbb{Z} \right\}$

↳ For functions with terms of the form $\tan(\varphi(x))$ we require: $\forall k \in \mathbb{Z} : \varphi(x) \neq k\pi + \pi/2$.
Note that the domain of $f(x) = \tan x$ is $A = \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}$. On a trigonometric circle this means excluding the points at the top and bottom of the circle.



e) Find the implied domain of $f(x) = \cot(\pi x + \pi/4)$.

Solution

Require $\forall k \in \mathbb{Z}: \pi x + \pi/4 \neq k\pi$.

Solve:

$$\pi x + \pi/4 = k\pi \iff x + 1/4 = k \iff$$

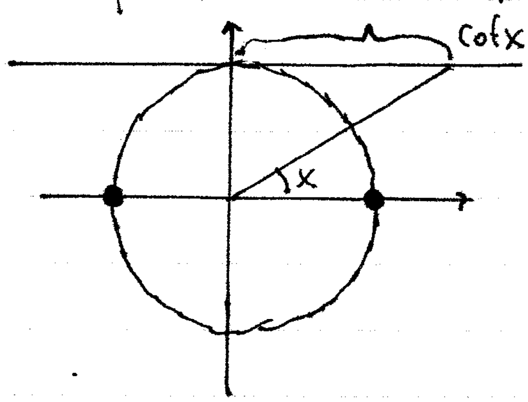
$$\iff x = k - \frac{1}{4} = \frac{4k-1}{4}$$

It follows that $A = \mathbb{R} - \left\{ \frac{4k-1}{4} \mid k \in \mathbb{Z} \right\}$

↳ For functions with terms of the form $\cot(\varphi(x))$ we require: $\forall k \in \mathbb{Z}: \varphi(x) \neq k\pi$.

Note that the domain of $f(x) = \cot x$ is $A = \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}$

On a trigonometric circle this means excluding the points on the left and right of the circle.



↳ See my College Algebra notes for review of solution methods on equations, inequalities, systems of inequalities, etc.

EXERCISES

① Find the domain of the following functions.

a) $f(x) = \dfrac{x}{x^2-6x+9}$

b) $f(x) = \dfrac{x-1}{x^2+5x+6}$

c) $f(x) = \dfrac{(x+1)(x-2)}{(x-2)(x+2)}$

d) $f(x) = \dfrac{x+1}{x+2}$

e) $f(x) = \sqrt{2x-5}$

f) $f(x) = \sqrt{x^2-x-12}$

g) $f(x) = \sqrt{x^2+x+3}$

h) $f(x) = \sqrt{x^2+5x+1}$

i) $f(x) = \sqrt{x-x^3}$

j) $f(x) = -\dfrac{2}{\sqrt{3-x}}$

k) $f(x) = \sqrt{\dfrac{x+2}{x-3}}$

l) $f(x) = \dfrac{\sqrt{x+2}}{\sqrt{x-3}}$

m) $f(x) = \dfrac{1}{\sqrt{9-x^2}} + \sqrt{x^2-4}$

n) $f(x) = \sqrt{-x} + \dfrac{1}{\sqrt{5+x}}$

o) $f(x) = \sqrt{(x^3-1)(x^2-5)}$

p) $f(x) = \sqrt{x^2-4} + \sqrt{x-1}$

q) $f(x) = \dfrac{2}{|x-2|-1}$

r) $f(x) = \sqrt{3-|x|}$

s) $f(x) = \sqrt{|x-2|-1}$

t) $f(x) = \sqrt{|x|+3}$

(2) Likewise, find the domain of the following functions.

a) $f(x) = \tan(\pi x + x/3)$

b) $f(x) = \tan\left(\dfrac{\pi(x+1)}{2} + \dfrac{3\pi}{2}\right)$

c) $f(x) = \cot\left(\pi x + \dfrac{\pi(x+3)}{3}\right)$

d) $f(x) = \cot\left(\dfrac{\pi}{6} + \dfrac{\pi x + 2\pi(x+2)}{5}\right)$

CAL1.2: Limits and Continuity

LIMITS

▼ Definition of limits

- Let $f: A_f \to \mathbb{R}$ be a function. Informally, the statement $\lim_{x\to\sigma} f(x) = L$ means that "when x approaches σ, then $f(x)$ approaches L".
- Possibilities for $x \to \sigma$:
  $x \to x_0 \in \mathbb{R}$: approach x_0 from both sides
  $x \to x_0^-$: approach x_0 from $x < x_0$
  $x \to x_0^+$: approach x_0 from $x > x_0$
  $x \to +\infty$: x becomes arbitrarily large
  $x \to -\infty$: x becomes arbitrarily small.
- Possibilities for L:
  $L = \ell \in \mathbb{R}$: $f(x)$ approaches a real number ℓ.
  $L = +\infty$: $f(x)$ becomes arbitrarily large
  $L = -\infty$: $f(x)$ becomes arbitrarily small.
- We can give 15 formal definitions for all possible combinations or combine them into 1 abstract definition. Overview of the argument:
  a) Define neighborhoods (i.e. the concept $x \to \sigma$)
  b) Define limit points (i.e. that σ can be approached
  c) Abstract limit definition
  d) The 15 Weierstrass definitions.

Definition of neighborhoods

- Let $\delta > 0$ be given. We define the neighborhood $N(\sigma, \delta)$ to represent the set of numbers near σ, with δ controlling the meaning of "near":

$x \in N(x_0, \delta) \Leftrightarrow 0 < \lvert x - x_0 \rvert < \delta$	$x \to x_0$
$x \in N(x_0^+, \delta) \Leftrightarrow x_0 < x < x_0 + \delta$	$x \to x_0^+$
$x \in N(x_0^-, \delta) \Leftrightarrow x_0 - \delta < x < x_0$	$x \to x_0^-$
$x \in N(+\infty, \delta) \Leftrightarrow x > 1/\delta$	$x \to +\infty$
$x \in N(-\infty, \delta) \Leftrightarrow x < -1/\delta$	$x \to -\infty$

- Smaller δ makes the neighborhood tighter around σ.
- Note that $x_0 \notin N(x_0, \delta)$ and $x_0 \notin N(x_0^+, \delta)$ and $x_0 \notin N(x_0^-, \delta)$.
- Let $\varepsilon > 0$ be given. We define the interval $I(L, \varepsilon)$ to represent the set of numbers near L, including L itself if it is finite, with ε controlling the meaning of near.

$$y \in I(\ell, \varepsilon) \Leftrightarrow \lvert y - \ell \rvert < \varepsilon$$
$$y \in I(+\infty, \varepsilon) \Leftrightarrow y > 1/\varepsilon$$
$$y \in I(-\infty, \varepsilon) \Leftrightarrow y < -1/\varepsilon$$

- Note that:

$$I(+\infty, \varepsilon) = N(+\infty, \varepsilon)$$
$$I(-\infty, \varepsilon) = N(-\infty, \varepsilon)$$
$$I(\ell, \varepsilon) = N(\ell, \varepsilon) \cup \{\ell\}$$
$$\ell \in N(\ell, \varepsilon)$$

• Limit point

Let $f: A \to \mathbb{R}$ be a function with domain A.
We say that

$$\boxed{\sigma \text{ limit point of } A \Leftrightarrow \forall \delta > 0 : N(\sigma,\delta) \cap A \neq \emptyset}$$

i.e. no matter how much we "squeeze" $N(\sigma,\delta)$ it will always overlap with A. This means that we can get as close to σ as we want using numbers from A.

EXAMPLES

a) For $A = [1,6)$
$2, 2^+, 2^-, 1, 1^+, 6, 6^-$ are all limit points of A.
$1^-, 6^+, 7, 0, +\infty, -\infty$ are NOT limit points of A.

b) For $A = (3, +\infty)$
$3, 3^+, 5, 5^+, 5^-, +\infty$ are limit points of A.
$3^-, -\infty, 1, 1^-, 1^+$ are NOT limit points of A.

• Note from the examples that endpoints of intervals can be limit points even if the point does not belong to the interval.

- For endpoints of intervals, one of the side limits (+ or −) is NOT a limit point.

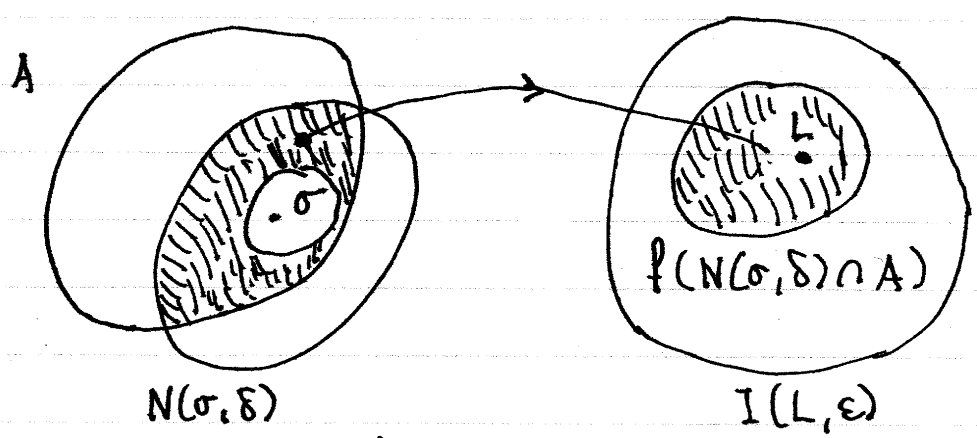
c) For $A = (1,3] \cup \{5\}$
 5 is NOT a limit point
 $5^+, 5^-$ are NOT limit points.

- Isolated points are not limit points either.

Abstract limit definition

- Let $f: A \to \mathbb{R}$ be a function. To define $\lim_{x \to \sigma} f(x)$ it is necessary for σ to be a limit point of A. Assume it is. Then

$$\boxed{\lim_{x\to\sigma} f(x) = L \Leftrightarrow \forall \varepsilon > 0 : \exists \delta > 0 : f(N(\sigma,\delta) \cap A) \subseteq I(L,\varepsilon)}$$



"For all $\varepsilon > 0$, there is a $\delta > 0$, such that $f(N(\sigma,\delta)\cap A)$ is contained in $I(L,\varepsilon)$"

- To interpret this statement we note that:

$f(N(\sigma,\delta)\cap A)$ = Values taken by $f(x)$ when x is NEAR σ but not equal to σ (with $x\in A$)

$I(L,\varepsilon)$ = Values near L, including L if finite

ε = How close we want $f(x)$ to be to L.

δ = How close x must be brought to σ so that $f(x)$ will be as close to L as has been required by our choice of ε.

- Thus: as we make ε smaller, it should be possible to find a smaller δ that will squeeze $f(N(\sigma,\delta)\cap A)$ back inside $I(L,\varepsilon)$.

• Weierstrass definitions of limit

To construct Weierstrass definitions for the 15 possible cases, we first rewrite the general definition as follows:

$$\boxed{\begin{aligned}&\lim_{x\to\sigma} f(x) = L \iff\\ &\iff \forall\varepsilon>0 : \exists\delta>0 : \forall x\in A : (x\in N(\sigma,\delta) \Rightarrow f(x)\in I(L,\varepsilon))\end{aligned}}$$

by using:

$$f(N(\sigma,\delta)\cap A)\subseteq I(L,\varepsilon) \iff$$
$$\iff \forall x\in A : (x\in N(\sigma,\delta)\Rightarrow f(x)\in I(L,\varepsilon)).$$

The definition reads:
"For all $\varepsilon > 0$, there is a $\delta > 0$, such that for all $x \in A$, if $x \in N(\sigma, \delta)$ then $f(x) \in I(L, \varepsilon)$"
To construct specific Weierstrass definitions we replace the two belonging conditions $x \in N(\sigma, \delta)$ and $f(x) \in I(L, \varepsilon)$ with the corresponding inequalities.

EXAMPLES

a) $\lim_{x \to x_0} f(x) = \ell \Leftrightarrow \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)$

"For all $\varepsilon > 0$, there is a $\delta > 0$, such that for all $x \in A$, if $0 < |x - x_0| < \delta$ then $|f(x) - \ell| < \varepsilon$".

b) $\lim_{x \to 3^+} f(x) = +\infty \Leftrightarrow$

$\Leftrightarrow \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (3 < x < 3 + \delta \Rightarrow f(x) > 1/\varepsilon)$

c) $\lim_{x \to -\infty} f(x) = 4 \Leftrightarrow$

$\Leftrightarrow \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (x < -1/\delta \Rightarrow |f(x) - 4| < \varepsilon)$

d) $\lim_{x \to 0^-} f(x) = 0 \Leftrightarrow$

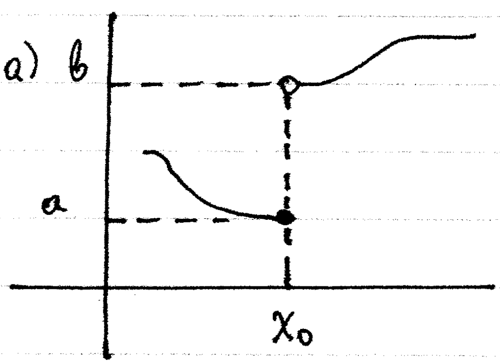
$\Leftrightarrow \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (-\delta < x < 0 \Rightarrow |f(x)| < \varepsilon)$

Limit does not exist

Let $f: A \to \mathbb{R}$ be a function with σ a limit point of A.
We say that:

$$\lim_{x\to\sigma} f(x) \text{ does not exist} \Leftrightarrow \begin{cases} \forall \ell \in \mathbb{R}: \lim_{x\to\sigma} f(x) \neq \ell \\ \lim_{x\to\sigma} f(x) \neq +\infty \\ \lim_{x\to\sigma} f(x) \neq -\infty \end{cases}$$

Geometric examples

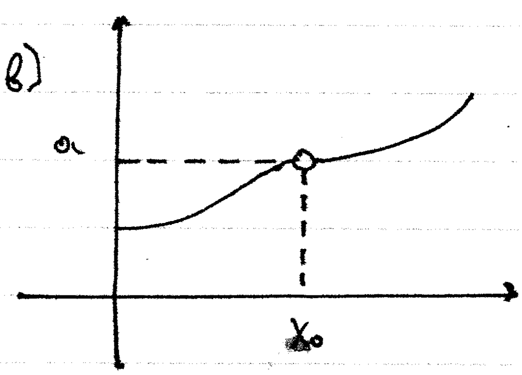
a) 

$\lim_{x\to x_0^+} f(x) = b$

$\lim_{x\to x_0^-} f(x) = a$

$\lim_{x\to x_0} f(x)$ does not exist

$f(x_0) = a$

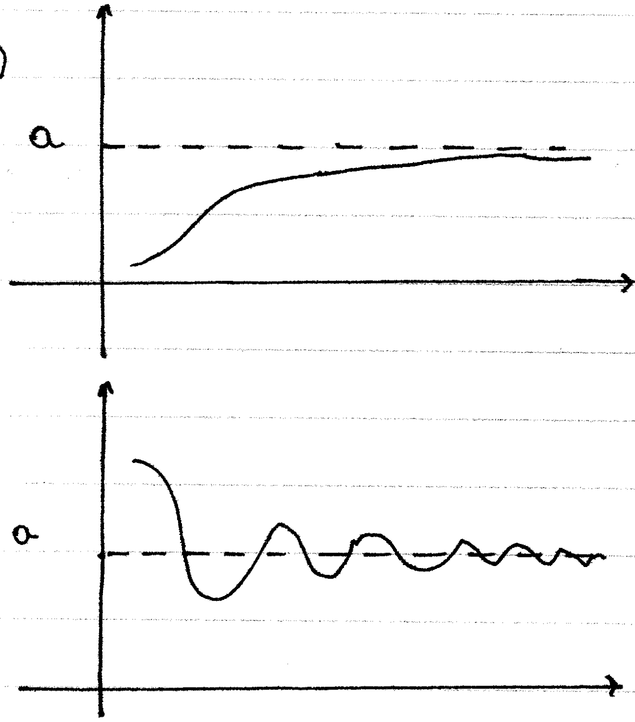
b) 

$\lim_{x\to x_0} f(x) = a$

$f(x_0)$ not defined!

$\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = a$

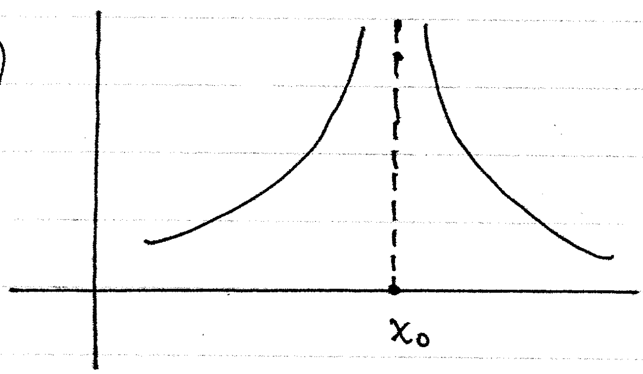
c)



In both cases:
$\lim_{x\to+\infty} f(x)=a$

Note that it is possible that $f(x)\neq a$, $\forall x\in A$ as in the top figure but $f(x)$ is allowed to satisfy $f(x)=a$ for a finite or infinite number of points x as $x\to+\infty$.

d)



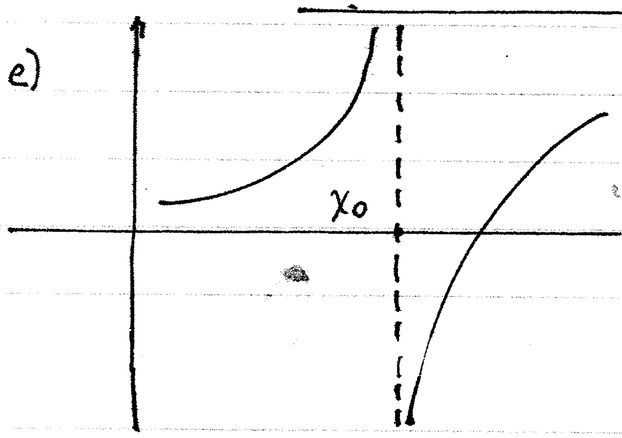
$f(x_0)$ not defined

$\lim_{x\to x_0} f(x)=+\infty$

$\lim_{x\to x_0^+} f(x)=+\infty$

$\lim_{x\to x_0^-} f(x)=+\infty$

e)



$f(x_0)$ not defined

$\lim_{x\to x_0^+} f(x)=-\infty$

$\lim_{x\to x_0^-} f(x)=+\infty$

$\lim_{x\to x_0} f(x)$ does not exist.

EXERCISES

① Write the definition with quantifiers and also in English for the following statements

a) $\lim\limits_{x\to 2} f(x) = 3$

b) $\lim\limits_{x\to -\infty} f(x) = 9$

c) $\lim\limits_{x\to +\infty} f(x) = -\infty$

d) $\lim\limits_{x\to 4^+} f(x) = 3$

e) $\lim\limits_{x\to 1^-} f(x) = +\infty$

f) $\lim\limits_{x\to 2^-} f(x) = -\infty$

g) $\lim\limits_{x\to 1} f(x) = +\infty$

h) $\lim\limits_{x\to -\infty} f(x) = +\infty$

i) $\lim\limits_{x\to +\infty} f(x) = 0$

j) $\lim\limits_{x\to 3} f(x) = -\infty$.

Solution to 1

a) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (0 < |x-2| < \delta \Rightarrow |f(x) - 3| < \varepsilon)$

b) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (x < -1/\delta \Rightarrow |f(x) - 9| < \varepsilon)$

c) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (x > 1/\delta \Rightarrow f(x) < -1/\varepsilon)$

d) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (4 < x < 4 + \delta \Rightarrow |f(x) - 3| < \varepsilon)$

e) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (1 - \delta < x < 1 \Rightarrow f(x) > 1/\varepsilon)$

f) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (2 - \delta < x < 2 \Rightarrow f(x) < -1/\varepsilon)$

g) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (0 < |x-1| < \delta \Rightarrow f(x) > 1/\varepsilon)$

h) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (x < -1/\delta \Rightarrow f(x) > 1/\varepsilon)$

i) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (x > 1/\delta \Rightarrow |f(x)| < \varepsilon)$

j) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (0 < |x-3| < \delta \Rightarrow f(x) < -1/\varepsilon)$

▼ Limits and operations

① Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ be two functions and let σ be a limit point of both A and B. We assume that: $\lim_{x \to \sigma} f(x) = \ell_1$ and $\lim_{x \to \sigma} g(x) = \ell_2$. Then:

a) $\lim_{x \to \sigma} [f(x) + g(x)] = \ell_1 + \ell_2 = \lim_{x \to \sigma} f(x) + \lim_{x \to \sigma} g(x)$

b) $\lim_{x \to \sigma} [f(x) g(x)] = \ell_1 \ell_2 = [\lim_{x \to \sigma} f(x)][\lim_{x \to \sigma} g(x)]$

c) $\forall a \in \mathbb{R} : \lim_{x \to \sigma} [a f(x)] = a \ell_1 = a \lim_{x \to \sigma} f(x)$

d) $\lim_{x \to \sigma} g(x) \neq 0 \implies \lim_{x \to \sigma} \left[\frac{f(x)}{g(x)} \right] = \frac{\ell_1}{\ell_2} = \frac{\lim_{x \to \sigma} f(x)}{\lim_{x \to \sigma} g(x)}$

e) $\lim_{x \to \sigma} |f(x)| = |\ell_1| = |\lim_{x \to \sigma} f(x)|$

f) $\lim_{x \to \sigma} f(x) > 0 \implies \lim_{x \to \sigma} \sqrt{f(x)} = \sqrt{\ell_1} = \sqrt{\lim_{x \to \sigma} f(x)}$

Trivial limits

- $P(x)$ polynomial $\Rightarrow \forall x_0 \in \mathbb{R} : \lim\limits_{x \to x_0} P(x) = P(x_0)$

- $\left.\begin{array}{l} P(x),\, Q(x) \text{ polynomials} \\ Q(x_0) \neq 0 \end{array}\right\} \Rightarrow \lim\limits_{x \to x_0} \dfrac{P(x)}{Q(x)} = \dfrac{P(x_0)}{Q(x_0)}$

- $\lim\limits_{x \to x_0} f(x) = a > 0 \Rightarrow \lim\limits_{x \to x_0} \sqrt{f(x)} = \sqrt{a}$

EXAMPLES

1) $f(x) = \sqrt{x^2 + 3x} \leftarrow \lim\limits_{x \to 1} f(x)$.

Rigorous proof:

$$\lim_{x \to 1} (x^2 + 3x) = 1^2 + 3 \cdot 1 = 4 > 0 \Rightarrow$$

$$\Rightarrow \lim_{x \to 1} f(x) = \lim_{x \to 1} \sqrt{x^2 + 3x} = \sqrt{4} = 2.$$

Brief version:

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \sqrt{x^2 + 3x} = \sqrt{1^2 + 3 \cdot 1} = \sqrt{4} = 2.$$

2) $f(x) = \dfrac{2x+1}{\sqrt{x^2+1}} \leftarrow \lim\limits_{x \to 2} f(x)$.

Rigorous proof.

$$\lim_{x \to 2} (x^2+1) = 2^2 + 1 = 4+1 = 5 > 0 \Rightarrow$$

$$\Rightarrow \lim_{x \to 2} \sqrt{x^2+1} = \sqrt{5} \quad (1)$$

$$\lim_{x \to 2} (2x+1) = 2 \cdot 2 + 1 = 5 \quad (2)$$

From (1) and (2):

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{2x+1}{\sqrt{x^2+1}} = \frac{5}{\sqrt{5}} = \frac{5\sqrt{5}}{5} = \sqrt{5}.$$

Brief version

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{2x+1}{\sqrt{x^2+1}} = \frac{2 \cdot 2+1}{\sqrt{2^2+1}} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

Indeterminent forms

① Form 0/0 : You need to find the cancelation that removes 0 from the numerator and denominator.

EXAMPLES

a) $f(x) = \dfrac{x^2+6x+9}{x^2+5x+6} \longleftarrow \lim_{x\to -3} f(x).$

Solution

▶ Note that 0/0. Thus

$$f(x) = \frac{x^2+6x+9}{x^2+5x+6} = \frac{(x+3)^2}{(x+2)(x+3)} = \frac{x+3}{x+2} \Rightarrow$$

$$\Rightarrow \lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{x+3}{x+2} = \frac{-3+3}{-3+2} = \frac{0}{-1} = 0$$

b) $f(x) = \dfrac{x^2-3x+2}{|x-1|} \longleftarrow \lim_{x\to 1^-} f(x), \lim_{x\to 1^+} f(x).$

Solution

▶ Note 0/0 form.

$$f(x) = \frac{x^2-3x+2}{|x-1|} = \frac{(x-1)(x-2)}{|x-1|}.$$

It follows that:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{(x-1)(x-2)}{|x-1|} = \lim_{x\to 1^+} \frac{(x-1)(x-2)}{x-1}$$

$$= \lim_{x \to 1^+} (x-2) = 1-2 = -1$$

and

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{(x-1)(x-2)}{|x-1|} = \lim_{x \to 1^-} \frac{(x-1)(x-2)}{-(x-1)}$$

$$= \lim_{x \to 1^-} (2-x) = 2-1 = 1.$$

② → Form 0/0 with radicals

We use the identity

$$a^2 - b^2 = (a-b)(a+b) \Rightarrow \boxed{a-b = \frac{a^2-b^2}{a+b}}$$

to eliminate the square root and expose the 0/0 cancelation.

EXAMPLES

a) $f(x) = \dfrac{\sqrt{x-1} - 2}{x-5}$ ← $\lim_{x \to 5} f(x)$

Solution

$$f(x) = \frac{\sqrt{x-1} - 2}{x-5} = \frac{1}{x-5} \frac{(\sqrt{x-1})^2 - 2^2}{\sqrt{x-1} + 2} =$$

$$= \frac{1}{x-5}\,\frac{(x-1)-4}{\sqrt{x-1}+2} = \frac{1}{x-5}\,\frac{x-5}{\sqrt{x-1}+2} =$$

$$= \frac{1}{\sqrt{x-1}+2} \Rightarrow$$

$$\Rightarrow \lim_{x\to 5} f(x) = \lim_{x\to 5} \frac{1}{\sqrt{x-1}+2} = \frac{1}{\sqrt{5-1}+2} =$$

$$= \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}.$$

β) $f(x) = \dfrac{x^2-\sqrt{x}}{\sqrt{x}-1} \leftarrow \lim_{x\to 1} f(x)$

Solution

$$f(x) = \frac{x^2-\sqrt{x}}{\sqrt{x}-1} = \frac{\dfrac{x^4-(\sqrt{x})^2}{x^2+\sqrt{x}}}{\dfrac{(\sqrt{x})^2-1^2}{\sqrt{x}+1}} =$$

$$= \frac{(x^4-x)(\sqrt{x}+1)}{(x-1)(x^2+\sqrt{x})} = \frac{x(x^3-1)(\sqrt{x}+1)}{(x-1)(x^2+\sqrt{x})} =$$

$$= \frac{x(x-1)(x^2+x+1)(\sqrt{x}+1)}{(x-1)(x^2+\sqrt{x})} =$$

$$= \frac{x(x^2+x+1)(\sqrt{x}+1)}{x^2+\sqrt{x}} \Rightarrow$$

$$\Rightarrow \lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x(x^2+x+1)(\sqrt{x}+1)}{x^2+\sqrt{x}} =$$

$$= \frac{1\cdot(1^2+1+1)(\sqrt{1}+1)}{1^2+\sqrt{1}} = \frac{1\cdot 3\cdot 2}{2} = 3.$$

③ → Using side limits : $x \to x_0^+$, $x \to x_0^-$

From the definition we prove the following properties of side limits:

• $\lim\limits_{x\to x_0} f(x) = L \Leftrightarrow \lim\limits_{x\to x_0^+} f(x) = L \wedge \lim\limits_{x\to x_0^-} f(x) = L$
• $\left.\begin{array}{l} \lim\limits_{x\to x_0^+} f(x) = L_1 \\ \lim\limits_{x\to x_0^-} f(x) = L_2 \\ L_1 \neq L_2 \end{array}\right\} \Rightarrow \lim\limits_{x\to x_0} f(x)$ does not exist

- Does not exist means that $\lim\limits_{x\to x_0} f(x) = L$ is a false statement for all possible choices of L (i.e. $L \in \mathbb{R}$, $L = +\infty$, $L = -\infty$).
- We use the above properties in conjunction with side limits.

EXAMPLES

a) $f(x) = \dfrac{x^2+2|x|}{x^2-2|x|} \leftarrow \lim\limits_{x\to 0} f(x)$.

Solution

- We use side limits to simplify $|x|$.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2+2|x|}{x^2-2|x|} = \lim_{x\to 0^+} \frac{x^2+2x}{x^2-2x} =$$

$$= \lim_{x\to 0^+} \frac{x(x+2)}{x(x-2)} = \lim_{x\to 0^+} \frac{x+2}{x-2} =$$

$$= \frac{0+2}{0-2} = -1 \qquad (1)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^2+2|x|}{x^2-2|x|} = \lim_{x\to 0^-} \frac{x^2-2x}{x^2+2x} =$$

$$= \lim_{x\to 0^-} \frac{x(x-2)}{x(x+2)} = \lim_{x\to 0^-} \frac{x-2}{x+2}$$

$$= \frac{0-2}{0+2} = -1 \qquad (2)$$

From (1) and (2): $\lim\limits_{x\to 0} f(x) = -1$.

b) $f(x) = \dfrac{x^2+x}{|x|} \longleftarrow \lim\limits_{x\to 0} f(x)$

<u>Solution</u>

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2+x}{|x|} = \lim_{x\to 0^+} \frac{x(x+1)}{x} =$$

$$= \lim_{x\to 0^+} (x+1) = 0+1 = 1 \qquad (1)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^2+x}{|x|} = \lim_{x\to 0^-} \frac{x(x+1)}{-x} =$$

$$= \lim_{x\to 0^-} (-x-1) = -0-1 = -1 \qquad (2)$$

From (1) and (2): $\lim\limits_{x\to 0} f(x)$ does not exist.

EXERCISES

② Show that

a) $\lim\limits_{x\to 3} \dfrac{x^2-9}{x^2-4x+3} = 3$

b) $\lim\limits_{x\to \sqrt{2}} \dfrac{x^4-4}{x^2-2} = 4$

c) $\lim\limits_{x\to 1} \dfrac{x^3+2x-3}{x-1}$

d) $\lim\limits_{x\to 1} \dfrac{x^2+x-2}{x^2-1} = \dfrac{3}{2}$

e) $\lim\limits_{x\to 1^-} \dfrac{x^2-2x+1}{|x|-1} = 0$

f) $\lim\limits_{x\to 1^+} \dfrac{x^2-2x+1}{|x|-1} = 0$

g) $\lim\limits_{x\to 5^-} \dfrac{|x-5|+x^2-4x-5}{x-5} = 5$

h) $\lim\limits_{x\to 1^+} \dfrac{(x^2-1)^2}{x^3-x^2-x+1} = 2$

i) $\lim\limits_{x\to -1^+} \dfrac{3x^2-3}{|x+1|} = -6$

j) $\lim\limits_{x\to 3^-} \dfrac{x^2-6x+9}{|x|-3} = 0$

③ Evaluate, if they exist, the following limits:

a) $\lim\limits_{x\to 0} \dfrac{\sqrt{9+x}-3}{x}$

b) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x^2}-1}{x}$

c) $\lim\limits_{x\to 5} \dfrac{\sqrt{x-1}-2}{x-5}$

d) $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$

e) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x^2}-1}{x}$

f) $\lim\limits_{x\to 1} \dfrac{x^2-\sqrt{x}}{\sqrt{x}-1}$

g) $\lim\limits_{x\to 2} \dfrac{\sqrt{x+3}-2}{x-1}$

h) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+1}-1}{\sqrt{x+4}-2}$,

(4) Evaluate the following limits, if they exist:

a) $\lim\limits_{x \to 1} \dfrac{x^2-2x+1}{|x|-1}$

b) $\lim\limits_{x \to -1} \dfrac{x^2+2x+1}{|x|-1}$

c) $\lim\limits_{x \to 2} \dfrac{|x-2|+x^2-3x+2}{x-2}$

d) $\lim\limits_{x \to 3} \dfrac{x^2-5x+6+|x-3|}{x-3}$

e) $\lim\limits_{x \to 1} \dfrac{x^2+x-2}{|x|-1}$

f) $\lim\limits_{x \to 0} \dfrac{x^2+2|x|}{x^2-2|x|}$

g) $\lim\limits_{x \to 3} \dfrac{|x-3|+x^2-x-6}{x-3}$

h) $\lim\limits_{x \to 0} \dfrac{3x^2}{|x|}$

Functions with limits going to infinity

Addition and product of two functions where one of them goes to infinity can be handled by the following theorem

② Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ be two functions and let σ be a limit point of A and B. Let $\delta > 0$ and $a \in \mathbb{R}$. Then:

a) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) > a \\ \lim\limits_{x \to \sigma} f(x) = +\infty \end{cases} \Rightarrow \lim\limits_{x \to \sigma} [f(x) + g(x)] = +\infty$

b) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) < a \\ \lim\limits_{x \to \sigma} f(x) = -\infty \end{cases} \Rightarrow \lim\limits_{x \to \sigma} [f(x) + g(x)] = -\infty$

c) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) > a > 0 \\ \lim\limits_{x \to \sigma} f(x) = \pm\infty \end{cases} \Rightarrow \lim\limits_{x \to \sigma} [f(x) g(x)] = \pm\infty$

d) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) < a < 0 \\ \lim\limits_{x \to \sigma} f(x) = \pm\infty \end{cases} \Rightarrow \lim\limits_{x \to \sigma} [f(x) g(x)] = \mp\infty$

Note that it is not necessary for the limit $\lim\limits_{x \to \sigma} g(x)$ to exist. It is sufficient to know how $g(x)$ is bounded. However, if $\lim\limits_{x \to \sigma} g(x)$ does exist, we can infer the following deductions, which we summarize in the following operation tables.

$f(x)\downarrow \setminus g(x)\rightarrow$	a	$+\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	?
$-\infty$	$-\infty$	?	$-\infty$

$$\underbrace{\qquad\qquad\qquad}_{\lim\limits_{x\to\sigma}[f(x)+g(x)]}$$

$f(x)\downarrow \setminus g(x)\rightarrow$	0	$p>0$	$n<0$	$+\infty$	$-\infty$
$+\infty$	?	$+\infty$	$-\infty$	$+\infty$	$-\infty$
$-\infty$	?	$-\infty$	$+\infty$	$-\infty$	$+\infty$

$$\underbrace{\qquad\qquad\qquad}_{\lim\limits_{x\to\sigma}[f(x)g(x)]}$$

↳ The "?" correspond to indeterminate forms. It means that the limit cannot be determined without more information, and the limit may or may not exist.

Form k/0

- Let $f: A \to \mathbb{R}$ be a function, let $\delta > 0$, and let σ be a limit point of A. Then:

a) $\begin{cases} \forall x \in N(\sigma, \delta) \cap A : f(x) > 0 \\ \lim\limits_{x \to \sigma} f(x) = 0 \end{cases} \Rightarrow \lim\limits_{x \to \sigma} \dfrac{1}{f(x)} = +\infty$

b) $\begin{cases} \forall x \in N(\sigma, \delta) \cap A : f(x) < 0 \\ \lim\limits_{x \to \sigma} f(x) = 0 \end{cases} \Rightarrow \lim\limits_{x \to \sigma} \dfrac{1}{f(x)} = -\infty$

c) $\lim\limits_{x \to \sigma} f(x) \in \{+\infty, -\infty\} \Rightarrow \lim\limits_{x \to \sigma} \dfrac{1}{f(x)} = 0$

- From this theorem, we can show that

$$\forall a \in \mathbb{R} : \lim_{x \to a^+} \frac{1}{x-a} = +\infty$$

$$\forall a \in \mathbb{R} : \lim_{x \to a^-} \frac{1}{x-a} = -\infty$$

$$\forall k \in \mathbb{N}^* : \forall a \in \mathbb{R} : \lim_{x \to a} \frac{1}{(x-a)^{2k}} = +\infty$$

These results by themselves are sufficient for handling limits of functions that are defined as a ratio of two polynomials and yield a k/0 form.

EXAMPLES

a) $f(x) = \dfrac{1-3x}{(x-2)^2} \leftarrow \lim\limits_{x \to 2} f(x)$

Solution

$$f(x) = \frac{1-3x}{(x-2)^2} = (1-3x)\frac{1}{(x-2)^2} \qquad (1)$$

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = +\infty \qquad (2)$$

$$\lim_{x \to 2} (1-3x) = 1-3\cdot 2 = 1-6 = -5 \qquad (3)$$

From Eq. (1), (2), (3): $\lim\limits_{x \to 2} f(x) = (-5)(+\infty) = -\infty$

b) $f(x) = \dfrac{2x+1}{2x-1} \leftarrow \lim\limits_{x \to 1/2^-} f(x).$

Solution

$$f(x) = \frac{2x+1}{2x-1} = (2x+1)\frac{1}{2x-1} = \frac{2x+1}{2}\,\frac{1}{x-1/2} \qquad (1)$$

$$\lim_{x \to 1/2^-} \frac{1}{x-1/2} = -\infty \qquad (2)$$

$$\lim_{x \to 1/2^-} \frac{2x+1}{2} = \frac{2(1/2)+1}{2} = \frac{1+1}{2} = 1 \qquad (3)$$

From Eq. (1), (2), (3):

$$\lim_{x\to 1/2^-} f(x) = 1\cdot(-\infty) = -\infty$$

c) $f(x) = \dfrac{x^2+3x+2}{x^2+4x+4} \longleftarrow \lim_{x\to -2^+} f(x)$

Solution : Note that initially this is a 0/0 limit.

$$f(x) = \frac{x^2+3x+2}{x^2+4x+4} = \frac{(x+1)(x+2)}{(x+2)^2} = \frac{x+1}{x+2} =$$

$$= (x+1)\frac{1}{x+2} = (x+1)\frac{1}{x-(-2)} . \qquad (1)$$

$$\lim_{x\to -2^+} \frac{1}{x-(-2)} = +\infty \qquad (2)$$

$$\lim_{x\to -2^+} (x+1) = (-2)+1 = -1 \qquad (3)$$

From Eq.(1),(2),(3): $\lim_{x\to -2^+} f(x) = (-1)(+\infty) = -\infty$

d) $f(x) = \dfrac{3-4x}{(x+1)^2(x^2+6x+5)} \longleftarrow \lim_{x\to -1} f(x).$

Solution

$$f(x) = \frac{3-4x}{(x+1)^2(x^2+6x+5)} = \frac{3-4x}{(x+1)^2(x+5)(x+1)} =$$

$$= \frac{1}{(x+1)^3}\,\frac{3-4x}{x+5} = \frac{1}{[x-(-1)]^3}\,\frac{3-4x}{x+5} \qquad (1)$$

$$\lim_{x\to -1}\frac{3-4x}{x+5}=\frac{3-4(-1)}{(-1)+5}=\frac{3+4}{4}=\frac{7}{4}>0 \quad (2)$$

$$\lim_{x\to -1^+}\frac{1}{x-(-1)}=+\infty \Rightarrow \lim_{x\to -1^+}\frac{1}{[x-(-1)]^3}=+\infty \quad (3)$$

$$\lim_{x\to -1^-}\frac{1}{x-(-1)}=-\infty \Rightarrow \lim_{x\to -1^-}\frac{1}{[x-(-1)]^3}=-\infty \quad (4)$$

From Eq. (1), (2), (3): $\lim_{x\to -1^+} f(x)=(7/4)(+\infty)=+\infty$ (5)

From Eq. (1), (2), (4): $\lim_{x\to -1^-} f(x)=(7/4)(-\infty)=-\infty$ (6)

From Eq. (5), (6):

$\lim_{x\to -1^+} f(x)\neq \lim_{x\to -1^-} f(x) \Rightarrow \lim_{x\to -1} f(x)$ does not exist.

EXERCISES

⑤ Evaluate the following limits, if they exist.

a) $\lim\limits_{x \to 2^+} \dfrac{3x+1}{2-x}$

b) $\lim\limits_{x \to 0^+} \dfrac{1-2x}{x}$

c) $\lim\limits_{x \to -2} \dfrac{x^2+3x+2}{x^2+4x+4}$

d) $\lim\limits_{x \to 3} \dfrac{2-5x}{x^2-6x+9}$

e) $\lim\limits_{x \to 1^-} \dfrac{x^2-2x+1}{x^3-3x^2+3x-1}$

f) $\lim\limits_{x \to 1/2^+} \dfrac{x^2+1}{2x-1}$

g) $\lim\limits_{x \to 2^-} \dfrac{2x}{x^2-4}$

h) $\lim\limits_{x \to 2^+} \dfrac{2x-7}{8-x^3}$

Methods for limits $x \to \pm\infty$

For limits with $x \to +\infty$ and $x \to -\infty$ we rely on the following methodology:

Trivial limits

① Monomials

For limits involving monomials, we use the following results:

a) $\forall k \in \mathbb{N}^*: \lim\limits_{x \to +\infty} x^k = +\infty$

b) $\forall k \in \mathbb{N}^*: \lim\limits_{x \to -\infty} x^{2k} = +\infty$

c) $\forall k \in \mathbb{N}^*: \lim\limits_{x \to -\infty} x^{2k+1} = -\infty$

d) $\forall k \in \mathbb{N}^*: \lim\limits_{x \to \pm\infty} x^{-k} = 0$

EXAMPLES

a) $\lim\limits_{x \to +\infty} (3x^5) = 3 \cdot (+\infty) = +\infty$

b) $\lim\limits_{x \to -\infty} (2x^4) = 2 \cdot (+\infty) = +\infty$

c) $\lim\limits_{x \to -\infty} (-7x^3) = -7(-\infty) = +\infty$

d) $\lim\limits_{x \to +\infty} \dfrac{3}{2x^2} = \dfrac{3}{2} \cdot 0 = 0$

② Polynomials

If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, then:

$$\boxed{\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n}$$

EXAMPLE

a) $\lim\limits_{x \to -\infty} (3x - 2x^3 + 1) = \lim\limits_{x \to -\infty} (-2x^3) = -2(-\infty) = +\infty$

b) $f(x) = (1-3x^2)^3 (x + 3x^2 - x^3) \leftarrow \lim\limits_{x \to -\infty} f(x)$.

Solution

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left[(1-3x^2)^3 (x+3x^2-x^3)\right] =$$

$$= \lim_{x \to -\infty} \left[(-3x^2)^3(-x^3)\right] = \lim_{x \to -\infty} \left[(-27x^6)(-x^3)\right]$$

$$= \lim_{x \to -\infty} (27x^9) = 27(-\infty) = -\infty.$$

3 Rational functions

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, and
$Q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0$,
then:

$$\boxed{\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)} = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}}$$

EXAMPLES

a) $f(x) = \dfrac{x + x^3 + 1}{2x - x^2} \leftarrow \lim\limits_{x \to -\infty} f(x)$

Solution

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x + x^3 + 1}{2x - x^2} = \lim_{x \to -\infty} \frac{x^3}{-x^2} =$$

$$= \lim_{x \to -\infty} (-x) = -(-\infty) = +\infty.$$

b) $f(x) = \dfrac{3x^2 + 3x - 1}{x^2 - 2} \leftarrow \lim\limits_{x \to +\infty} f(x)$

Solution

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{3x^2 + 3x - 1}{x^2 - 2} = \lim_{x \to +\infty} \frac{3x^2}{x^2} = 3$$

c) $f(x) = \dfrac{x^2+1}{2x^4-x} \longleftarrow \lim\limits_{x\to-\infty} f(x)$

<u>Solution</u>

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+1}{2x^4-x} = \lim_{x\to-\infty} \frac{x^2}{2x^4} = \lim_{x\to-\infty} \frac{1}{2x^2}$$

$$= \frac{1}{2} \cdot 0 = 0$$

d) $f(x) = \dfrac{3x^2(2x-1)^3(x^2-2)}{(3x+2)(5x^2+4x+2)^2} \longleftarrow \lim\limits_{x\to+\infty} f(x).$

<u>Solution</u>

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{3x^2(2x-1)^3(x^2-2)}{(3x+2)(5x^2+4x+2)^2} =$$

$$= \lim_{x\to+\infty} \frac{(3x^2)(2x)^3\,x^2}{(3x)(5x^2)^2} =$$

$$= \lim_{x\to+\infty} \frac{(3x^2)(8x^3)x^2}{(3x)(25x^4)} = \lim_{x\to+\infty} \frac{24x^7}{75x^5}$$

$$= \lim_{x\to+\infty} \frac{24x^2}{75} = \frac{24}{75} \cdot (+\infty) = +\infty$$

Indeterminate forms

1) Form ∞/∞

$$f(x)=\frac{\sqrt{g(x)}}{h(x)}\ ,\quad f(x)=\frac{g(x)}{\sqrt{h(x)}}\ ,\quad f(x)=\sqrt{\frac{g(x)}{h(x)}}$$

- Factor highest-order term and simplify.

CAUTION: $\boxed{\begin{array}{l}\sqrt{x^2}=|x| \\ (\sqrt{x})^2=x\end{array}}$

EXAMPLE

$$\lim_{x\to-\infty}\frac{\sqrt{9x^2-2x+5}}{x+4}=\lim_{x\to-\infty}\frac{\sqrt{x^2}\sqrt{9-2/x+5/x^2}}{x(1+4/x)}$$

$$=\lim_{x\to-\infty}\frac{\sqrt{9-2/x+5/x^2}}{1+4/x}\cdot\lim_{x\to-\infty}\frac{\sqrt{x^2}}{x}=$$

$$=\frac{\sqrt{9-0+0}}{1+0}\cdot\lim_{x\to-\infty}\frac{|x|}{x}\underset{*}{=}\sqrt{9}\cdot\lim_{x\to-\infty}\frac{-x}{x}$$

$$=3\lim_{x\to-\infty}(-1)=-3.$$

* Since $x\to-\infty$, assume $x<0 \Rightarrow \sqrt{x^2}=|x|=-x$.

2) Form $\infty - \infty$

A) $f(x) = Q_1(x) - Q_2(x)$
with Q_1, Q_2 rational functions.

▶ Method: Combine to one fraction, and use previous methods.

EXAMPLE

a) $f(x) = \dfrac{x^3}{x^2+1} - \dfrac{2x^2}{x-1} \longrightarrow \lim\limits_{x\to+\infty} f(x)$.

▶ Gives $\infty - \infty$. Thus:

$$f(x) = \frac{x^3}{x^2+1} - \frac{2x^2}{x-1} = \frac{x^3(x-1) - 2x^2(x^2+1)}{(x^2+1)(x-1)}$$

$$= \frac{x^4 - x^3 - 2x^4 - 2x^2}{(x^2+1)(x-1)} = \frac{-x^4 - x^3 - 2x^2}{(x^2+1)(x-1)} \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{-x^4 - x^3 - 2x^2}{(x^2+1)(x-1)} = \lim_{x\to+\infty} \frac{-x^4}{x^3}$$

$$= \lim_{x\to+\infty} (-x) = -(+\infty) = -\infty$$

b) $f(x) = \dfrac{x^4+2x^2}{x^2-1} - \dfrac{x^5+3x+1}{x^2+1} \longrightarrow \lim\limits_{x\to-\infty} f(x)$

▶ Gives $\infty + \infty$! Not indeterminate form so we can use a constructive argument

$$\lim_{x\to-\infty} \frac{x^4+2x^2}{x^2-1} = \lim_{x\to-\infty} \frac{x^4}{x^2} = \lim_{x\to-\infty} x^2 = +\infty \quad (1)$$

$$\lim_{x\to-\infty} \frac{x^5+3x+1}{x^2+1} = \lim_{x\to-\infty} \frac{x^5}{x^2} = \lim_{x\to-\infty} x^3 = -\infty \quad (2)$$

Thus, from (1) and (2):

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left[\frac{x^4+2x^2}{x^2-1} - \frac{x^5+3x+1}{x^2+1} \right] =$$

$$= (+\infty) - (-\infty) = (+\infty) + (+\infty) = +\infty$$

B) $f(x) = \sqrt{g(x)} - \sqrt{h(x)}$

$f(x) = g(x) - \sqrt{h(x)}$

$f(x) = \sqrt{g(x)} - h(x)$

- Use the identity $\boxed{a-b = \dfrac{a^2-b^2}{a+b}}$

to eliminate the square roots from the $\infty - \infty$ factor.

EXAMPLE

$f(x) = \sqrt{9x^2+x+1} - 3x \longrightarrow \lim_{x\to+\infty} f(x), \lim_{x\to-\infty} f(x)$

a) $f(x) = \sqrt{9x^2+x+1} - 3x = \dfrac{(\sqrt{9x^2+x+1})^2 - (3x)^2}{\sqrt{9x^2+x+1} + 3x} =$

$$= \frac{(9x^2+x+1) - 9x^2}{\sqrt{9x^2+x+1} + 3x} = \frac{x+1}{\sqrt{9x^2+x+1} + 3x}$$

Since

$$\sqrt{9x^2+x+1} = \sqrt{x^2}\ \sqrt{9+1/x+1/x^2} =$$
$$= |x|\sqrt{9+1/x+1/x^2} =$$
$$= x\sqrt{9+1/x+1/x^2}$$

it follows that

$$f(x) = \frac{x+1}{x\sqrt{9+1/x+1/x^2}+3x} =$$
$$= \frac{x(1+1/x)}{x\left[\sqrt{9+1/x+1/x^2}+3\right]} =$$
$$= \frac{1+1/x}{\sqrt{9+1/x+1/x^2}+3} \Rightarrow$$
$$\Rightarrow \lim_{x\to+\infty} f(x) = \frac{1+0}{\sqrt{9+0+0}+3} = \frac{1}{3+3} = \frac{1}{6}$$

b) $\lim\limits_{x\to-\infty}(9x^2+x+1) = \lim\limits_{x\to-\infty}(9x^2) = 9(+\infty) = +\infty$

$$\Rightarrow \lim_{x\to-\infty}\sqrt{9x^2+x+1} = +\infty. \qquad (1)$$

$$\lim_{x\to-\infty}(-3x) = -3(-\infty) = +\infty \qquad (2)$$

From Eq. (1), (2):

$$\lim_{x\to-\infty} f(x) = (+\infty)+(+\infty) = +\infty.$$

↳ In the absense of the indeterminate form $\infty-\infty$, the limit can be evaluated directly as shown above.

EXAMPLES

⑥ Evaluate the following limits, if they exist:

a) $\lim\limits_{x \to -\infty} (2x^4 + x^3 - 2x + 3)$

b) $\lim\limits_{x \to +\infty} (-2x^3 + 5x^2 - 3)$

c) $\lim\limits_{x \to -\infty} (-x^2 + 5x)$

d) $\lim\limits_{x \to -\infty} (5x^5 + x^2 - x - 1)$

e) $\lim\limits_{x \to +\infty} \dfrac{9x^3 - 5x + 1}{3x^3 - 2x^2}$

f) $\lim\limits_{x \to -\infty} \dfrac{x^4 - x + 3}{x^2 + 2}$

g) $\lim\limits_{x \to +\infty} \dfrac{2x^3 - 5x + 1}{x^5 + 3x^4}$

h) $\lim\limits_{x \to -\infty} \dfrac{x^2 - 4}{x - 2}$

i) $\lim\limits_{x \to -\infty} \dfrac{3}{x^2 - 9}$

j) $\lim\limits_{x \to +\infty} \dfrac{-x^2 + x + 2}{2x - 1}$

⑦ Evaluate the following limits, if they exist:

a) $\lim\limits_{x \to +\infty} \dfrac{\sqrt{4x^2 - 2x + 1}}{2x + 3} = 1$

b) $\lim\limits_{x \to +\infty} \dfrac{6x^2 - x + 1}{\sqrt{x^4 + 3}} = 6$

c) $\lim\limits_{x \to -\infty} \dfrac{\sqrt{9x^2 - 2x + 5}}{x + 4} = -3$

d) $\lim\limits_{x \to +\infty} \dfrac{4x^3 - x + 1}{\sqrt{x^2 + x + 5}} = +\infty$

e) $\lim\limits_{x \to +\infty} \dfrac{\sqrt{x^3 + 4} + x - 2}{x\sqrt{x} - 3\sqrt{x^3 + 1}} = -\dfrac{1}{2}$

f) $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x^2 + 2} - x + 5}{x + 4} = -2$

g) $\lim\limits_{x \to -\infty} \sqrt{\dfrac{x - 2}{x - 1}}$

h) $\lim\limits_{x \to +\infty} \dfrac{5x}{3x - 1 + \sqrt{9x^2 + x + 1}}$

(8) Evaluate the following limits, if they exist:

a) $\lim\limits_{x \to -\infty} \sqrt{x^2-9}$

b) $\lim\limits_{x \to -\infty} \left(\sqrt{16x^2+x+5} - 4x\right)$

c) $\lim\limits_{x \to -\infty} \left(x+4-\sqrt{x^2-x+3}\right)$

d) $\lim\limits_{x \to +\infty} \left(\sqrt{4x^2-3x} + 2x\right)$

e) $\lim\limits_{x \to +\infty} \sqrt{2x^4+3x^2+1}$

f) $\lim\limits_{x \to -\infty} \sqrt{x^2+4}$

(9) Evaluate the following limits, if they exist:

a) $\lim\limits_{x \to +\infty} \left(\frac{x}{x-2} - \frac{1}{x+2}\right)$

b) $\lim\limits_{x \to -\infty} \left(\frac{1-x}{2x} - \frac{3+x}{x^2}\right)$

c) $\lim\limits_{x \to +\infty} \left(\frac{x^3}{x^2+1} - \frac{2x^2}{x-1}\right)$

d) $\lim\limits_{x \to -\infty} \left(\frac{-x^4-2}{2x} + \frac{x^3+1}{x}\right)$

(10) Evaluate the following limits, if they exist:

a) $\lim\limits_{x \to +\infty} \left(\sqrt{x^2+1} - x\right)$

b) $\lim\limits_{x \to -\infty} \left(\sqrt{x^2+1} - x\right)$

c) $\lim\limits_{x \to +\infty} \left(\sqrt{9x^2+x+1} - 3x\right)$

d) $\lim\limits_{x \to +\infty} \left(\sqrt{4x^2+2} - \sqrt{4x^2-1}\right)$

e) $\lim\limits_{x \to +\infty} \left(x+4-\sqrt{x^2-x+3}\right)$

f) $\lim\limits_{x \to +\infty} \left(\sqrt{x+\sqrt{x}} - \sqrt{x-\sqrt{x}}\right)$

g) $\lim\limits_{x \to +\infty} \left(x+1+\sqrt{x^2+x+1}\right)$

h) $\lim\limits_{x \to +\infty} \frac{\sqrt{x^2+1} - x}{\sqrt{x} - \sqrt{x+1}}$

Trigonometric limits

To establish the basic theory we use the following inequality:

$\forall x \in (-\pi/2, \pi/2): |\sin x| \leq |x| \leq |\tan x|$

in conjunction with the "squeeze to zero" theorem

Squeeze to zero theorem

Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ be two functions and let σ be a limit point of A. Then:

$$\left\{ \begin{array}{l} \forall x \in N(\sigma, \delta) \cap A: |f(x)| \leq g(x) \\ \lim_{x \to \sigma} g(x) = 0 \end{array} \right. \Rightarrow \lim_{x \to \sigma} f(x) = 0$$

Proof

It is sufficient to show that

$\forall \varepsilon > 0: \exists \delta > 0: \forall x \in A: (x \in N(\sigma, \delta) \Rightarrow |f(x)| < \varepsilon)$

Let $\varepsilon > 0$ be given. Since, by hypothesis,

$\lim_{x \to \sigma} g(x) = 0 \Rightarrow \exists \delta > 0: \forall x \in A: (x \in N(\sigma, \delta) \Rightarrow |g(x)| < \varepsilon)$

Choose a $\delta > 0$ such that $\forall x \in A: (x \in N(\sigma, \delta) \Rightarrow |g(x)| < \varepsilon)$ (1)

Let $x \in A$ be given and assume that $x \in N(\sigma, \delta)$. Then:

$$\begin{aligned} x \in N(\sigma, \delta) &\Rightarrow |g(x)| < \varepsilon \Rightarrow \\ &\Rightarrow |f(x)| \leq g(x) && \text{[hypothesis]} \\ &\qquad \leq |g(x)| && \text{[algebra]} \\ &\qquad < \varepsilon && \text{[Eq. (1)]} \end{aligned}$$

$\Rightarrow |f(x)| < \varepsilon$

From the above argument it follows that

$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (x \in N(\sigma, \delta) \Rightarrow |f(x)| < \varepsilon)$

$\Rightarrow \lim_{x \to \sigma} f(x) = 0$ $\square$

Limit of sinx

$$\boxed{\forall x_0 \in \mathbb{R} : \lim_{x \to x_0} \sin x = \sin x_0}$$

Proof

Let $x_0 \in \mathbb{R}$ be given and define $f(x) = \sin x - \sin x_0$. Then:

$$|f(x)| = |\sin x - \sin x_0| = \left| 2 \sin\left(\frac{x-x_0}{2}\right) \cos\left(\frac{x+x_0}{2}\right)\right| =$$

$$= 2\left|\sin\left(\frac{x-x_0}{2}\right)\right| \cdot \left|\cos\left(\frac{x+x_0}{2}\right)\right| \leq$$

$$\leq 2\left|\sin\left(\frac{x-x_0}{2}\right)\right| \leq 2\left|\frac{x-x_0}{2}\right| =$$

$$= 2\frac{|x-x_0|}{2} = |x-x_0|, \forall x \in \mathbb{R} \Rightarrow$$

$$\Rightarrow \forall x \in \mathbb{R} : |f(x)| \leq |x - x_0| \quad (1).$$

and

$$\lim_{x \to x_0} (x - x_0) = x_0 - x_0 = 0 \Rightarrow \lim_{x \to x_0} |x - x_0| = 0 \quad (2)$$

From Eq.(1) and Eq.(2):

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (\sin x - \sin x_0) = 0 \Rightarrow \lim_{x \to x_0} \sin x = \sin x_0 \quad \square$$

$$\Rightarrow \lim_{x\to x_0} \sin x = \lim_{x\to x_0} \left[(\sin x - \sin x_0) + \sin x_0 \right] =$$

$$= \lim_{x\to x_0} (\sin x - \sin x_0) + \sin x_0 =$$

$$= 0 + \sin x_0 = \sin x_0$$

Composition theorem

Thm: Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ be two functions and let $h: C \to \mathbb{R}$ with $C = \{x \in B \mid g(x) \in A\}$ be defined as $\forall x \in C: h(x) = f(g(x))$. Let σ be a limit point of B and C and let $a \in A$. Then:

$$\boxed{\left.\begin{array}{l} \lim_{x\to\sigma} g(x) = a \in \mathbb{R} \\ \lim_{x\to a} f(x) = f(a) \end{array}\right\} \Rightarrow \lim_{x\to\sigma} f(g(x)) = f(a) = f\left(\lim_{x\to\sigma} g(x)\right)}$$

Proof

It is sufficient to show that

$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in C : (x \in N(\sigma, \delta) \Rightarrow |f(g(x)) - f(a)| < \varepsilon)$

Let $\varepsilon > 0$ be given. Since:

$\lim_{x\to a} f(x) = f(a) \Rightarrow$

$\Rightarrow \forall \varepsilon_0 > 0 : \exists \delta > 0 : \forall x \in A : \quad (0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$

For $\varepsilon_0 = \varepsilon$ we have:

$\exists \delta > 0 : \forall x \in A : (0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$

Choose a $\delta_0 > 0$ such that

$\forall x \in A : (0 < |x - a| < \delta_0 \Rightarrow |f(x) - f(a)| < \varepsilon)$ (1)

Since $\lim_{x \to \sigma} g(x) = a \in \mathbb{R} \Rightarrow$

$\Rightarrow \forall \varepsilon_0 > 0 : \exists \delta > 0 : \forall x \in B : (x \in N(\sigma, \delta) \Rightarrow |g(x) - a| < \varepsilon_0)$

For $\varepsilon_0 = \delta_0$, we have:

$\exists \delta > 0 : \forall x \in B : (x \in N(\sigma, \delta) \Rightarrow |g(x) - a| < \delta_0)$

Choose a $\delta_1 > 0$ such that

$\forall x \in B : (x \in N(\sigma, \delta_1) \Rightarrow |g(x) - a| < \delta_0)$. (2)

Let $x \in G$ be given and assume that $x \in N(\sigma, \delta_1)$. Then:

$x \in G \wedge x \in N(\sigma, \delta_1) \Rightarrow x \in A \wedge x \in N(\sigma, \delta_1)$ [via $G \subseteq A$]

$\Rightarrow |g(x) - a| < \delta_0$ [via Eq. (2)]

We need a stronger condition $0 < |g(x) - a| < \delta_0$, so we distinguish between the following cases:

<u>Case 1</u>: Assume that $g(x) = a$. Then

$|f(g(x)) - f(a)| = |f(a) - f(a)| = |0| = 0 < \varepsilon$.

<u>Case 2</u>: Assume that $g(x) \neq a$. Then:

$x \in G \wedge 0 < |g(x) - a| < \delta_0 \Rightarrow g(x) \in B \wedge 0 < |g(x) - a| < \delta_0$

$\Rightarrow |f(g(x)) - f(a)| < \varepsilon$ [via Eq. (1)]

In both cases we show: $|f(g(x)) - f(a)| < \varepsilon$.

and from the above argument we have shown that

$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in G : (x \in N(\sigma, \delta) \Rightarrow |f(g(x)) - f(a)| < \varepsilon)$

$\Rightarrow \lim_{x \to \sigma} f(g(x)) = f(a)$.

If we replace the statement $\lim_{x\to a} f(x) = f(a)$ with $\lim_{x\to a} f(x) = b$ we get the following corollary of the composition theorem:

<u>Corollary</u>: Let $f: A\to\mathbb{R}$ and $g: B\to\mathbb{R}$ be two functions and let $h: G\to\mathbb{R}$ with $G = \{x\in B \mid g(x)\in A\}$ be defined as $\forall x\in G: h(x) = f(g(x))$. Let σ be a limit point of B and G and let $a\in A$. Then:

$$\boxed{\left.\begin{array}{l} \lim_{x\to\sigma} g(x) = a\in\mathbb{R} \\ \lim_{x\to a} f(x) = b\in\mathbb{R} \\ \forall x\in N(\sigma,\delta)\cap B: g(x)\neq a \end{array}\right\} \Rightarrow \lim_{x\to\sigma} f(g(x)) = b}$$

<u>Proof</u>

We define

$$F(x) = \begin{cases} f(x), & \text{if } x\in A-\{a\} \\ b, & \text{if } x = a \end{cases}$$

Then: $\lim_{x\to a} F(x) = \lim_{x\to a} f(x) = b = F(a)$

Using the composition theorem:

$$\left\{\begin{array}{l} \lim_{x\to\sigma} g(x) = b \\ \lim_{x\to a} F(x) = F(a) \end{array}\right. \Rightarrow \lim_{x\to\sigma} F(g(x)) = F(a) = b. \quad (1)$$

Since

$$(\forall x \in N(\sigma,\delta) \cap B : g(x) \neq a) \Rightarrow (\forall x \in N(\sigma,\delta) \cap G : F(g(x)) = f(g(x)))$$

$$\Rightarrow \lim_{x\to\sigma} f(g(x)) = \lim_{x\to\sigma} F(g(x)) = b \qquad \square$$

Limits of $\cos x$, $\tan x$, $\cot x$

From the composition theorem combined with the result

$$\lim_{x\to x_0} \sin x = \sin x_0$$

it follows that:

$$\lim_{x\to\sigma} g(x) = a \Rightarrow \lim_{x\to\sigma} \sin(g(x)) = \sin a = \sin\left(\lim_{x\to\sigma} g(x)\right)$$

We use this result along with the co-factor identity:

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

to show that:

- $$\boxed{\forall x_0 \in \mathbb{R} : \lim_{x\to x_0} \cos x = \cos x_0}$$

Proof

Let $x_0 \in \mathbb{R}$ be given. Then:

$$\lim_{x\to x_0} \cos x = \lim_{x\to x_0} \sin\left(\frac{\pi}{2} - x\right) \qquad \text{[co-factor identity]}$$

$$= \sin\left(\lim_{x\to x_0}\left(\frac{\pi}{2} - x\right)\right) \qquad \text{[composition theorem]}$$

$$= \sin\left(\frac{\pi}{2} - x_0\right)$$

$$= \cos x_0 \qquad \text{[co-factor identity]} \qquad \square$$

From this result, via basic limit properties we get:

- $$\boxed{\begin{aligned} &\forall x_0 \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}: \lim_{x \to x_0} \tan x = \tan x_0 \\ &\forall x_0 \in \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}: \lim_{x \to x_0} \cot x = \cot x_0 \end{aligned}}$$

Combining these results with the composition theorem, we get:

$$\boxed{\begin{aligned} &\lim_{x \to \sigma} g(x) = a \Rightarrow \lim_{x \to \sigma} \sin(g(x)) = \sin a = \sin\left(\lim_{x \to \sigma} g(x)\right) \\ &\lim_{x \to \sigma} g(x) = a \Rightarrow \lim_{x \to \sigma} \cos(g(x)) = \cos a = \cos\left(\lim_{x \to \sigma} g(x)\right) \\ &\lim_{x \to \sigma} g(x) = a \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\} \Rightarrow \lim_{x \to \sigma} \tan(g(x)) = \tan a \\ &\lim_{x \to \sigma} g(x) = a \in \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\} \Rightarrow \lim_{x \to \sigma} \cot(g(x)) = \cot a \end{aligned}}$$

EXAMPLES

a) $f(x) = \tan\left(\dfrac{\pi x(2x+3)}{12x^2}\right) \longleftarrow \lim\limits_{x\to+\infty} f(x)$

Solution

Since,

$$\lim_{x\to-\infty} \frac{\pi x(2x+3)}{12x^2} = \lim_{x\to+\infty} \frac{\pi x(2x)}{12x^2} = \lim_{x\to+\infty} \frac{2\pi x^2}{12x^2} =$$

$$= \frac{2\pi}{12} = \frac{\pi}{6} \notin \mathbb{R} - \{k\pi + \pi/2 \mid k\in\mathbb{Z}\} \Rightarrow$$

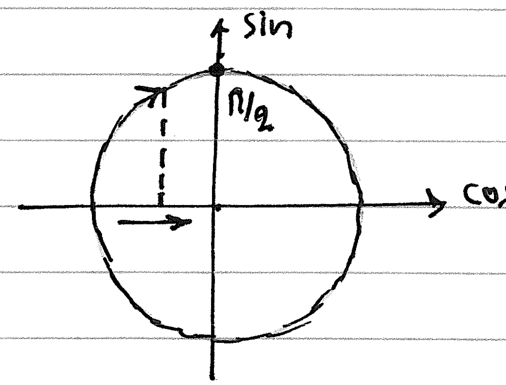
$$\Rightarrow \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \tan\left(\frac{\pi x(2x+3)}{12x^2}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

b) $f(x) = \tan x \longleftarrow \lim\limits_{x\to\pi/2^+} f(x)$

Solution

Since

$$f(x) = \tan x = \frac{\sin x}{\cos x} = (\sin x)\frac{1}{\cos x}$$



and

$$\left.\begin{array}{l} \cos x < 0,\ \forall x\in(\pi/2, \pi/2+\pi/6) \\ \lim\limits_{x\to\pi/2^+} \cos x = \cos(\pi/2) = 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \lim_{x\to\pi/2^+} \frac{1}{\cos x} = -\infty \qquad (1)$$

and $\lim\limits_{x\to\pi/2^+} \sin x = \sin(\pi/2) = 1 \qquad (2)$

From Eq.(1) and Eq.(2)

$$\lim_{x\to \pi/2^+} f(x) = \lim_{x\to \pi/2^+} \left[(\sin x) \frac{1}{\cos x} \right] = 1(-\infty) = -\infty$$

c) $f(x) = \dfrac{x^2 - \cos x}{x \sin x} \leftarrow \lim_{x\to 0} f(x)$

Solution

Since,

$$\lim_{x\to 0} (x^2 - \cos x) = 0^2 - \cos 0 = 0 - 1 = -1 \qquad (1)$$

and

$$\left\{ \begin{array}{l} \forall x \in (0, \pi/2) : (x > 0 \wedge \sin x > 0) \\ \forall x \in (-\pi/2, 0) : (x < 0 \wedge \sin x < 0) \end{array} \right. \Rightarrow$$

$$\left. \begin{array}{l} \Rightarrow \forall x \in (-\pi/2, 0) \cup (0, \pi/2) : x \sin x > 0 \\ \lim_{x\to 0} x \sin x = 0 \sin 0 = 0 \end{array} \right\} \Rightarrow$$

$$\Rightarrow \lim_{x\to 0} \frac{1}{x \sin x} = +\infty \qquad (2)$$

it follows from Eq.(1) and Eq.(2):

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^2 - \cos x}{x \sin x} = -1(+\infty) = -\infty$$

Zero-bounded limits

Def: Let $f : A \to \mathbb{R}$ be a function and let $S \subseteq A$. Then:

f bounded on $S \iff \exists a \in (0, +\infty) : \forall x \in S : |f(x)| \leq a$

- For example:

  $\forall x \in \mathbb{R} : |\sin x| \leq 1 \implies \sin$ bounded on $\mathbb{R}$

  $\forall x \in \mathbb{R} : |\cos x| \leq 1 \implies \cos$ bounded on $\mathbb{R}$

- To show that a function is bounded, it is useful to recall the following properties of absolute values:

  $\forall a, b \in \mathbb{R} : |a+b| \leq |a| + |b|$

  $\forall a, b \in \mathbb{R} : |a-b| \leq |a| + |b|$

  $\forall a, b \in \mathbb{R} : |ab| = |a||b|$

  $\forall a \in \mathbb{R} : \forall b \in \mathbb{R} - \{0\} : \left| \dfrac{a}{b} \right| = \dfrac{|a|}{|b|}$

Thm: Let $f : A \to \mathbb{R}$ and $b : A \to \mathbb{R}$ be two functions and let σ be a limit point of A. Then:

$$\boxed{\left.\begin{array}{l} b \text{ bounded on } A \cap N(\sigma, \delta) \\ \lim\limits_{x \to \sigma} f(x) = 0 \end{array}\right\} \implies \lim_{x \to \sigma} [f(x) b(x)] = 0}$$

Proof

Since b bounded on $A \cap N(\sigma, \delta) \implies$

$$\implies \exists a \in (0, +\infty) : \forall x \in A \cap N(\sigma, \delta) : |b(x)| \leq a$$

and it follows that

$\forall x \in A \cap N(\sigma, \delta): |f(x)b(x)| = |f(x)||b(x)| \leqslant a|f(x)|$ (1)

We also note that

$$\lim_{x \to \sigma} f(x) = 0 \Rightarrow \lim_{x \to \sigma} |f(x)| = 0 \Rightarrow \lim_{x \to \sigma} (a|f(x)|) = 0 \quad (2).$$

From Eq. (1) and Eq. (2), via the squeeze to zero theorem we get:

$$\lim_{x \to \sigma} [f(x)\, b(x)] = 0.$$

$\square$

EXAMPLES

a) $f(x) = \dfrac{\sin x \cos x - 3\sin(2x)\cos(2x)}{x^2+5x+3} \leftarrow \lim\limits_{x \to -\infty} f(x)$.

Solution

Note that

$$f(x) = \frac{\sin x \cos x - 3\sin(2x)\cos(2x)}{x^2+5x+3} =$$

$$= \frac{1}{x^2+5x+3} \cdot \left[\sin x \cos x - 3\sin(2x)\cos(2x)\right]$$

Define $b(x) = \sin x \cos x - 3\sin(2x)\cos(2x)$, $\forall x \in \mathbb{R}$.

$$|b(x)| = |\sin x \cos x - 3\sin(2x)\cos(2x)| \leq$$

$$\leq |\sin x \cos x| + |3\sin(2x)\cos(2x)|$$

$$= |\sin x||\cos x| + 3|\sin(2x)|\cdot|\cos(2x)|$$

$$\leq 1\cdot 1 + 3\cdot 1\cdot 1 = 1+3 = 4 \quad, \forall x \in \mathbb{R} \Rightarrow$$

$\Rightarrow b$ bounded on $\mathbb{R}$. (1).

and

$$\lim_{x \to -\infty} \frac{1}{x^2+5x+3} = \lim_{x \to -\infty} \frac{1}{x^2} = 0 \qquad (2)$$

From Eq.(1) and Eq.(2): $\lim\limits_{x \to -\infty} f(x) = 0$.

$\hookrightarrow$ It would have been sufficient to show that b is bounden on $(-\infty, -1/8)$ but in this case it is easy to show that b is bounded on $\mathbb{R}$.

b) $f(x) = x[1-\sin(1/x)] \longleftarrow \lim_{x\to 0} f(x)$

Solution

We note that $f(x) = x\,b(x)$, $\forall x \in \mathbb{R}-\{0\}$

where we define $\forall x \in \mathbb{R}-\{0\}: b(x) = 1-\sin(1/x)$. Since:

$$\forall x \in \mathbb{R}-\{0\}: |b(x)| = |1-\sin(1/x)| \leq |1| + |\sin(1/x)|$$

$$\leq 1+1 = 2 \Rightarrow$$

$\Rightarrow b$ bounded on $\mathbb{R}-\{0\}$ (1).

Also: $\lim_{x\to 0} x = 0$ (2)

From Eq.(1) and Eq.(2): $\lim_{x\to 0} f(x) = 0$

$\hookrightarrow$ It would have been sufficient to show that b is bounded on $(-\delta, 0)\cup(0,\delta)$. Note that using the zero-bounded theorem becomes necessary because $f(0)$ is not defined. It is possible to have zero-bounded limits that are trivial limits.

EXERCISES

(11) Evaluate the following limits, if they exist.

a) $\lim\limits_{x \to +\infty} \dfrac{\sin 4x}{x^2+5}$

b) $\lim\limits_{x \to -\infty} \dfrac{\sin 5x}{x}$

c) $\lim\limits_{x \to -\infty} \dfrac{2x \cos x}{x^2-9}$

d) $\lim\limits_{x \to +\infty} \dfrac{(2x-1)\sin 2x}{x^2+2}$

e) $\lim\limits_{x \to +\infty} \dfrac{3x^2 \sin x}{x^3+2}$

f) $\lim\limits_{x \to 0} \left(x^3 \sin(1/x)\right)$

g) $\lim\limits_{x \to +\infty} \left(\dfrac{\cos x}{x^3}\right)$

h) $\lim\limits_{x \to 1} \left[(x-1)\cos 3x\right]$

i) $\lim\limits_{x \to -\infty} \dfrac{x(\sin x + 2\cos 3x \sin 2x)}{2x^2+x+1}$

j) $\lim\limits_{x \to 0} \sin x \left[\cos(1/x^2) + \sin(1/x^2)\right]$

k) $\lim\limits_{x \to +\infty} \dfrac{x^3\left[(\sin x + \cos x)^2 + \sin x \cos x\right]}{(2x+1)^2(x^2-x+1)}$

l) $\lim\limits_{x \to +\infty} \left[\sin(1/x)\cos x + \cos(1/x)\sin x\right]$

m) $\lim\limits_{x \to 3} \dfrac{x^2-2x-3}{x+1}\left[\cos\left(\dfrac{1}{x-3}\right) + 3\sin x \sin\left(\dfrac{1}{x-3}\right)\right]$

Trigonometric 0/0 limits

Some 0/0 trigonometric limits can be handled via results that we establish via the squeeze theorem. We use the squeeze to 0 theorem to prove the squeeze theorem.

▶ Squeeze theorem

Thm: Let $f: A \to \mathbb{R}$, $g_1: A \to \mathbb{R}$, $g_2: A \to \mathbb{R}$ be three functions and let σ be a limit point of A. Then:

$$\left.\begin{array}{l} \forall x \in A \cap N(\sigma, \delta): g_1(x) \leq f(x) \leq g_2(x) \\ \lim_{x \to \sigma} g_1(x) = \lim_{x \to \sigma} g_2(x) = \ell \in \mathbb{R} \end{array}\right\} \Rightarrow \lim_{x \to \sigma} f(x) = \ell.$$

Proof

Let $x \in A \cap N(\sigma, \delta)$ be given. Then:

$$g_1(x) \leq f(x) \leq g_2(x) \Rightarrow 0 \leq f(x) - g_1(x) \leq g_2(x) - g_1(x) \Rightarrow$$
$$\Rightarrow 0 \leq |f(x) - g_1(x)| \leq g_2(x) - g_1(x)$$

and it follows that

$$\forall x \in A \cap N(\sigma, \delta): 0 \leq |f(x) - g_1(x)| \leq g_2(x) - g_1(x). \quad (1)$$

We also note that

$$\lim_{x \to \sigma} [g_2(x) - g_1(x)] = \lim_{x \to \sigma} g_2(x) - \lim_{x \to \sigma} g_1(x) = \ell - \ell = 0 \quad (2)$$

From Eq.(1) and Eq.(2), via the squeeze to zero theorem:

$$\lim_{x \to \sigma} (f(x) - g_1(x)) = 0 \Rightarrow$$
$$\Rightarrow \lim_{x \to \sigma} f(x) = \lim_{x \to \sigma} [(f(x) - g_1(x)) + g_1(x)] =$$

$$= \lim_{x\to\sigma} (f(x) - g_1(x)) + \lim_{x\to\sigma} g_1(x) =$$

$$= 0 + l = l \qquad \blacksquare$$

▶ Limits of $\sin x/x$ and $\tan x/x$

We now use the squeeze theorem combined with the inequality:

$\forall x \in (-\pi/2, 0) \cup (0, \pi/2): |\sin x| \leq |x| \leq |\tan x|$

to show the following basic results:

① $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$

Proof

Define $f(x) = (\sin x)/x$, $\forall x \in \mathbb{R} - \{0\}$.

We note that

$$\forall x \in (-\pi/2, 0) \cup (0, \pi/2): f(x) = \frac{\sin x}{x} \leq \left|\frac{\sin x}{x}\right| = \frac{|\sin x|}{|x|}$$

$$\leq \frac{|x|}{|x|} = 1 \qquad (1)$$

and

$$\begin{cases} \forall x \in (0, \pi/2): (x > 0 \wedge \sin x > 0) \\ \forall x \in (-\pi/2, 0): (x < 0 \wedge \sin x < 0) \end{cases} \Rightarrow$$

$$\Rightarrow \forall x \in (-\pi/2, 0) \cup (0, \pi/2): f(x) = \frac{\sin x}{x} > 0$$

$$\Rightarrow \forall x \in (-\pi/2, 0) \cup (0, \pi/2): f(x) = \frac{\sin x}{x} = \left|\frac{\sin x}{x}\right| =$$

$$= \frac{|\sin x|}{|x|} \geqslant \frac{|\sin x|}{|\tan x|} = \left|\frac{\sin x}{\tan x}\right| =$$

$$= \left|\frac{\sin x}{\left(\frac{\sin x}{\cos x}\right)}\right| = |\cos x| \qquad (2)$$

From Eq. (1) and Eq. (2):

$$\forall x \in (-\pi/2, 0) \cup (0, \pi/2): |\cos x| \leqslant f(x) \leqslant 1 \qquad (3)$$

and since:

$$\lim_{x \to 0} \cos x = \cos 0 = 1 \Rightarrow \lim_{x \to 0} |\cos x| = |1| = 1 \qquad (4)$$

From Eq. (3) and Eq. (4), via the squeeze theorem, we have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1$. □

② $\boxed{\lim_{x \to 0} \frac{\tan x}{x} = 1}$

Proof

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left[\frac{\sin x}{x} \frac{1}{\cos x}\right] =$$

$$= \left[\lim_{x \to 0} \frac{\sin x}{x}\right]\left[\lim_{x \to 0} \frac{1}{\cos x}\right] =$$

$$= 1 \cdot \frac{1}{\cos 0} = 1 \cdot \frac{1}{1} = 1. \qquad □$$

Combining these results with the composition corrolary it follows that:

Prop: Let $f: A \to \mathbb{R}$ be a function and let σ be a limit point of A. Then:

$$\left\{\begin{array}{l} \lim_{x\to\sigma} f(x) = 0 \\ \forall x \in A \cap N(\sigma, \delta) : f(x) \neq 0 \end{array}\right. \Rightarrow \lim_{x\to\sigma} \frac{\sin(f(x))}{f(x)} = 1$$

$$\left\{\begin{array}{l} \lim_{x\to\sigma} f(x) = 0 \\ \forall x \in A \cap N(\sigma, \delta) : f(x) \neq 0 \end{array}\right. \Rightarrow \lim_{x\to\sigma} \frac{\tan(f(x))}{f(x)} = 1$$

EXAMPLES

a) $f(x) = \dfrac{\sin(2x)}{\sin(5x)} \leftarrow \lim\limits_{x \to 0} f(x).$

Solution

$$f(x) = \frac{\sin(2x)}{\sin(5x)} = \frac{\sin(2x)}{2x} \frac{5x}{\sin(5x)} \frac{2}{5}, \quad \forall x \in \mathbb{R} - \{0\}. \quad (1)$$

$$\lim_{x \to 0} \frac{\sin(2x)}{2x} = 1 \quad (2)$$

$$\lim_{x \to 0} \frac{\sin(5x)}{5x} = 1 \Rightarrow \lim_{x \to 0} \frac{5x}{\sin(5x)} = 1 \quad (3)$$

From Eq. (1), Eq. (2), Eq. (3):

$$\lim_{x \to 0} f(x) = 1 \cdot 1 \cdot (2/5) = \frac{2}{5}.$$

b) $f(x) = \dfrac{1 - \cos x}{x^2} \leftarrow \lim\limits_{x \to 0} f(x)$

Solution

$$f(x) = \frac{1 - \cos x}{x^2} = \frac{2\sin^2(x/2)}{x^2} = \frac{2(1/2)^2 \sin^2(x/2)}{(x/2)^2}$$

$$= \frac{1}{2}\left[\frac{\sin(x/2)}{x/2}\right]^2, \quad \forall x \in \mathbb{R} - \{0\}$$

Since $\lim\limits_{x \to 0} \dfrac{\sin(x/2)}{x/2} = 1 \Rightarrow$

c) $f(x) = \dfrac{\sin x - \sin(5x)}{x\cos(3x)} \leftarrow \lim\limits_{x \to \pi/6} f(x).$

Solution

- Note that

$\cos(3(\pi/6)) = \cos(\pi/2) = 0$

$\sin(\pi/6) - \sin(5\pi/6) = \sin(\pi/6) - \sin(\pi/6) = 0$

thus this is a 0/0 limit.

$$f(x) = \frac{\sin x - \sin(5x)}{x\cos(3x)} = \frac{2\sin\left(\frac{x-5x}{2}\right)\cos\left(\frac{x+5x}{2}\right)}{x\cos(3x)}$$

$$= \frac{2\sin(-2x)\cos(3x)}{x\cos(3x)} = \frac{-2\sin(2x)}{x} \Rightarrow$$

$$\Rightarrow \lim_{x\to\pi/6} f(x) = \lim_{x\to\pi/6} \frac{-2\sin(2x)}{x} = \frac{-2\sin(\pi/3)}{\pi/6} =$$

$$= \frac{-2(\sqrt{3}/2)}{\pi/6} = \frac{-\sqrt{3}}{\pi/6} = \frac{-6\sqrt{3}}{\pi}$$

EXERCISES

(12) Evaluate the following limits, if they exist

a) $\lim\limits_{x \to 0} \dfrac{\sin 4x}{x}$

b) $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x}$

c) $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x}$

d) $\lim\limits_{x \to 0} \dfrac{\cos x - \cos 3x}{x \sin 3x}$

e) $\lim\limits_{x \to 0} \dfrac{1+\sin x - \cos x}{1 - \sin x - \cos x}$

f) $\lim\limits_{x \to 0} \dfrac{\sin 2x}{\tan 3x}$

g) $\lim\limits_{x \to \pi/4} \dfrac{1-\cos x}{\sin x}$

h) $\lim\limits_{x \to 0} \left(x \cdot \sin(1/x)\right)$

i) $\lim\limits_{x \to 0} \dfrac{\sin 5x}{\tan 6x}$

j) $\lim\limits_{x \to \pi/4} \dfrac{\cos x - \sin x}{x - \pi/4}$

Continuity

- Let $f: A \to \mathbb{R}$ be a function. We say that

a) f continuous at $x_0 \in A \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)$

b) f continuous at $I \Leftrightarrow \forall x_0 \in I$: f continuous at x_0

- By the limit definition, it follows that:

$$\boxed{\begin{array}{l} f \text{ continuous at } x_0 \in A \Leftrightarrow \\ \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon) \end{array}}$$

- There are three ways a function f can fail to be continuous at x_0
  1) $f(x_0)$ is not defined (i.e. $x_0 \notin A$)
  2) $\lim_{x \to x_0} f(x)$ does not exist.
  3) Both $f(x_0)$ and $\lim_{x \to x_0} f(x)$ exist but $\lim_{x \to x_0} f(x) \neq f(x_0)$.

Operations and continuity.

Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ with f, g continuous at $x_0 \in A$, and let $\lambda \in \mathbb{R}$. Then

a) $h_1(x) = f(x) + g(x)$ continuous at $x_0 \in A$

$h_2(x) = f(x) g(x)$ continuous at $x_0 \in A$

$h_3(x) = \lambda f(x)$ continuous at $x_0 \in A$

b) $g(x_0) \neq 0 \Rightarrow h(x) = f(x)/g(x)$ continuous at x_0
c) $\forall x \in N(x_0, \delta) : f(x) \geq 0 \Rightarrow h(x) = \sqrt{f(x)}$ continuous at x_0.

Continuity of basic functions

1) Every polynomial function
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
is continuous at $\mathbb{R}$.

2) Every rational function
$$f(x) = \frac{P(x)}{Q(x)}$$
with P, Q polynomials is continuous at each domain $A = \mathbb{R} - \{x \in \mathbb{R} \mid Q(x) = 0\}$

3) $\sin, \cos$ continuous at $\mathbb{R}$
$\tan$ continuous at $\mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}$
$\cot$ continuous at $\mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}$

Continuity of function composition

$$\boxed{\left.\begin{array}{l} g \text{ continuous at } x_0 \\ f \text{ continuous at } g(x_0) \end{array}\right\} \Rightarrow h(x) = f(g(x)) \text{ continuous at } x_0}$$

EXAMPLE

a) Find all $a, b \in \mathbb{R}$ such that

$$f(x) = \begin{cases} x^2 - 3x + 1 & , \text{ if } x \in (-\infty, 2) \\ ax + b & , \text{ if } x \in [2, 3) \\ x^2 + 5x + 2 & , \text{ if } x \in [3, +\infty) \end{cases}$$

is continuous on $\mathbb{R}$.

Solution

First, we note that f continuous on $\mathbb{R} - \{2, 3\}$, $\forall a \in \mathbb{R}$ (1)

At $x = 2$:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 - 3x + 1) = 2^2 - 3 \cdot 2 + 1 = 4 - 6 + 1 = -1. \quad (2)$$

$$f(2) = \lim_{x \to 2^-} f(x) = -1. \quad (3)$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax + b) = 2a + b \quad (4)$$

At $x = 3$:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax + b) = 3a + b \quad (5)$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^2 + 5x + 2) = 3^2 + 5 \cdot 3 + 2 = 9 + 15 + 2 = 26 \quad (6)$$

$$f(3) = 3^2 + 5 \cdot 3 + 2 = 26 \quad (7)$$

It follows that:

f continuous on $\mathbb{R} \Leftrightarrow f$ continuous on $\{2,3\} \Leftrightarrow$

$$\Leftrightarrow \lim_{x\to 2} f(x) = f(2) \wedge \lim_{x\to 3} f(x) = f(3) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2) \\ \lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = f(3) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2a+b=-1 \\ 3a+b=26 \end{cases} \Leftrightarrow \begin{cases} 2a+b=-1 \\ a+(2a+b)=26 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2a+b=-1 \\ a+(-1)=26 \end{cases} \Leftrightarrow \begin{cases} 2\cdot 27+b=-1 \\ a=26+1=27 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a=27 \\ b=-1-2\cdot 27=-1-54=-55 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow (a,b)=(27,-55)$$

We have thus shown that

f continuous on $\mathbb{R} \Leftrightarrow (a,b)=(27,-55)$

b) Find all $a \in \mathbb{R}$ such that

$$f(x) = \begin{cases} x-2 & , \text{ if } x \in (2, +\infty) \\ x^2-(a+1)x+(a^2-1) & , \text{ if } x \in (-\infty, 2] \end{cases}$$

is continuous on $\mathbb{R}$.

Solution

We note that f continuous on $\mathbb{R}-\{2\}$, $\forall a \in \mathbb{R}$.

At $x=2$:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x-2) = 2-2=0$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} [x^2-(a+1)x+(a^2-1)] = 2^2-(a+1)2+(a^2-1)$$

$$= 4-2a-2+a^2-1 = a^2-2a+1 = (a-1)^2$$

$f(2) = 2^2-(a+1)2+(a^2-1) = (a-1)^2$.

It follows that

f continuous on $\mathbb{R} \Leftrightarrow f$ continuous at $x=2 \Leftrightarrow$

$$\Leftrightarrow \lim_{x\to 2} f(x) = f(2) \Leftrightarrow$$

$$\Leftrightarrow \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$$

$$\Leftrightarrow (a-1)^2 = 0 \Leftrightarrow a-1=0 \Leftrightarrow a=1.$$

We have thus shown that

f continuous on $\mathbb{R} \Leftrightarrow a=1$.

c) Show that the function

$$f(x) = \begin{cases} (\sin x)[1+\cos(1/x)] & , \text{ if } x \in \mathbb{R}-\{0\} \\ 0 & , \text{ if } x = 0 \end{cases}$$

is continuous on $\mathbb{R}$.

Solution

We note that f is continuous at $\mathbb{R}-\{0\}$. (1)

At $x=0$: we define $b(x) = 1+\cos(1/x)$, $\forall x \in \mathbb{R}-\{0\}$.

Then:

$|b(x)| = |1+\cos(1/x)| \leq |1| + |\cos(1/x)| \leq 1+1 = 2$, $\forall x \in \mathbb{R}-\{0\}$

$\Rightarrow$ b bounded on $\mathbb{R}-\{0\}$ (2)

and

$$\lim_{x\to 0} \sin x = \sin 0 = 0 \qquad (3)$$

From Eq.(2) and Eq.(3) it follows that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (\sin x)[1+\cos(1/x)] =$$

$$= 0 = f(0) \Rightarrow$$

$\Rightarrow$ f continuous at $x=0$ $\Rightarrow$

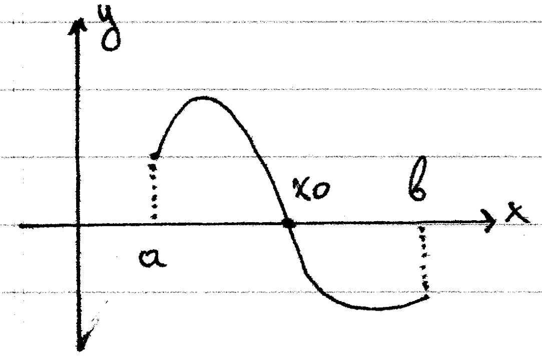
$\Rightarrow$ f continuous on $\mathbb{R}$ [via Eq. (1)].

V Consequences of continuity

1 → Bolzano theorem

Thm: Let $f: A \to \mathbb{R}$ be a function with $[a,b] \subseteq A$. Then:

$$\boxed{\left.\begin{array}{l} f \text{ continuous on } [a,b] \\ f(a)f(b) < 0 \end{array}\right\} \Longrightarrow \exists x_0 \in [a,b] : f(x_0) = 0}$$



The condition $f(a)f(b) < 0$ means that $f(a)$ and $f(b)$ have opposite signs. According to the Bolzano theorem, in order for the function to change sign, from $x=a$ to $x=b$, it has to be 0 for some $x = x_0 \in (a,b)$. A proof of the Bolzano theorem is omitted as it requires the theory of sequences.

2 → Intermediate value theorem

Let $f: A \to \mathbb{R}$ be a function and let $S \subseteq A$. Recall that we defined $f(S)$ as

$$f(S) = \{f(x) \mid x \in S\}$$

or via the belonging condition

$$y \in f(S) \Longleftrightarrow \exists x \in S : f(x) = y$$

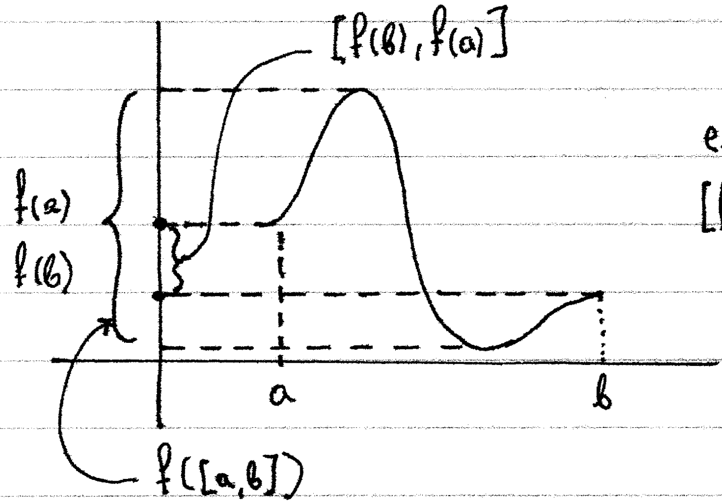
We also recall that for any two sets A and B:

$$A \subseteq B \iff \forall x \in A : x \in B$$

An immediate consequence of the Bolzano theorem is the intermediate value theorem:

<u>Thm</u>: Let $f : A \to \mathbb{R}$ be a function and let $[a,b] \subseteq A$. Then:

$$\boxed{\begin{array}{l} \left\{\begin{array}{l} f \text{ continuous on } [a,b] \\ f(a) \leq f(b) \end{array}\right. \implies [f(a), f(b)] \subseteq f([a,b]) \\ \left\{\begin{array}{l} f \text{ continuous on } [a,b] \\ f(a) \geq f(b) \end{array}\right. \implies [f(b), f(a)] \subseteq f([a,b]) \end{array}}$$



e.g.: $[f(b), f(a)] \subseteq f([a,b])$

▶ <u>Interpretation</u>: $f([a,b])$ represents all the values taken by the function f over the interval $[a,b]$. The interval $[f(a), f(b)]$ obviously contains all numbers y such that $f(a) \leq y \leq f(b)$. Thus the statement $[f(a), f(b)] \subseteq f([a,b])$ means that the function f takes all the values y with $f(a) \leq y \leq f(b)$.

Proof

Assume that f continuous on $[a,b]$ and $f(a)<f(b)$.

Since: $a\in[a,b] \Rightarrow f(a)\in f([a,b])$

$b\in[a,b] \Rightarrow f(b)\in f([a,b])$

we let $y\in(f(a),f(b))$ be given and we define

$g(x)=f(x)-y\ ,\ \forall x\in[a,b]$

It follows that:

f continuous on $[a,b] \Rightarrow g$ continuous on $[a,b]$ (1)

and

$$y\in(f(a),f(b)) \Rightarrow f(a)<y<f(b) \Rightarrow \begin{cases} f(a)<y \\ y<f(b) \end{cases} \Rightarrow \begin{cases} f(a)-y<0 \\ f(b)-y>0 \end{cases}$$

$$\Rightarrow \begin{cases} g(a)<0 \\ g(b)>0 \end{cases} \Rightarrow g(a)g(b)<0 \qquad (2)$$

From Eq.(1) and Eq.(2), via the Bolzano theorem,

$$(\exists x_0\in(a,b):\ g(x_0)=0) \Rightarrow (\exists x_0\in(a,b):\ f(x_0)-y=0)$$

$$\Rightarrow \exists x_0\in(a,b):\ f(x_0)=y$$

$$\Rightarrow y\in f((a,b)) \Rightarrow y\in f([a,b])$$

We have thus shown that

$$(\forall y\in[f(a),f(b)]:\ y\in f([a,b])) \Rightarrow \underline{[f(a),f(b)]\subseteq f([a,b])}$$

EXAMPLES

a) Show that the equation

$$\sin(\cos 3x) = 0$$

has at least one solution on $(0, \pi)$.

Solution

Define $f(x) = \sin(\cos(3x))$, $\forall x \in \mathbb{R}$.

We note that f continuous on $[0, \pi]$ (1).

and also:

$f(0) = \sin(\cos(3 \cdot 0)) = \sin(\cos 0) = \sin 1$ (2)

$f(\pi) = \sin(\cos(3\pi)) = \sin(\cos \pi) = \sin(-1) = -\sin 1$ (3)

From Eq.(2) and Eq.(3):

$f(0)f(\pi) = (\sin 1)(-\sin 1) = -\sin^2 1 < 0$ (3)

From Eq.(1) and Eq.(3):

$(\exists x_0 \in (0, \pi) : f(x_0) = 0) \Rightarrow x_0$ solves $\sin(\cos(3x)) = 0.$

b) If $a, b \in \mathbb{R}$ with $0 < a < b < \pi/2$, show that the equation

$$\frac{\sin x}{x-a} + \frac{\cos x}{x-b} = 0$$

has at least one solution $x_0 \in (a, b)$.

Solution

We note that for $x \in (a, b)$, we have $(x-a)(x-b) \neq 0$, and therefore:

$$\frac{\sin x}{x-a} + \frac{\cos x}{x-b} = 0 \Leftrightarrow (x-b)\sin x + (x-a)\cos x = 0$$

Define $f(x) = (x-b)\sin x + (x-a)\cos x$, $\forall x \in \mathbb{R}$

Then: f continuous on $[a, b]$ (1)

$f(a) = (a-b)\sin a + (a-a)\cos a = (a-b)\sin a$ (2)

$f(b) = (b-b)\sin b + (b-a)\cos b = (b-a)\cos b$ (3)

From Eq.(2) and Eq.(3):

$$f(a)f(b) = [(a-b)\sin a][(b-a)\cos b] = (a-b)(b-a)\sin a \cos b$$
$$= -(a-b)^2 \sin a \cos b.$$

We note that $a \neq b \Rightarrow (a-b)^2 > 0$

and $0 < a < \pi/2 \Rightarrow \sin a > 0$

and $0 < b < \pi/2 \Rightarrow \cos b > 0$.

It follows that

$f(a)f(b) = -(a-b)^2 \sin a \cos b < 0$ (4)

From Eq.(1) and Eq.(4), via Bolzano theorem,

$$(\exists x_0 \in (a, b) : f(x_0) = 0) \Rightarrow x_0 \text{ solves } \frac{\sin x}{x-a} + \frac{\cos x}{x-b} = 0$$

(3) Continuity and Boundedness

Thm: Let $f: A \to \mathbb{R}$ be a function with $[a,b] \subseteq A$.
Then:

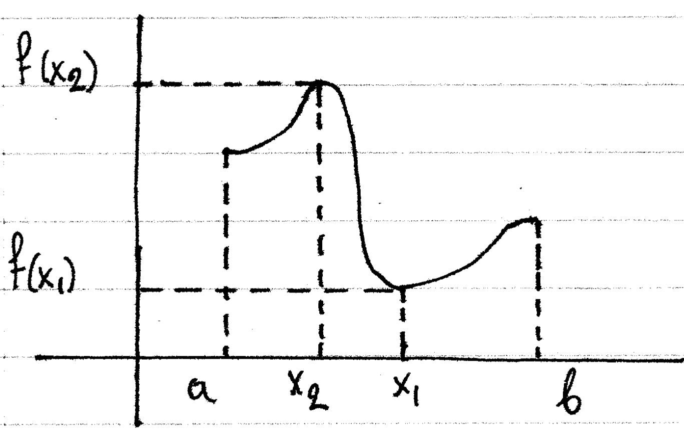
$$\boxed{f \text{ continuous on } [a,b] \Rightarrow f \text{ bounded on } [a,b]}$$

Although this result is very intuitive, geometrically, a proof requires the theory of sequences and is therefore omitted.

(4) Extremum value theorem

Thm: Let $f: A \to \mathbb{R}$ be a function with $[a,b] \subseteq A$. Then:

$$\boxed{\begin{array}{l} f \text{ continuous on } [a,b] \Rightarrow \\ \Rightarrow \exists x_1, x_2 \in [a,b] : \forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2) \end{array}}$$



EXERCISES

(13) Show that the following functions are continuous in $\mathbb{R}$.

a) $f(x) = \begin{cases} x\sin(1/x) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

b) $f(x) = \begin{cases} (\sin x)/x & , x \neq 0 \\ 1 & , x = 0 \end{cases}$

(14) Find all $a \in \mathbb{R}$ for which the following functions are continuous in $\mathbb{R}$.

a) $f(x) = \begin{cases} \frac{|x|}{x} + x & , x < 0 \\ a & , x \geq 0 \end{cases}$

b) $f(x) = \begin{cases} 2x+3 & , x \leq 1 \\ ax^2 - a^2x + 7 & , x > 1 \end{cases}$

c) $f(x) = \begin{cases} 2x^2+1 & , x \leq 1 \\ \frac{x^2+ax-3}{x+1} & , x > 1 \end{cases}$

(15) Find all $a, b \in \mathbb{R}$ for which the following functions are continuous at $\mathbb{R}$:

a) $f(x) = \begin{cases} 1+2\sin x & , x < -\pi \\ a\cos x & , -\pi \leq x < 0 \\ b - 4\cos^2 x & , x \geq 0 \end{cases}$

8) $f(x)=\begin{cases} -\sin 2x & , x \leqslant -\pi/4 \\ |a\sin x + b| & , x \in (-\pi/4, \pi/4) \\ \cos 2x & , x \geqslant \pi/4 \end{cases}$

(16) Show that the equation $\sin(\cos 3x) = 0$ has at least one solution in the interval $(0, \pi)$

(17) Show that $f(x) = ax^3 + x^2 + x - 1$ with $a \neq -1$ has at least one zero in the interval $(-1, 1)$. What happens when $a = -1$?

(18) Show that $f(x) = x^3/4 + \sin(\pi x) + 3$ takes the value $5/3$ within the interval $(-2, 2)$

(19) Show that $f(x) = x^3 - \cos(\pi x) + 1$ takes the value 5 within the interval $(-2, 3)$

(20) Let a function $f : [0,1] \longrightarrow (0,1)$ be given. Show that if f is continuous at $[0,1]$ then the equation $f(x) = x$ has at least one solution at the interval $(0,1)$.

(21) Let two functions f, g be given such that they are both continuous at $[a,b]$ and $f(a) = g(b)$ and $f(b) = g(a)$. Show that there is a $c \in [a,b]$ such that $f(c) = g(c)$.

(22) Show that the equation

$$\frac{x^2+1}{x-a} + \frac{x^6+1}{x-b} = 0$$

with $a<b$ has a solution in (a,b)

(23) Let f be a function that is continuous at $[0,2\pi]$ with $f(0)=f(2\pi)$. Show that there is an $x_0 \in [0,\pi]$ such that $f(x_0+\pi)=f(x_0)$.

CAL1.3: Asymptotes

Asymptotes

Asymptotes are lines that are approached by the graph of the function under the limits $x \to x_0^\pm$ or $x \to +\infty$ or $x \to -\infty$. The graph may or may not make contact with an asymptote. The precise definitions are as follows: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ be a function

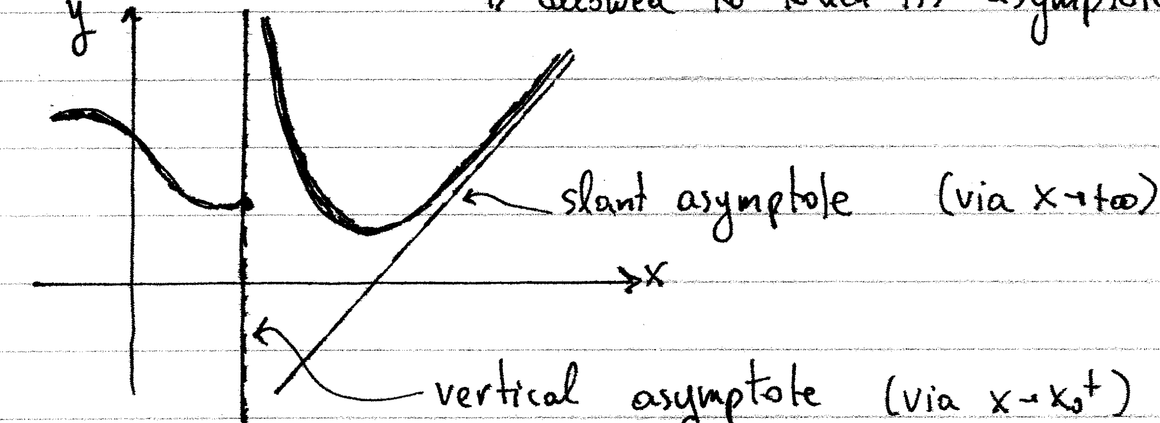
1. $\boxed{\left.\begin{array}{l}(\ell): y = ax+b \text{ is} \\ \text{asymptote of } f(x) \\ \text{at } \pm\infty\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}\lim\limits_{x\to\pm\infty} \dfrac{f(x)}{x} = a \\ \lim\limits_{x\to\pm\infty} [f(x) - ax] = b\end{array}\right.}$

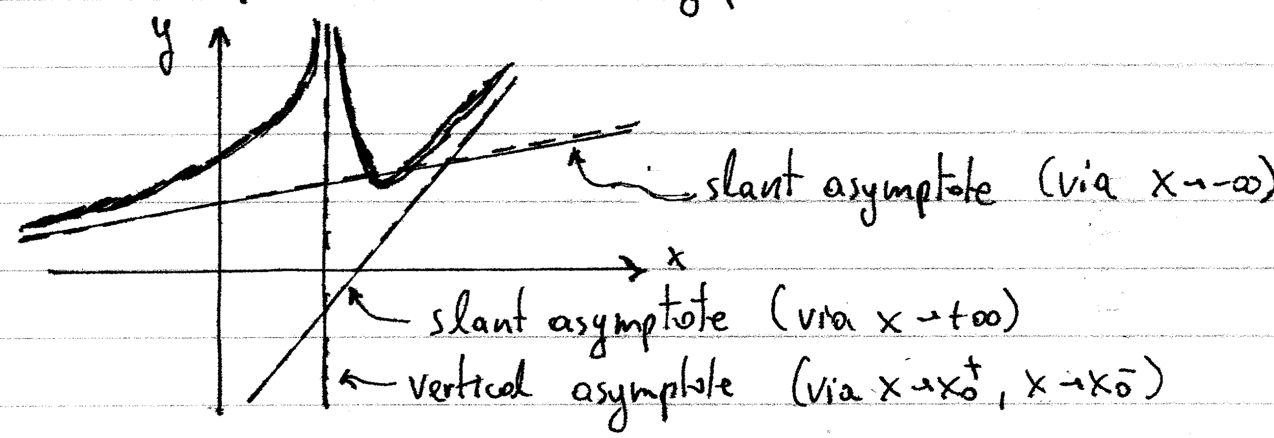
<u>notation</u>: $f(x) \sim ax+b$ with $x \to \pm\infty \Leftrightarrow$
$\Leftrightarrow (\ell): y = ax+b$ is asymptote of $f(x)$ at $\pm\infty$

<u>remark</u>: If $a \neq 0$, then (ℓ) is a <u>slant asymptote</u>.
If $a = 0$, then (ℓ) is a <u>horizontal asymptote</u>.

2. $\boxed{\left.\begin{array}{l}(\ell): x = x_0 \text{ is} \\ \text{a vertical asymptote} \\ \text{of } f(x)\end{array}\right\} \Leftrightarrow \begin{array}{l}\lim\limits_{x\to x_0^+} f(x) \in \{+\infty, -\infty\} \vee \\ \vee \lim\limits_{x\to x_0^-} f(x) \in \{+\infty, -\infty\}\end{array}}$

<u>Graphic Examples</u>: Note that the graph of a function is allowed to touch its asymptotes.





Continuity can be used to rule out the existence of vertical asymptotes:

<u>Prop:</u> $\boxed{f \text{ continuous at } x_0 \in A \Rightarrow (\ell): x = x_0 \text{ NOT asymptote}}$

<u>Proof</u>

Assume that f continuous at x_0. Then:

f continuous at $x_0 \Rightarrow \lim\limits_{x \to x_0} f(x) = f(x_0) \Rightarrow$

$$\Rightarrow \lim_{x \to x_0^+} f(x) = f(x_0) \wedge \lim_{x \to x_0^-} f(x) = f(x_0) \Rightarrow$$

$$\Rightarrow \lim_{x\to x_0^+} f(x) \notin \{+\infty, -\infty\} \wedge \lim_{x\to x_0^-} f(x) \notin \{+\infty, -\infty\}$$

$\Rightarrow (\ell): x = x_0$ is NOT vertical asymptote.

▶ Methodology

• 1 To check for slant or horizontal asymptotes, we calculate

$$a_+ = \lim_{x\to+\infty} \frac{f(x)}{x} \qquad b_+ = \lim_{x\to+\infty} [f(x) - a_+ x]$$

$$a_- = \lim_{x\to-\infty} \frac{f(x)}{x} \qquad b_- = \lim_{x\to-\infty} [f(x) - a_- x]$$

If $a_+, b_+ \in \mathbb{R}$ then $(\ell): y = a_+ x + b_+$ is slant asymptote when $x\to+\infty$.

If $a_-, b_- \in \mathbb{R}$ then $(\ell): y = a_- x + b_-$ is slant asymptote when $x\to-\infty$

If $f(x)$ is a rational function, we can do the limits $x\to+\infty$ and $x\to-\infty$ simultaneously, as shown in the examples.

• 2 To find all vertical asymptotes:

a) Use continuity to eliminate all points $x_0 \in A$ where the function has no vertical asymptote.

b) Any points $x_0 \notin A$ where x_0 is NOT a limit point can be also ruled out.

c) We check the remaining points on a case by case basis.

EXAMPLES

a) Find all asymptotes of
$f(x) = x^3 + 5x^2 + 3x + 1$.

Solution

$$a_{\pm} = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^3+5x^2+3x+1}{x} =$$

$$= \lim_{x\to\pm\infty} \frac{x^3}{x} = \lim_{x\to\pm\infty} (x^2) = +\infty \Rightarrow$$

$\Rightarrow$ no slant or horizontal asymptotes when $x\to\pm\infty$.
Also: f continuous in $\mathbb{R} \Rightarrow \forall x_0 \in \mathbb{R}$: $(\ell): x = x_0$ not asymptote
$\Rightarrow$ no vertical asymptotes.

b) Find all asymptotes of $f(x) = \dfrac{x^2(x+1)}{x^2+3x+2}$

Solution

- Slant asymptotes

$$a = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^2(x+1)}{x(x^2+3x+2)} = \lim_{x\to\pm\infty} \frac{x^2 x}{x x^2}$$

$$= \lim_{x\to\pm\infty} \frac{x^3}{x^3} = 1$$

$$b = \lim_{x\to\pm\infty} [f(x) - ax] = \lim_{x\to\pm\infty} [f(x) - x] = \lim_{x\to\pm\infty} \left[\frac{x^2(x+1)}{x^2+3x+2} - x \right]$$

$$= \lim_{x \to \pm\infty} \frac{x^2(x+1) - x(x^2+3x+2)}{x^2+3x+2} =$$

$$= \lim_{x \to \pm\infty} \frac{x^3+x^2-x^3-3x^2-2x}{x^2+3x+2} = \lim_{x \to \pm\infty} \frac{-2x^2-2x}{x^2+3x+2} =$$

$$= \lim_{x \to \pm\infty} \frac{-2x^2}{x^2} = -2$$

It follows that $(\ell): y = x - 2$ is slant asymptote as $x \to \pm\infty$.

▶ Vertical asymptotes.

- Solve $x^2+3x+2 = 0 \Leftrightarrow (x+1)(x+2) = 0 \Leftrightarrow x+1 = 0 \vee x+2 = 0$
$$\Leftrightarrow x = -1 \vee x = -2$$

thus $\operatorname{dom}(f) = \mathbb{R} - \{-1, -2\}$.

- Since f continuous on $\mathbb{R} - \{-1, -2\} \Rightarrow$
$\Rightarrow \forall x_0 \in \mathbb{R} - \{-1, -2\}: (\ell): x = x_0$ is NOT a vertical asymptotes
- Note that $f(x) = \dfrac{x^2(x+1)}{x^2+3x+2} = \dfrac{x^2(x+1)}{(x+1)(x+2)} = \dfrac{x^2}{x+2}$
- At $x = -1$:

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2}{x+2} = \frac{(-1)^2}{-1+2} = \frac{1}{1} = 1 \Rightarrow$$

$$\Rightarrow \lim_{x \to -1^+} f(x) \notin \{+\infty, -\infty\} \wedge \lim_{x \to -1^-} f(x) \notin \{+\infty, -\infty\} \Rightarrow$$

$\Rightarrow (\ell): x = -1$ is <u>NOT</u> a vertical asymptote.

- At $x = -2$:

$$\left.\begin{array}{l}\lim\limits_{x\to -2^+} \dfrac{1}{x+2} = \lim\limits_{x\to -2^+} \dfrac{1}{x-(-2)} = +\infty \\ \lim\limits_{x\to -2^+} x^2 = (-2)^2 = 4\end{array}\right\} \Rightarrow$$

$$\Rightarrow \lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{x^2}{x+2} = +\infty \Rightarrow$$

$\Rightarrow (\ell): x = -2$ is vertical asymptote.

Thus: we have two asymptotes:

$(\ell_1): y = x-2$

$(\ell_2): x = -2.$

c) $f(x) = \sqrt{3x^2 - x}$ ← Find all asymptotes.

Solution

• **Domain**

Require $3x^2 - x \geq 0 \iff x(3x-1) \geq 0 \iff x \in (-\infty, 0] \cup [1/3, +\infty)$

x		0		$1/3$	
x	$-$	0	$+$		$+$
$3x-1$	$-$		$-$	0	$+$
ineq	$+$	0	$-$	0	$+$

thus $\mathrm{dom}(f) = (-\infty, 0] \cup [1/3, +\infty)$.

• Slant asymptotes:

$$\frac{f(x)}{x} = \frac{\sqrt{3x^2 - x}}{x} = \frac{\sqrt{x^2}\sqrt{3 - 1/x}}{x} = \frac{|x|\sqrt{3 - 1/x}}{x}.$$

$$= \begin{cases} \sqrt{3 - 1/x}, & x \in [1/3, +\infty) \\ -\sqrt{3 - 1/x}, & x \in (-\infty, 0] \end{cases} \Rightarrow$$

therefore:

$$a_+ = \lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \sqrt{3 - 1/x} = \sqrt{3 - 0} = \sqrt{3}$$

$$a_- = \lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} \left(-\sqrt{3 - 1/x}\right) = -\sqrt{3 - 0} = -\sqrt{3}$$

$\forall x \in [1/3, +\infty)$: $f(x) - a_+ x = \sqrt{3x^2 - x} - (\sqrt{3})x =$

$$= \frac{(\sqrt{3x^2 - x})^2 - (x\sqrt{3})^2}{\sqrt{3x^2 - x} + x\sqrt{3}} = \frac{(3x^2 - x) - 3x^2}{|x|\sqrt{3 - 1/x} + x\sqrt{3}} =$$

$$= \frac{-x}{x\sqrt{3 - 1/x} + x\sqrt{3}} = \frac{-1}{\sqrt{3 - 1/x} + \sqrt{3}} \Rightarrow$$

$$b_+ = \lim_{x\to+\infty} [f(x) - a_+ x] = \lim_{x\to+\infty} \frac{-1}{\sqrt{3-1/x} + \sqrt{3}} =$$

$$= \frac{-1}{\sqrt{3}+\sqrt{3}} = \frac{-1}{2\sqrt{3}} = \frac{-\sqrt{3}}{2\cdot 3} = \frac{-\sqrt{3}}{6}$$

$$\forall x \in (-\infty, 0]: f(x) - a_- x = \sqrt{3x^2 - x} - (-\sqrt{3})x =$$

$$= \sqrt{3x^2-x} + (\sqrt{3})x = \frac{(\sqrt{3x^2-x})^2 - (x\sqrt{3})^2}{\sqrt{3x^2-x} - x\sqrt{3}} =$$

$$= \frac{(3x^2-x) - 3x^2}{|x|\sqrt{3-1/x} - x\sqrt{3}} = \frac{-x}{-x\sqrt{3-1/x} - x\sqrt{3}} =$$

$$= \frac{1}{\sqrt{3-1/x} + \sqrt{3}} \Rightarrow$$

$$\Rightarrow b_- = \lim_{x\to-\infty} [f(x) - a_- x] = \lim_{x\to-\infty} \frac{1}{\sqrt{3-1/x} + \sqrt{3}} =$$

$$= \frac{1}{\sqrt{3}+\sqrt{3}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$$

It follows that f has the following slant asymptotes:

$(\ell_1): y = x\sqrt{3} - \frac{\sqrt{3}}{6}$ for $x \to +\infty$

$(\ell_2): y = -x\sqrt{3} + \frac{\sqrt{3}}{6}$ for $x \to -\infty$

- Vertical asymptotes.

Since f continuous on $(-\infty, 0] \cup [1/3, +\infty) \Rightarrow$

$\Rightarrow \forall x_0 \in (-\infty, 0] \cup [1/3, +\infty): (\ell): x = x_0$ is NOT vertical asymptote.

and

$(\forall x_0 \in (0, 1/3):\ x_0$ not a limit point of $\mathrm{dom}(f)) \Rightarrow$

$\Rightarrow \forall x_0 \in (0, 1/3):\ (\ell): x = x_0$ not a vertical asymptote.

It follows that f does not have any vertical asymptotes.

EXERCISES

① Find all the asymptotes for the following functions

a) $f(x) = \dfrac{x-2}{x+3}$

b) $f(x) = \dfrac{x^2+5}{3x}$

c) $f(x) = \dfrac{x^2+x+1}{x-2}$

d) $f(x) = \dfrac{2}{x-1}$

e) $f(x) = \dfrac{(x+2)^3}{8x^3}$

f) $f(x) = \dfrac{x^2+1}{2x-1}$

g) $f(x) = \dfrac{x^3+2x-1}{x^2-1}$

h) $f(x) = \dfrac{x^4}{x^2+1}$

② Find all the asymptotes for the following functions

a) $f(x) = \begin{cases} \dfrac{1}{x-2}, & \text{if } x<2 \\ 1, & \text{if } x=2 \\ \dfrac{1}{x-2} - 2, & \text{if } x>2 \end{cases}$

b) $f(x) = \begin{cases} \dfrac{x+1}{x-2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$

c) $f(x) = \sqrt{x^2-25}$

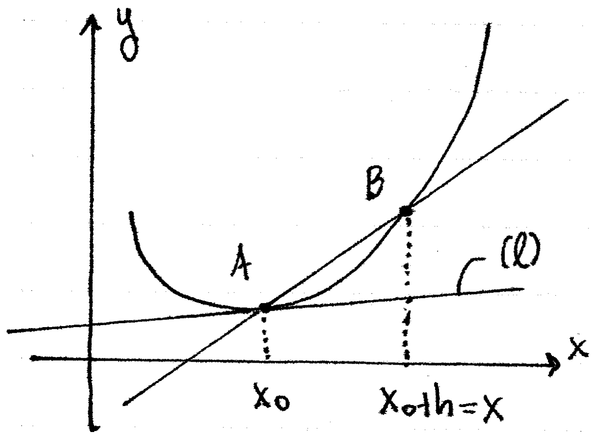
d) $f(x) = \sqrt{x^2+2x+5}$

e) $f(x) = \sqrt{x^2-x+1}$

CAL1.4: Derivatives

DIFFERENTIAL CALCULUS

▼ Differentiability – Tangent line problem



• Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and let $(c): y = f(x)$ be the graph of the function. We assume that f continuous at x_0. Consider the points $A(x_0, f(x_0))$ and $B(x, f(x))$. As B approaches A, the line (AB) approaches a line (ℓ) given by

$$(\ell):\ y - y_A = a(x - x_A) \Leftrightarrow y - f(x_0) = a(x - x_0)$$

with a the slope of the line (ℓ).

• To calculate the slope of (ℓ) we note that the slope of the line (AB) is given by:

$$\lambda(f|x, x_0) = \frac{y_B - y_A}{x_B - x_A} = \frac{f(x) - f(x_0)}{x - x_0}$$

and it follows that

$$a = \lim_{x \to x_0} \lambda(f|x, x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =$$

$$= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

• We will see below that the assumption that f is continuous at x_0 is not sufficient to ensure the existence of the above limit. This motivates the following definition of differentiability:

$$\boxed{f \text{ differentiable at } x_0 \Leftrightarrow \exists \ell \in \mathbb{R}: \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \ell}$$

↦ Differentiability and continuity

Thm: Let $f: A \to \mathbb{R}$ with $x_0 \in A$. Then:

$$\boxed{f \text{ differentiable at } x_0 \Rightarrow f \text{ continuous at } x_0}$$

Proof

Assume that f differentiable at x_0. Then:

$$f \text{ differentiable at } x_0 \Rightarrow \exists \ell \in \mathbb{R}: \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \ell$$

and therefore:

$$\lim_{x \to x_0} f(x) = f(x_0) + \lim_{x \to x_0} [f(x)-f(x_0)] =$$

$$= f(x_0) + \lim_{x \to x_0} \left[\frac{f(x)-f(x_0)}{x-x_0} \cdot (x-x_0) \right] =$$

$$= f(x_0) + \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \cdot \lim_{x \to x_0} (x-x_0)$$

$$= f(x_0) + \ell (x_0 - x_0) = f(x_0) + \ell \cdot 0 = f(x_0) \Rightarrow$$

$\Rightarrow f$ continuous at x_0.

- The contrapositive statement reads:
  f NOT continuous at $x_0 \Rightarrow f$ NOT differentiable at x_0.

Corner points

- We say that

$$\boxed{x_0 \text{ corner point of } f \Leftrightarrow \begin{cases} f \text{ continuous at } x_0 \\ f \text{ NOT differentiable at } x_0 \end{cases}}$$

- To show that f NOT differentiable at x_0, we use the negation of the definition of differentiability:

$$\boxed{f \text{ NOT differentiable at } x_0 \Leftrightarrow \forall \ell \in \mathbb{R} : \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \neq \ell}$$

- Corner points can emerge from
  a) Sudden change of direction in the graph of the function
  b) When the graph of the function becomes momentarily vertical.

EXAMPLE

a) For $f(x)=|x|, \forall x \in \mathbb{R}$ show that $x_0=0$ is a corner point.

Solution

Since

$$f(0)=|0|=0 \quad (1)$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0 \quad (2)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0 \quad (3)$$

it follows that

$$\lim_{x\to 0} f(x) = 0 \qquad [\text{from Eq. (2) and Eq. (3)}]$$

$$= f(0) \qquad [\text{from Eq. (1)}]$$

$\Rightarrow$ f continuous at $x_0=0$. (4)

Furthermore:

$$\lambda(f|x,0) = \frac{f(x)-f(0)}{x-0} = \frac{|x|-|0|}{x-0} = \frac{|x|}{x} =$$

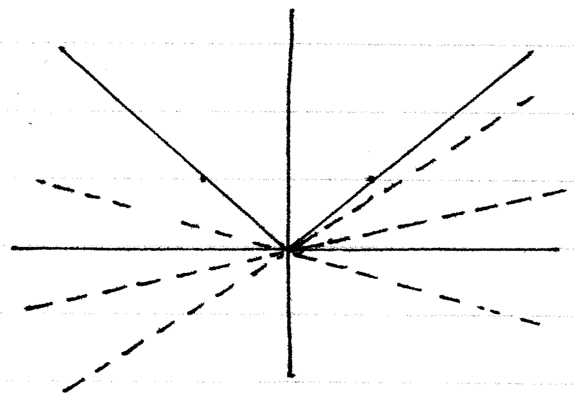
$$= \begin{cases} x/x, & \text{if } x>0 \\ -x/x, & \text{if } x<0 \end{cases} = \begin{cases} 1, & \text{if } x>0 \\ -1, & \text{if } x<0 \end{cases} \Rightarrow$$

$$\Rightarrow \lim_{x\to 0^+} \lambda(f|x,0) = 1 \wedge \lim_{x\to 0^-} \lambda(f|x,0) = -1$$

$\Rightarrow \lim_{x\to 0} \lambda(f|x,0)$ does not exist $\Rightarrow \forall \ell \in \mathbb{R}: \lim \dfrac{f(x)-f(0)}{x-0} \neq \ell$

$\Rightarrow$ f not differentiable at $x_0=0$. (5)

From Eq.(4) and Eq.(5): $x_0=0$ corner point of f.



From the graph of $f(x)=|x|$, $\forall x \in \mathbb{R}$ we see that the corner point $x_0=0$, the function suddenly changes direction. As a result, we cannot draw a unique tangent line at $x_0=0$.

b) Show that $f(x)=\sqrt{x}$, $\forall x \in [0,+\infty)$ has a corner point at $x_0=0$.

Solution

$$f(0)=\sqrt{0}=0 \qquad (1)$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\sqrt{x} = \lim_{x\to 0^+}\sqrt{x} = \sqrt{0}=0 \qquad (2)$$

From Eq.(1) and Eq.(2):

$$\lim_{x\to 0} f(x) = f(0) \Rightarrow f \text{ continuous at } x_0=0 \qquad (3)$$

Furthermore:

$$\lambda(f|x,0) = \frac{f(x)-f(0)}{x-0} = \frac{\sqrt{x}-\sqrt{0}}{x-0} = \frac{\sqrt{x}}{x} =$$

$$= \frac{\sqrt{x}}{\sqrt{x}\sqrt{x}} = \frac{1}{\sqrt{x}}, \quad \forall x \in (0,+\infty)$$

Since:

$$\begin{cases} \sqrt{x} > 0, \ \forall x \in (0, +\infty) \\ \lim\limits_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0 \end{cases} \Rightarrow \lim_{x \to 0^+} \frac{1}{\sqrt{x}} = +\infty \Rightarrow$$

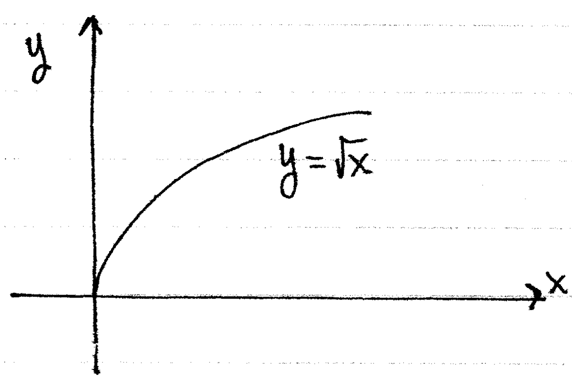
$$\Rightarrow \lim_{x \to 0^+} \lambda(f \mid x, 0) = +\infty$$

$$\Rightarrow \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} \neq \ell, \ \forall \ell \in \mathbb{R}$$

$\Rightarrow f$ not differentiable at $x_0 = 0$. (4)

From Eq. (3) and Eq. (4): $x_0 = 0$ corner point of f.

$\mapsto$ From the graph of $f(x) = \sqrt{x}$, $\forall x \in [0, +\infty)$ we see that the graph becomes vertical at $x_0 = 0$. This is the second way one may lose differentiability without losing continuity.



c) Consider the function

$$f(x) = \begin{cases} x^3 & , \text{ if } x \in (-\infty, 1] \\ x^2 + x - 1 & , \text{ if } x \in (1, +\infty) \end{cases}$$

Show that f is differentiable at $x_0 = 1$.

<u>Solution</u>

Since:

$$\forall x \in (-\infty, 1) : \lambda(f|x,1) = \frac{f(x) - f(1)}{x-1} = \frac{x^3 - 1^3}{x-1}$$

$$= \frac{(x-1)(x^2+x+1)}{x-1} = x^2+x+1 \Rightarrow$$

$$\Rightarrow \lim_{x \to 1^-} \lambda(f|x,1) = \lim_{x \to 1^-} (x^2+x+1) = 1^2+1+1 = 3. \qquad (1)$$

$$\forall x \in (1, +\infty) : \lambda(f|x,1) = \frac{f(x) - f(1)}{x-1} = \frac{(x^2+x-1) - 1^3}{x-1} =$$

$$= \frac{x^2+x-2}{x-1} = \frac{(x-1)(x+2)}{x-1} =$$

$$= x+2 \Rightarrow$$

$$\Rightarrow \lim_{x \to 1^+} \lambda(f|x,1) = \lim_{x \to 1^+} (x+2) = 1+2 = 3 \qquad (2)$$

From Eq. (1) and Eq. (2):

$$\lim_{x \to 1} \lambda(f|x,1) = 3 \Rightarrow \exists a \in \mathbb{R} : \lim_{x \to 1} \lambda(f|x,1) = a$$

$\Rightarrow f$ differentiable at $x_0 = 1$.

d) Consider the function

$$f(x) = \begin{cases} x^2\left[\sin(\pi/x) + \cos(\pi/x)\right] & , \ x \in \mathbb{R} - \{0\} \\ 0 & , \ x = 0 \end{cases}$$

Show that f is differentiable at $x_0 = 0$

<u>Solution</u>

Let $x \in \mathbb{R} - \{0\}$ be given. Then

$$\lambda(f \mid x, 0) = \frac{f(x) - f(0)}{x - 0} = \frac{x^2\left[\sin(\pi/x) + \cos(\pi/x)\right] - 0}{x - 0}$$

$$= x\left[\sin(\pi/x) + \cos(\pi/x)\right]$$

Define $b(x) = \sin(\pi/x) + \cos(\pi/x), \ \forall x \in \mathbb{R} - \{0\}$. Then:

$$|b(x)| = |\sin(\pi/x) + \cos(\pi/x)| \leq$$

$$\leq |\sin(\pi/x)| + |\cos(\pi/x)| \leq$$

$$\leq 1 + 1 = 2, \ \forall x \in \mathbb{R} - \{0\} \Rightarrow$$

$\Rightarrow b$ bounded at $\mathbb{R} - \{0\}$. (1)

Also note that $\lim\limits_{x \to 0} x = 0$ (2)

From Eq. (1) and Eq. (2):

$$\lim_{x \to 0} x\,b(x) = 0 \Rightarrow \lim_{x \to 0} \lambda(f \mid x, 0) = 0$$

$$\Rightarrow \exists a \in \mathbb{R} : \lim_{x \to 0} \lambda(f \mid x, 0) = a$$

$\Rightarrow f$ differentiable at $x_0 = 0$.

c) Consider the function

$$f(x) = \begin{cases} x^2+2x & , x \in [0,+\infty) \\ ax+b & , x \in (-\infty,0) \end{cases}$$

Find all $a,b \in \mathbb{R}$ for which f differentiable at $x_0=0$.

Solution

We note that

$$\forall x \in (0,+\infty): \lambda(f|x,0) = \frac{f(x)-f(0)}{x-0} = \frac{(x^2+2x)-(0^2+2\cdot 0)}{x}$$

$$= \frac{x^2+2x}{x} = \frac{x(x+2)}{x} = x+2$$

$$\forall x \in (-\infty,0): \lambda(f|x,0) = \frac{f(x)-f(0)}{x-0} = \frac{ax+b-0}{x} = \frac{ax+b}{x}$$

$$\lim_{x\to 0^+} \lambda(f|x,0) = \lim_{x\to 0^+}(x+2) = 0+2 = 2$$

The limit $\lim_{x\to 0^-} \lambda(f|x,0)$ may or may not exist depending on whether $b=0$ or $b\neq 0$, so we leverage continuity but must do, as a result, a split argument:

($\Rightarrow$): Assume that f differentiable at $x_0=0$. Since:

$$f(0) = 0^2+2\cdot 0 = 0$$

$$\lim_{x\to 0^+} f(x) = 0^2+2\cdot 0 = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(ax+b) = a\cdot 0+b = b$$

it follows that:

$$f \text{ differentiable at } x_0=0 \Rightarrow f \text{ continuous at } x_0=0$$

$$\Rightarrow \lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) \Rightarrow \underline{b=0}$$

For $b=0$:

$$\lim_{x\to 0^-} \lambda(f|x,0) = \lim_{x\to 0^-} \frac{ax+0}{x} = \lim_{x\to 0^-} a = a$$

and therefore:

f differentiable at $x_0=0 \Rightarrow \exists\, l\in\mathbb{R} : \lim_{x\to 0} \lambda(f|x,0) = l$

$$\Rightarrow \lim_{x\to 0^-} \lambda(f|x,0) = \lim_{x\to 0^+} \lambda(f|x,0)$$

$$\Rightarrow a = 2.$$

We have thus shown that

f differentiable at $x_0=0 \Rightarrow (a=2 \wedge b=0)$

($\Leftarrow$): Assume that $a=2\wedge b=0$. Then:

$$a=2\wedge b=0 \Rightarrow \forall x\in(-\infty,0): \lambda(f|x,0) = \frac{2x+0}{x} = \frac{2x}{x} = 2$$

$$\Rightarrow \lim_{x\to 0^-} \lambda(f|x,0) = 2 = \lim_{x\to 0^+} \lambda(f|x,0)$$

$$\Rightarrow \lim_{x\to 0} \lambda(f|x,0) = 2 \Rightarrow \exists\, l\in\mathbb{R} : \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow f \text{ differentiable at } x_0=0.$$

We have thus shown that:

f differentiable at $x_0=0 \Leftrightarrow a=2\wedge b=0$. $\square$

$\hookrightarrow$ Note that a direct argument of the form

f differentiable at $x_0=0 \Leftrightarrow \cdots \Leftrightarrow \cdots \Leftrightarrow$

$\Leftrightarrow a=2\wedge b=0$

is not possible if we wish to use continuity.

Consequently the forward ($\Rightarrow$) and backward ($\Leftarrow$) arguments need to be done separately.

EXERCISES

① Show that the function

$$f(x) = \begin{cases} x^2 + 4x & , x \in [0, +\infty) \\ x^2 - 4x & , x \in (-\infty, 0) \end{cases}$$

is continuous on $\mathbb{R}$ but not differentiable at $x_0 = 0$.

② Show that the function

$$f(x) = (x + |x|)^2 \quad , \forall x \in \mathbb{R}$$

is continuous and differentiable at $x_0 = 0$

③ Define the function

$$f(x) = \begin{cases} x \sin(2x) \cos(\pi/x)[1 + \sin(\pi/x)] & , \text{ if } x \in \mathbb{R} - \{0\} \\ 0 & , \text{ if } x = 0 . \end{cases}$$

Show that f is differentiable at $x_0 = 0$.

④ Let $f : A \to \mathbb{R}$ be a function and define

$\forall x \in A : g(x) = x f(x)$

Show that: f continuous at $x = 0 \Rightarrow g$ differentiable at $x = 0$.

⑤ Find all $a, b \in \mathbb{R}$ such that the following functions are differentiable at x_0

a) $f(x) = \begin{cases} ax + b & , x \in (-\infty, 3) \\ x^2 & , x \in [3, +\infty) \end{cases}$ at $x_0 = 3$

b) $f(x) = \begin{cases} ax^2 + 2bx & , \text{ if } x \in [1, +\infty) \\ bx - a & , \text{ if } x \in (-\infty, 1) \end{cases}$ at $x_0 = 1$.

⑥ Let $f: A \to \mathbb{R}$ be a function and define $g: A \to \mathbb{R}$ such that

$\forall x \in A: g(x) = |f(x)|$

Show that:

$\begin{cases} f \text{ differentiable at } x_0 \in A \\ f(x_0) \neq 0 \end{cases} \Rightarrow g$ differentiable at x_0.

(Hint: We write:

$$\lambda(g|x, x_0) = \ldots = \frac{(|f(x)| - |f(x_0)|)(|f(x)| + |f(x_0)|)}{(x - x_0)(|f(x)| + |f(x_0)|)}$$

$= \ldots$

use $|x|^2 = x^2$ and take it from there)

▼ Derivative function

- Let $f : A \to \mathbb{R}$ be a function and let $S \subseteq A$. We say that
  f differentiable at $S \Leftrightarrow \forall x_0 \in S : f$ differentiable at x_0
- If $f : A \to \mathbb{R}$ is differentiable at S, then we define the derivative function $f' : S \to \mathbb{R}$ as:
  $$\forall x_0 \in S : f'(x_0) = \lim_{x \to x_0} \lambda(f \mid x, x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$
- The notation $f'(x)$ is attributed to Newton. The Leibnitz notation of the derivative is:
  $$\frac{df}{dx} = f' \quad \text{and} \quad \left.\frac{df}{dx}\right|_{x = x_0} = f'(x_0)$$
- If f' is also differentiable at S, then the derivative of f' is denoted as f'' and is called the <u>2nd derivative</u> of f. Likewise we define
  $$f'' = \frac{df'}{dx} = \frac{d^2 f}{dx^2}$$
  $$f''' = \frac{df''}{dx} = \frac{d^3 f}{dx^3}$$

  Beyond the 3rd derivative, we use the notation $f^{(4)}, f^{(5)}, \dots, f^{(n)}$ and write:
  $$f^{(n)} = \frac{df^{(n-1)}}{dx} = \frac{d^n f}{dx^n}$$

- If we can define $f^{(n)}$ at x_0 we say that f is n-times differentiable at x_0. Likewise, for $S \subseteq A$, we say that f n-times differentiable at $S \iff \forall x_0 \in S : f$ n-times differentiable at x_0.

Derivatives of basic functions

(1) $$\boxed{f(x) = ax+b, \forall x \in \mathbb{R} \implies f'(x) = a, \forall x \in \mathbb{R}}$$

Proof

Since

$$\forall x, x_0 \in \mathbb{R} : \lambda(f|x,x_0) = \frac{f(x)-f(x_0)}{x-x_0} = \frac{(ax+b)-(ax_0+b)}{x-x_0}$$

$$= \frac{ax-ax_0}{x-x_0} = \frac{a(x-x_0)}{x-x_0} = a \implies$$

$$\implies \forall x_0 \in \mathbb{R} : f'(x_0) = \lim_{x \to x_0} \lambda(f|x,x_0) = a. \quad \square$$

For the next result we use the identity

$$\boxed{\forall a,b \in \mathbb{R} : \forall n \in \mathbb{N}-\{0\} : a^n - b^n = (a-b)\sum_{k=0}^{n-1} (a^{n-k-1} b^k)}$$

Note that:

$n=2$: $\quad a^2 - b^2 = (a-b)(a+b)$

$n=3$: $\quad a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$n=4$: $\quad a^4 - b^4 = (a-b)(a^3+a^2b+ab^2+b^3)$

Proof

$$(a-b)\sum_{k=0}^{n-1} a^{n-k-1} b^k = \sum_{k=0}^{n-1} (a-b) a^{n-k-1} b^k =$$

$$= \sum_{k=0}^{n-1} \left(a^{n-k} b^k - a^{n-k-1} b^{k+1}\right) =$$

$$= \sum_{k=0}^{n-1} a^{n-k} b^k - \sum_{k=0}^{n-1} a^{n-k-1} b^{k+1} =$$

$$= a^n + \sum_{k=1}^{n-1} a^{n-k} b^k - \sum_{k=0}^{n-2} a^{n-k-1} b^{k+1} - a^{n-(n-1)-1} b^{(n-1)+1}$$

$$= a^n + \sum_{k=1}^{n-1} a^{n-k} b^k - \sum_{k=1}^{n-1} a^{n-k} b^k - b^n =$$

$$= a^n - b^n \qquad \square$$

② $\boxed{f(x) = ax^n,\ \forall x \in \mathbb{R} \Rightarrow f'(x) = nax^{n-1},\ \forall x \in \mathbb{R}}$

Proof

Since:

$$\lambda(f \mid x, x_0) = \frac{f(x) - f(x_0)}{x - x_0} = \frac{ax^n - ax_0^n}{x - x_0} = \frac{a(x^n - x_0^n)}{x - x_0} =$$

$$= \frac{a(x - x_0)\sum_{k=0}^{n-1} x^{n-k-1} x_0^k}{x - x_0} =$$

$$= a\sum_{k=0}^{n-1} x^{n-k-1} x_0^k \Longrightarrow$$

$$\Rightarrow f'(x_0) = \lim_{x\to x_0} \lambda(f|x,x_0) = \lim_{x\to x_0}\left[a\sum_{k=0}^{n-1} x^{n-k-1}x_0^k\right] =$$

$$= a\lim_{x\to x_0}\sum_{k=0}^{n-1} x^{n-k-1}x_0^k = a\sum_{k=0}^{n-1}\lim_{x\to x_0}(x^{n-k-1}x_0^k)$$

$$= a\sum_{k=0}^{n-1} x_0^{n-k-1}x_0^k = a\sum_{k=0}^{n-1} x_0^{n-1} = anx_0^{n-1} =$$

$$= nax_0^{n-1},\ \forall x_0 \in \mathbb{R}. \qquad \square$$

(3) $\boxed{f(x) = \sqrt{x},\ \forall x\in[0,+\infty) \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x}},\ \forall x\in(0,+\infty)}$

<u>Proof</u>

$$\forall x, x_0 \in [0,+\infty) : \lambda(f|x,x_0) = \frac{f(x)-f(x_0)}{x-x_0} = \frac{\sqrt{x}-\sqrt{x_0}}{x-x_0} =$$

$$= \frac{\sqrt{x}-\sqrt{x_0}}{(\sqrt{x})^2-(\sqrt{x_0})^2} = \frac{\sqrt{x}-\sqrt{x_0}}{(\sqrt{x}-\sqrt{x_0})(\sqrt{x}+\sqrt{x_0})} = \frac{1}{\sqrt{x}+\sqrt{x_0}} \Rightarrow$$

$$\Rightarrow \forall x_0\in(0,+\infty) : f'(x_0) = \lim_{x\to x_0}\lambda(f|x,x_0) = \lim_{x\to x_0}\frac{1}{\sqrt{x}+\sqrt{x_0}} =$$

$$= \frac{1}{\sqrt{x_0}+\sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}. \qquad \square$$

↳ Note that, as was shown previously, although the function $f(x)=\sqrt{x}$ is defined at $x=0$, it is <u>not</u> differentiable at $x=0$.

Basic differentiation rules

Let f, g be functions differentiable at a set $A \subseteq \mathbb{R}$ and let $a \in \mathbb{R}$. Then:

$$\boxed{\begin{array}{l} h(x) = f(x) + g(x),\ \forall x \in A \Rightarrow h'(x) = f'(x) + g'(x),\ \forall x \in A \\ h(x) = af(x),\ \forall x \in A \Rightarrow h'(x) = af'(x),\ \forall x \in A \\ h(x) = f(x)g(x),\ \forall x \in A \Rightarrow h'(x) = f'(x)g(x) + f(x)g'(x),\ \forall x \in A \end{array}}$$

Proof

a) Assume that $h(x) = f(x) + g(x), \forall x \in A$. Then

$$\forall x, x_0 \in A : \lambda(h|x, x_0) = \frac{h(x) - h(x_0)}{x - x_0} =$$

$$= \frac{[f(x) + g(x)] - [f(x_0) + g(x_0)]}{x - x_0} =$$

$$= \frac{[f(x) - f(x_0)] + [g(x) - g(x_0)]}{x - x_0} =$$

$$= \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} = \lambda(f|x, x_0) + \lambda(g|x, x_0)$$

$$\Rightarrow \forall x_0 \in A : h'(x_0) = \lim_{x \to x_0} \lambda(h|x, x_0) =$$

$$= \lim_{x \to x_0} [\lambda(f|x, x_0) + \lambda(g|x, x_0)]$$

$$= \lim_{x \to x_0} \lambda(f|x, x_0) + \lim_{x \to x_0} \lambda(g|x, x_0)$$

$$= f'(x_0) + g'(x_0).$$

b) Assume that $h(x) = af(x)$, $\forall x \in A$. Then

$$\forall x, x_0 \in A: \lambda(h|x, x_0) = \frac{h(x) - h(x_0)}{x - x_0} = \frac{af(x) - af(x_0)}{x - x_0} =$$

$$= \frac{a[f(x) - f(x_0)]}{x - x_0} = a\lambda(f|x, x_0) \Rightarrow$$

$$\Rightarrow \forall x_0 \in A: h'(x_0) = \lim_{x \to x_0} \lambda(h|x, x_0) = \lim_{x \to x_0} [a\lambda(f|x, x_0)]$$

$$= a \lim_{x \to x_0} \lambda(f|x, x_0) = af'(x_0)$$

c) Assume that $h(x) = f(x)g(x)$, $\forall x \in A$. Then

$$\forall x, x_0 \in A: \lambda(h|x, x_0) = \frac{h(x) - h(x_0)}{x - x_0} = \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} =$$

$$= \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} =$$

$$= \frac{g(x)[f(x) - f(x_0)] + f(x_0)[g(x) - g(x_0)]}{x - x_0} =$$

$$= f(x_0)\frac{g(x) - g(x_0)}{x - x_0} + g(x)\frac{f(x) - f(x_0)}{x - x_0} =$$

$$= f(x_0)\lambda(g|x, x_0) + \lambda(f|x, x_0)g(x)$$

We note that:

g differentiable at $x_0 \Rightarrow g$ continuous at x_0

$$\Rightarrow \lim_{x \to x_0} g(x) = g(x_0)$$

and therefore:

$$\forall x_0 \in A : h'(x_0) = \lim_{x \to x_0} \lambda(h|x, x_0) =$$

$$= \lim_{x \to x_0} \left[f(x_0)\, \lambda(g|x, x_0) + \lambda(f|x, x_0)\, g(x) \right]$$

$$= f(x_0) \lim_{x \to x_0} \lambda(g|x, x_0) + \lim_{x \to x_0} \lambda(f|x, x_0) \lim_{x \to x_0} g(x)$$

$$= f(x_0)\, g'(x_0) + f'(x_0)\, g(x_0)$$

$$= f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0). \qquad \square$$

EXAMPLES

a) Evaluate f' and f'' for the function
$f(x) = x^4 + 2x^3 + 5x^2$, $\forall x \in \mathbb{R}$.

Solution

$$\begin{aligned} f'(x) &= (x^4 + 2x^3 + 5x^2)' \\ &= (x^4)' + (2x^3)' + (5x^2)' \qquad \text{[addition rule]} \\ &= 4x^3 + 2(3x^2) + 5(2x) \qquad [\,(x^n)' = nx^{n-1}\,] \\ &= 4x^3 + 6x^2 + 10x, \quad \forall x \in \mathbb{R} \end{aligned}$$

and

$$\begin{aligned} f''(x) &= (4x^3 + 6x^2 + 10x)' \\ &= (4x^3)' + (6x^2)' + (10x)' \\ &= 4(3x^2) + 6(2x) + 10 \\ &= 12x^2 + 12x + 10, \quad \forall x \in \mathbb{R}. \end{aligned}$$

↦ In evaluating derivatives of polynomials it is ok to skip steps, including the addition rule, and evaluate the derivative quickly. However, you should NEVER skip the application of the product rule.

↦ Tangent line : The tangent line (ℓ) to the graph $(c): y = f(x)$ of a function f at $x = x_0$ is given by $(\ell): y - f(x_0) = f'(x_0)(x - x_0)$

b) Find the tangent line to the graph of

$$f(x) = (x^2+2x)\sqrt{x} \quad , \forall x \in [0,+\infty)$$

at $x_0 = 2$.

Solution

Note that

$$f'(x) = [(x^2+2x)\sqrt{x}]' =$$

$$= (x^2+2x)'\sqrt{x} + (x^2+2x)(\sqrt{x})' =$$

$$= (2x+2)\sqrt{x} + (x^2+2x)\left(\frac{1}{2\sqrt{x}}\right) =$$

$$= \frac{(2x+2)\sqrt{x}\,(2\sqrt{x}) + (x^2+2x)}{2\sqrt{x}} = \frac{2x(2x+2)+(x^2+2x)}{2\sqrt{x}}$$

$$= \frac{4x^2+4x+x^2+2x}{2\sqrt{x}} = \frac{5x^2+6x}{2\sqrt{x}} = \frac{x(5x+6)}{2\sqrt{x}} =$$

$$= \frac{1}{2}\sqrt{x}\,(5x+6) \quad , \forall x \in (0,+\infty)$$

and therefore

$$f(2) = (2^2+2\cdot 2)\sqrt{2} = (4+4)\sqrt{2} = 8\sqrt{2}$$

$$f'(2) = \frac{1}{2}\sqrt{2}\,(5\cdot 2+6) = \frac{\sqrt{2}}{2}\cdot 16 = 8\sqrt{2}$$

It follows that

$$(\ell): y - f(2) = f'(2)(x-2) \Leftrightarrow y - 8\sqrt{2} = 8\sqrt{2}(x-2) \Leftrightarrow$$

$$\Leftrightarrow y - 8\sqrt{2} = 8\sqrt{2}x - (8\sqrt{2})\cdot 2 \Leftrightarrow 8\sqrt{2}x - y - 8\sqrt{2} = 0$$

and therefore

$$(\ell): 8\sqrt{2}x - y - 8\sqrt{2} = 0.$$

Derivatives of multiple formula functions

If a function f follows a given formula at a closed interval $[a,b]$, then the corresponding side limit of $\lambda(f|x,x_0)$ with $x\to x_0^+$ or $x\to x_0^-$ at $x_0=a$ and $x_0=b$ can be evaluated directly from the differentiation rules. The same is true for intervals of the form $[a,+\infty)$ or $(-\infty,a]$ at $x_0=a$, and also for points in the interior of the interval. HOWEVER, at the boundary points $x_0=a$ or $x_0=b$ of OPEN intervals of the form (a,b) or $(a,+\infty)$ or $(-\infty,a)$, etc., we have to use the limit definition directly.

c) Evaluate the derivative of the function

$$f(x)=\begin{cases} 3x^2+x, & \text{if } x\in(-\infty,1] \\ 2(x-1)^2+3, & \text{if } x\in(1,+\infty). \end{cases}$$

Solution

- We note that

$$\forall x\in(-\infty,1): f(x)=3x^2+x \Rightarrow \forall x\in(-\infty,1): f'(x)=6x+1$$

$$\forall x\in(1,+\infty): f(x)=2(x-1)^2+3=2(x^2-2x+1)+3=$$
$$=2x^2-4x+2+3=2x^2-4x+5\Rightarrow$$
$$\Rightarrow \forall x\in(1,+\infty): f'(x)=4x-4$$

- At $x=1$:

$$\forall x\in(-\infty,1]: f(x)=3x^2+x\Rightarrow$$
$$\Rightarrow \lim_{x\to 1^-}\lambda(f|x,1)=\frac{d}{dx}(3x^2+x)\Big|_{x=1}=(6x+1)\Big|_{x=1}=$$

$$= 6\cdot 1 + 1 = 7. \qquad (1)$$

and

$$\forall x \in (1, +\infty): \lambda(f|x,1) = \frac{f(x) - f(1)}{x-1} = \frac{[2(x-1)^2+3] - (3\cdot 1^2+1)}{x-1}$$

$$= \frac{2(x-1)^2+3-4}{x-1} = \frac{2(x-1)^2-1}{x-1} =$$

$$= 2(x-1) + \frac{-1}{x-1}.$$

Since $\left.\begin{array}{l} \lim\limits_{x\to 1^+} 2(x-1) = 2(1-1) = 0 \\ \lim\limits_{x\to 1^+} \dfrac{-1}{x-1} = -\infty \end{array}\right\} \Rightarrow \lim\limits_{x\to 1^+} \lambda(f|x,1) = -\infty \qquad (2)$

From Eq.(1) and Eq.(2):

$$\lim_{x\to 1^-} \lambda(f|x,1) \neq \lim_{x\to 1^+} \lambda(f|x,1) \Rightarrow$$

$$\Rightarrow \lim_{x\to 1} \lambda(f|x,1) \text{ does not exist } \Rightarrow$$

$$\Rightarrow \forall \ell \in \mathbb{R}: \lim_{x\to 1} \lambda(f|x,1) \neq \ell \Rightarrow$$

$\Rightarrow f$ not differentiable at $x = 1$.

It follows that

$$f'(x) = \begin{cases} 6x+1, & x \in (-\infty, 1) \\ 4x-4, & x \in (1, +\infty) \end{cases}$$

↳ Sometimes, but not always, lack of continuity can be used to deduce lack of differentiability

<u>2nd method</u>

Since

$$\left.\begin{aligned}\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+}\left[2(x-1)^2+3\right] = 2(1-1)^2+3=3\\ \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-}(3x^2+x) = 3\cdot 1^2+1 = 3+1=4\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow \lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x) \Rightarrow \lim_{x\to 1} f(x) \text{ does not exist} \Rightarrow$$

$\Rightarrow f$ not continuous at $x=1 \Rightarrow$

$\Rightarrow f$ not differentiable at $x=1$.

d) Evaluate the derivative of the function

$$f(x) = \begin{cases} x^2 - x - 1 & , x \in (1, +\infty) \\ x - 2 & , x \in (-\infty, 1] \end{cases}$$

<u>Solution</u>

Since

$\forall x \in (1, +\infty): f(x) = x^2 - x - 1 \Rightarrow \forall x \in (1, +\infty): f'(x) = 2x - 1$

$\forall x \in (-\infty, 1): f(x) = x - 2 \Rightarrow \forall x \in (-\infty, 1): f'(x) = 1$

At $x = 1$:

$\forall x \in (-\infty, 1]: f(x) = x - 2 \Rightarrow$

$$\Rightarrow \lim_{x \to 1^-} \lambda(f|x,1) = \frac{d}{dx}(x-2)\Big|_{x=1} = 1\Big|_{x=1} = 1 \quad (1)$$

and

$$\forall x \in (1, +\infty): \lambda(f|x,1) = \frac{f(x) - f(1)}{x-1} = \frac{(x^2 - x - 1) - (1-2)}{x-1} =$$

$$= \frac{x^2 - x - 1 + 1}{x-1} = \frac{x^2 - x}{x-1} = \frac{x(x-1)}{x-1} = x \Rightarrow$$

$$\Rightarrow \lim_{x \to 1^+} \lambda(f|x,1) = \lim_{x \to 1^+} x = 1 \quad (2)$$

From Eq. (1) and Eq. (2):

$$f'(1) = \lim_{x \to 1} \lambda(f|x,1) = 1$$

and it follows that:

$$f'(x) = \begin{cases} 2x - 1 & , \text{if } x \in (1, +\infty) \\ 1 & , \text{if } x = 1 \\ 1 & , \text{if } x \in (-\infty, 1) \end{cases} = \begin{cases} 2x - 1 & , x \in (1, +\infty) \\ 1 & , x \in (-\infty, 1] \end{cases}$$

EXERCISES

(7) Find the derivatives of the following functions and show the implications below.

a) $f(x) = (x+1)(x^2-x+1) \Rightarrow f'(x) = 3x^2$

b) $f(x) = (x^2+1)(2x+1)(2x+3) \Rightarrow f'(x) = 2(8x^3+12x^2+7x+4)$

c) $f(x) = (2x+1)\sqrt{x} \Rightarrow f'(x) = \dfrac{6x+1}{2\sqrt{x}}$

d) $f(x) = (3x+2)^2\sqrt{x} \Rightarrow f'(x) = \dfrac{(3x+2)(15x+2)}{2\sqrt{x}}$

e) $f(x) = (x^2+5x+6)\sqrt{x} \Rightarrow f'(x) = \dfrac{5x^2+15x+6}{2\sqrt{x}}$

(8) Find the tangent line $(\ell): y = ax+b$ to the graph $(C): y = f(x)$ at $x = x_0$ for the following choices of f and x_0:

a) $f(x) = x^2+3x+1$ at $x_0 = \sqrt{2}+\sqrt{3}$

b) $f(x) = x^3+2x^2+2x+1$ at $x_0 = 1-\sqrt{3}$

c) $f(x) = x^5+4x^3$ at $x_0 = 3\sqrt{2}+2\sqrt{3}$

d) $f(x) = (x^2+3x)\sqrt{x}$ at $x_0 = 2$

(9) Show the following more generalized implications.

a) $f(x) = (ax^2+bx+c)\sqrt{x} \Rightarrow f'(x) = \dfrac{5ax^2+3bx+c}{2\sqrt{x}}$

b) $f(x) = (ax+b)^2\sqrt{x} \Rightarrow f'(x) = \dfrac{(ax+b)(5ax+b)}{2\sqrt{x}}$

(10) Consider the functions

$f(x) = x^3 + ax^2 + bx + 1$, $\forall x \in \mathbb{R}$

$g(x) = x^2 + x + 2$, $\forall x \in \mathbb{R}$

Find all $a, b \in \mathbb{R}$ such that the graphs $(C_1): y = f(x)$ and $(C_2): y = g(x)$ have the same tangent line at $x_0 = 1$

(11) Find all $a \in \mathbb{R}$ such that the line $(\ell): y = 2x+1$ is the tangent line to $(C): y = f(x)$ with

$f(x) = x^2 + 2ax + 1$, $\forall x \in \mathbb{R}$

at $x = x_0$ and find the corresponding contact point x_0.

(12) Define the widest possible subset of $\mathbb{R}$ over which the following functions are differentiable and define the derivative $f'(x)$:

a) $f(x) = \begin{cases} 2x-1, & x \in [1, +\infty) \\ x^2, & x \in (-\infty, 1) \end{cases}$

b) $f(x) = \begin{cases} x^2+3x, & x \in (1, +\infty) \\ (x-1)^2+4, & x \in (-\infty, 1] \end{cases}$

c) $f(x) = |x^2+3x|$, $\forall x \in \mathbb{R}$

(Hint: rewrite as a multiple formula function)

(13) Let $f: A \to \mathbb{R}$ be a function that is differentiable at $x_0 = a$. Show that

$$\lim_{x \to a} \frac{xf(a) - af(x)}{x-a} = f(a) - af'(a)$$

(14) Let $f:\mathbb{R}\to\mathbb{R}$ be a function differentiable at $x_0=0$ with $f(0)=0$ and $f'(0)=0$. Show that

$$g(x)=\begin{cases} f(x)\cos(1/x) & , x\in\mathbb{R}-\{0\} \\ 0 & , x=0 \end{cases}$$

is differentiable at $x_0=0$

(15) Let $f:A\to\mathbb{R}$ be a function with $x_0=0\in A$. Show that

$$\left.\begin{array}{l} f \text{ differentiable at } x_0=0 \\ \lim\limits_{x\to 0}\dfrac{f(x)}{x}=1 \end{array}\right\} \Rightarrow f'(0)=1$$

(Hint: You need to show $f(0)=0$ using the implied continuity at $x_0=0$ first).

Chain rule

- The chain rule is a superrule that is used to generate differentiation rules that are then used in problems. We seldomly use the chain rule directly.
- Recall the definition of function composition:
  For $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$, we define $f \circ g : G \to \mathbb{R}$ with

$$\begin{cases} \mathrm{dom}(f \circ g) = \{x \in \mathrm{dom}(g) \mid g(x) \in \mathrm{dom}(f)\} \\ \qquad\qquad\quad = \{x \in B \mid g(x) \in A\} = G \\ \forall x \in G : (f \circ g)(x) = f(g(x)) \end{cases}$$

  Note that by definition, the belonging condition for $\mathrm{dom}(f \circ g)$ is:

$$x \in \mathrm{dom}(f \circ g) \Longleftrightarrow \begin{cases} x \in \mathrm{dom}(g) \\ g(x) \in \mathrm{dom}(f) \end{cases}$$

- The chain rule claims that:

$$\boxed{\begin{cases} g \text{ differentiable at } x_0 \\ f \text{ differentiable at } g(x_0) \end{cases} \Longrightarrow \begin{cases} f \circ g \text{ differentiable at } x_0 \\ (f \circ g)'(x_0) = f'(g(x_0))\, g'(x_0) \end{cases}}$$

We postpone the proof. Every choice of f generates a new generalized differentiation rule. For example:

1) For $f(x) = x^n$ with $n \in \mathbb{N}^*$, using

$$(x^n)' = n x^{n-1}$$

we obtain:

$$\boxed{([g(x)]^n)' = n\,[g(x)]^{n-1}\, g'(x)}$$

2) For $f(x) = \sqrt{x}$, using $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$, we obtain:

$$\boxed{\left(\sqrt{g(x)}\right)' = \frac{g'(x)}{2\sqrt{g(x)}}}$$

↳ Note that for each generalization, starting from the initial differentiation rule:
(a) All x are replaced with $g(x)$
(b) The entire result is then multiplied with $g'(x)$.
Step (a) corresponds to the $f'(g(x_0))$ factor
Step (b) corresponds to the $g'(x_0)$ factor.
We see therefore that every basic differentiation rule can give a more powerful generalized differentiation rule via the chain rule.

EXAMPLES

a) Find and factor the derivative of
$f(x) = (x^3+2x^2+x+2)^3$, $\forall x \in \mathbb{R}$.

Solution

$$f'(x) = \left[(x^3+2x^2+x+2)^3\right]' =$$
$$= 3(x^3+2x^2+x+2)^2\,(x^3+2x^2+x+2)'$$
$$= 3(x^3+2x^2+x+2)^2\,(3x^2+4x+1)$$
$$= 3\left[x^2(x+2)+(x+2)\right]^2(3x^2+4x+1)$$
$$= 3(x^2+1)^2(x+2)^2(3x^2+4x+1), \quad \forall x \in \mathbb{R}$$

b) Find and factor the derivative of
$f(x) = (x^2-2)^3(3x+2)^4$, $\forall x \in \mathbb{R}$.

Solution

$$f'(x) = \left[(x^2-2)^3(3x+2)^4\right]' =$$
$$= \left[(x^2-2)^3\right]'(3x+2)^4 + (x^2-2)^3\left[(3x+2)^4\right]'$$
$$= 3(x^2-2)^2(x^2-2)'(3x+2)^4 + (x^2-2)^3\cdot 4(3x+2)^3(3x+2)'$$
$$= 3(x^2-2)^2(2x)(3x+2)^4 + (x^2-2)^3\cdot 4(3x+2)^3\cdot 3$$
$$= 3\cdot 2\cdot (x^2-2)^2(3x+2)^3\left[(x)(3x+2)+2(x^2-2)\right]$$
$$= 6(x^2-2)^2(3x+2)^3(3x^2+2x+2x^2-4)$$
$$= 6(x^2-2)^2(3x+2)^3(5x^2+2x-4), \quad \forall x \in \mathbb{R}.$$

c) Find and factor the derivative of

$f(x) = (3x-5)^2\sqrt{x^2-2x}$

Solution

$$f'(x) = \left[(3x-5)^2\sqrt{x^2-2x}\,\right]' =$$

$$= \left[(3x-5)^2\right]'\sqrt{x^2-2x} + (3x-5)^2\left(\sqrt{x^2-2x}\right)' =$$

$$= 2(3x-5)(3x-5)'\sqrt{x^2-2x} + (3x-5)^2\,\frac{(x^2-2x)'}{2\sqrt{x^2-2x}} =$$

$$= 2(3x-5)\cdot 3\sqrt{x^2-2x} + (3x-5)^2\,\frac{2x-2}{2\sqrt{x^2-2x}} =$$

$$= \frac{2(3x-5)\cdot 3\sqrt{x^2-2x}\cdot 2\sqrt{x^2-2x} + (3x-5)^2(2x-2)}{2\sqrt{x^2-2x}} =$$

$$= \frac{12(3x-5)(x^2-2x) + (3x-5)^2(2x-2)}{2\sqrt{x^2-2x}} =$$

$$= \frac{(3x-5)\left[12(x^2-2x) + (3x-5)(2x-2)\right]}{2\sqrt{x^2-2x}} =$$

$$= \frac{(3x-5)(12x^2-24x+6x^2-6x-10x+10)}{2\sqrt{x^2-2x}} =$$

$$= \frac{(3x-5)((12+6)x^2+(-24-6-10)x+10)}{2\sqrt{x^2-2x}}$$

$$= \frac{(3x-5)(18x^2-40x+10)}{2\sqrt{x^2-2x}} = \frac{(3x-5)\cdot 2(9x^2-20x+5)}{2\sqrt{x^2-2x}}$$

$$= \frac{(3x-5)(9x^2-20x+5)}{\sqrt{x^2-2x}}$$

Proof of chain rule

Assume that g differentiable at x_0 and f differentiable at $g(x_0)$. It follows that $f\circ g$ can be defined on a neighborhood $N(x_0,\delta)$ for some $\delta>0$.
We define $y_0=g(x_0)$ and

$$F(y)=\begin{cases}\lambda(f\mid y,y_0) &, \text{ if } y\neq y_0\\ f'(y_0) &, \text{ if } y=y_0\end{cases}$$

We claim that $\lambda(f\circ g\mid x,x_0)=F(g(x))\lambda(g\mid x,x_0)$, $\forall x\in N(x_0,\delta)$ (1)
To show the claim, let $x\in N(x_0,\delta)$ be given. We distinguish between the following cases:

Case 1: If $g(x)\neq g(x_0)$ then:

$$\lambda(f\circ g\mid x,x_0)=\frac{(f\circ g)(x)-(f\circ g)(x_0)}{x-x_0}=\frac{f(g(x))-f(g(x_0))}{x-x_0}=$$

$$=\frac{f(g(x))-f(g(x_0))}{g(x)-g(x_0)}\,\frac{g(x)-g(x_0)}{x-x_0}=$$

$$=\lambda(f\mid g(x),y_0)\,\lambda(g\mid x,x_0)=F(g(x))\,\lambda(g\mid x,x_0)$$

Case 2: If $g(x)=g(x_0)$, then:

$$\lambda(f\circ g\mid x,x_0)=\frac{(f\circ g)(x)-(f\circ g)(x_0)}{x-x_0}=\frac{f(g(x))-f(g(x_0))}{x-x_0}$$

$$=\frac{f(g(x_0))-f(g(x_0))}{x-x_0}=0$$

and

$$\lambda(g|x,x_0) = \frac{g(x)-g(x_0)}{x-x_0} = \frac{g(x_0)-g(x_0)}{x-x_0} = 0$$

and therefore $\lambda(f\circ g|x,x_0) = F(g(x))\lambda(g|x,x_0)$ holds trivially since both sides are zero.

This proves the claim.

Now, we note that

$$g \text{ differentiable at } x_0 \Rightarrow g \text{ continuous at } x_0 \Rightarrow$$

$$\Rightarrow \lim_{x\to x_0} g(x) = g(x_0) = y_0 \quad (2)$$

and

$$\begin{aligned}
\lim_{y\to y_0} F(y) &= \lim_{y\to y_0} \lambda(f|y,y_0) = && [\text{def of } F(y)]\\
&= f'(y_0) = && [f \text{ differentiable at } y_0]\\
&= F(y_0) \Rightarrow && [\text{def of } F(y)]
\end{aligned}$$

$$\Rightarrow F \text{ continuous at } y_0. \quad (3)$$

Via the composition theorem, from Eq. (2) and Eq. (3):

$$\begin{aligned}
\lim_{x\to x_0} F(g(x)) &= F(\lim_{x\to x_0} g(x)) && [\text{via composition thm}]\\
&= F(y_0) && [\text{via eq. (2)}]\\
&= f'(y_0) && [\text{def of } F(y)]\\
&= f'(g(x_0)) \quad (4) && [\text{def of } y_0]
\end{aligned}$$

and it follows that

$$[f(g(x_0))]' = \lim_{x\to x_0} \lambda(f\circ g|x,x_0) = \lim_{x\to x_0} [F(g(x))\lambda(g|x,x_0)] =$$

$$= \lim_{x\to x_0} F(g(x)) \cdot \lim_{x\to x_0} \lambda(g|x,x_0) = f'(g(x_0))\, g'(x_0). \quad \square$$

EXERCISES

(16) Find and factor the derivatives of the following functions.

a) $f(x) = (x^2-3)^4$

b) $f(x) = (x^2-5x+6)^{30}$

c) $f(x) = (1+2x)^3$

d) $f(x) = (1-x^2)^2$

e) $f(x) = \sqrt{x^2+3x-1}$

f) $f(x) = \sqrt{(x^2-1)(2x+1)}$

(17) Show the following implications

a) $f(x) = (2x-1)^3(3x+2)^5 \Rightarrow f'(x) = 3(2x-1)^2(3x+2)^4(16x-1)$

b) $f(x) = (x+3)^2(5x+2)^4 \Rightarrow f'(x) = 2(x+3)(5x+2)^3(15x+32)$

c) $f(x) = (x^2-4)^3(x^2-1)^2 \Rightarrow f'(x) = 2x(x^2-1)(x^2-4)^2(5x^2-11)$

d) $f(x) = (x^2+3x+2)^3(x^2-5x+6)^5 \Rightarrow$

$\Rightarrow f'(x) = 4(x-3)^4(x-2)^4(x+1)^3(x+2)^2(4x^3-4x^2-16x+1)$

e) $f(x) = (2x+1)\sqrt{x^2+1} \Rightarrow f'(x) = \dfrac{4x^2+x+2}{\sqrt{x^2+1}}$

f) $f(x) = (x^2+3x)\sqrt{x^2-1} \Rightarrow f'(x) = \dfrac{3x^3+6x^2-2x-3}{\sqrt{x^2-1}}$

g) $f(x) = 2x\sqrt{x+1}\,\sqrt{2x+1} \Rightarrow f'(x) = \dfrac{8x^2+9x+2}{\sqrt{x+1}\,\sqrt{2x+1}}$

h) $f(x) = (5x+7)\sqrt{x^2+3x+5} \Rightarrow f'(x) = \dfrac{20x^2+59x+71}{2\sqrt{x^2+3x+5}}$

(18) Likewise show the following more generalized implications

a) $f(x) = (ax+b)\sqrt{cx+d} \Rightarrow f'(x) = \dfrac{3acx + (2ad+bc)}{2\sqrt{cx+d}}$

b) $f(x) = x^2\sqrt{ax+b} \Rightarrow f'(x) = \dfrac{x(5ax+4b)}{\sqrt{ax+b}}$

c) $f(x) = (ax+b)\sqrt{cx^2+dx+e} \Rightarrow$

$$\Rightarrow f'(x) = \frac{4acx^2 + (3ad+2bc)x + (2ae+bd)}{2\sqrt{cx^2+dx+e}}$$

▼ The quotient rule

The quotient rule is derived from the chain rule as follows.

•1 First we show that

$$\boxed{\left(\forall x\in\mathbb{R}^*: f(x)=\frac{1}{x}\right) \Rightarrow \forall x\in\mathbb{R}^*: f'(x)=\frac{-1}{x^2}}$$

Proof

Since

$$\forall x, x_0\in\mathbb{R}-\{0\}: \lambda(f\mid x, x_0)=\frac{f(x)-f(x_0)}{x-x_0}=\frac{\frac{1}{x}-\frac{1}{x_0}}{x-x_0}=$$

$$=\frac{\left(\frac{x_0-x}{xx_0}\right)}{x-x_0}=\frac{-(x-x_0)}{xx_0(x-x_0)}=\frac{-1}{xx_0}\Rightarrow$$

$$\Rightarrow \forall x_0\in\mathbb{R}-\{0\}: f'(x_0)=\lim_{x\to x_0}\lambda(f\mid x, x_0)=\lim_{x\to x_0}\left(\frac{-1}{xx_0}\right)=$$

$$=\frac{-1}{x_0x_0}=\frac{-1}{x_0^2} \qquad \square$$

•2 Via the chain rule, this result immediately generalizes to the <u>reduced quotient rule</u>:

$$\boxed{h(x)=\frac{1}{g(x)}, \forall x\in A \Rightarrow h'(x)=\frac{-g'(x)}{[g(x)]^2}, \forall x\in A}$$

• 3 Combined with the product rule, the reduced quotient rule gives the <u>quotient rule</u>:

$$\boxed{h(x) = \frac{f(x)}{g(x)}, \forall x \in A \Rightarrow h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$$

<u>Proof</u>

$$h'(x) = \left[\frac{f(x)}{g(x)}\right]' = \left[f(x) \cdot \frac{1}{g(x)}\right]' =$$

$$= f'(x)\frac{1}{g(x)} + f(x) \cdot \left[\frac{1}{g(x)}\right]' = \qquad \text{[product rule]}$$

$$= \frac{f'(x)}{g(x)} + f(x)\frac{-g'(x)}{[g(x)]^2} = \qquad \text{[reduced quotient rule]}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \qquad \square$$

EXAMPLES

a) $f(x) = \dfrac{1}{(x^2+3x)\sqrt{x}}$ ← Evaluate $f'(x)$.

Solution

$$f'(x) = \left[\frac{1}{(x^2+3x)\sqrt{x}}\right]' = \frac{-\left[(x^2+3x)\sqrt{x}\right]'}{\left[(x^2+3x)\sqrt{x}\right]^2} =$$

$$= \frac{-(x^2+3x)'\sqrt{x} - (x^2+3x)(\sqrt{x})'}{x(x^2+3x)^2} =$$

$$= \frac{-(2x+3)\sqrt{x} - (x^2+3x)\left(\frac{1}{2\sqrt{x}}\right)}{x(x^2+3x)^2} =$$

$$= \frac{-(2x+3)\sqrt{x}\,(2\sqrt{x}) - (x^2+3x)}{x(x^2+3x)^2(2\sqrt{x})} =$$

$$= \frac{-2x(2x+3) - (x^2+3x)}{x(x^2+3x)^2(2\sqrt{x})} =$$

$$= \frac{-4x^2-6x-x^2-3x}{2x^3(x+3)^2\sqrt{x}} = \frac{-5x^2-9x}{2x^3(x+3)^2\sqrt{x}} =$$

$$= \frac{-x(5x+9)}{2x^3\sqrt{x}\,(x+3)^2} = \frac{-(5x+9)}{2x^2\sqrt{x}\,(x+3)^2}$$

b) $f(x) = \dfrac{x^2+3x+2}{x^2-2x+5} \longleftarrow$ Evaluate $f'(x)$.

Solution

$$f'(x) = \left[\frac{x^2+3x+2}{x^2-2x+5}\right]' =$$

$$= \frac{(x^2+3x+2)'(x^2-2x+5) - (x^2+3x+2)(x^2-2x+5)'}{(x^2-2x+5)^2}$$

$$= \frac{(2x+3)(x^2-2x+5) - (x^2+3x+2)(2x-2)}{(x^2-2x+5)^2}$$

$$= \frac{2x^3 - 4x^2 + 10x + 3x^2 - 6x + 15 - 2x^3 + 2x^2 - 6x^2 + 6x - 4x + 4}{(x^2-2x+5)^2}$$

$$= \frac{(2-2)x^3 + (-4+3+2-6)x^2 + (10-6+6-4)x + (15+4)}{(x^2-2x+5)^2}$$

$$= \frac{-5x^2+6x+19}{(x^2-2x+5)^2} = \frac{-(5x^2-6x-19)}{(x^2-2x+5)^2}$$

EXERCISES

(19) Show the following implications

a) $f(x) = \dfrac{x^2}{x-1} \Rightarrow f'(x) = \dfrac{x(x-2)}{(x-1)^2}$

b) $f(x) = \dfrac{2x+1}{3x-2} \Rightarrow f'(x) = \dfrac{-7}{(3x-2)^2}$

c) $f(x) = \dfrac{x^2+x+1}{x^2-3x+2} \Rightarrow f'(x) = \dfrac{-(4x^2-2x-5)}{(x-2)^2(x-1)^2}$

d) $$f(x) = \dfrac{x + \dfrac{x+1}{2x+1}}{x - \dfrac{x-1}{2x-1}} \Rightarrow f'(x) = \dfrac{-8x^2(2x^2-3)}{(2x+1)^2(2x^2-2x+1)^2}$$

e) $f(x) = \dfrac{x^2-1}{\sqrt{x}} \Rightarrow f'(x) = \dfrac{3x^2+1}{2x\sqrt{x}}$

f) $f(x) = \dfrac{(3x-2)^3}{(2x-5)^2} \Rightarrow f'(x) = \dfrac{(3x-2)^2(6x-37)}{(2x-5)^3}$

g) $f(x) = \dfrac{(x+3)^2}{(3x-1)^4} \Rightarrow f'(x) = \dfrac{-2(x+3)(3x+19)}{(3x-1)^5}$

(20) Likewise, show that

a) $f(x) = \dfrac{\sqrt{2x+1}}{3x+2} \Rightarrow f'(x) = \dfrac{-(3x+1)}{(3x+2)^2\sqrt{2x+1}}$

b) $f(x)=\dfrac{\sqrt{2x+1}}{3x+2} \Rightarrow f'(x)=\dfrac{-(3x+1)}{(3x+2)^2\sqrt{2x+1}}$

c) $f(x)=\dfrac{\sqrt{x^2+2x}}{x^2-1} \Rightarrow f'(x)=\dfrac{-(x^3+3x^2+x+1)}{(1-x^2)^2\sqrt{x(x+2)}}$

d) $f(x)=\dfrac{x^2+1}{\sqrt{3x+2}} \Rightarrow f'(x)=\dfrac{9x^2+8x-3}{2(3x+2)\sqrt{3x+2}}$

e) $f(x)=\dfrac{(5x+6)^3}{\sqrt{x^2+3x}} \Rightarrow f'(x)=\dfrac{(5x+6)^2(20x^2+63x-18)}{2x(x+3)\sqrt{x(x+3)}}$

f) $f(x)=\dfrac{(x+1)^2}{(2x-1)^3\sqrt{2x+3}} \Rightarrow f'(x)=\dfrac{-(x+1)(6x^2+23x+23)}{(2x-1)^4(2x+3)\sqrt{2x+3}}$

(21) Given the function

$$f(x)=\frac{x\cos\omega-\sin\omega}{x\sin\omega+\cos\omega}$$

with $0<\omega<\pi/4$, show that

$$\frac{f'(x)}{1+[f(x)]^2}=\frac{1}{x^2+1}$$

(22) Given the function $f(x)=\sqrt{x+\sqrt{1+x^2}}$, $\forall x\in\mathbb{R}$ show that:

a) $f(x)=2\sqrt{1+x^2}\,f'(x)$

b) $4(1+x^2)f''(x)+4xf'(x)=f(x)$

▼ Trigonometric derivatives

- The derivative of $\sin x$ can be derived via the result

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

and the trigonometric identity for factoring the sum/difference of sine functions:

$$\sin a \pm \sin b = 2 \sin\left(\frac{a \pm b}{2}\right) \cos\left(\frac{a \mp b}{2}\right)$$

The main result is:

(1) $\boxed{\forall x \in \mathbb{R} : (\sin x)' = \cos x}$

Proof

Let $x, x_0 \in \mathbb{R}$ be given with $x \neq x_0$.

$$\lambda(\sin \mid x, x_0) = \frac{\sin x - \sin x_0}{x - x_0} = \frac{2 \sin\left(\frac{x - x_0}{2}\right) \cos\left(\frac{x + x_0}{2}\right)}{x - x_0} =$$

$$= \frac{\sin\left(\frac{x - x_0}{2}\right)}{\frac{x - x_0}{2}} \cos\left(\frac{x + x_0}{2}\right), \quad \forall x, x_0 \in \mathbb{R} \quad (1)$$

Since

$$\left. \begin{array}{l} \lim_{x \to x_0} \frac{x + x_0}{2} = \frac{x_0 + x_0}{2} = x_0 \\ \cos \text{ continuous on } \mathbb{R} \end{array} \right\} \Rightarrow \lim_{x \to x_0} \cos\left(\frac{x + x_0}{2}\right) = \cos x_0 \quad (2)$$

and

$$\left.\begin{array}{l} \lim\limits_{x \to x_0} \dfrac{x-x_0}{2} = 0 \\[2ex] \dfrac{x-x_0}{2} \neq 0, \forall x \in N(x_0, \delta) \\[2ex] \lim\limits_{y \to 0} \dfrac{\sin y}{y} = 1 \end{array}\right\} \Rightarrow \lim_{x \to 0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} = 1 \quad (3)$$

From Eq. (1), Eq. (2), Eq. (3):

$$(\sin x_0)' = \lim_{x \to x_0} \Delta(\sin | x, x_0) =$$

$$= \lim_{x \to x_0} \left[\frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} \cdot \cos\left(\frac{x+x_0}{2}\right) \right]$$

$$= \lim_{x \to x_0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} \lim_{x \to x_0} \cos\left(\frac{x+x_0}{2}\right) =$$

$$= 1 \cdot \cos\left(\frac{x_0+x_0}{2}\right) = \cos x_0 \qquad \square$$

↳ Note that the proof of this result depends on the continuity of cos and the limit $\lim\limits_{x \to 0} (\sin x)/x$. Consequently continuity has to be established first before establishing differentiability.

- For the derivative of cos we use the chain rule generalization of the above result

$$[\sin(g(x))]' = g'(x) \cos(g(x))$$

and the cofactor identities:

$$\forall x \in \mathbb{R}: \ \sin(\pi/2 - x) = \cos x$$
$$\forall x \in \mathbb{R}: \ \cos(\pi/2 - x) = \sin x$$

as follows:

(2) $\boxed{(\cos x)' = -\sin x \ , \forall x \in \mathbb{R}}$

<u>Proof</u>

$$(\cos x)' = [\sin(\pi/2 - x)]' = (\pi/2 - x)' \cos(\pi/2 - x)$$
$$= -\cos(\pi/2 - x) = -\sin x \ , \ \forall x \in \mathbb{R}.$$

(3) $\boxed{(\tan x)' = \dfrac{1}{\cos^2 x} = 1 + \tan^2 x \ , \ \forall x \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}}$

<u>Proof</u>

$$(\tan x)' = \left[\frac{\sin x}{\cos x}\right]' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} =$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} \qquad (1)$$

From Eq. (1):

$$(\tan x)' = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$(\tan x)' = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} =$$

$$= 1 + \left(\frac{\sin x}{\cos x}\right)^2 = 1 + \tan^2 x \qquad \square$$

- Via the chain rule, we obtain the following generalized differentiation rules:

$(\sin x)' = \cos x$	$[\sin(g(x))]' = g'(x)\cos(g(x))$
$(\cos x)' = -\sin x$	$[\cos(g(x))]' = -g'(x)\sin(g(x))$
$(\tan x)' = \dfrac{1}{\cos^2 x}$	$[\tan(g(x))]' = \dfrac{g'(x)}{\cos^2(g(x))}$
$(\tan x)' = 1+\tan^2 x$	$[\tan(g(x))]' = [1+\tan^2(g(x))]\,g'(x)$

EXAMPLE

a) $f(x) = \sin^2(\cos^3(2x)) \longleftarrow$ Evaluate $f'(x)$.

Solution

$$f'(x) = [\sin^2(\cos^3(2x))]' =$$
$$= 2\sin(\cos^3(2x))[\sin(\cos^3(2x))]'$$
$$= 2\sin(\cos^3(2x))\cos(\cos^3(2x))[\cos^3(2x)]'$$
$$= 2\sin(\cos^3(2x))\cos(\cos^3(2x))[3\cos^2(2x)][\cos(2x)]'$$
$$= 6\sin(\cos^3(2x))\cos(\cos^3(2x))\cos^2(2x)[-\sin(2x)](2x)'$$
$$= -12\sin(\cos^3(2x))\cos(\cos^3(2x))\cos^2(2x)\sin(2x)$$

b) Show that:

$$f(x) = \cos x\sqrt{\sin(3x)} \Rightarrow f'(x) = \frac{5\cos(4x) + \cos(2x)}{4\sqrt{\sin(3x)}}$$

Solution

$$f'(x) = [\cos x\sqrt{\sin(3x)}]' =$$
$$= (\cos x)'\sqrt{\sin(3x)} + \cos x(\sqrt{\sin(3x)})'$$
$$= -\sin x\sqrt{\sin(3x)} + \cos x\,\frac{(\sin(3x))'}{2\sqrt{\sin(3x)}} =$$
$$= -(\sin x)\sqrt{\sin(3x)} + \cos x\,\frac{(3x)'\cos(3x)}{2\sqrt{\sin(3x)}} =$$
$$= -(\sin x)\sqrt{\sin(3x)} + \cos x\,\frac{3\cos(3x)}{2\sqrt{\sin(3x)}} =$$

$$= \frac{-(\sin x)\sqrt{\sin(3x)}\left(2\sqrt{\sin(3x)}\right) + \cos x\,(3\cos(3x))}{2\sqrt{\sin(3x)}}$$

$$= \frac{-2\sin x\sin(3x) + 3\cos x\cos(3x)}{2\sqrt{\sin(3x)}} =$$

$$= \frac{-(\cos(x-3x) - \cos(x+3x)) + 3\cdot(1/2)[\cos(x-3x) + \cos(x+3x)]}{2\sqrt{\sin(3x)}}$$

$$= \frac{-2(\cos 2x - \cos 4x) + 3(\cos 2x + \cos 4x)}{4\sqrt{\sin(3x)}} =$$

$$= \frac{-2\cos 2x + 2\cos 4x + 3\cos 2x + 3\cos 4x}{4\sqrt{\sin(3x)}} =$$

$$= \frac{5\cos 4x + \cos 2x}{4\sqrt{\sin(3x)}} \qquad \square$$

↳ For the above exercise we used the following trigonometric identities:

$\cos(-x) = \cos x$

$2\cos a\cos b = \cos(a-b) + \cos(a+b)$

$2\sin a\sin b = \cos(a-b) - \cos(a+b)$.

EXERCISES

(23) Find the derivatives of the following functions

a) $f(x) = \sin(3x^2+2)$

b) $f(x) = \tan(1/x)$

c) $f(x) = \sin^3(x^2+x-5)$

d) $f(x) = \cos(\sqrt{x-1})$

e) $f(x) = \sin(\tan(x))$

f) $f(x) = \cos(\sin(x))$

g) $f(x) = \sin(\cot^2 x)$

h) $f(x) = \sin(\sqrt{2x+1})$

(24) Show that

a) $f(x) = \sin^2 x \cos x \Rightarrow f'(x) = \sin x\,(\cos(2x) + \cos^2 x)$

b) $f(x) = \tan x - \cot x \Rightarrow f'(x) = \dfrac{4}{\sin^2(2x)}$

c) $f(x) = \sqrt{x}\tan^3 x \Rightarrow f'(x) = \dfrac{\tan^2 x\,[\sin x \cos x + 6x]}{2\sqrt{x}\cos^2 x}$

d) $f(x) = \dfrac{\sin x}{\sin x + \cos x} \Rightarrow f'(x) = \dfrac{1}{1+\cos x}$

e) $f(x) = \sin(\cos^2 x)\cos(\sin^2 x) \Rightarrow f'(x) = -\sin(2x)\cos(\cos(2x))$

f) $f(x) = \dfrac{\sin x - \cos x}{\sin x + \cos x} \Rightarrow f'(x) = \dfrac{2}{\sin(2x)+1}$

g) $f(x) = (x - \sin x \cos x)^2 \Rightarrow f'(x) = 4\sin^2 x\,(x - \sin x \cos x)$

h) $f(x) = 3\sin x\sqrt{\cos(2x)} \Rightarrow f'(x) = \dfrac{3\cos(3x)}{\sqrt{\cos(2x)}}$

i) $f(x) = \dfrac{\tan(x)}{\sqrt{x}} \Rightarrow f'(x) = \dfrac{4x - \sin(2x)}{4x\sqrt{x}\cos^2 x}$

j) $f(x) = \dfrac{\tan x - \cot x}{\tan x + \cot x} \Rightarrow f'(x) = 2\sin(2x)$

k) $f(x) = \cos x \sqrt{\cos(2x)} \Rightarrow f'(x) = \dfrac{-\sin(3x)}{\sqrt{\cos(2x)}}$

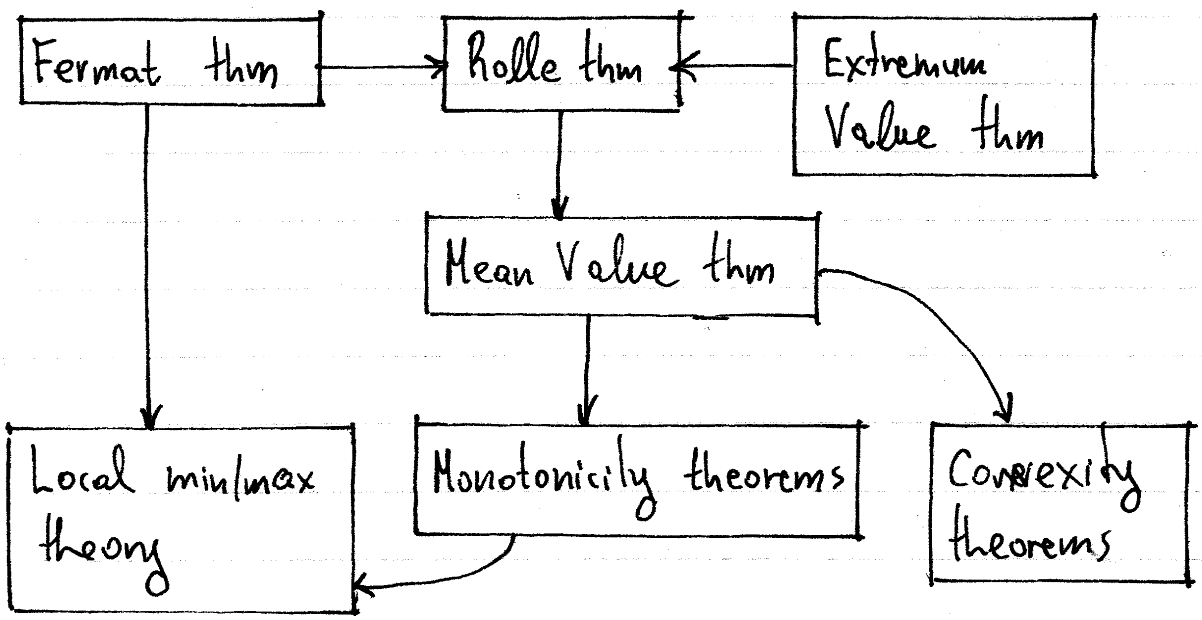
l) $f(x) = \tan x \sqrt{\cot x} \Rightarrow f'(x) = \dfrac{1}{\sin(2x)\sqrt{\cot x}}$

CAL1.5: Differential Calculus

DIFFERENTIAL CALCULUS

▼ Foundation of Differential Calculus

The applications of derivatives are based on a collection of theorems that have the following interdependence amongst themselves



① → Fermat theorem

Def: (Interior points)
Let A be a set $A \subseteq \mathbb{R}$. We say that
x_0 interior point of $A \iff \exists \delta \in (0, +\infty) : (x_0 - \delta, x_0 + \delta) \subseteq A$

<u>notation</u>: The set of all interior points of a set A is denoted as

$$\begin{aligned} \mathrm{int}(A) &= \{x_0 \in A \mid x_0 \text{ interior to } A\} \\ &= \{x_0 \in A \mid \exists \delta \in (0,+\infty) : (x_0-\delta, x_0+\delta) \subseteq A\} \end{aligned}$$

▸ In general, given a set defined as a union of intervals, int(A) can be obtained by changing all closed intervals to open intervals

<u>example</u>: For $A = (1,3] \cup [5,+\infty)$, we have

$$\mathrm{int}(A) = (1,3) \cup (5,+\infty).$$

Consequently, 2 is interior to A but for $x_0 \in \{1,3,5\}$, x_0 is <u>not</u> interior to A.

<u>Def</u>: (<u>Local min/max</u>)

Let $f: A \to \mathbb{R}$ be a function and let $x_0 \in A$.

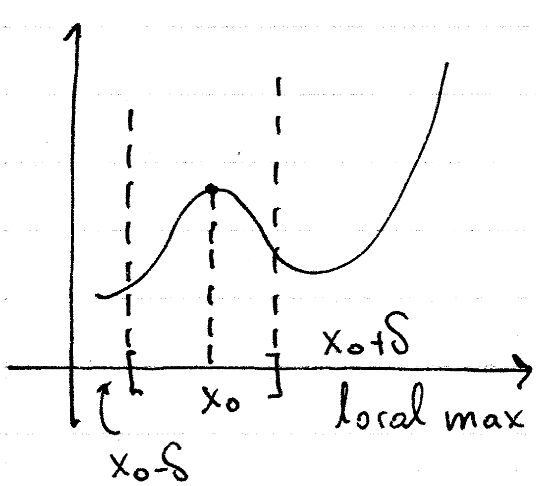
We say that

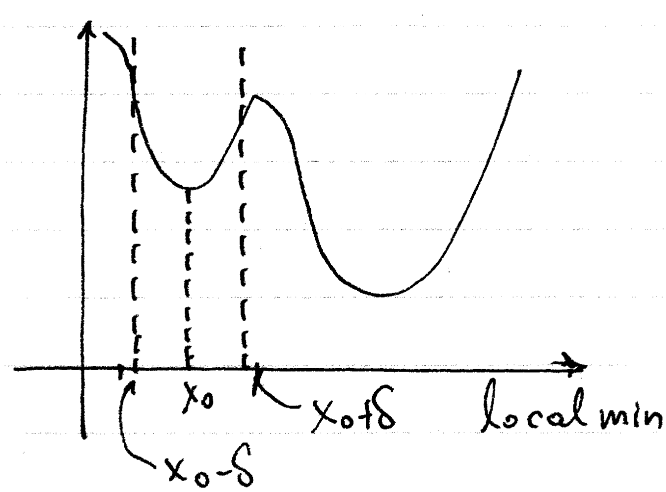
a) x_0 <u>local max</u> of $f \iff$

$$\iff \exists \delta \in (0,+\infty) : \forall x \in (x_0-\delta, x_0+\delta) \cap A : f(x) \leq f(x_0)$$

b) x_0 <u>local min</u> of $f \iff$

$$\iff \exists \delta \in (0,+\infty) : \forall x \in (x_0-\delta, x_0+\delta) \cap A : f(x) \geq f(x_0)$$



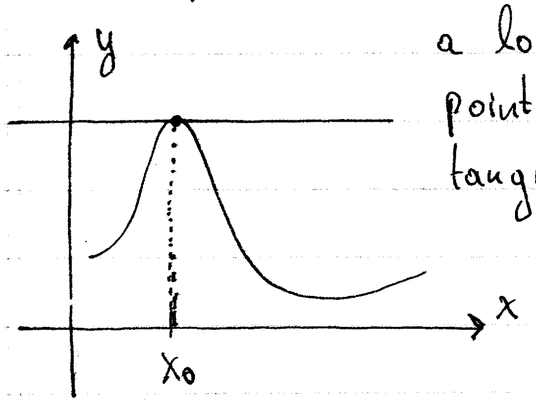


<u>interpretation</u>: A point $x_0 \in A$ is local min of $f: A \to \mathbb{R}$ if and only if $f(x_0)$ is the minimum value of f in a small enough interval around the point x_0. Likewise, a point $x_0 \in A$ is local max of $f: A \to \mathbb{R}$ if and only if $f(x_0)$ is the maximum value of f in a small enough interval around the point x_0.

<u>Thm</u>: (<u>Fermat theorem</u>)
Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ be a function and let $x_0 \in A$.
We have:

$$\left\{\begin{array}{l} x_0 \in \operatorname{int}(A) \\ x_0 \text{ local min or max of } f \\ f \text{ differentiable on } x_0 \end{array}\right. \Rightarrow f'(x_0) = 0$$

▸ <u>interpretation</u>: If a function is differentiable and has a local max or min at an interior point x_0 of its domain, then the tangent line (ℓ) to the graph of f at the point x_0 is horizontal.



<u>Proof</u>
With no loss of generality, assume that

$$\left\{\begin{array}{l} x_0 \in \operatorname{int}(A) \wedge x_0 \text{ local max of } f \\ f \text{ differentiable on } x_0 \end{array}\right.$$

It follows that

$x_0 \in \operatorname{int}(A) \Rightarrow \exists \delta_1 \in (0,+\infty): (x_0-\delta_1, x_0+\delta_1) \subseteq A$

x_0 local max of $f \Rightarrow$

$\Rightarrow \exists \delta_2 \in (0,+\infty): \forall x \in (x_0-\delta_2, x_0+\delta_2) \cap A: f(x) \leq f(x_0)$

Choose $\delta_1, \delta_2 \in (0,+\infty)$ such that

$$\begin{cases} (x_0-\delta_1, x_0+\delta_1) \subseteq A \\ \forall x \in (x_0-\delta_2, x_0+\delta_2) \cap A: f(x) \leq f(x_0) \end{cases}$$

Define $\delta = \min\{\delta_1, \delta_2\}$ and define

$$\forall x, x_0 \in A: \lambda(x, x_0) = \frac{f(x)-f(x_0)}{x-x_0}$$

Since

$(x_0-\delta, x_0+\delta) \subseteq (x_0-\delta_1, x_0+\delta_1) \subseteq A \Rightarrow$

$\Rightarrow (x_0-\delta, x_0+\delta) \subseteq A \Rightarrow (x_0-\delta, x_0+\delta) \cap A = (x_0-\delta, x_0+\delta)$

$\Rightarrow \forall x \in (x_0-\delta, x_0+\delta): f(x) \leq f(x_0)$

$\Rightarrow \forall x \in (x_0-\delta, x_0+\delta): f(x) - f(x_0) \leq 0$

$$\Rightarrow \begin{cases} \forall x \in (x_0-\delta, x_0): \lambda(x, x_0) = \dfrac{f(x)-f(x_0)}{x-x_0} \geq 0 & (1) \\ \forall x \in (x_0, x_0+\delta): \lambda(x, x_0) = \dfrac{f(x)-f(x_0)}{x-x_0} \leq 0 & (2) \end{cases}$$

Since f differentiable at x_0

$$f'(x_0) = \lim_{x \to x_0^-} \lambda(x, x_0) \geq 0 \quad \text{, from Eq. (1)}$$

$$f'(x_0) = \lim_{x \to x_0^+} \lambda(x, x_0) \leq 0 \quad \text{, from Eq. (2)}$$

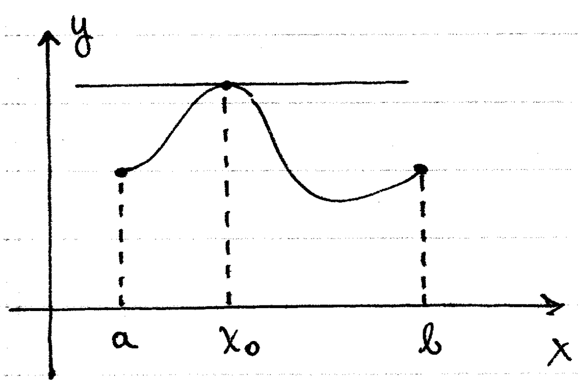
and it follows that $f'(x_0) = 0$. $\square$

(2) → Rolle theorem

Thm: Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and let $\alpha, b \in A$ with $[a,b] \subseteq A$. Then,

$$\left.\begin{array}{l} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \\ f(a) = f(b) \end{array}\right\} \Rightarrow \exists\, x_0 \in (a,b) : f'(x_0) = 0$$

interpretation :



If a function f is continuous on $[a,b]$ and differentiable on (a,b) and if $f(a) = f(b)$, then there is a point $x_0 \in (a,b)$ where the tangent line to the graph of the function becomes horizontal.

Proof

Assume that

$$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \\ f(a) = f(b) \end{cases}$$

Using the Extremum Value Theorem,

f continuous on $[a,b] \Rightarrow$

$\Rightarrow \exists\, x_1, x_2 \in [a,b] : \forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2)$.

Choose $x_1, x_2 \in [a,b]$ such that

$\forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2)$

We distinguish between the following cases.

Case 1: Assume that $x_1 \in (a,b)$. Then

$(\forall x \in [a,b] : f(x) \geq f(x_1)) \Rightarrow x_1$ local min of f (1)

We also know that

$\begin{cases} x_1 \text{ interior to } (a,b) \\ f \text{ differentiable on } (a,b) \end{cases}$ (2)

From Eq.(1) and Eq.(2), via the Fermat theorem:

$f'(x_1) = 0 \Rightarrow \exists x_0 \in (a,b) : f'(x_0) = 0$ (for $x_0 = x_1$)

Case 2: Assume that $x_2 \in (a,b)$. Then

$(\forall x \in [a,b] : f(x) \leq f(x_2)) \Rightarrow x_2$ local max of f (3)

We also know that

$\begin{cases} x_2 \text{ interior to } (a,b) \\ f \text{ differentiable on } (a,b) \end{cases}$ (4)

From Eq.(3) and Eq.(4), via the Fermat theorem:

$f'(x_2) = 0 \Rightarrow \exists x_0 \in (a,b) : f'(x_0) = 0$ (for $x_0 = x_2$).

Case 3: Assume that $x_1 = a \wedge x_2 = b$.

We define $c = f(a) = f(b)$. Then:

$\forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2)$

$\Rightarrow \forall x \in [a,b] : f(a) \leq f(x) \leq f(b)$

$\Rightarrow \forall x \in [a,b] : c \leq f(x) \leq c$

$\Rightarrow \forall x \in [a,b] : f(x) = c$

$\Rightarrow \forall x \in [a,b] : f'(x) = c$

$\Rightarrow \exists x_0 \in [a,b] : f'(x_0) = c.$

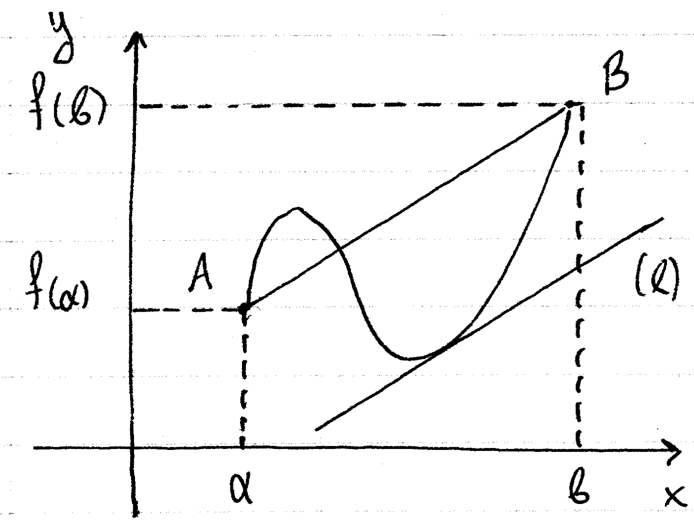
In all cases we conclude that $\exists x_0 \in [a,b] : f'(x_0) = c$.

(3) → Mean Value Theorem

Thm: (Lagrange's Mean Value Theorem)
Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ be a function and let $a, b \in A$ such that $[a,b] \subseteq A$. Then

$$\left.\begin{array}{l} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{array}\right\} \Rightarrow \exists\, x_0 \in (a,b) : f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

Interpretation:



If the function f is continuous on $[a,b]$ and differentiable on (a,b), then given the points $A(a, f(a))$ and $B(b, f(b))$ on the graph of f, there is at least one $x_0 \in (a,b)$ such that the tangent line (ℓ) at $x = x_0$ to the graph of f satisfies $(\ell) // (AB)$.

Proof

Assume that

$$\left\{\begin{array}{l} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{array}\right.$$

Define

$\forall x \in [a,b]: F(x) = (a-b)f(x) + [f(b)-f(a)]x + [bf(a) - af(b)]$

and note that

f continuous on $[a,b] \Rightarrow F$ continuous on $[a,b]$ (1)

and

f differentiable on $(a,b) \Rightarrow F$ differentiable on (a,b) (2)

with $\forall x \in (a,b): F'(x) = (a-b)f'(x) - [f(a)-f(b)]$ (3)

We also have

$$\begin{aligned}
F(a) &= (a-b)f(a) + [f(b)-f(a)]a + [bf(a) - af(b)] = \\
&= (a-b)f(a) + af(b) - af(a) + bf(a) - af(b) = \\
&= (a-b-a+b)f(a) + (a-a)f(b) = \\
&= 0f(a) + 0f(b) = 0 \qquad (4)
\end{aligned}$$

and

$$\begin{aligned}
F(b) &= (a-b)f(b) + [f(b)-f(a)]b + [bf(a) - af(b)] = \\
&= (a-b)f(b) + bf(b) - bf(a) + bf(a) - af(b) = \\
&= (-b+b)f(a) + (a-b+b-a)f(b) = \\
&= 0f(a) + 0f(b) = 0 \qquad (5)
\end{aligned}$$

From Eq.(4) and Eq.(5): $F(a) = F(b) = 0$ (6).

From Eq. (1) and Eq.(2) and Eq.(6), via the Rolle theorem:

$$\left\{\begin{array}{l} F \text{ continuous on } [a,b] \\ F \text{ differentiable on } (a,b) \\ F(a) = F(b) \end{array}\right. \Rightarrow \exists x_0 \in (a,b): F'(x_0) = 0$$

$$\begin{aligned}
&\Rightarrow \exists x_0 \in (a,b): (a-b)f'(x_0) - [f(a)-f(b)] = 0 \\
&\Rightarrow \exists x_0 \in (a,b): (b-a)f'(x_0) = f(b)-f(a) \\
&\Rightarrow \exists x_0 \in (a,b): f'(x_0) = \frac{f(b)-f(a)}{b-a} \qquad \square
\end{aligned}$$

<u>Remark</u>: During the early development of Calculus, many arguments were based on the concept of the <u>linear approximation</u>

$$f(x+\Delta x) \approx f(x) + \Delta x f'(x)$$

where Δx is very small relative to x (i.e. $\Delta x \ll x$). The linear approximation assumes that the graph of the function f in the interval $[x, x+\Delta x]$ is approximately a straight line as long as Δx is small enough, and can be therefore represented by a linear function with respect to Δx. The linear approximation can be used to argue, e.g. that if a function has $f'(x) > 0$, then it is increasing from x to $x+\Delta x$. The problem is that such arguments are not rigorous because they are based on a statement that is true only approximately.

According to the Mean Value Theorem, if f satisfies

$$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{cases} \quad \text{with } a=x \text{ and } b=x+\Delta x$$

then we conclude that

$$\exists x_0 \in (x, x+\Delta x) : f(x+\Delta x) = f(x) + \Delta x f'(x_0)$$

It follows that the linear approximation statement becomes exact if we replace $f'(x)$ with $f'(x_0)$ for some choice of $x_0 \in (x, x+\Delta x)$. This in turn makes it possible to formulate rigorous arguments based on the overall linear approximation concept.

Immediate corollaries of the Mean Value Theorem

The following theorems are immediate consequences of the Mean Value Theorem. We use the assumption that a set $I \subseteq \mathbb{R}$ is an interval, as opposed to a union of disjoint intervals (e.g. $I = [a, b]$ or $I = (a, b]$ or $I = [a, b)$ etc....). A practical definition that encompasses all possibilities is the following:

Def: Let $I \subseteq \mathbb{R}$. We say that
$$I \text{ interval} \Leftrightarrow \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow [x_1, x_2] \subseteq I)$$

We also define the concept of a constant function:

Def: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $I \subseteq A$. We say that
$$f \text{ constant on } I \Leftrightarrow \forall x_1, x_2 \in I : f(x_1) = f(x_2)$$

We will now show that

Thm: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $I \subseteq A$. Then:
$$\begin{cases} I \text{ interval} \\ f \text{ differentiable on } I \\ \forall x \in I : f'(x) = 0 \end{cases} \Rightarrow f \text{ constant on } I.$$

Proof

Assume that

$$\begin{cases} I \text{ interval} \\ f \text{ differentiable on } I \\ \forall x \in I : f'(x) = 0 \end{cases}$$

▸ We will show that $\forall x_1, x_2 \in I : f(x_1) = f(x_2)$.

Let $x_1, x_2 \in I$ be given and assume with no loss of generality that $x_1 < x_2$. Then

$$\begin{cases} I \text{ interval} \\ x_1, x_2 \in I \wedge x_1 < x_2 \end{cases} \Rightarrow [x_1, x_2] \subseteq I$$

and therefore:

f differentiable on $I \Rightarrow f$ differentiable on $[x_1, x_2] \Rightarrow$

$$\Rightarrow \begin{cases} f \text{ continuous on } [x_1, x_2] \\ f \text{ differentiable on } (x_1, x_2) \end{cases}$$

$$\Rightarrow \exists x_0 \in (x_1, x_2) : f'(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Choose $x_0 \in (x_1, x_2)$ such that $f'(x_0) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

It follows that

$$\begin{aligned} f(x_2) - f(x_1) &= f'(x_0)(x_2 - x_1) \\ &= 0(x_2 - x_1) \qquad [\text{via } \forall x \in I : f'(x) = 0] \\ &= 0 \Rightarrow f(x_1) = f(x_2) \end{aligned}$$

and therefore:

$(\forall x_1, x_2 \in I : f(x_1) = f(x_2)) \Rightarrow$

$\Rightarrow f$ constant on I.

<u>Thm</u>: Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $I \subseteq A$. Then:

$$\left.\begin{cases} I \text{ interval} \\ f, g \text{ differentiable on } I \\ \forall x \in I: f'(x) = g'(x) \end{cases}\right. \Rightarrow \exists c \in \mathbb{R}: \forall x \in I: f(x) = g(x) + c$$

<u>Proof</u>

Assume that

$$\begin{cases} I \text{ interval} & (1) \\ f, g \text{ differentiable on } I \\ \forall x \in I: f'(x) = g'(x) \end{cases}$$

Define $\forall x \in I: h(x) = f(x) - g(x)$. Then

f, g differentiable on $I \Rightarrow h$ differentiable on I (2)

with

$$\forall x \in I: h'(x) = [f(x) - g(x)]' = f'(x) - g'(x)$$
$$= f'(x) - f'(x) = 0 \qquad (3)$$

From Eq.(1), Eq.(2), Eq.(3):

h constant on $I \Rightarrow \exists c \in \mathbb{R}: \forall x \in I: h(x) = c$

$\Rightarrow \exists c \in \mathbb{R}: \forall x \in I: f(x) - g(x) = c$

$\Rightarrow \exists c \in \mathbb{R}: \forall x \in I: f(x) = g(x) + c$

Method - Examples

① To show that an equation has a unique solution (i.e. $f(x)=0$) in (a,b).

- •1 Use the Bolzano theorem to establish EXISTENCE of a solution $x_0 \in (a,b)$.
- •2 Show that $f'(x) \neq 0$, $\forall x \in (a,b)$
- •3 Assume there are two solutions $x_0, x_1 \in (a,b)$ with $x_0 \neq x_1$ and use the Rolle theorem to reach a contradiction.

EXAMPLES

a) Show that $x^3-3x+1=0$ has a unique solution at $(-1,1)$

Solution

• Existence: Let $f(x)=x^3-3x+1$. Then

$$\left.\begin{aligned} f(-1) &= (-1)^3-3(-1)+1=-1+3+1=3 \\ f(1) &= 1^3-3\cdot 1+1=-1 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow f(-1)f(1)=3\cdot(-1)<0 \qquad (1)$$

$$f \text{ continuous at } [-1,1] \qquad (2)$$

From (1) and (2):

$$\exists\, x_0 \in (-1,1) : f(x_0)=0$$

• <u>Uniqueness</u> : Assume that the equation is satisfied by $x_0, x_1 \in (-1,1)$ with $x_0 < x_1$

We note that

$f'(x) = (x^3-3x+1)' = 3x^2-3 = 3(x^2-1) < 0,\ \forall x \in (-1,1) \Rightarrow$

$\Rightarrow f'(x) \neq 0,\ \forall x \in (-1,1).$ (3)

$$\left.\begin{array}{l} \text{Since } f(x_0) = f(x_1) = 0 \\ f \text{ continuous at } [x_0, x_1] \\ f \text{ differentiable at } (x_0, x_1) \end{array}\right\} \Rightarrow$$

$\Rightarrow \exists\, x_2 \in (x_0, x_1) : \underline{f'(x_2) = 0.}$

From (3): $f'(x_2) \neq 0$, thus we have a contradiction.

It follows that the solution x_0 is unique.

B) Show that $x^5 + 2x^3 + 7x + 12 = 0$ has a unique solution in $\mathbb{R}$.

<u>Solution</u>

• <u>Existence</u> : Let $f(x) = x^5 + 2x^3 + 7x + 12$.

We note that:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} (x^5 + 2x^3 + 7x + 12) = \lim_{x\to+\infty} x^5 = +\infty \Rightarrow$$

$\Rightarrow \exists\, b \in (0, +\infty) : f(b) > 0$ (1)

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x^5 + 2x^3 + 7x + 12) = \lim_{x\to-\infty} x^5 = -\infty \Rightarrow$$

$\Rightarrow \exists\, a \in (-\infty, 0) : f(a) < 0$ (2)

From (1) and (2):

$$\left.\begin{array}{l} f(a)f(b)<0 \\ f \text{ continuous at } [a,b] \end{array}\right\} \Rightarrow \exists x_0 \in (a,b): f(x_0)=0 \Rightarrow$$

$\Rightarrow$ x_0 solves the equation.

- <u>Uniqueness</u>: Assume that $x_0, x_1 \in \mathbb{R}$ solve the equation with $x_0 < x_1$. We note that

$$f'(x) = (x^5+2x^3+7x+12)' = 5x^4+6x^2+7 >$$
$$> 5x^4+6x^2 \geq 0, \ \forall x \in \mathbb{R} \Rightarrow$$
$$\Rightarrow \forall x \in \mathbb{R}: f'(x) > 0 \qquad (3)$$

Furthermore:

$$\left.\begin{array}{l} f(x_0)=f(x_1)=0 \\ f \text{ continuous at } [x_0,x_1] \\ f \text{ differentiable at } (x_0,x_1) \end{array}\right\} \Rightarrow \exists x_2 \in (x_0,x_1): \underline{f'(x_2)=0}.$$

From (3): $\underline{f'(x_2)>0}$, so we have a contradiction. It follows that the equation cannot have more than one solution in $\mathbb{R}$.

$\hookrightarrow$ In the above solution we have used the statements:

$$\lim_{x\to+\infty} f(x) = +\infty \Rightarrow \exists a \in (0,+\infty): f(a) > 0$$

$$\lim_{x\to-\infty} f(x) = -\infty \Rightarrow \exists a \in (-\infty,0): f(a) < 0$$

which are immediate consequences of the limit definition. More generally:

$$\lim_{x\to\sigma} f(x) = L \Rightarrow \exists a \in N(\sigma,\delta): f(a) \in I(L,\varepsilon)$$

② Inequalities : In general, using the Mean Value Theorem, an inequality satisfied by $f'(x)$ implies an inequality satisfied by $f(x)$.

EXAMPLES

a) Let f be a function differentiable in $\mathbb{R}$. Show that if $\forall x \in \mathbb{R}: 3 \leq f'(x) \leq 5$, then $18 \leq f(8) - f(2) \leq 30$.

Solution

f differentiable in $\mathbb{R} \Rightarrow$ MVT applies on $[2,8] \Rightarrow$

$\Rightarrow \exists x_0 \in (2,8) : f(8) - f(2) = f'(x_0)(8-2) = 6f'(x_0)$ (1)

It follows that

$3 \leq f'(x) \leq 5, \forall x \in \mathbb{R} \Rightarrow 3 \leq f'(x_0) \leq 5 \Rightarrow$

$\Rightarrow 18 \leq 6f'(x_0) \leq 30 \Rightarrow 18 \leq f(8) - f(2) \leq 30.$

→ Inequalities involving $\boxed{\text{two}}$ variables can be proved via the Mean Value Theorem if it is possible, with or without, some manipulation, to produce an expression of the form $f(b) - f(a)$. Then we can use:

$$f(b) - f(a) = f'(x_0)(b-a)$$

for some $x_0 \in (a,b)$.

b) Show that:

$$0 < a < b < \pi/2 \Rightarrow \frac{a}{b} < \frac{\sin a}{\sin b}$$

Solution

Since $0 < a < b < \pi/2 \Rightarrow b\sin b > 0$ and $ab > 0$.
It follows that

$$\frac{a}{b} < \frac{\sin a}{\sin b} \overset{*}{\Leftrightarrow} \frac{a}{b}(b\sin b) < \frac{\sin a}{\sin b}(b\sin b) \Leftrightarrow$$

$$\Leftrightarrow a\sin b < b\sin a \Leftrightarrow a\sin b - b\sin a < 0 \overset{*}{\Leftrightarrow}$$

$$\Leftrightarrow \frac{a\sin b - b\sin a}{ab} < 0 \Leftrightarrow \frac{\sin b}{b} - \frac{\sin a}{a} < 0 \quad (1).$$

Define $f(x) = \frac{\sin x}{x}$. It follows that:

$$f'(x) = \left(\frac{\sin x}{x}\right)' = \frac{(\sin x)'x - \sin x\,(x)'}{x^2} =$$

$$= \frac{x\cos x - \sin x}{x^2}$$

Since:

$\left.\begin{array}{l} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } [a,b] \end{array}\right\} \Rightarrow$ The Mean-Value-Theorem applies on $[a,b] \Rightarrow$

$$\Rightarrow \exists x_0 \in (a,b) : f(b) - f(a) = f'(x_0)(b-a) \Rightarrow$$

$$\Rightarrow \frac{\sin b}{b} - \frac{\sin a}{a} = f(b) - f(a) = f'(x_0)(b-a) =$$

$$= \frac{x_0\cos x_0 - \sin x_0}{x_0^2} \cdot (b-a) =$$

$$= \frac{(x_0 \cos x_0 - \sin x_0)(b-a)}{x_0^2} \quad (2)$$

Note that

$a < b \Rightarrow b - a > 0 \quad (3)$

and $x_0^2 > 0 \quad (4)$

and

$$\left.\begin{array}{l} |\tan x_0| > |x_0| \\ x_0 \in (0, \pi/2) \end{array}\right\} \Rightarrow \tan x_0 > x_0 \Rightarrow \frac{\sin x_0}{\cos x_0} > x_0 \Rightarrow$$

$$\Rightarrow \sin x_0 > x_0 \cos x_0 \Rightarrow x_0 \cos x_0 - \sin x_0 < 0 \quad (5)$$

From (2),(3),(4),(5):

$$\frac{\sin b}{b} - \frac{\sin a}{a} < 0 \underset{(1)}{\Rightarrow} \frac{a}{b} < \frac{\sin a}{\sin b} \quad \square$$

↳ Note the 3-step process:

- 1 Reduce the inequality to be shown to an equivalent simpler inequality that exposes the $f(b) - f(a)$ expression
- 2 Define $f(x)$ and calculate $f'(x)$.
- 3 Apply the MVT and establish a relation between f and f'.
- 4 Determine if $f'(x_0)$ is positive or negative and backtrack your way back to the original inequality.

↳ Also recall the inequalities:

$|\tan x| > |x|, \forall x \in (-\pi/2, 0) \cup (0, \pi/2)$

$|\sin x| < |x|, \forall x \in \mathbb{R} - \{0\}$.

EXERCISES

① Use the Bolzano and Rolle theorems to show that the following equations have a unique solution in the corresponding sets:

a) $\frac{\cos x}{2} + \frac{1}{(1+x)^2} = 0$ at $(2\pi, 3\pi)$

b) $\cos x = x$ at $(0, \pi)$

c) $x^3 - 9x^2 + 24x - 1 = 0$ at $(0, 2)$

d) $x^5 + x^3 + x = a^2(b-x) + b^2(c-x) + c^2(a-x)$ at $\mathbb{R}$

e) $x^5 + 2x^3 + 7x + 12 = 0$ at $\mathbb{R}$

f) $x^3 - 3x + 1 = 0$ at $(-1, 1)$

② Show that the equations $x^2 = x\sin x + \cos x$ has only 2 solutions, that are distinct, in the interval $(-\pi, \pi)$.

③ Let f be a function that is twice-differentiable in $\mathbb{R}$, with

$$\forall x \in \mathbb{R} : f''(x) \neq 0.$$

Show that the equation $f(x) = 0$ cannot have more than two distinct solutions in $\mathbb{R}$.

(4) Show that the equation $x^n + ax + b = 0$ has
a) at most 2 real solutions when n even
b) no more that 3 real solutions when n is odd.

(5) Show that the equation $x^n + nx + 1 = 0$ has
a) only one real solution when n is odd
b) at most 2 real solutions when n is even.

(6) Use the mean value theorem to prove the following inequalities:

a) $|\sin a - \sin b| \le |a-b|$ for $a, b \in \mathbb{R}$
b) $n(b-a)a^{n-1} \le b^n - a^n \le n(b-a)b^{n-1}$, for $0<a<b$ and $n \in \mathbb{N}$.
c) $\dfrac{a-b}{\cos^2 b} \le \tan a - \tan b \le \dfrac{a-b}{\cos^2 a}$, for $0<a\le b<\pi/2$
d) $\sin(a+b) < \sin a + b\cos a$, for $0<a<a+b<\pi/2$
e) $\dfrac{\tan a}{\tan b} < \dfrac{b}{a}$, for $0<a<b<\pi/2$
(Hint: Use $f(x) = x\tan x$)

(7) Let f be a function continues with $[a,b]$, differentiable in (a,b), and with $f(a) = f(b)$. Show that there exist $c_1, c_2 \in (a,b)$ such that $f'(c_1) + f'(c_2) = 0$.

Monotonicity and local min/max

Derivatives can be used to determine whether a function f is increasing or decreasing in specific intervals I. We give the following definitions for functions that are strictly increasing or decreasing or weakly increasing or decreasing on some interval

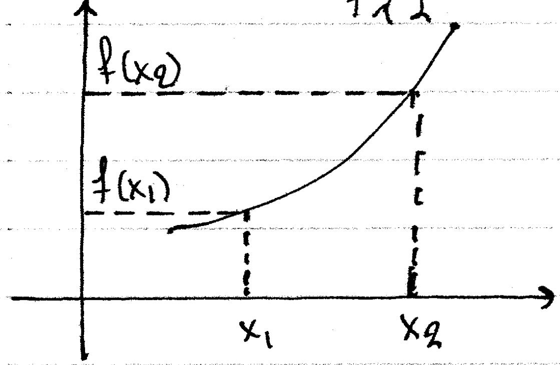
Def: Let $f: A \to \mathbb{R}$ be a function and let $I \subseteq A$ be an interval. Then:

$$f \uparrow I \Leftrightarrow \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2))$$
$$f \downarrow I \Leftrightarrow \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow f(x_1) > f(x_2))$$
$$f \nearrow I \Leftrightarrow \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2))$$
$$f \searrow I \Leftrightarrow \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2))$$

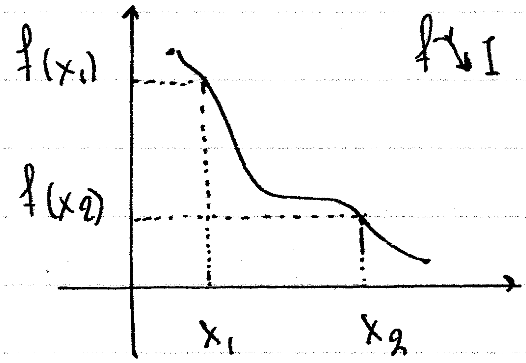
terminology: $f \uparrow I \longrightarrow f$ strictly increasing on I
$f \downarrow I \longrightarrow f$ strictly decreasing on I
$f \nearrow I \longrightarrow f$ increasing on I
$f \searrow I \longrightarrow f$ decreasing on I

interpretation

$f \uparrow I$



If $f \uparrow I$ then for any $x_1, x_2 \in I$, if $x_1 < x_2$ then $f(x_1) < f(x_2)$. Similar statements apply to the other 3 definitions.



Preliminaries

Recall that the general characterization of an interval $I \subseteq \mathbb{R}$ is given by the equivalence

$$I \text{ interval} \Leftrightarrow \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow [x_1, x_2] \subseteq I)$$

In the context of discussing the Fermat theorem, we also introduced the interior $\text{int}(A)$ of a set A as:

$$\text{int}(A) = \{x_0 \in A \mid \exists \delta \in (0, +\infty) : (x_0 - \delta, x_0 + \delta) \subseteq A\}$$

Applying this definition to intervals, it is easy to show that:

$$(A = [a,b] \lor A = [a,b) \lor A = (a,b] \lor A = (a,b)) \Rightarrow \text{int}(A) = (a,b)$$

$$(A = [a,+\infty) \lor A = (a,+\infty)) \Rightarrow \text{int}(A) = (a,+\infty)$$

$$(A = (-\infty, a] \lor A = (-\infty, a)) \Rightarrow \text{int}(A) = (-\infty, a)$$

An important property of the interior, needed to prove the main theorems, is:

Lemma: Let A, B be two sets such that $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. Then:

$$A \subseteq B \Rightarrow \text{int}(A) \subseteq \text{int}(B).$$

(I) → Monotonicity theorem

Thm: Let $f: A \to \mathbb{R}$ be a function and let $I \subseteq A$ be an interval. Then:

$$\left.\begin{array}{l} f \text{ continuous on } I \\ f \text{ differentiable on } \operatorname{int}(I) \\ \forall x \in \operatorname{int}(I): f'(x) > 0 \end{array}\right\} \Rightarrow f \nearrow I$$

$$\left.\begin{array}{l} f \text{ continuous on } I \\ f \text{ differentiable on } \operatorname{int}(I) \\ \forall x \in \operatorname{int}(I): f'(x) < 0 \end{array}\right\} \Rightarrow f \searrow I$$

Remark: Note that differentiability implies continuity. Also, note that there are 8 possible types of intervals, and therefore 16 corresponding statements. For example, for $I = (a, b]$ we obtain:

$$\left.\begin{array}{l} f \text{ differentiable on } (a,b) \\ f \text{ continuous on } x_0 = b \\ \forall x \in (a,b): f'(x) > 0 \end{array}\right\} \Rightarrow f \nearrow (a,b]$$

There are 15 other similar statements.

Proof

Assume that, given an interval I:

$$\left\{\begin{array}{l} f \text{ continuous on } I \\ f \text{ differentiable on } \operatorname{int}(I) \\ \forall x \in \operatorname{int}(I): f'(x) > 0 \end{array}\right.$$

Let $x_1, x_2 \in I$ be given and assume that $x_1 < x_2$. Then:

$$\begin{cases} x_1 < x_2 \land I \text{ interval} \\ f \text{ continuous on } I \end{cases} \Rightarrow \begin{cases} [x_1, x_2] \subseteq I \\ f \text{ continuous on } I \end{cases} \Rightarrow$$

$$\Rightarrow f \text{ continuous on } [x_1, x_2] \quad (1)$$

and

$$\begin{cases} [x_1, x_2] \subseteq I \\ f \text{ differentiable on } \mathrm{int}(I) \end{cases} \Rightarrow \begin{cases} \mathrm{int}([x_1, x_2]) \subseteq \mathrm{int}(I) \\ f \text{ differentiable on } \mathrm{int}(I) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} (x_1, x_2) \subseteq \mathrm{int}(I) \\ f \text{ differentiable on } I \end{cases}$$

$$\Rightarrow f \text{ differentiable on } (x_1, x_2) \quad (2)$$

From Eq.(1) and Eq.(2), via the Mean Value Theorem, we have:

$$\exists x_0 \in (x_1, x_2) : f'(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Choose some $x_0 \in (x_1, x_2)$ such that $f'(x_0) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

Then:

$$\begin{aligned} x_0 \in (x_1, x_2) &\Rightarrow x_0 \in \mathrm{int}([x_1, x_2]) \\ &\Rightarrow x_0 \in \mathrm{int}(I) \quad [\text{via } [x_1, x_2] \subseteq I \text{ and lemma}] \\ &\Rightarrow f'(x_0) > 0 \quad [\text{hypothesis}] \\ &\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \\ &\Rightarrow f(x_2) - f(x_1) > 0 \quad [\text{via } x_2 - x_1 > 0] \\ &\Rightarrow f(x_1) < f(x_2) \end{aligned}$$

We have thus shown:

$$(\forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2))) \Rightarrow f \nearrow I$$

A similar argument proves the second statement.

② → Local min/max — 1st derivative test

Thm: Let $f: A \to \mathbb{R}$ be a function, let $\delta \in (0, +\infty)$, and let $x_0 \in \text{int}(A)$ such that

$$\begin{cases} f \text{ differentiable on } (x_0-\delta, x_0) \cup (x_0, x_0+\delta) \\ f \text{ continuous on } x_0 \end{cases}$$

Then, it follows that

$$\begin{cases} \forall x \in (x_0-\delta, x_0): f'(x) < 0 \\ \forall x \in (x_0, x_0+\delta): f'(x) > 0 \end{cases} \Rightarrow x_0 \text{ local minimum of } f$$

$$\begin{cases} \forall x \in (x_0-\delta, x_0): f'(x) > 0 \\ \forall x \in (x_0, x_0+\delta): f'(x) < 0 \end{cases} \Rightarrow x_0 \text{ local maximum of } f$$

$$\begin{cases} \forall x \in (x_0-\delta, x_0): f'(x) < 0 \\ \forall x \in (x_0, x_0+\delta): f'(x) < 0 \end{cases} \Rightarrow x_0 \text{ \underline{not} local min/max of } f$$

$$\begin{cases} \forall x \in (x_0-\delta, x_0): f'(x) > 0 \\ \forall x \in (x_0, x_0+\delta): f'(x) > 0 \end{cases} \Rightarrow x_0 \text{ \underline{not} local min/max of } f.$$

Remark: Fermat theorem can be used to eliminate all points that are not local min/max, leaving us with a small set of candidates, that we call critical points. We use the following contrapositive form of the Fermat theorem:

$$\begin{cases} x_0 \in \text{int}(A) \\ f \text{ differentiable on } x_0 \\ f'(x_0) \neq 0 \end{cases} \Rightarrow x_0 \text{ not local min or max}$$

It eliminates most points $x_0 \in A$ from the domain A of the function f. The surviving points that require further investigation are:

a) All x_0 not interior to A (i.e. $x_0 \notin \mathrm{int}(A)$)
b) All x_0 such that f not differentiable on x_0
c) All x_0 with $f'(x_0) = 0$.

These are the <u>critical points</u> of f.

<u>Remark</u>: The above theorem handles all points $x_0 \in \mathrm{int}(A)$ where f continuous on x_0. It covers most critical points for cases (b) and (c). However, if f not continuous on x_0 OR if x_0 not interior to A, then the definition of local min/max needs to be used directly, to determine what happens.

<u>Proof</u>

We will prove the first statement. Assume that

$$\begin{cases} f \text{ differentiable on } (x_0-\delta, x_0) \cup (x_0, x_0+\delta) \\ f \text{ continuous on } x_0 \\ \forall x \in (x_0-\delta, x_0) : f'(x) < 0 \\ \forall x \in (x_0, x_0+\delta) : f'(x) \geq 0 \end{cases}$$

It is sufficient to show that $\forall x \in (x_0-\delta, x_0+\delta) : f(x) \geq f(x_0)$.

Note that from the preceding theorem:

$$\begin{cases} f \text{ differentiable on } (x_0-\delta, x_0) \\ f \text{ continuous on } x_0 \\ \forall x \in (x_0-\delta, x_0) : f'(x) < 0 \end{cases} \Longrightarrow$$

$$\Rightarrow \begin{cases} f \text{ differentiable on } (x_0-\delta, x_0) \\ f \text{ continuous on } (x_0-\delta, x_0] \\ \forall x \in (x_0-\delta, x_0) : f'(x) < 0 \end{cases} \Rightarrow \underline{f \searrow (x_0-\delta, x_0]}$$

and

$$\begin{cases} f \text{ differentiable on } (x_0, x_0+\delta) \\ f \text{ continuous on } x_0 \\ \forall x \in (x_0, x_0+\delta) : f'(x) > 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f \text{ differentiable on } (x_0, x_0+\delta) \\ f \text{ continuous on } [x_0, x_0+\delta) \\ \forall x \in (x_0, x_0+\delta) : f'(x) > 0 \end{cases} \Rightarrow \underline{f \nearrow [x_0, x_0+\delta)}$$

Let $x \in (x_0-\delta, x_0+\delta)$ be given. We distinguish between the following cases:

<u>Case 1</u>: Assume $x = x_0$. Then

$$f(x) = f(x_0) \Rightarrow f(x) \geq f(x_0)$$

<u>Case 2</u>: Assume $x \in (x_0-\delta, x_0)$. Then:

$$\begin{aligned} x \in (x_0-\delta, x_0) &\Rightarrow x < x_0 \\ &\Rightarrow f(x) > f(x_0) \quad [\text{via } f \searrow (x_0-\delta, x_0]] \\ &\Rightarrow f(x) \geq f(x_0) \end{aligned}$$

<u>Case 3</u>: Assume $x \in (x_0, x_0+\delta)$. Then:

$$\begin{aligned} x \in (x_0, x_0+\delta) &\Rightarrow x > x_0 \\ &\Rightarrow f(x) > f(x_0) \quad [\text{via } f \nearrow [x_0, x_0+\delta)] \\ &\Rightarrow f(x) \geq f(x_0). \end{aligned}$$

From the above argument, it follows that

$\forall x \in (x_0-\delta, x_0+\delta) : f(x) \geq f(x_0)$

$\Rightarrow x_0$ local min of f.

Similar proofs apply to the other 3 statements.

▸ <u>Methodology</u>: In general, to find the monotonicity and local max/min of a function,

we work as follows:

- 1 Find the domain A of f (if not given)
- 2 Calculate <u>and factor</u> $f'(x)$. Check whether the domain of f' includes all points of A.
- 3 Make a table of signs for $f'(x)$.
- 4 Use the sign table to deduce the monotonicity of f.
- 5 Use the monotonicity to deduce the local min/max.

<u>Critical points</u>

Recall that possible local min/max can be found only among the following critical points:

1) Points x_0 with $f'(x_0) \stackrel{?}{=} 0$
2) Endpoints of closed intervals of A (i.e. $x_0 \notin \text{int}(A)$)
   example: For $A = [1,2) \cup [3,+\infty)$
   possible local max/min may exist at $x_0 = 1$ and $x_0 = 3$.
3) Points x_0 where $f'(x_0)$ is not defined
   (i.e. f not differentiable on x_0).

EXAMPLE

a) Determine the monotonicity and local min/max for the function

$\forall x \in \mathbb{R}: f(x) = (x-1)^2(x+2)^3$

Solution

• Domain: No restrictions, so domain of f is $A = \mathbb{R}$.

$\forall x \in \mathbb{R}: f'(x) = [(x-1)^2(x+2)^3]' =$

$= [(x-1)^2]'(x+2)^3 + (x-1)^2[(x+2)^3]'$

$= 2(x-1)(x-1)'(x+2)^3 + (x-1)^2 3(x+2)^2(x+2)' =$

$= 2(x-1)(x+2)^3 + 3(x-1)^2(x+2)^2 =$

$= (x-1)(x+2)^2[2(x+2) + 3(x-1)] =$

$= (x-1)(x+2)^2(2x+4+3x-3) =$

$= (x-1)(x+2)^2(5x+1)$

Since:

x		-2		$-1/5$		1	
$x-1$	$-$		$-$		$-$	0	$+$
$(x+2)^2$	$+$	0	$+$		$+$		$+$
$5x+1$	$-$		$-$	0	$+$		$+$
$f'(x)$	$+$	0	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$		$\nearrow$	max	$\searrow$	min	$\nearrow$

$f \nearrow$ on $(-\infty,-2)$, $(-2,-1/5)$, and $(1,+\infty)$

$f \searrow$ on $(-1/5, 1)$

local max at $x_0 = -1/5$ and local min at $x_0 = 1$.

▸ Note that $x_0 = -2$ is NOT local min or max!

EXAMPLE

Determine the monotonicity and the local min/max for the function $f(x) = x\sqrt{1-2x}$

Solution

- Domain of f.

We require $1-2x \geq 0 \Leftrightarrow -2x \geq -1 \Leftrightarrow x \leq 1/2$ and therefore the domain of f is $A = (-\infty, 1/2]$

- Derivative of f

$$f'(x) = [x\sqrt{1-2x}\,]' =$$
$$= (x)'\sqrt{1-2x} + x\left(\sqrt{1-2x}\right)' =$$
$$= \sqrt{1-2x} + x\,\frac{(1-2x)'}{2\sqrt{1-2x}} =$$
$$= \sqrt{1-2x} + \frac{-2x}{2\sqrt{1-2x}} = \sqrt{1-2x} - \frac{x}{\sqrt{1-2x}} =$$
$$= \frac{(\sqrt{1-2x})^2 - x}{\sqrt{1-2x}} = \frac{(1-2x)-x}{\sqrt{1-2x}} = \frac{1-3x}{\sqrt{1-2x}}, \ \forall x \in (-\infty, 1/2).$$

- Monotonicity and local min/max

x		$1/3$		$1/2$	
$1-3x$	$+$	0	$-$		$-$
$\sqrt{1-2x}$	$+$		$+$	0	
f'	$+$	0	$-$		
f	$\nearrow$	max	$\searrow$	min	

$f \nearrow (-\infty, 1/3)$

$f \searrow (1/3, 1/2)$

local max at $x_0 = 1/3$

local min at $x_0 = 1/2$.

EXAMPLE

Determine the monotonicity and local min/max of the function $f(x) = \dfrac{\sqrt{4-x^2}}{(x+1)^2}$

Solution

• Domain: We require $\begin{cases} 4-x^2 \geq 0 \\ x+1 \neq 0 \end{cases} \Leftrightarrow \begin{cases} (2-x)(2+x) \geq 0 \quad (1) \\ x \neq -1 \end{cases}$

x		-2		$+2$	
$2-x$	$+$		$+$	0	$-$
$2+x$	$-$	0	$+$		$+$
	$-$	0	$+$	0	$-$

and therefore

Eq.(1) $\Leftrightarrow \begin{cases} x \in [-2,2] \\ x \neq -1 \end{cases} \Leftrightarrow x \in [-2,2] - \{-1\}$

so the domain of f is:

$A = [-2,2] - \{-1\} = [-2,-1) \cup (-1,2]$.

• Derivative

$$f'(x) = \left[\frac{\sqrt{4-x^2}}{(x+1)^2}\right]' = \frac{(\sqrt{4-x^2})'(x+1)^2 - \sqrt{4-x^2}\,[(x+1)^2]'}{(x+1)^4}$$

$$= \frac{1}{(x+1)^4}\left[\frac{(4-x^2)'}{2\sqrt{4-x^2}}(x+1)^2 - \sqrt{4-x^2}\; 2(x+1)(x+1)'\right]$$

$$= \frac{1}{(x+1)^4}\left[\frac{-2x}{2\sqrt{4-x^2}}(x+1)^2 - 2(x+1)\sqrt{4-x^2}\right] =$$

$$= \frac{1}{(x+1)^4}\left[\frac{-x(x+1)^2}{\sqrt{4-x^2}} - 2(x+1)\sqrt{4-x^2}\right] =$$

$$= \frac{1}{(x+1)^4}\left[\frac{-x(x+1)^2 - 2(x+1)\left(\sqrt{4-x^2}\right)^2}{\sqrt{4-x^2}}\right] =$$

$$= \frac{1}{(x+1)^4}\,\frac{(x+1)\left[-x(x+1) - 2(4-x^2)\right]}{\sqrt{4-x^2}} =$$

$$= \frac{1}{(x+1)^3}\,\frac{-x^2-x-8+2x^2}{\sqrt{4-x^2}} = \frac{x^2-x-8}{(x+1)^3\sqrt{4-x^2}}, \ \forall x \in (-2,2)-\{-1\}$$

$(x+1)^3$ has zero at -1

$4-x^2$ has zeroes $2, -2$

For x^2-x-8:

$$\Delta = (-1)^2 - 4\cdot 1\cdot(-8) = 1+32 = 33 \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{-(-1)\pm\sqrt{33}}{2\cdot 1} = \frac{1\pm\sqrt{33}}{2} = \begin{cases} (1+\sqrt{33})/2 = x_1 \\ (1-\sqrt{33})/2 = x_2 \end{cases}$$

and note that

$$x_1 - 2 = \frac{1+\sqrt{33}}{2} - 2 = \frac{1+\sqrt{33}-4}{2} = \frac{\sqrt{33}-3}{2} =$$

$$= \frac{\sqrt{33}-\sqrt{9}}{2} > 0 \Rightarrow x_1 > 2$$

and

$$x_2 - (-2) = \frac{1-\sqrt{33}}{2} - (-2) = \frac{1-\sqrt{33}+4}{2} =$$

$$= \frac{5-\sqrt{33}}{2} = \frac{\sqrt{25}-\sqrt{33}}{2} < 0 \Rightarrow x_2 < -2.$$

Thus: $\frac{1-\sqrt{33}}{2} < -2 < -1 < 2 < \frac{1+\sqrt{33}}{2}$

- Monotonicity

x		$(1-\sqrt{33})/2$		-2		-1		2		$(1+\sqrt{33})/2$	
$\sqrt{4-x^2}$				0	+		+	0			
$(x+1)^3$	−		−		−	0	+		+		+
x^2-x-8	+	0	−		−		−		−	0	+
$f'(x)$				‖	+	‖	−	‖			
$f(x)$				min	↗	‖	↘	min			

$f \nearrow (-2,-1)$
$f \searrow (-1,2)$
$x_0 = -2$ local min
$x_0 = 2$ local min

↳ Note that
a) $-2, 2 \in A$ but $f'(-2)$, $f'(2)$ are not defined. Never theless, the function f has local min/max at $-2, 2$!
b) f has singularity at $x_0 = -1$, so there is no local min/max on that point.

(2) → 2nd derivative test

Thm: Let $f: A \to \mathbb{R}$ and $x_0 \in \text{int}(A)$ and $\delta \in (0, +\infty)$ such that

$$\begin{cases} f \text{ twice differentiable on } x_0 \\ f \text{ differentiable on } (x_0-\delta, x_0) \cup (x_0, x_0+\delta) \end{cases}$$

Then:

$f'(x_0) = 0 \wedge f''(x_0) > 0 \Rightarrow x_0$ local min of f

$f'(x_0) = 0 \wedge f''(x_0) < 0 \Rightarrow x_0$ local max of f

Remark: The second derivative test has the major disadvantage that if $f'(x_0) = 0 \wedge f''(x_0) = 0$, then the test is inconclusive and cannot classify x_0 as local maximum or minimum. Furthermore, in some problems calculating f'' may involve a lot of computation. An example that easily breaks the 2nd derivative test is

$f(x) = (x-1)^4, \forall x \in \mathbb{R}$

Then,

$f'(x) = 4(x-1)^3 (x-1)' = 4(x-1)^3$

$f''(x) = 12(x-1)^2 (x-1)' = 12(x-1)^2$

and $f'(1) = 0 \wedge f''(1) = 0$, so $x_0 = 1$ cannot be classified as local minimum and maximum. On the other hand, using the first derivative test:

x		1	
$4(x-1)^3$	$-$	0	$+$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	min	$\nearrow$

thus:

$x_0=1$ is local minimum

<u>Remark</u>: Consequently, the 2nd derivative test is not recommended for polynomials, rational functions, and functions with square roots. It does afford us however with a practical advantage for trigonometric functions that can have an infinite number of critical points.

<u>Proof</u>

Assume that

$$\begin{cases} f \text{ twice differentiable on } x_0 \\ f \text{ differentiable on } (x_0-\delta, x_0)\cup(x_0, \cancel{x_0}+\delta) \\ f'(x_0)=0 \wedge f''(x_0)>0 \end{cases}$$

Since f differentiable on $(x_0-\delta, x_0)\cup(x_0, x_0+\delta)$, we can consider the following limit:

$$\begin{aligned} \lim_{x\to x_0} \frac{f'(x)}{x-x_0} &= \lim_{x\to x_0} \frac{f'(x)-f'(0)}{x-x_0} && [\text{via } f'(x_0)=0] \\ &= f''(x_0) && [\text{via } f \text{ twice differentiable on } x_0] \\ &> 0 && [\text{via } f''(x_0)>0] \end{aligned}$$

$$\Rightarrow \exists\, \delta_1 \in (0,+\infty) : \forall x \in N(x_0;\delta_1) : \frac{f'(x)}{x-x_0} > 0$$

Choose a $\delta_1 \in (0,+\infty)$ that satisfies the above statement and define $\delta_2 = \min\{\delta, \delta_1\}$. It follows that

$$\begin{cases} \forall x \in (x_0 - \delta_2, x_0) : (f'(x)/(x - x_0) > 0 \wedge x - x_0 < 0) \\ \forall x \in (x_0, x_0 + \delta_2) : (f'(x)/(x - x_0) > 0 \wedge x - x_0 > 0) \end{cases}$$

$$\Rightarrow \begin{cases} \forall x \in (x_0 - \delta_2, x_0) : f'(x) < 0 \\ \forall x \in (x_0, x_0 + \delta_2) : f'(x) > 0 \end{cases} \quad (1)$$

and

f differentiable on $(x_0 - \delta_2, x_0) \cup (x_0, x_0 + \delta_2)$ (2)

f twice differentiable on $x_0 \Rightarrow f$ continuous on x_0 (3)

From Eq. (1), Eq. (2), Eq. (3), using the first derivative test, x_0 local minimum of f.

▸ The other case can be proved by a similar argument.

→ The 2nd derivative test is usually avoided because
  a) Calculating $f''(x)$ may be tedious
  b) If $f''(x_0) = 0$, then the test is inconclusive (i.e. totally worthless)

An important EXCEPTION is with trigonometric functions where we have to deal with infinite sets of local min or max points.

EXAMPLE

Find the local min and local max of the function $f(x) = x - \sin(2x)$.

Solution

- 1 No constraints, so domain of f is $A = \mathbb{R}$.
- 2 We have

$$f'(x) = 1 - \cos(2x)(2x)' = 1 - 2\cos(2x), \ \forall x \in \mathbb{R}$$

$$f''(x) = [1 - 2\cos(2x)]' = +2\sin(2x)(2x)' = 4\sin(2x), \ \forall x \in \mathbb{R}.$$

- 3 All possible local min/max are zeroes of f', since f differentiable on $\mathbb{R}$. Note that

$$f'(x) = 0 \Leftrightarrow 1 - 2\cos(2x) = 0 \Leftrightarrow 2\cos(2x) = 1 \Leftrightarrow$$

$$\Leftrightarrow \cos(2x) = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right) \Leftrightarrow$$

$$\Leftrightarrow \exists k \in \mathbb{Z} : (2x = 2k\pi + \pi/3 \lor 2x = 2k\pi - \pi/3)$$

$$\Leftrightarrow \exists k \in \mathbb{Z} : (x = k\pi + \pi/6 \lor x = k\pi - \pi/6)$$

$\Leftrightarrow x \in \{k\pi + \pi/6\ ,\ k\pi - \pi/6 \mid k \in \mathbb{Z}\}$

For $x_0 = k\pi \pm \pi/6$:

$f''(x_0) = 4\sin\left[2(k\pi \pm \pi/6)\right] = 4\sin(2k\pi \pm \pi/3) =$

$= 4\sin(\pm\pi/3) = \pm 4\sin(\pi/3) = \pm 4(\sqrt{3}/2) = \pm 2\sqrt{3}$

and therefore:

for $x_0 = k\pi + \pi/6$; $k \in \mathbb{Z}$:

$f'(x_0) = 0 \wedge f''(x_0) = 2\sqrt{3} > 0 \Rightarrow \forall k \in \mathbb{Z}$: $x_0 = k\pi + \pi/6$ local min

for $x_0 = k\pi - \pi/6$; $k \in \mathbb{Z}$:

$f'(x_0) = 0 \wedge f''(x_0) = -2\sqrt{3} < 0 \Rightarrow$

$\Rightarrow \forall k \in \mathbb{Z}$: $x_0 = k\pi - \pi/6$ local max.

EXERCISES

(8) Analyze the following functions with respect to monotonicity

a) $f(x) = x^3 - 6x$

b) $f(x) = \dfrac{2x+3}{x-2}$

c) $f(x) = \dfrac{x^2+1}{x-1}$

d) $f(x) = x^4 - x^2 + 1$

e) $f(x) = \sqrt{x^2-1}$

f) $f(x) = \sqrt{x^2}$

g) $f(x) = \dfrac{x^3}{x^2-1}$

h) $f(x) = \dfrac{x}{4-x^2}$

(9) Analyze the following functions with respect to monotonicity and find all local min and max points.

a) $f(x) = x^3 - 9x^2 + 5$

b) $f(x) = 2 - 3x^4$

c) $f(x) = (x-2)^4 + 3$

d) $f(x) = (x-5)^3$

e) $f(x) = x + 1/x$

f) $f(x) = \dfrac{x+1}{x^2-2}$

g) $f(x) = x\sqrt{4-x^2}$

h) $f(x) = x^2\sqrt{2x-1}$

i) $f(x) = \dfrac{x^2-3x+2}{x^2+2x+1}$

j) $f(x) = (x-1)^3(2x+1)^2$

k) $f(x) = \dfrac{(3x+1)^2}{(x-2)^3}$

(10) Similarly for the following functions:

a) $f(x) = (3x+2)^3(x-2)^4$
b) $f(x) = x^2(x^2-1)^2$
c) $f(x) = \dfrac{\sqrt{x+1}}{2x-1}$
d) $f(x) = \dfrac{3x+5}{\sqrt{4x+1}}$
e) $f(x) = \sin^2 x$
f) $f(x) = 2\sin x + \cos 2x$
g) $f(x) = 2\sin^2 x - 2\sin x + 3$

(11) Use monotonicity to show that

a) $\ln(1+x) \leqslant x - x^2/2 + x^3/3$, $\forall x \in [0,+\infty)$
b) $\dfrac{1}{3}\tan x + \dfrac{2}{3}\sin x > x$, $\forall x \in (0,\pi/2)$
c) $\dfrac{\sin x}{x} \geqslant \dfrac{2}{\pi}$, $\forall x \in (0,\pi/2]$
d) $x\sin x + \cos x > 1$, $\forall x \in (0,\pi/2]$
e) $\sin x \geqslant x - x^3/6$, $\forall x \in [0,+\infty)$

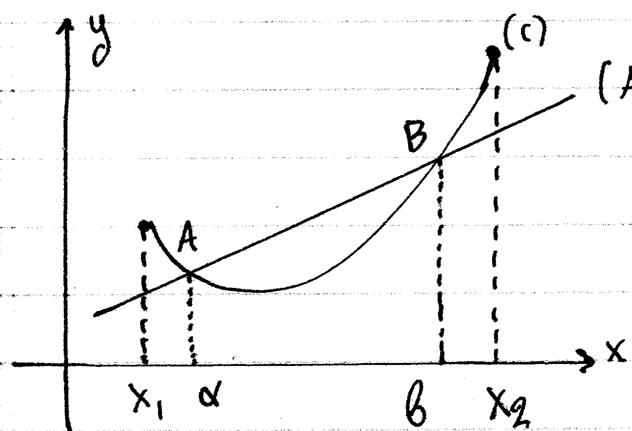
Convexity

The general definition of convexity can be stated geometrically as follows:

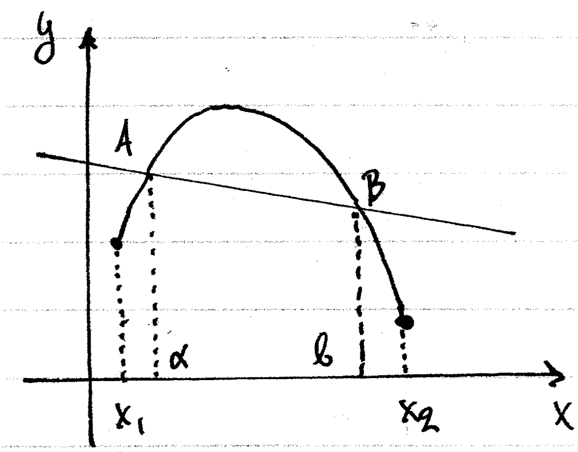
Def : (Geometric definition of convexity)
Let $f : A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and let $[x_1, x_2] \subseteq A$ be an interval. We say that

a) f convex up on $[x_1, x_2] \iff$ for any points $a, b \in [x_1, x_2]$ with $a < b$, the secant line (AB) passing through $A(a, f(a))$ and $B(b, f(b))$ is ABOVE the curve $(c): y = f(x)$ for all $x \in (a, b)$

b) f convex down on $[x_1, x_2] \iff$ for any points $a, b \in [x_1, x_2]$ with $a < b$, the secant line (AB) passing through $A(a, f(a))$ and $B(b, f(b))$ is BELOW the curve $(c): y = f(x)$ for all $x \in (a, b)$.



f convex up on $[x_1, x_2]$



f convex down on $[x_1, x_2]$.

To rewrite an equivalent geometric definition, we note that for all $t \in (0,1)$, a parametric representation of the line segment AB is given by
$x = a + t(b-a)$
and
$$y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a) = f(a) + \frac{f(b)-f(a)}{b-a}t(b-a) =$$
$$= f(a) + t[f(b)-f(a)] = (1-t)f(a) + tf(b)$$
and therefore
$$AB: \begin{cases} x = a + t(b-a) \\ y = (1-t)f(a) + tf(b) \end{cases}, \forall t \in [0,1]$$
Consequently, the claims can be rewritten as follows:
AB is above $(c) \iff \forall t \in (0,1): f(a+t(b-a)) < (1-t)f(a) + tf(b)$
AB is below $(c) \iff \forall t \in (0,1): f(a+t(b-a)) > (1-t)f(a) + tf(b)$
We may therefore write the following equivalent, definition algebraic as follows:

<u>Def</u>: (<u>Algebraic definition of convexity</u>)
Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $[x_1, x_2] \subseteq A$. We say that
f <u>convex up</u> on $[x_1, x_2] \iff$
$$\iff \forall a \in [x_1, x_2): \forall b \in (a, x_2]: \forall t \in (0,1):$$
$$: f(a+t(b-a)) < (1-t)f(a) + tf(b)$$
f <u>convex down</u> on $[x_1, x_2] \iff$
$$\iff \forall a \in [x_1, x_2): \forall b \in (a, x_2]: \forall t \in (0,1):$$
$$: f(a+t(b-a)) > (1-t)f(a) + tf(b)$$

• Properties of convexity

(1) → Convexity and monotonicity of f'

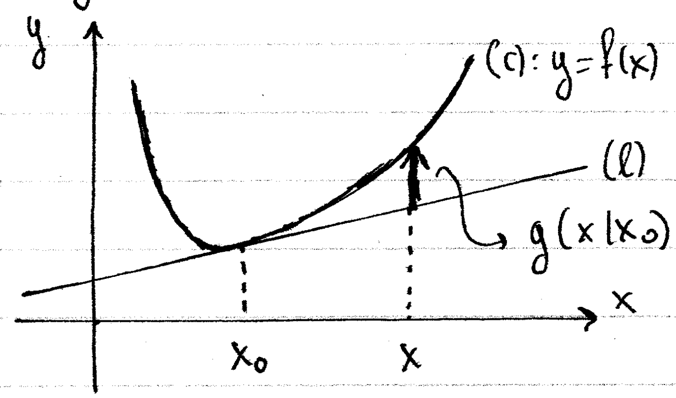
The definition of convexity does not require the function f to be differentiable or even continuous. However, if the function f is in fact differentiable, then we can show that;

> **Thm**: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and $[x_1, x_2] \subseteq A$ such that f differentiable on $[x_1, x_2]$. Then:
> f convex up on $[x_1, x_2] \Leftrightarrow f' \nearrow [x_1, x_2]$
> f convex down on $[x_1, x_2] \Leftrightarrow f' \searrow [x_1, x_2]$

(2) → Tangent line characterization of convexity

Given the function f, let us define $g(x|x_0)$ as:

$$g(x|x_0) = f(x) - [f'(x_0)(x - x_0) + f(x_0)]$$



The interpretation of $g(x|x_0)$ is that it measures the difference in y coordinates between a point $(x, f(x))$ on the graph of the function f and a point with the same x coordinate on the tangent line of the function f

with contact point chosen at $(x_0, f(x_0))$. It follows that:

(a) The graph (c) of the function f is ABOVE the tangent line (l) if and only if
$$\forall x \in [x_1, x_2] - \{x_0\}: g(x|x_0) > 0$$

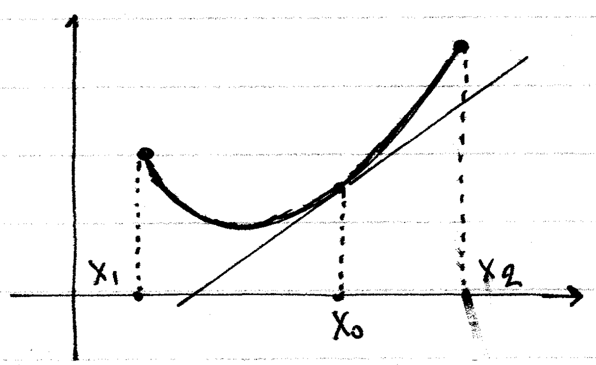
(b) The graph (c) of the function f is BELOW the tangent line (l) if and only if
$$\forall x \in [x_1, x_2] - \{x_0\}: g(x|x_0) < 0$$

The corresponding theorem gives an equivalent characterization of convexity for differentiable functions:

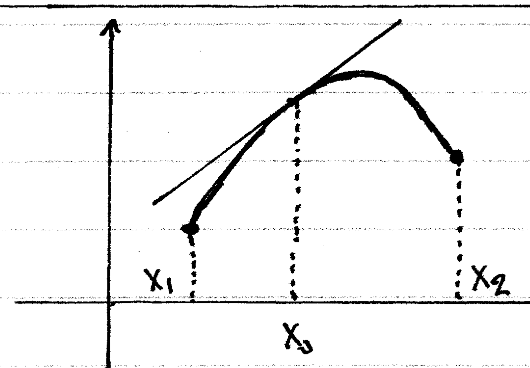
Thm: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $[x_1, x_2] \subseteq A$. Then

f convex up on $[x_1, x_2] \Leftrightarrow \forall x, x_0 \in [x_1, x_2]: (x \neq x_0 \Rightarrow g(x|x_0) > 0)$

f convex down on $[x_1, x_2] \Leftrightarrow \forall x, x_0 \in [x_1, x_2]: (x \neq x_0 \Rightarrow g(x|x_0) < 0)$



f convex up on $[x_1, x_2]$



f convex down on $[x_1, x_2]$.

(3) → Convexity and the 2nd derivative

Thm: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $[x_1, x_2] \subseteq A$.

Then:

$$\left\{\begin{array}{l} f \text{ twice differentiable on } [x_1, x_2] \\ \forall x \in (x_1, x_2): f''(x) > 0 \end{array}\right. \Rightarrow f \text{ convex up on } [x_1, x_2]$$

$$\left\{\begin{array}{l} f \text{ twice differentiable on } [x_1, x_2] \\ \forall x \in (x_1, x_2): f''(x) < 0 \end{array}\right. \Rightarrow f \text{ convex down on } [x_1, x_2]$$

▸ Method for determining convexity

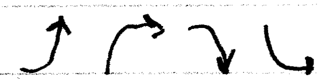
- 1 Calculate and FACTOR $f''(x)$
- 2 Make a sign table for $f''(x)$
- 3 Indicate convexity on table
- 4 Inflection points arises at the points where the convexity changes.

▸ Variation table

The variation table indicates the shape of the curve of the function in more detail.

- 1 We make a monotonicity and convexity table separately.
- 2 We merge the two tables into a table of the form

x
f'
f''
f

where for f we use the notations: 
and where we indicate both the local min/max and the inflection points.

EXAMPLE

Determine the monotonicity, convexity, and make a variation table for the function
$$f(x) = \frac{x^3}{x^2-1}$$

Solution

• Domain.

We require $x^2-1 \neq 0$. Note that
$$x^2-1=0 \Leftrightarrow (x-1)(x+1)=0 \Leftrightarrow x-1=0 \vee x+1=0 \Leftrightarrow$$
$$\Leftrightarrow x=1 \vee x=-1 \Leftrightarrow x \in \{-1,1\}$$
and therefore the domain of f is $A = \mathbb{R} - \{-1,1\}$.

• Derivatives.

We have
$$f'(x) = \left[\frac{x^3}{x^2-1}\right]' = \frac{(x^3)'(x^2-1) - x^3(x^2-1)'}{(x^2-1)^2} =$$
$$= \frac{3x^2(x^2-1) - x^3(2x)}{(x-1)^2(x+1)^2} = \frac{3x^4-3x^2-2x^4}{(x-1)^2(x+1)^2} =$$
$$= \frac{x^4-3x^2}{(x-1)^2(x+1)^2} = \frac{x^2(x^2-3)}{(x-1)^2(x+1)^2} =$$
$$= \frac{x^2(x-\sqrt{3})(x+\sqrt{3})}{(x-1)^2(x+1)^2}, \quad \forall x \in \mathbb{R} - \{-1,1\}$$

and

$$f''(x) = \left[\frac{x^4-3x^2}{(x-1)^2(x+1)^2} \right]' = \left[\frac{x^4-3x^2}{(x^2-1)^2} \right]' =$$

$$= \frac{(x^4-3x^2)'(x^2-1)^2 - (x^4-3x^2)\left[(x^2-1)^2\right]'}{(x^2-1)^4}$$

$$= \frac{(4x^3-6x)(x^2-1)^2 - (x^4-3x^2)\,2\,(x^2-1)(x^2-1)'}{(x^2-1)^4}$$

$$= \frac{(x^2-1)\left[(4x^3-6x)(x^2-1) - 2(x^4-3x^2)(2x)\right]}{(x^2-1)^4}$$

$$= \frac{4x^5-4x^3-6x^3+6x-4x^5+12x^3}{(x^2-1)^3}$$

$$= \frac{(4-4)x^5+(-4-6+12)x^3+6x}{(x-1)^3(x+1)^3}$$

$$= \frac{2x^3+6x}{(x-1)^3(x+1)^3} = \frac{2x(x^2+3)}{(x-1)^3(x+1)^3}, \quad \forall x \in \mathbb{R} - \{1,-1\}$$

- Monotonicity

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
x^2	+		+		+	0	+		+		+
$x-\sqrt{3}$	−		−		−		−		−	0	+
$x+\sqrt{3}$	−	0	+		+		+		+		+
$(x-1)^2$	+		+		+		+	0	+		+
$(x+1)^2$	+		+	0	+		+		+		+
$f'(x)$	+	0	−	∥	−	0	−	∥	−	0	+
$f(x)$	↗	max	↘ $-\infty$	∥	$+\infty$ ↘		↘ $-\infty$	∥	$+\infty$ ↘	min	↗

Local max at $x_0 = -\sqrt{3}$

Local min at $x_0 = \sqrt{3}$

- Convexity

x		-1		0		1	
$2x$	$-$		$-$	0	$+$		$+$
x^2+3	$+$		$+$		$+$		$+$
$(x-1)^3$	$-$		$-$		$-$	0	$+$
$(x+1)^3$	$-$	0	$+$		$+$		$+$
$f''(x)$	$-$	$\|\|$	$+$	0	$-$	$\|\|$	$+$
$f(x)$	$\cap$		$\cup$	infl.	$\cap$		$\cup$

- Variation table

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
$f'(x)$	$+$	0	$-$	$\|\|$	$-$	0	$-$	$\|\|$	$-$	0	$+$
$f''(x)$	$-$		$-$	$\|\|$	$+$	0	$-$	$\|\|$	$+$		$+$
$f(x)$	$\nearrow$	max	$\searrow -\infty$	$\|\|$	$+\infty \searrow$	infl.	$\searrow -\infty$	$\|\|$	$+\infty \searrow$	min	$\nearrow$
				vertical asymptote				vertical asymptote			

EXERCISES

(12) Analyze the following functions with respect to monotonicity and convexity and build the variation table. Locate all local min/max and all inflection points.

a) $f(x) = 4x^3 - 8x^2 + 2$

b) $f(x) = x^3 - 2x^2 + x - 5$

c) $f(x) = x^3 - 6x^2 - 15x$

d) $f(x) = x^4 - 6x^2$

e) $f(x) = (x-2)^5 + 3x + 1$

f) $f(x) = \dfrac{x^3}{x^2 - 1}$

g) $f(x) = \dfrac{x^2 - x}{x^2 + 1}$

h) $f(x) = \dfrac{x^3 - 9x}{x^2 - 1}$

(13) Show that the inflection points of the function

$$f(x) = \frac{a - x}{x^2 + a^2}$$

are all on the same line, for $a \neq 0$.

(14) Find all $a \in \mathbb{R}$ such that the line tangent to the graph of the function $f(x) = x^3 - ax^2$ at its inflection point passes through the point $(0,0)$ (i.e. the origin).

CAL1.6: Exponentials and Logarithms

EXPONENTIALS AND LOGARITHMS

Definition of powers

Although the concept of raising a number to a power is oftentimes taken for granted, a rigorous definition is not easy to construct and can only be done in multiple steps as follows:

① → Powers on $\mathbb{N}$

Let $a \in \mathbb{R}$ and $x \in \mathbb{N}$, noting that $\mathbb{N} = \{0, 1, 2, \ldots\}$. We define a^x as follows:

$$\boxed{\begin{cases} a^0 = 1 \\ \forall x \in \mathbb{N}: \ a^{x+1} = a \cdot a^x \end{cases}}$$

It follows that for $x > 0$:

$$\boxed{a^x = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{x \text{ times}}}$$

Although there is some controversy on whether 0^0 should be presumed to be undefined, no mathematical inconsistencies emerge if we assume that $0^0 = 1$.

An immediate consequence of this definition is that powers satisfy the following properties:

$$\boxed{\begin{aligned}&\forall a\in\mathbb{R}:\forall x_1,x_2\in\mathbb{N}: a^{x_1}a^{x_2}=a^{x_1+x_2}\\&\forall a,b\in\mathbb{R}:\forall x\in\mathbb{N}: (ab)^x=a^x b^x\\&\forall a\in\mathbb{R}:\forall x_1,x_2\in\mathbb{N}: (a^{x_1})^{x_2}=(a^{x_2})^{x_1}=a^{x_1x_2}\end{aligned}}$$

→ Our goal is to now expand the definition of powers while preserving the validity of these three fundamental properties

2 → Powers on $\mathbb{Z}$

Let $a\in\mathbb{R}-\{0\}$ and $x\in\mathbb{N}-\{0\}$. We define negative integer powers via

$$\boxed{\forall a\in\mathbb{R}-\{0\}:\forall x\in\mathbb{N}-\{0\}: a^{-x}=\frac{1}{a^x}}$$

Note that the fundamental 3 properties continue to hold as follows:

$$\boxed{\begin{aligned}&\forall a\in\mathbb{R}-\{0\}:\forall x_1,x_2\in\mathbb{Z}: a^{x_1}a^{x_2}=a^{x_1+x_2}\\&\forall a,b\in\mathbb{R}-\{0\}:\forall x\in\mathbb{Z}: (ab)^x=a^x b^x\\&\forall a\in\mathbb{R}-\{0\}:\forall x_1,x_2\in\mathbb{Z}: (a^{x_1})^{x_2}=(a^{x_2})^{x_1}=a^{x_1x_2}\end{aligned}}$$

We also note that negative powers of 0 cannot be defined. For example, assume that $a = 0^{-x}$ for any $x \in \mathbb{N} - \{0\}$. Then, it follows that

$$a0 = 0^{-x}0^{x} = 0^{-x+x} = 0^0 = 1$$

which is a contradiction. It is possible, however, to define negative integer powers of any nonzero real number $a \in \mathbb{R} - \{0\}$. We stress that this extension of powers to negative integers is unique No other possible extensions exist that would retain consistency with the three fundamental properties given above.

3 → Powers on $\mathbb{Q}$

Let $a \in (0, +\infty)$ and $x = p/q \in \mathbb{Q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N} - \{0\}$. We define a^x as follows:

- 1 We use the Bolzano and Rolle theorems to show that
  q even $\Rightarrow x^q - a = 0$ has a unique solution on $(0, +\infty)$
  q odd $\Rightarrow x^q - a = 0$ has a unique solution on $\mathbb{R}$
  This unique solution is denoted as $x = \sqrt[q]{a}$ thus defining radicals of order q.
- 2 We then use radicals to define

$$\boxed{a^x = a^{p/q} = [\sqrt[q]{a}]^p}$$

This extended definition continues to satisfy the three fundamental properties of powers as follows:

$$\boxed{\begin{array}{l}\forall a \in (0,+\infty) : \forall x_1, x_2 \in \mathbb{Q} : a^{x_1} a^{x_2} = a^{x_1+x_2} \\ \forall a, b \in (0,+\infty) : \forall x \in \mathbb{Q} : (ab)^x = a^x b^x \\ \forall a \in (0,+\infty) : \forall x_1, x_2 \in \mathbb{Q} : (a^{x_1})^{x_2} = (a^{x_2})^{x_1} = a^{x_1 x_2}\end{array}}$$

This is the only possible definition of rational powers that satisfies the above properties. Extending the definition of rational powers to negative numbers results in inconsistency with the fundamental properties of powers. For example:

$$1 = 1^{1/2} = [(-1)(-1)]^{1/2} = (-1)^{1/2}(-1)^{1/2} = (-1)^{1/2+1/2} = (-1)^1 = -1$$

is a contradiction. For this reason, for rational powers a^x, we limit the base a to $a \in (0,+\infty)$.

④ → Real powers

Let $a \in (0,+\infty)$ and $x \in \mathbb{R}$. The final challenge is to define a^x where the exponent x is an arbitrary real number.

We begin by noting that every real number $x \in \mathbb{R}$ can be approximated by a sequence of rational numbers $x_1, x_2, \dots, x_n, \dots \in \mathbb{Q}$. We then say that $\lim_{n \in \mathbb{N}^*} x_n = x$.

e.g. The number $x=\sqrt{2}$ can be approximated by the following sequence of rational numbers:

$x_1 = 1 \qquad x_4 = 1.414$

$x_2 = 1.4 \qquad x_5 = 1.4142$

$x_3 = 1.41 \qquad x_6 = 1.41421$

and we write $\lim\limits_{n\in\mathbb{N}^*} x_n = \sqrt{2}$.

▶ Limit of sequences

Let a_n be a sequence and let $\ell \in \mathbb{R}$. We say that

$$\boxed{\begin{array}{l} \lim\limits_{n\in\mathbb{N}^*} a_n = \ell \Leftrightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N}^* - [n_0] : |a_n - \ell| < \varepsilon \\ (a_n) \text{ convergent} \Leftrightarrow \exists \ell \in \mathbb{R} : \lim\limits_{n\in\mathbb{N}^*} a_n = \ell \end{array}}$$

with $[n_0] = \{1,2,3,\dots,n_0\}$ and $\mathbb{N}^* = \{1,2,3,\dots\}$.
We also note that

$$\boxed{\left.\begin{array}{l} \lim\limits_{n\in\mathbb{N}^*} a_n = x \\ \lim\limits_{x\to x_0} f(x) = y_0 \end{array}\right\} \Rightarrow \lim\limits_{n\in\mathbb{N}^*} f(a_n) = \ell}$$

Let $x_1, x_2, \ldots, x_n, \ldots \in \mathbb{Q}$ be a sequence of rational numbers that approximate $x \in \mathbb{R}$ such that

$$\lim_{n \in \mathbb{N}^*} x_n = x$$

and let $a \in (0, +\infty)$. It can be shown that a^{x_n}, which consists of previously defined rational powers, is a convergent sequence and we define

$$\boxed{a^x = \lim_{n \in \mathbb{N}^*} a^{x_n}}$$

We conclude that real powers satisfy the following properties:

$$\forall x \in \mathbb{R} : a^x > 0$$
$$\forall x_1, x_2 \in \mathbb{R} : a^{x_1} a^{x_2} = a^{x_1 + x_2}$$
$$\forall x \in \mathbb{R} : (ab)^x = a^x b^x$$
$$\forall x_1, x_2 \in \mathbb{R} : (a^{x_1})^{x_2} = (a^{x_2})^{x_1} = a^{x_1 x_2}$$
$$a > 1 \Rightarrow \begin{cases} a^x > 1 & , \forall x \in (0, +\infty) \\ a^x = 1 & , \text{for } x = 0 \\ a^x < 1 & , \forall x \in (-\infty, 0) \end{cases}$$
$$0 < a < 1 \Rightarrow \begin{cases} 0 < a^x < 1 & , \forall x \in (0, +\infty) \\ a^x = 1 & , \text{for } x = 0 \\ a^x > 1 & , \forall x \in (-\infty, 0) \end{cases}$$
$$a > b > 0 \wedge x > 0 \Rightarrow a^x > b^x$$
$$a > b > 0 \wedge x < 0 \Rightarrow a^x < b^x$$

▼ Napier's constant

Recall that we defined Napier's constant as

$$\boxed{e = \lim \left(1 + \frac{1}{n}\right)^n}$$

We now show that e satisfies the following properties:

1) $$\boxed{\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e}$$

Proof

Assume, with no loss of generality, that $x \in (1, +\infty)$.
Define $[x] = \max\{n \in \mathbb{Z} \mid n \leq x\}$.
It follows that $[x] \leq x < [x]+1 \Rightarrow$

$$\Rightarrow \frac{1}{[x]+1} < \frac{1}{x} \leq \frac{1}{[x]}$$

Now, we note that

$$\left(1 + \frac{1}{x}\right)^x \geq \left(1 + \frac{1}{x}\right)^{[x]} > \left(1 + \frac{1}{[x]+1}\right)^{[x]} \quad (1)$$

and

$$\left(1+\frac{1}{x}\right)^x \leqslant \left(1+\frac{1}{[x]}\right)^x < \left(1+\frac{1}{[x]}\right)^{[x]+1} \qquad (2)$$

and

$$\lim_{x\to+\infty}\left(1+\frac{1}{1+[x]}\right)^{[x]} = \lim\left(1+\frac{1}{n+1}\right)^n =$$

$$= \lim \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)} = \frac{\lim\left(1+\frac{1}{n+1}\right)^{n+1}}{\lim\left(1+\frac{1}{n+1}\right)} =$$

$$= \frac{e}{1+0} = e \qquad (3)$$

and

$$\lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]+1} = \lim\left(1+\frac{1}{n}\right)^{n+1} =$$

$$= \lim\left[\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)\right] =$$

$$= \lim\left(1+\frac{1}{n}\right)^n \cdot \lim\left(1+\frac{1}{n}\right) =$$

$$= e\cdot(1+0) = e \qquad (4)$$

Using the squeeze theorem, from (1), (2), (3), (4) we get

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e \qquad \square$$

2) $$\boxed{\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e}$$

Proof

Let $x=-(y+1) \Leftrightarrow y=-x-1$. Then

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y\to+\infty}\left(1-\frac{1}{y+1}\right)^{-(y+1)} =$$

$$= \lim_{y\to+\infty}\left(\frac{y+1-1}{y+1}\right)^{-(y+1)} = \lim_{y\to+\infty}\left(\frac{y}{y+1}\right)^{-(y+1)}$$

$$= \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y+1} = \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y}\cdot\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)$$

$$= e\cdot(1+0) = e \quad \square$$

3) $$\boxed{\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^x = e^a, \ \forall a\in\mathbb{R}}$$

Proof

Distinguish three cases:

Case 1: If $a=0$, then

$$\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^x = \lim_{x\to+\infty} 1^x = 1 = e^0 = e^a.$$

Case 2 : If $a>0$, then

$$\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^{x}=\lim_{x\to+\infty}\left[\left(1+\frac{a}{x}\right)^{x/a}\right]^{a}=$$

$$=\left[\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^{x/a}\right]^{a}=$$

$$=\left[\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y}\right]^{a}=e^{a}$$

for $y=x/a$.

Case 3 : If $a<0$, then

$$\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^{x}=\lim_{x\to+\infty}\left[\left(1+\frac{a}{x}\right)^{x/a}\right]^{a}=$$

$$=\left[\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^{x/a}\right]^{a}= \quad (!!)$$

$$=\left[\lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^{y}\right]^{a}=$$

$$=e^{a} \qquad \square$$

It follows from (3) that e^x can be written as the limit of a sequence:

$$\boxed{e^{x}=\lim\left(1+\frac{x}{n}\right)^{n}}$$

Recall the Bernoulli inequality:

$$\boxed{1+a>0 \Rightarrow \forall n\in\mathbb{N} : (1+a)^n \geq 1+na}$$

We now use it with (3) to show that:

4) $\boxed{e^x \geq x+1 \ , \ \forall x\in\mathbb{R}}$

<u>Proof</u>

Let $x\in\mathbb{R}$ be given. Choose $n_0\in\mathbb{N}$ such that $n_0>-x$. It follows that for

$n>n_0 \Rightarrow n>-x \Rightarrow n+x>0 \Rightarrow 1+\frac{x}{n}>0 \Rightarrow$

$$\Rightarrow \left(1+\frac{x}{n}\right)^n \geq 1+n\cdot\frac{x}{n} = 1+x, \ \forall n>n_0$$

$$\Rightarrow e^x = \lim\left(1+\frac{x}{n}\right)^n \geq 1+x \Rightarrow$$

$$\Rightarrow e^x \geq 1+x. \qquad \square$$

5) $\boxed{\lim\limits_{x\to 0}\frac{e^x-1}{x} = 1}$

<u>Proof</u>

Let $x \in (-1,0) \cup (0,1)$ be given. Then

$$e^x \geq x+1 \Rightarrow e^{-x} \geq 1-x \Rightarrow e^x \leq \frac{1}{1-x} \Rightarrow$$

$$\Rightarrow e^x - 1 \leq \frac{1}{1-x} - 1 = \frac{1-1+x}{1-x} = \frac{x}{1-x}$$

and $e^x - 1 \geq (x+1) - 1 = x$. It follows that

$$x \leq e^x - 1 \leq \frac{x}{1-x} \quad (1)$$

Note that $\lim\limits_{x \to 0} \frac{1}{1-x} = \frac{1}{1-0} = 1$ (2).

For $x \in (0,1)$, from (1) we get

$$1 \leq \frac{e^x - 1}{x} \leq \frac{1}{1-x} \underset{(2)}{\Rightarrow} \lim_{x \to 0^+} \frac{e^x - 1}{x} = 1 \quad (3)$$

For $x \in (-1,0)$, from (1) we get

$$1 \geq \frac{e^x - 1}{x} \geq \frac{1}{1-x} \underset{(2)}{\Rightarrow} \lim_{x \to 0^-} \frac{e^x - 1}{x} = 1 \quad (4)$$

From (3) and (4): $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = 1$

▼ Napier's constant – Flowchart

Definition $\longrightarrow$ $\boxed{e = \lim \left(1 + \frac{1}{n}\right)^n}$

$$\boxed{\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e} \longrightarrow \boxed{\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e}$$

$$\boxed{\lim_{x \to +\infty} \left(1 + \frac{a}{x}\right)^x = e^a}$$

$$\boxed{\lim \left(1 + \frac{a}{n}\right)^n = e^a}$$

$$\boxed{e^x \geqslant x + 1 \,, \quad \forall x \in \mathbb{R}}$$

$$\boxed{\lim_{x \to 0} \frac{e^x - 1}{x} = 1}$$

The natural exponential function

Using the Napier constant e, we define the natural exponential function $\exp : \mathbb{R} \to (0, +\infty)$ such that

$$\forall x \in \mathbb{R} : \exp(x) = e^x = \lim_{n \in \mathbb{N}^*} \left(1 + \frac{a}{n}\right)^n$$

Derivative

$$\boxed{\forall x \in \mathbb{R} : [\exp(x)]' = \exp(x)}$$

Proof

Let $x \in \mathbb{R}$ be given. Then,

$$[\exp(x)]' = \lim_{h \to 0} \frac{\exp(x+h) - \exp(x)}{h} =$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x e^h - e^x}{h} =$$

$$= \lim_{h \to 0} \frac{e^x (e^h - 1)}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x \cdot 1 = e^x \qquad \square$$

Using the chain rule, this differentiation rule can be extended to read:

$$\boxed{[e^{g(x)}]' = g'(x) e^{g(x)}}$$

Limits

Since exp is differentiable on $\mathbb{R}$, it is also continuous on $\mathbb{R}$, therefore:

$$\boxed{\forall x_0 \in \mathbb{R} : \lim_{x \to x_0} e^x = e^{x_0}}$$

We can also show that

$$\boxed{\lim_{x \to +\infty} e^x = +\infty \quad \Big| \quad \lim_{x \to -\infty} e^x = 0}$$

Proof

$$\left.\begin{array}{l} \forall x \in \mathbb{R} : e^x \geq x+1 \\ \lim\limits_{x \to +\infty} (x+1) = \lim\limits_{x \to +\infty} x = +\infty \end{array}\right\} \Rightarrow \lim_{x \to +\infty} e^x = +\infty .$$

Let $x \in (-\infty, 0)$ be given. Then

$$e^{-x} \geq (-x)+1 > 0 \Rightarrow 0 < \frac{1}{1-x} < \frac{1}{e^{-x}} \Rightarrow$$

$$\Rightarrow 0 < e^x \leq \frac{1}{1-x}$$

and therefore $\forall x \in (-\infty, 0) : 0 < e^x \leq \dfrac{1}{1-x}$ (1)

Since

$$\lim_{x \to -\infty} \frac{1}{1-x} = \lim_{x \to -\infty} \frac{1}{-x} = 0 \qquad (2)$$

from Eq.(1) and Eq.(2) it follows that $\lim_{x \to -\infty} e^x = 0$. $\square$

Combining these results with the composition theorem, we obtain:

$$\boxed{\begin{aligned} &\lim_{x\to\sigma} g(x) = a \in \mathbb{R} \Rightarrow \lim_{x\to\sigma} e^{g(x)} = e^a \\ &\lim_{x\to\sigma} g(x) = +\infty \Rightarrow \lim_{x\to\sigma} e^{g(x)} = +\infty \\ &\lim_{x\to\sigma} g(x) = -\infty \Rightarrow \lim_{x\to\sigma} e^{g(x)} = 0 \end{aligned}}$$

Likewise, via the composition theorem, the result that $\lim_{x\to 0} (e^x - 1)/x = 1$, generalizes to:

$$\boxed{\left.\begin{aligned} &\lim_{x\to\sigma} g(x) = 0 \\ &\forall x \in N(\sigma,\delta) \cap \mathrm{dom}(g) : g(x) \neq 0 \end{aligned}\right\} \Rightarrow \lim_{x\to\sigma} \frac{e^{g(x)} - 1}{g(x)} = 1}$$

EXAMPLES

a) Determine the monotonicity and local min/max of the function $f(x) = x^3 \exp(1/x)$.

Solution

- Domain: We require $x \neq 0$, thus $A = \mathbb{R} - \{0\}$
- Derivative

$$
\begin{aligned}
f'(x) &= [x^3 \exp(1/x)]' = \\
&= (x^3)' \exp(1/x) + x^3 [\exp(1/x)]' \\
&= 3x^2 \exp(1/x) + x^3 \exp(1/x) (1/x)' \\
&= 3x^2 \exp(1/x) + x^3 \exp(1/x)(-1/x^2) = \\
&= (3x^2 - x)\exp(1/x) = x(3x-1)\exp(1/x).
\end{aligned}
$$

- Monotonicity

x		0		$1/3$	
x	$-$	0	$+$		$+$
$3x-1$	$-$		$-$	0	$+$
$\exp(1/x)$	$+$	$\parallel$	$+$		$+$
$f'(x)$	$+$	$\parallel$	$-$	0	$+$
$f(x)$	$\nearrow$	$\parallel$	$\searrow$	min	$\nearrow$

Local min at $x = 1/3$

Singular point at $x = 0$.

b) $f(x) = e^{-x}\cos x \sin(3x) \leftarrow$ Evaluate $\lim_{x\to+\infty} f(x)$

Solution

Define $b(x) = \cos x \sin(3x)$, $\forall x \in \mathbb{R}$.

$|b(x)| = |\cos x \sin(3x)| = |\cos x||\sin(3x)| \leqslant 1 \cdot 1 = 1$, $\forall x \in \mathbb{R}$

$\Rightarrow b$ bounded on $\mathbb{R}$. (1)

$\lim_{x\to+\infty} e^{-x} = 0$ (2)

From Eq. (1) and Eq. (2), via the zero-bounded theorem, it follows that

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} e^{-x} b(x) = 0$

c) $f(x) = \dfrac{e^x + 2e^{-x}}{3e^x - e^{-2x}} \leftarrow \lim_{x\to+\infty} f(x)$

Solution

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{e^x + 2e^{-x}}{3e^x - e^{-2x}} = \lim_{x\to+\infty} \frac{e^x[1+2e^{-2x}]}{e^x[3-e^{-3x}]}$$

$$= \lim_{x\to+\infty} \frac{1+2e^{-2x}}{3-e^{-3x}} = \frac{1+2\cdot 0}{3-0} = \frac{1}{3}$$

d) $f(x) = \dfrac{e^{2x} - e^{-2x}}{x} \leftarrow \lim_{x\to 0} f(x)$

Solution

$$f(x) = \frac{e^{2x} - e^{-2x}}{x} = \frac{e^{-2x}\left[e^{4x} - 1\right]}{x} =$$

$$= \left(4e^{-2x}\right)\frac{e^{4x} - 1}{4x} \qquad (1)$$

$$\lim_{x \to 0}\left(4e^{-2x}\right) = 4e^{0} = 4 \qquad (2)$$

$$\left.\begin{array}{l} \lim\limits_{x \to 0}(4x) = 4 \cdot 0 = 0 \\ \forall x \in \mathbb{R} - \{0\} : 4x \neq 0 \end{array}\right\} \Rightarrow \lim_{x \to 0}\frac{e^{4x} - 1}{4x} = 1 \qquad (3)$$

From Eq. (1), Eq. (2), and Eq. (3):

$$\lim_{x \to 0} f(x) = \lim_{x \to 0}\left[\left(4e^{-2x}\right)\frac{e^{4x} - 1}{4x}\right] = 4 \cdot 1 = 4$$

EXERCISES

① Evaluate the following limits, if they exist.

a) $\lim\limits_{x\to 0} e^{\cos x}$

b) $\lim\limits_{x\to -\infty} e^{x^2-3x}$

c) $\lim\limits_{x\to -\infty} e^{x^3-x^2}$

d) $\lim\limits_{x\to 3^-} \exp\left(\dfrac{2x+1}{x-3}\right)$

e) $\lim\limits_{x\to 2} \exp\left(\dfrac{x-3}{(x-2)^2}\right)$

f) $\lim\limits_{x\to +\infty} e^{-2x+1}(\cos 3x+\sin 2x)$

g) $\lim\limits_{x\to -\infty} e^{-x^2}(\sin x\cos x+1)$

h) $\lim\limits_{x\to +\infty} \dfrac{2e^{2x}+3}{e^{2x}+5}$

i) $\lim\limits_{x\to +\infty} \dfrac{e^{x}+e^{-x}-2e^{-2x}}{3e^{x}+1+e^{-x}}$

j) $\lim\limits_{x\to -\infty} \exp\left(\dfrac{x^3+3x}{x^3-3x}\right)$

② Similarly, with the following limits:

a) $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{5x}$

b) $\lim\limits_{x\to 0} \dfrac{e^{3x}-1}{\sin x}$

c) $\lim\limits_{x\to 0} \dfrac{e^{x}-e^{-x}}{\tan x}$

d) $\lim\limits_{x\to 0} \dfrac{e^{ax}-e^{bx}}{x}$, with $a>b\geq 0$

e) $\lim\limits_{x\to 0} \dfrac{e^{\sin x}-1}{2x}$

f) $\lim\limits_{x\to 0} \dfrac{e^{\tan x}-1}{3x}$

(3) Evaluate and FACTOR the derivatives of the following functions:

a) $f(x) = e^{x+\sin x}$
b) $f(x) = e^{\cot(x^2)}$
c) $f(x) = e^{x\tan x}$
d) $f(x) = e^x(x^2+3x+1)$
e) $f(x) = (x\sin x+1)e^{-x^2}$
f) $f(x) = (x+1)^2(x-2)^3e^{-x^2}$
g) $f(x) = \dfrac{(2x+1)^2e^{-x}}{(x-3)^4}$
h) $f(x) = x^2\sqrt{x+1}\cdot e^x$

(4) Analyze the following functions with respect to monotonicity, concavity, and locate all local min/max and inflection points. Show the variation table.

a) $f(x) = x^3e^x$
b) $f(x) = xe^{-x^2}$
c) $f(x) = \dfrac{x^2}{e^x}$
d) $f(x) = xe^{1/x}$
e) $f(x) = \dfrac{e^x-1}{e^x+1}$
f) $f(x) = (2x-1)^3(x+2)^2e^x$
g) $f(x) = \dfrac{(x+1)^2e^{-x}}{(x-1)^2}$

(5) Use monotonicity to show that
$x>0 \Rightarrow e^x(1+x)>1$

(6) Use the Rolle and Bolzano theorems to show that

a) The equation $e^{2x} - e + 2 = 0$ has a unique solution in $\mathbb{R}$.

b) The equation $x^2 e^{2x} = 1 - xe^x$ has a unique solution in $[0, +\infty)$

c) The equation $e^{2x}(2x-1) + x^4 = 0$ has at least one solution and no more than two solutions in $\mathbb{R}$.

(7) Use the mean-value theorem to show that

$$a < b \Rightarrow e^a(b-a) < e^b - e^a < e^b(b-a)$$

(8) Show that the function $f(x) = e^x / x^n$ with $n \in \mathbb{N} - \{0\}$ and domain $A = (0, +\infty)$ has a unique minimum at $x = n$. Use this result to show that

$$e^x \geqslant \left(\frac{ex}{n}\right)^n, \quad \forall x \in [0, +\infty)$$

(9) Let $f : [0, +\infty) \to \mathbb{R}$ with $f(0) = 1$ and $\forall x \in [0, +\infty) : f'(x) \geqslant f(x)$.

a) Analyze the function $g(x) = f(x)/e^x$ with respect to monotonicity.

b) Use (a) to show that

$$\forall x \in [0, +\infty) : f(x) \geqslant e^x .$$

Inverse functions

- Let $f: A \to \mathbb{R}$ be a function. We say that:

$$\boxed{f \text{ one-to-one} \iff \forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)}$$

<u>interpretation</u>: A function f is one-to-one if and only if no horizontal line intersects its graph more than once.

<u>Thm</u>:
$$\boxed{\begin{array}{l} f \nearrow A \Rightarrow f \text{ one-to-one} \\ f \searrow A \Rightarrow f \text{ one-to-one} \end{array}}$$

<u>Proof</u>

Assume, without loss of generality that, $f \nearrow A$.
Let $x_1, x_2 \in A$ be given such that $f(x_1) = f(x_2)$.
If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \longleftarrow$ contradiction
If $x_1 > x_2 \Rightarrow f(x_1) > f(x_2) \longleftarrow$ contradiction.
It follows that $x_1 = x_2$. $\square$

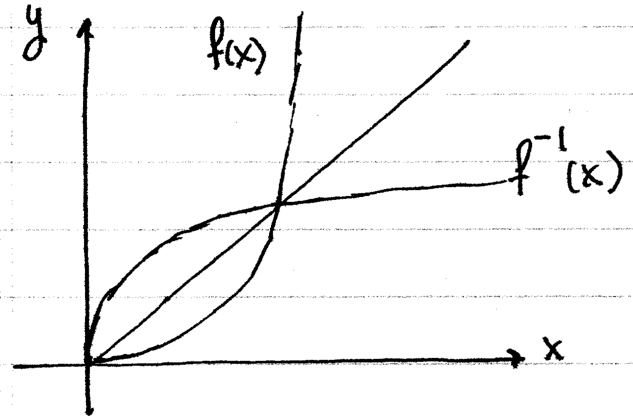
<u>Def</u>: Let $f: A \to \mathbb{R}$ be a one-to-one function. We define the inverse function $f^{-1}: f(A) \to A$ such that

$$\boxed{f^{-1}(x) = y \iff f(y) = x}$$

The immediate consequence of this definition is that

$$\boxed{\begin{array}{l} \forall x \in A: f^{-1}(f(x)) = x \\ \forall x \in f(A): f(f^{-1}(x)) = x \end{array}}$$

The graph of f^{-1} is the reflection of the graph of f across the line $(\ell): y = x$



Method : To find the inverse of a function $f : A \to \mathbb{R}$ we work as follows:

- 1 We setup the equation
$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow \cdots$
- 2 It may be necessary to require restrictions on y to evaluate $f(y)$. If that is the case, then do so.
- 3 Solve for y. During the process, it may be necessary to require restrictions on x to ensure that at least one solution exists. These restrictions define the domain of the inverse function f^{-1}.
- 4 When you show that, under possible restrictions on x, that your equation has a unique solution $y = y_0(x)$, you implicitly prove that both f is one-to-one and that $f^{-1}(x) = y_0(x)$. Thus you have the formula of the inverse function.
- 5 If applicable, check the constraints on y from step 2. They may or may not introduce further restrictions on the variable x and therefore on the domain of the inverse function.

EXAMPLES

a) Find the inverse function of $f(x) = \dfrac{x+3}{2x-5}$

Solution

$$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow \frac{y+3}{2y-5} = x \quad (\text{Require } \underline{2y-5 \neq 0})$$

$$\Leftrightarrow y+3 = x(2y-5) \Leftrightarrow y+3 = 2xy - 5x \Leftrightarrow (1-2x)y = -3-5x \quad (1)$$

For $1-2x=0$: $x = 1/2$, and therefore

$(1) \Leftrightarrow 0y = -3-5\cdot(1/2) \Leftrightarrow 0y = -3-5/2 \longleftarrow$ inconsistent

thus $\underline{x = 1/2 \notin \text{dom}(f^{-1})}$.

For $1-2x \neq 0$:

$$(1) \Leftrightarrow y = \frac{-3-5x}{1-2x}$$

Now we must check the requirement $2y-5 \neq 0$.

We note that:

$$2y-5 = 2\cdot\left(\frac{-3-5x}{1-2x}\right) - 5 = \frac{2(-3-5x)-5}{1-2x} =$$

$$= \frac{-6-10x-5(1-2x)}{1-2x} = \frac{-6-10x-5+10x}{1-2x} =$$

$$= \frac{-11}{1-2x} \neq 0$$

thus $2y-5 \neq 0$ is satisfied.

Thus $f^{-1}(x) = \dfrac{-3-5x}{1-2x}$ with $\mathrm{dom}(f^{-1}) = \mathbb{R} - \{1/2\}$.

↳ In the above example we see that the domain of f^{-1} coincides with the widest possible domain. However, this is not always true, as seen in the next example.

b) Find the inverse function of $f(x) = 2 + \dfrac{\sqrt{3x+1}}{3}$

Solution

$$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow 2 + \frac{\sqrt{3y+1}}{3} = x \Leftrightarrow$$

$$\Leftrightarrow 6 + \sqrt{3y+1} = 3x \Leftrightarrow \sqrt{3y+1} = 3x - 6 \Leftrightarrow \sqrt{3y+1} = 3(x-2) \quad (1)$$

Require $3(x-2) \geqslant 0 \Leftrightarrow x \geqslant 2$, otherwise equation (1) is inconsistent. For $x \geqslant 2$:

$$(1) \Leftrightarrow 3y + 1 = 9(x-2)^2 \Leftrightarrow 3y = 9(x-2)^2 - 1 \Leftrightarrow$$

$$\Leftrightarrow y = 3(x-2)^2 - \frac{1}{3}$$

It follows that

$$f^{-1}(x) = 3(x-2)^2 - 1/3 \quad \text{with } \mathrm{dom}(f^{-1}) = [2, +\infty)$$

↳ In this example we see that the domain $\mathrm{dom}(f^{-1})$ is restricted from the widest possible domain of the polynomial formula for $f^{-1}(x)$ which is $\mathbb{R}$.

Thus, to determine the domain of the inverse function f^{-1}, it is necessary to keep track of all constraints, as I suggested in the methodology.

c) Find the inverse function of $f(x) = 4x - 3$.

Solution

$$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow 4y - 3 = x \Leftrightarrow 4y = x + 3 \Leftrightarrow$$
$$\Leftrightarrow y = \frac{x+3}{4}$$

It follows that:

$f^{-1}(x) = \dfrac{x+3}{4}$ with $\mathrm{dom}(f^{-1}) = \mathbb{R}$ (no constraints).

A property of one-to-one functions

The following property is used later to establish the continuity property of f^{-1}.

$$\boxed{\left.\begin{array}{l} f \text{ one-to-one} \\ I \text{ interval} \\ f \text{ continuous at } I \end{array}\right\} \Rightarrow f \nearrow I \vee f \searrow I}$$

Proof

Assume that not $f \nearrow I$ and not $f \searrow I$. Then there are $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$ such that $f(x_2)$ is not between $f(x_1)$ and $f(x_3)$. Assume, with no loss of generality that $f(x_1) < f(x_3)$. It follows that $f(x_2) \notin [f(x_1), f(x_3)]$. Distinguish two cases:

<u>Case 1</u>: If $\left.\begin{array}{l} f(x_2) < f(x_1) < f(x_3) \\ f \text{ continuous at } [x_2, x_3] \end{array}\right\} \Rightarrow$

$\Rightarrow \exists c \in [x_2, x_3] : f(c) = f(x_1)$ [intermediate value thm]

$\Rightarrow c = x_1$ [f one-to-one]

But $c \geq x_2 > x_1 \Rightarrow c \neq x_1 \longleftarrow$ contradiction.

<u>Case 2</u>: If $\left.\begin{array}{l} f(x_1) < f(x_3) < f(x_2) \\ f \text{ continuous at } [x_1, x_2] \end{array}\right\} \Rightarrow$

$\Rightarrow \exists c \in [x_1, x_2] : f(c) = f(x_3)$ [intermediate value thm]

$\Rightarrow c = x_3$. But $c \leq x_2 < x_3 \Rightarrow c \neq x_3 \longleftarrow$ contradiction.

Thus $f \nearrow I \vee f \searrow I$. $\square$

Properties of inverse functions

1) <u>Monotonicity</u>: Let $f: A \to \mathbb{R}$ be a function.

$$\boxed{\begin{array}{l} f \nearrow A \Rightarrow f^{-1} \nearrow f(A) \\ f \searrow A \Rightarrow f^{-1} \searrow f(A) \end{array}}$$

<u>Proof</u>

Assume, without loss of generality, that $f \nearrow A$.
Let $y_1, y_2 \in f(A)$ be given with $y_1 < y_2$.
Define $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$.
Sufficient to show that $x_1 < x_2$.
Assume that $x_1 \geq x_2$. Then,

$$\begin{aligned} x_1 \geq x_2 &\Rightarrow f(x_1) \geq f(x_2) \quad [\text{because } f \nearrow A] \\ &\Rightarrow f(f^{-1}(y_1)) \geq f(f^{-1}(y_2)) \\ &\Rightarrow y_1 \geq y_2 \leftarrow \text{contradiction.} \end{aligned}$$

It follows that: $x_1 < x_2 \Rightarrow \underline{f^{-1}(y_1) < f^{-1}(y_2)}$

and therefore $f^{-1} \nearrow A$. $\square$

2) <u>Continuity</u>: Let $f: A \to \mathbb{R}$ be a function.

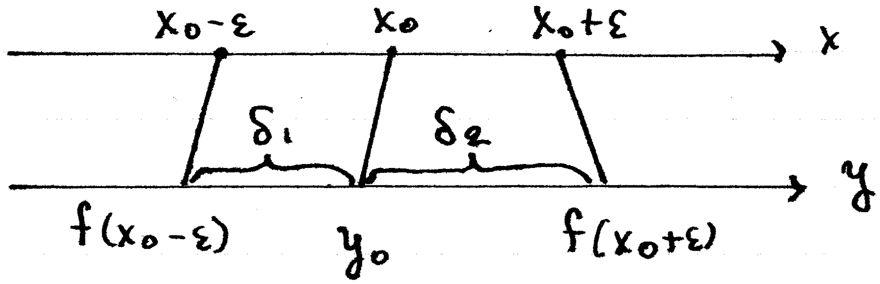
$$\boxed{\left.\begin{array}{l} f \text{ continuous at } A \\ A \text{ interval} \\ f \text{ one-to-one} \end{array}\right\} \Rightarrow f^{-1} \text{ continuous at } f(A)}$$

<u>Proof</u>

Since $\left.\begin{array}{l} f \text{ one-to-one} \\ f \text{ continuous at } A \end{array}\right\} \Rightarrow f \nearrow A \vee f \searrow A$

Assume, with no loss of generality, that $f \nearrow A$.
Let $y_0 \in f(A)$ be given and define $x_0 = f^{-1}(y_0)$
To show $\lim_{y \to y_0} f^{-1}(y) = f^{-1}(y_0)$, it is sufficient
to show that
$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall y \in f(A) : (0 < |y - y_0| < \delta \Rightarrow |f^{-1}(y) - f^{-1}(y_0)| < \varepsilon)$$

Let $\underline{\varepsilon > 0}$ be given such that $(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq A$.
Since $\left.\begin{array}{l} f \text{ continuous at } A \\ f \nearrow A \end{array}\right\} \Rightarrow$
$\Rightarrow f((x_0 - \varepsilon, x_0 + \varepsilon)) = (f(x_0 - \varepsilon), f(x_0 + \varepsilon)) \Rightarrow$
$\Rightarrow f^{-1}((f(x_0 - \varepsilon), f(x_0 + \varepsilon))) = (x_0 - \varepsilon, x_0 + \varepsilon)$



Let $\delta_1 = y_0 - f(x_0-\varepsilon)$ and
$\delta_2 = f(x_0+\varepsilon) - y_0$ and
$\underline{\delta = \min\{\delta_1, \delta_2\}}$

It follows (see figure) that

$$(y_0-\delta, y_0+\delta) \subseteq (f(x_0-\varepsilon), f(x_0+\varepsilon)) \Rightarrow$$
$$\Rightarrow f^{-1}((y_0-\delta, y_0+\delta)) \subseteq f^{-1}((f(x_0-\varepsilon), f(x_0+\varepsilon)))$$
$$= (x_0-\varepsilon, x_0+\varepsilon)$$

consequently:

$$\underline{0 < |y-y_0| < \delta} \Rightarrow y \in (y_0-\delta, y_0+\delta) \Rightarrow$$
$$\Rightarrow f^{-1}(y) \in (x_0-\varepsilon, x_0+\varepsilon)$$
$$\Rightarrow \underline{|f^{-1}(y) - f^{-1}(y_0)|} = |f^{-1}(y) - x_0| \underline{< \varepsilon}$$

It follows that $\forall y_0 \in f(A): \lim\limits_{y \to y_0} f^{-1}(y) = f^{-1}(y_0) \Rightarrow$

$\Rightarrow f^{-1}$ continuous at $f(A)$ $\square$

3) Differentiability : Let $f: A \to \mathbb{R}$, with $f'(f^{-1}(x)) \neq 0$.

$$\boxed{\left.\begin{array}{l} f \text{ differentiable at } A \\ f \text{ one-to-one} \\ A \text{ union of intervals} \end{array}\right\} \Rightarrow f^{-1} \text{ differentiable at } f(A) \quad \text{with} \quad (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}, \ \forall x \in f(A)}$$

Proof

Assume with no loss of generality that A is an interval. We note that

$$\left.\begin{array}{l} f \text{ differentiable at } A \Rightarrow f \text{ continuous at } A \\ \qquad\qquad\qquad\qquad\qquad f \text{ one-to-one} \end{array}\right\} \Rightarrow$$

$\Rightarrow f^{-1}$ continuous at $f(A)$ [using (2)-previous result]

Let $x_0 \in f(A)$ be given. Define $y_0 = f^{-1}(x_0)$.
It follows that $\lim\limits_{x \to x_0} f^{-1}(x) = f^{-1}(x_0) = y_0$.

and consequently:

$$(f^{-1})'(x_0) = \lim_{x \to x_0} \frac{f^{-1}(x) - f^{-1}(x_0)}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{f^{-1}(x) - f^{-1}(x_0)}{f(f^{-1}(x)) - f(f^{-1}(x_0))} =$$

$$= \lim_{x \to x_0} \frac{f^{-1}(x) - y_0}{f(f^{-1}(x)) - f(y_0)} \underset{\substack{\uparrow \\ \lim\limits_{x \to x_0} f^{-1}(x) = f^{-1}(x_0) = y_0}}{=}$$

$$= \lim_{y \to y_0} \frac{y - y_0}{f(y) - f(y_0)} = \frac{1}{\lim\limits_{y \to y_0} \dfrac{f(y) - f(y_0)}{y - y_0}} =$$

$$= \frac{1}{f'(y_0)} = \frac{1}{f'(f^{-1}(x_0))} \qquad \square$$

Properties of inverse functions - Flowchart

$$\left.\begin{array}{l} f \text{ one-to-one} \\ A \text{ interval} \\ f \text{ continuous at } A \end{array}\right\} \Rightarrow f \nearrow A \vee f \searrow A$$

$$\begin{array}{l} f \nearrow A \Rightarrow f^{-1} \nearrow f(A) \\ f \searrow A \Rightarrow f^{-1} \searrow f(A) \end{array}$$

$$\left.\begin{array}{l} f \text{ continous at } A \\ f \text{ one-to-one} \\ A \text{ interval} \end{array}\right\} \Rightarrow f^{-1} \text{ continuous at } f(A)$$

$$\left.\begin{array}{l} f \text{ differentiable at } A \\ f \text{ one-to-one} \\ A \text{ interval} \\ f'(x) \neq 0 \ , \forall x \in A \end{array}\right\} \Rightarrow \begin{array}{l} f^{-1} \text{ differentiable} \\ \text{at } f(A) \text{ with} \end{array}$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

EXAMPLE

Given the function $f(x) = x^3 + x + 1$, evaluate $(f^{-1})'(1)$

Solution

We note that

$\forall x \in \mathbb{R} : f'(x) = (x^3+x+1)' = 3x^2 + 1 \geq 1 \geq 0$

$\Rightarrow (\forall x \in \mathbb{R} : f'(x) \geq 0) \Rightarrow f \nearrow \mathbb{R} \Rightarrow f$ one-to-one.

therefore f^{-1} can be defined. Since:

$$f^{-1}(1) = x \Leftrightarrow f(x) = 1 \Leftrightarrow x^3 + x + 1 = 1 \Leftrightarrow$$

$$\Leftrightarrow x^3 + x = 0 \Leftrightarrow x(x^2+1) = 0 \Leftrightarrow$$

$$\Leftrightarrow x = 0 \vee x^2 + 1 = 0 \Leftrightarrow x = 0$$

it follows that $f^{-1}(1) = 0$, and therefore

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} =$$

$$= \frac{1}{3 \cdot 0^2 + 1} = 1$$

EXERCISES

(10) Show that the following functions are one-to-one and find their inverse

a) $f(x)=3x+2$

b) $f(x)=1-2x$

c) $f(x)=2x^3+7$

d) $f(x)=3+\sqrt{x-1}$

e) $f(x)=-1-\sqrt{2-3x}$

f) $f(x)=\dfrac{2}{3x}$

g) $f(x)=\dfrac{2x+3}{x-2}$

h) $f(x)=\dfrac{x+4}{3x-1}$

(11) Use monotonicity to show that the following functions are one-to-one and then calculate $(f^{-1})'(a)$ at the value of a given below:

a) $f(x)=e^x+(x-2)(x+2)$ at $a=e-3$ (with $A_f=(0,+\infty)$)

b) $f(x)=x^3+4x+2$ at $a=2$

c) $f(x)=x(\tan^2 x+1)+1$, $\forall x\in[0,\pi/2)$ at $a=1$

d) $f(x)=(x^2+1)e^x$ at $a=2e$

(12) Let $f:\mathbb{R}\to\mathbb{R}$ be a one-to-one function which is differentiable in $\mathbb{R}$. Let $g:\mathbb{R}\to\mathbb{R}$ be the function $g(x)=af(x)+b$ with $a\neq 0$. Show that

$$(g^{-1})'(x)=\frac{1}{a}(f^{-1})'(x).$$

▼ The natural logarithm

The natural logarithm function is defined as the inverse of the natural exponential function exp.
Since:

$[\exp(x)]' = e^x > 0, \ \forall x \in \mathbb{R} \Rightarrow \exp \uparrow \mathbb{R} \Rightarrow$
$\Rightarrow \exp$ one-to-one

it follows that exp has an inverse function that we will denote: $\ln = \exp^{-1}$.

▸ Domain and definition of ln

Given a known x, we have:

$\ln x = y \Leftrightarrow \exp(y) = x \Leftrightarrow e^y = x \Leftrightarrow e^y - x = 0. \quad (1)$

We claim that Eq.(1) has no solution if $x \in (-\infty, 0]$ and a unique solution if $x \in (0, +\infty)$. It follows that the domain of ln is

$$\mathrm{dom}(\ln) = (0, +\infty)$$

and consequently; ln is defined via:

$$\boxed{\forall x \in (0, +\infty): \ \ln x = y \Leftrightarrow e^y = x}$$

Proof of claim

Since

$\forall x \in (-\infty, 0]: \ e^y - x \geq e^y > 0$

$\Rightarrow \forall x \in (-\infty, 0]: \ e^y - x \neq 0$

$\Rightarrow$ The equation (1) has no solution with respect to y for all $x \in (-\infty, 0]$.

Let $x \in (0, +\infty)$ be given and define $g(y) = e^y - x$. Then,

$$\lim_{y \to -\infty} g(y) = \lim_{y \to -\infty} (e^y - x) = 0 - x = -x < 0 \Rightarrow$$

$$\Rightarrow \exists a \in (-\infty, 0) : g(a) < 0$$

and

$$\lim_{y \to +\infty} g(y) = \lim_{y \to +\infty} (e^y - x) = +\infty \Rightarrow$$

$$\Rightarrow \exists b \in (0, +\infty) : g(b) > 0$$

Choose $a \in (-\infty, 0)$ and $b \in (0, +\infty)$ such that $g(a) < 0$ and $g(b) > 0$. Then:

$$\left.\begin{array}{l} g(a)g(b) < 0 \\ g \text{ continuous on } [a, b] \end{array}\right\} \Rightarrow \exists x_0 \in (a, b) : g(x_0) = 0$$

$\Rightarrow$ The equation (1) has at least one solution $y_0 \in \mathbb{R}$.

We will now show this solution is unique. To show a contradiction, assume that $y_0, y_1 \in \mathbb{R}$ are solutions to equation (1). Then it follows that:

$$\left.\begin{array}{l} g(y_0) = g(y_1) = 0 \\ g \text{ continous on } [y_0, y_1] \\ g \text{ differentiable on } (y_0, y_1) \end{array}\right\} \Rightarrow \exists z \in (y_0, y_1) : g'(z) = 0$$

$$\Rightarrow \exists z \in (y_0, y_1) : \exp(z) = 0$$

which is a contradiction, since

$$\forall z \in \mathbb{R} : g'(z) = \exp(z) > 0$$

We conclude that Eq. (1) has a unique solution on $\mathbb{R}$. $\square$

▸ Algebraic identities

Since $e^0 = 1 \Rightarrow$ $\boxed{\ln 1 = 0}$
$e^1 = e \Rightarrow$ $\boxed{\ln e = 1}$

Furthermore:

$$\boxed{\begin{array}{l} \forall x_1, x_2 \in (0, +\infty): \ln(x_1 x_2) = \ln x_1 + \ln x_2 \\ \forall x_1, x_2 \in (0, +\infty): \ln(x_1 / x_2) = \ln x_1 - \ln x_2 \\ \forall x \in (0, +\infty): \forall a \in \mathbb{R}: \ln(x^a) = a \ln x \end{array}}$$

Proof

Let $x_1, x_2 \in (0, +\infty)$ be given. Then:

$$\begin{aligned} \ln(x_1 x_2) &= \ln(\exp(\ln x_1)\exp(\ln x_2)) \\ &= \ln(\exp(\ln x_1 + \ln x_2)) \\ &= \ln x_1 + \ln x_2 \end{aligned}$$

and

$$\begin{aligned} \ln(x_1 / x_2) &= \ln\left(\frac{\exp(\ln x_1)}{\exp(\ln x_2)}\right) = \ln(\exp(\ln x_1 - \ln x_2)) \\ &= \ln x_1 - \ln x_2. \end{aligned}$$

Let $x \in (0, +\infty)$ and $a \in \mathbb{R}$ be given. Then,

$$\begin{aligned} \ln(x^a) &= \ln(\exp(\ln x)^a) = \ln(\exp(a \ln x)) = \\ &= a \ln x \end{aligned}$$

□

▸ Derivative

$$\boxed{(\ln x)' = \frac{1}{x}, \quad \forall x \in (0, +\infty)}$$

Proof

Define $f(x) = \exp(x)$, $\forall x \in \mathbb{R}$.

Note that $f'(x) = [\exp(x)]' = \exp(x)$, $\forall x \in \mathbb{R}$

Since:

$$\left.\begin{array}{l} f \text{ differentiable on } \mathbb{R} \\ f \text{ one-to-one} \end{array}\right\} \Rightarrow \begin{array}{l} f^{-1} \text{ differentiable} \\ \text{on } (0, +\infty) \end{array}$$

with

$$(\ln x)' = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\ln x)} =$$

$$= \frac{1}{\exp(\ln x)} = \frac{1}{x} \qquad \square$$

Via the chain rule, we get:

$$\boxed{[\ln(f(x))]' = \frac{f'(x)}{f(x)}}$$

▸ Generalization

$\dfrac{d}{dx} \ln\lvert x\rvert = \dfrac{1}{x}, \quad \forall x \in \mathbb{R} - \{0\}$
$\dfrac{d}{dx} \ln\lvert f(x)\rvert = \dfrac{f'(x)}{f(x)}$

Monotonicity and convexity

Since for $f(x) = \ln x$:

$\forall x \in (0, +\infty): f'(x) = 1/x > 0$
$\forall x \in (0, +\infty): f''(x) = -1/x^2 < 0$ $\Rightarrow$ $\boxed{\begin{array}{l} f \nearrow (0,+\infty) \\ f \text{ convex down on } (0,+\infty) \end{array}}$

It follows that f is also <u>one-to-one</u>. It follows that

$$\boxed{\begin{array}{l} \ln x_1 = \ln x_2 \Leftrightarrow x_1 = x_2 > 0 \\ \ln x_1 > \ln x_2 \Leftrightarrow x_1 > x_2 > 0 \end{array}}$$

We may use these properties to solve equations and inequalities involving exponentials and logarithms.
We also note that

$$e^0 = 1 \Rightarrow \boxed{\ln 1 = 0}$$

and therefore

$$\boxed{\begin{array}{l} \ln x > 0 \Leftrightarrow x > 1 \\ \ln x < 0 \Leftrightarrow 0 < x < 1 \end{array}}$$

EXAMPLES

↳ When solving equations, inequalities or determining the domain of functions, the expression $\ln(f(x))$ should result in the restriction $f(x) > 0$.

a) Solve the equation $\ln(\ln(2x+1)) = 5$

Solution

We require $\begin{cases} 2x+1 > 0 \\ \ln(2x+1) > 0 \end{cases}$

and note that

$2x+1 > 0 \Leftrightarrow 2x > -1 \Leftrightarrow x > -1/2 \Leftrightarrow x \in (-1/2, +\infty)$

$\ln(2x+1) > 0 \Leftrightarrow \ln(2x+1) > \ln 1 \Leftrightarrow 2x+1 > 1 \Leftrightarrow$

$\Leftrightarrow 2x > 0 \Leftrightarrow x > 0 \Leftrightarrow x \in (0, +\infty)$

thus the domain of the equation is

$A = (-1/2, +\infty) \cap (0, +\infty) = (0, +\infty)$.

We have:

$\ln(\ln(2x+1)) = 5 \Leftrightarrow \ln(2x+1) = e^5 \Leftrightarrow$

$\Leftrightarrow 2x+1 = \exp(e^5) \Leftrightarrow 2x = \exp(e^5) - 1$

$\Leftrightarrow x = (1/2)[\exp(e^5) - 1]$

which is accepted since:

$x = (1/2)[\exp(e^5) - 1] \geq (1/2)[e^5 + 1 - 1] = e^5/2$

$\geq (5+1)/2 > 0 \Rightarrow x \in (0, +\infty)$.

b) Solve the equation $\ln(x-1)+\ln(x+1)=2$

<u>Solution</u>

We require

$$\begin{cases} x-1>0 \\ x+1>0 \end{cases} \Leftrightarrow \begin{cases} x>1 \\ x>-1 \end{cases} \Leftrightarrow x>1 \Leftrightarrow x\in(1,+\infty)$$

thus the domain of the equation is $A=(1,+\infty)$.

It follows that

$$\ln(x-1)+\ln(x+1)=2 \Leftrightarrow \ln[(x-1)(x+1)]=\ln e^2 \Leftrightarrow$$

$$\Leftrightarrow (x-1)(x+1)=e^2 \Leftrightarrow x^2-1=e^2 \Leftrightarrow x^2=e^2+1$$

$$\Leftrightarrow x=\sqrt{e^2+1}\in(1,+\infty) \vee x=-\sqrt{1+e^2}\notin(1,+\infty)$$

$$\Leftrightarrow x=\sqrt{e^2+1}$$

The solution $x=-\sqrt{1+e^2}$ is rejected.

c) Solve the inequality $3^{x+1}<5^{2x+1}$

<u>Solution</u>

There are no restrictions. Therefore

$$3^{x+1}<5^{2x+1} \Leftrightarrow \ln(3^{x+1})<\ln(5^{2x+1}) \Leftrightarrow$$

$$\Leftrightarrow (x+1)\ln 3=(2x+1)\ln 5 \Leftrightarrow$$

$$\Leftrightarrow (\ln 3)x+\ln 3=(2\ln 5)x+\ln 5 \Leftrightarrow$$

$$\Leftrightarrow (\ln 3-2\ln 5)x=\ln 5-\ln 3 \Leftrightarrow$$

$$\Leftrightarrow x=\frac{\ln 5-\ln 3}{\ln 3-2\ln 5}$$

d) Find the monotonicity and local min/max of
$$f(x) = \ln\left(\frac{x^2-1}{x^2+1}\right)$$

Solution

• Domain

We require $\frac{x^2-1}{x^2+1} > 0 \Leftrightarrow \frac{(x-1)(x+1)}{x^2+1} > 0 \Leftrightarrow$

x		-1		$+1$	
$x-1$	$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$
x^2+1	$+$		$+$		$+$
	$+$	0	$-$	0	$+$

$\Leftrightarrow x \in (-\infty,-1) \cup (1,+\infty)$

It follows that the domain of f is $A = (-\infty,-1)\cup(1,+\infty)$

• Derivative

$$f'(x) = \left[\ln\left(\frac{x^2-1}{x^2+1}\right)\right]' = \frac{x^2+1}{x^2-1}\left(\frac{x^2-1}{x^2+1}\right)' =$$

$$= \frac{x^2+1}{x^2-1}\,\frac{(x^2-1)'(x^2+1)-(x^2-1)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{x^2+1}{x^2-1}\,\frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}$$

$$= \frac{2x(x^2+1-x^2+1)}{(x^2-1)(x^2+1)} = \frac{4x}{(x^2-1)(x^2+1)}$$

$$= \frac{4x}{(x-1)(x+1)(x^2+1)}$$

- <u>Monotonicity</u>

x		-1		0		1	
$4x$	$-$		$-$	0	$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$		$+$
x^2+1	$+$		$+$		$+$		$+$
$f'(x)$	$-$	$\|\|$		0		$\|\|$	$+$
$f(x)$	$\searrow$	$\|\|$				$\|\|$	$\nearrow$

No local min or max.

Limits with ln function

From continuity of ln it follows that

$$\boxed{\forall x_0 \in (0,+\infty): \lim_{x\to x_0} \ln x = \ln x_0}$$

We can also show, using the definition of the limit, that:

$$\lim_{x\to+\infty} \ln x = +\infty$$

$$\lim_{x\to 0^+} \ln x = -\infty$$

To evaluate the limit of $\ln(f(x))$ we use the composition theorem to show that

$$\lim_{x\to\sigma} f(x) = a > 0 \Rightarrow \lim_{x\to\sigma} \ln(f(x)) = \ln a$$

$$\lim_{x\to\sigma} f(x) = +\infty \Rightarrow \lim_{x\to\sigma} \ln(f(x)) = +\infty$$

$$\left.\begin{array}{l} \lim\limits_{x\to\sigma} f(x) = 0 \\ \forall x \in N(\sigma,\delta)\cap \mathrm{dom}(f): f(x) > 0 \end{array}\right\} \Rightarrow \lim_{x\to\sigma} \ln(f(x)) = -\infty$$

which we can use to evaluate limits of functions involving natural logarithms.

EXAMPLES

a) $f(x) = \ln(\sin x + \cos x) \leftarrow \lim_{x \to 0} f(x)$

Solution

$$\lim_{x \to 0} (\sin x + \cos x) = \sin 0 + \cos 0 = 0 + 1 = 1 \Rightarrow$$

$$\Rightarrow \lim_{x \to 0} f(x) = \lim_{x \to 0} \ln(\sin x + \cos x) = \ln 1 = 0$$

b) $f(x) = \ln(\ln(3x+1)) \leftarrow \lim_{x \to 0^+} f(x)$

Solution

$$\lim_{x \to 0^+} (3x+1) = 3 \cdot 0 + 1 = 1 \Rightarrow$$

$$\Rightarrow \lim_{x \to 0^+} \ln(3x+1) = \ln 1 = 0 \qquad (1)$$

Let $x \in (0,1)$ be given. Then:

$x > 0 \Rightarrow 3x > 0 \Rightarrow 3x+1 > 1 \Rightarrow \ln(3x+1) > \ln 1$

$\Rightarrow \ln(3x+1) > 0$

and therefore: $\forall x \in (0,1): \ln(3x+1) > 0 \qquad (2)$

From Eq. (1) and Eq. (2):

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \ln(\ln(3x+1)) = -\infty.$$

c) $f(x) = \ln(2x^2+3x-1) - \ln(x^2-x+4) \leftarrow \lim\limits_{x\to+\infty} f(x)$

<u>Solution</u>

Since,

$$f(x) = \ln(2x^2+3x-1) - \ln(x^2-x+4) =$$
$$= \ln\left(\frac{2x^2+3x-1}{x^2-x+1}\right)$$

it follows that

$$\lim_{x\to+\infty} \frac{2x^2+3x-1}{x^2-x+1} = \lim_{x\to+\infty} \frac{2x^2}{x^2} = 2 \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \ln\left(\frac{2x^2+3x-1}{x^2-x+1}\right) = \ln 2$$

EXERCISES

(13) Find the default domain for the following functions:

a) $f(x) = \ln\left(\dfrac{x^2-4}{x^2-1}\right)$

b) $f(x) = \ln(x^2-4) - \ln(x^2-1)$

c) $f(x) = \dfrac{\ln(x^2+3x+2)}{\ln(1-x)}$

(14) Solve the following equations with respect to x. (First find the domain of the equations)

a) $\ln(4x-1) = 2\ln 2 + \ln(x^2-1)$

b) $\dfrac{1}{2}\ln(x+2) + \ln\left(\sqrt{x+3}\right) = 1 + \ln 3$

c) $2\ln x - \ln(x+1) = \ln 4 - \ln 3$

d) $\ln(\ln(3x+1)) = 0$

e) $\ln(\ln(x^2-x-3)) = 0$

(15) Similarly with the equations

a) $3^{x+1} = 5^{2x-1}$

b) $(\sqrt{2})^{x+1} = (\sqrt{3})^{1-2x}$

c) $(1+\sqrt{5})^{2x+3} = (2+\sqrt{5})^{x-4}$

(16) Evaluate and simplify the derivatives of the following functions

a) $f(x) = e^x \ln x$

b) $f(x) = \sqrt{\ln x}$

c) $f(x) = \ln(x^2+x+1)$

d) $f(x) = x^2 \ln(2x-1)$

e) $f(x) = (2+\ln(3x-1))^3$

f) $f(x) = [\ln(x^2+5)]^2$

g) $f(x) = [\ln(x^2+1)^2]$

h) $f(x) = \ln\left(\dfrac{x^2+2}{1-x}\right)$

i) $f(x) = \dfrac{1+\ln x}{1-\ln x}$

j) $f(x) = \dfrac{\ln x}{e^x(x^2+1)}$

(17) Use the mean value theorem to show that:

a) $0 < b \leqslant a \Rightarrow \dfrac{a-b}{a} \leqslant \ln\left(\dfrac{a}{b}\right) \leqslant \dfrac{a-b}{b}$

b) $\forall x \in (0,+\infty): \dfrac{x}{x+1} \leqslant \ln(1+x) \leqslant x$

c) $x > 1 \Rightarrow \dfrac{2}{x+1} < \ln\left(\dfrac{x+1}{x-1}\right) < \dfrac{2}{x-1}$

d) $0 < a < b < \pi/2 \Rightarrow (b-a)\tan a < \ln\left(\dfrac{\cos a}{\cos b}\right) < (b-a)\tan b$

(18) Show that the equation

$$\ln\left(\frac{1}{x}-1\right) + \frac{1}{x-1} = 0$$

has a unique solution in the interval $(1/5, 1/4)$, using Bolzano and Rolle theorems.

(19) Analyze the following functions with respect to monotonicity, convexity, find all local min/max, find all inflection points, and show the variation table.

a) $f(x) = \ln(1-x^2)$

b) $f(x) = \dfrac{x}{\ln x}$

c) $f(x) = \ln(x-1) - x$

d) $f(x) = x\ln(1/x)$

e) $f(x) = \ln x - x$

f) $f(x) = \dfrac{2-\ln x}{x}$

g) $f(x) = x^2+3x-\ln(x^2)$

h) $f(x) = \ln\left(\dfrac{x+2}{x-2}\right)$

(20) Analyze the function $f(x) = (\ln x)/x$ with respect to monotonicity and show that f has a global maximum.
Then show that $e^\pi > \pi^e$.

(21) Analyze the function
$$f(x) = \frac{\ln(x-1)}{\ln x}$$
with respect to monotonicity. Then show that

a) $\ln(e-1)\ln(e+1) < 1$

b) $\ln(e^\pi-1)\ln(e^\pi+1) < \pi^2$.

(22) Evaluate the following limits, if they exist.

a) $\lim_{x\to 1} \ln(|\ln x|)$

b) $\lim_{x\to -\infty} [2\ln(3x^2+1) - \ln(x^4-1)]$

c) $\lim_{x\to +\infty} [3\ln(2x+1) - 2\ln(3x+1)]$

d) $\lim_{x\to +\infty} [2\ln(x+1) - \ln(x-3)]$

e) $\lim_{x\to 6^+} [\ln(\sqrt{x+3} - 3) - \ln(x-6)]$

f) $\lim_{x\to 0^+} [\ln(\sqrt{x^2+1} - 1) - \ln x]$

g) $\lim_{x\to +\infty} [\ln(1+\ln(x^3-1) - \ln(x-1) - \ln(x^2+x+1))]$

The general exponential function

- Let $a \in (0,+\infty)$. The general exponential function is defined as
  $$f(x) = a^x \ , \ \forall x \in \mathbb{R}.$$
- All properties of the general exponential function are inherented from the natural exponential function and the natural logarithmic function, via the following key statement:

$$\boxed{\forall a \in (0,+\infty) : \forall x \in \mathbb{R} : \ a^x = \exp(x \ln a)}$$

Proof

Let $a \in (0,+\infty)$ and $x \in \mathbb{R}$ be given. Then:

$$\begin{aligned} a^x &= [\exp(\ln a)]^x = \\ &= [e^{\ln a}]^x = e^{x \ln a} \\ &= \exp(x \ln a) \qquad \square \end{aligned}$$

Derivatives with respect to x

$$\boxed{\begin{aligned} (a^x)' &= a^x \ln a \\ (x^a)' &= a x^{a-1} \end{aligned}}$$

Proof

$$(a^x)' = [\exp(x \ln a)]' = \exp(x \ln a)\,(x \ln a)'$$
$$= \exp(x \ln a)\,(\ln a) = a^x \ln a$$

and

$$(x^a)' = [\exp(a \ln x)]' = \exp(a \ln x)\,(a \ln x)' =$$
$$= \exp(a \ln x)\,(a x^{-1}) =$$
$$= x^a\,(a x^{-1}) = a x^{a-1} \qquad \square$$

▶ Combining these differentiation rules with the chain rule gives the following more powerful differentiation rules:

$$(a^x)' = a^x \ln a \longrightarrow \boxed{(a^{g(x)})' = a^{g(x)} \ln a}$$
$$(x^a)' = a x^{a-1} \longrightarrow \boxed{([g(x)]^a)' = a[g(x)]^{a-1} g'(x)}$$

▶ We also note that

$$[f(x)^{g(x)}]' = [\exp(g(x) \ln(f(x)))]' =$$
$$= \exp(g(x) \ln(f(x)))\,[g(x) \ln(f(x))]' =$$
$$= f(x)^{g(x)} [g'(x) \ln(f(x)) + g(x)\,[\ln(f(x))]'] =$$
$$= f(x)^{g(x)} \left[g'(x) \ln(f(x)) + \frac{g(x) f'(x)}{f(x)}\right]$$

and therefore

$$\boxed{[f(x)^{g(x)}]' = f(x)^{g(x)} \left[g'(x) \ln f(x) + \frac{g(x) f'(x)}{f(x)}\right]}$$

Limits

As a consequence of differentiability, we have

$$\boxed{\forall x_0 \in \mathbb{R} : \lim_{x \to x_0} a^x = a^{x_0}}$$

For limits $x \to +\infty$ or $x \to -\infty$, we note that

$0 < a < 1 \Rightarrow \ln a < 0$

$a > 1 \Rightarrow \ln a > 0$

and combine this with the identity

$a^x = \exp(x \ln a)$

We conclude that

$$\lim_{x \to +\infty} a^x = \begin{cases} +\infty & , \text{ if } a > 1 \\ 0 & , \text{ if } 0 < a < 1 \end{cases}$$

$$\lim_{x \to -\infty} a^x = \begin{cases} 0 & , \text{ if } a > 1 \\ +\infty & , \text{ if } 0 < a < 1 \end{cases}$$

EXAMPLES

a) Find and simplify the derivative of
$f(x) = \sqrt[3]{x}\,(x+1)$

Solution

$$f'(x) = [\sqrt[3]{x}\,(x+1)]' = [x^{1/3}(x+1)]' =$$
$$= (x^{4/3} + x^{1/3}]' = (4/3)x^{4/3-1} + (1/3)x^{1/3-1}$$
$$= (4/3)x^{1/3} + (1/3)x^{-2/3} =$$
$$= (1/3)x^{-2/3}[4x+1] =$$
$$= (1/3)\frac{4x+1}{(\sqrt[3]{x})^2} = \frac{4x+1}{3(\sqrt[3]{x})^2}$$

b) Find and simplify the derivative of
$f(x) = x^{\sin x}$

Solution

$$f'(x) = (x^{\sin x})' = [\exp(\sin x \ln x)]' =$$
$$= \exp(\sin x \ln x)(\sin x \ln x)' =$$
$$= x^{\sin x}[(\sin x)'\ln x + \sin x(\ln x)'] =$$
$$= x^{\sin x}[\cos x \ln x + \sin x \cdot (1/x)]$$
$$= x^{\sin x - 1}[x\cos x \ln x + \sin x].$$

c) Given the function $f(x) = (1+x)^{\sin(1/x)}$, evaluate $\lim_{x\to 0} f(x)$.

Solution

We note that

$$f(x) = (1+x)^{\sin(1/x)} = \exp(\ln(1+x)\sin(1/x))$$

and we have:

$$\lim_{x\to 0}(1+x) = 1+0 = 1 \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow \lim_{x\to 0} \ln(1+x) = \ln 1 = 0 \\ \forall x \in (-1,0)\cup(0,1): |\sin(1/x)| \leq 1 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \lim_{x\to 0}[\ln(1+x)\sin(1/x)] = 0 \Rightarrow$$

$$\Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0} \exp(\ln(1+x)\sin(1/x)) =$$

$$= \exp(0) = 1.$$

EXERCISES

(23) Find the default domain of the following functions:

a) $f(x) = (2x-1)^{\sin x}$
b) $f(x) = (x^3+1)^{x-1}$
c) $f(x) = \left(\dfrac{x^2-1}{x^2+3x+2}\right)^{x}$
d) $f(x) = \left(\dfrac{x^2+9x+18}{x-1}\right)^{\ln x}$

(24) Evaluate and simplify the derivatives of the following functions:

a) $f(x) = x^x$
b) $f(x) = (\cos x)^{\sin(2x)}$
c) $f(x) = x^{1/x}$
d) $f(x) = x^{\sqrt{x}}$
e) $f(x) = \left(1+\dfrac{1}{x}\right)^{x}$
f) $f(x) = \sin(x^x)$
g) $f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

(25) Same with the following functions

a) $f(x) = 2^x \ln x$
b) $f(x) = \dfrac{2^x+3^x}{3^x+5^x}$
c) $f(x) = \cos(2^x)$
d) $f(x) = \tan(3^x+7^x)$
e) $f(x) = 3^x(2x+1)^3$
f) $f(x) = \dfrac{x^2+3x+2}{5^x}$

(26) Show that the equation $3^x + 4^x = 5^x$ has a unique solution in $\mathbb{R}$.

(27) Use the mean-value theorem to prove that if $a > b$ and $x > 1$, then
$$(a-b)x^a \ln x < (x^a - x^b) < (a-b)x^b \ln x$$

The general logarithmic function

The general logarithmic function is defined as the inverse of the general exponential function

$$f(x) = a^x,\ \forall x \in \mathbb{R}$$

with $a \in (0,1) \cup (1,+\infty)$. We note that

$a \in (0,1) \cup (1,+\infty) \Rightarrow \ln a \neq 0 \Rightarrow$

$\Rightarrow (\forall x \in \mathbb{R} : f'(x) = a^x \ln a > 0) \vee (\forall x \in \mathbb{R} : f'(x) = a^x \ln a < 0)$

$\Rightarrow f \nearrow \mathbb{R} \vee f \searrow \mathbb{R} \Rightarrow f$ one-to-one $\Rightarrow$

$\Rightarrow f$ has an inverse

We define the general logarithmic function $\log_a$ as $\log_a = f^{-1}$, with f as defined above.

Note that for $a = 1$ we have $f(x) = 1^x = 1,\ \forall x \in \mathbb{R}$ which is not one-to-one, therefore has no inverse, therefore $\log_a$ cannot be defined for $a = 1$.

Domain and definition of $\log_a$

We will show that, given $a \in (0,1) \cup (1,+\infty)$

$$\boxed{\forall x \in (0,+\infty) : \log_a x = \frac{\ln x}{\ln a}}$$

with $\mathrm{dom}(\log_a) = (0,+\infty)$. It follows that the theory of the natural logarithmic function $\ln$ can be used to handle all mathematical problems that involve general logarithms.

Proof

Given any x, y, we note that:

$$\log_a x = y \Leftrightarrow a^y = x$$
$$\Leftrightarrow \exp(y \ln a) = x \leftarrow \text{Require } x > 0$$
$$\Leftrightarrow y \ln a = \ln x$$
$$\Leftrightarrow y = \frac{\ln x}{\ln a}$$

The requirement $x > 0$ indicates that the domain of $\log_a$ is the set $A = (0, +\infty)$, and we conclude that

$$\forall x \in (0, +\infty): \log_a x = \frac{\ln x}{\ln a} \qquad \square$$

→ Immediate consequences

$$\boxed{\begin{aligned} &\log_a 1 = 0, \quad \forall a \in (0,1) \cup (1, +\infty) \\ &\log_a a = 1, \quad \forall a \in (0,1) \cup (1, +\infty) \end{aligned}}$$

Decimal logarithms

For base $a = 10$, we define the decimal logarithm function $\log$ as:

$$\boxed{\forall x \in (0, +\infty): \log x = \log_{10} x = \frac{\ln x}{\ln 10}}$$

and note that:

$\log 1 = 0$	$\log 10000 = 4$
$\log 10 = 1$	$\log 100000 = 5$
$\log 100 = 2$	$\log 1000000 = 6$
$\log 1000 = 3$	$\log 10000000 = 7$
etc	

We see that $\log x$ gives the order of magnitude of x in the decimal system (i.e. the number of zeroes for $x = 10^n$).

▶ Domains with logarithm function

To find the domain of

$$f(x) = \log_{a(x)} g(x) = \frac{\ln g(x)}{\ln a(x)}$$

we require: $\begin{cases} g(x) > 0 \\ a(x) > 0 \\ a(x) \neq 1 \end{cases}$

EXAMPLE

For $f(x) = \log_{3x-1}(x^2+3x+2)$
we require

$3x-1>0$ and $x^2+3x+2>0$ and $3x-1 \neq 1$. (1)

Since

$3x-1>0 \Leftrightarrow 3x>1 \Leftrightarrow x>1/3$, and

$x^2+3x+2>0 \Leftrightarrow (x+2)(x+1)>0 \Leftrightarrow x \in (-\infty,-2) \cup (-1,+\infty)$

x		-2		-1	
x^2+3x+2	$+$	0	$-$	0	$+$

$3x-1 \neq 1 \Leftrightarrow 3x \neq 2 \Leftrightarrow x \neq 2/3$.

Thus the domain is

$A = [(1/3,+\infty) \cap [(-\infty,-2) \cup (-1,+\infty)]] - \{2/3\}$

$= (1/3,+\infty) - \{2/3\} = (1/3,2/3) \cup (2/3,+\infty)$

Derivatives with logarithms

From the chain rule, we easily get:

$$\boxed{[\log_a f(x)]' = \frac{f'(x)}{f(x)\ln a}}$$

For the most general case:

$$[\log_{a(x)} f(x)]' = \left[\frac{\ln f(x)}{\ln a(x)}\right]' =$$

$$= \frac{(\ln f(x))'(\ln a(x)) - (\ln f(x))(\ln a(x))'}{[\ln a(x)]^2} =$$

$$= \frac{\frac{f'(x)}{f(x)}\ln a(x) - \frac{a'(x)}{a(x)}\ln f(x)}{[\ln a(x)]^2} =$$

$$= \frac{a(x) f'(x) \ln a(x) - f(x) a'(x) \ln f(x)}{f(x) a(x) [\ln a(x)]^2}$$

EXAMPLES

a) Find and simplify the derivative of
$f(x) = \log_4 (x^2+3x+2)$

Solution

$$f'(x) = \left[\log_4 (x^2+3x+2)\right]' = \left[\frac{\ln(x^2+3x+2)}{\ln 4}\right]' =$$
$$= \frac{1}{\ln 4}\,\frac{(x^2+3x+2)'}{x^2+3x+2} = \frac{2x+3}{(x^2+3x+2)\ln 4}$$

b) Find and simplify the derivative of
$f(x) = \log_{2x-1} (\sin x)$.

Solution

$$f'(x) = \left[\log_{2x-1} (\sin x)\right]' = \left[\frac{\ln(\sin x)}{\ln(2x-1)}\right]' =$$
$$= \frac{[\ln(\sin x)]'\ln(2x-1) - \ln(\sin x)[\ln(2x-1)]'}{[\ln(2x-1)]^2}$$
$$= \frac{1}{[\ln(2x-1)]^2}\left[\frac{(\sin x)'}{\sin x}\ln(2x-1) - \ln(\sin x)\frac{(2x-1)'}{2x-1}\right]$$
$$= \frac{1}{[\ln(2x-1)]^2}\left[\frac{\cos x}{\sin x}\ln(2x-1) - \ln(\sin x)\frac{2}{2x-1}\right]$$
$$= \frac{(2x-1)\ln(2x-1)\cos x - 2\ln(\sin x)\sin x}{(2x-1)(\sin x)[\ln(2x-1)]^2}$$

▸ Limits with logarithms

EXAMPLES

a) $f(x) = \log_{1/3}(x^2+1) \leftarrow \lim_{x\to+\infty} f(x)$

Solution

Since

$$f(x) = \log_{1/3}(x^2+1) = \frac{\ln(x^2+1)}{\ln(1/3)} = \frac{\ln(x^2+1)}{-\ln 3}$$

we have

$$\lim_{x\to+\infty}(x^2+1) = \lim_{x\to+\infty} x^2 = +\infty \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} \ln(x^2+1) = +\infty \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\ln(x^2+1)}{-\ln 3} = -\infty \qquad \square$$

b) $f(x) = \log_x(2x^2+1) - \log_x(x^2-1) \leftarrow \lim_{x\to+\infty} f(x)$

Solution

Since

$$f(x) = \log_x(2x^2+1) - \log_x(x^2-1) =$$

$$= \frac{\ln(2x^2+1)}{\ln x} - \frac{\ln(x^2-1)}{\ln x} =$$

$$= \frac{\ln(2x^2+1) - \ln(x^2-1)}{\ln x} =$$

$$= \frac{1}{\ln x} \ln\left(\frac{2x^2+1}{x^2-1}\right)$$

we have

$$\lim_{x \to +\infty} \frac{2x^2+1}{x^2-1} = \lim_{x \to +\infty} \frac{2x^2}{x^2} = 2 \Rightarrow$$

$$\Rightarrow \lim_{x \to +\infty} \ln\left(\frac{2x^2+1}{x^2-1}\right) = \ln 2$$

and

$$\lim_{x \to +\infty} \ln x = +\infty \Rightarrow \lim_{x \to +\infty} \frac{1}{\ln x} = 0$$

and therefore

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \left[\frac{1}{\ln x} \ln\left(\frac{2x^2+1}{x^2-1}\right)\right]$$

$$= 0 \cdot \ln 2 = 0$$

EXERCISES

(28) Find the default domain for the following functions:

a) $f(x) = \log_3 (x^5 - x^3)$

b) $f(x) = \log_{x+1} (x^2 - 4)$

c) $f(x) = \log_{2x} (x^2 + 2x)$

d) $f(x) = \log_{x-1} (x+1) + \log_{x+1} (x-1)$

e) $f(x) = \log_{x^2-4} (x^2 + 3x + 2)$

(29) Find and simplify the derivatives of the following functions:

a) $f(x) = \log_{e^2} (x^2+1)$

b) $f(x) = \log_x 3$

c) $f(x) = \log_x (\cos x)$

d) $f(x) = \log_{x+1} (x-1)$

e) $f(x) = \log_x (3x)$

f) $f(x) = \log_x 3$

(30) Show that, for $a, b, c \in (0,1) \cup (1, +\infty)$:

a) $\log_a \left(\frac{1}{b^5}\right) \log_b (a^2) = -10$

b) $\log_a (bc) = \dfrac{1}{\log_b a} + \dfrac{1}{\log_c a}$

c) $\log_{ab}(c) = \dfrac{\log_b(c)}{1+\log_b(a)}$

(31) Evaluate the following limits, if they exist:

a) $\lim\limits_{x\to \pi/2^-} \log_3(\cos x)$

b) $\lim\limits_{x\to+\infty} \left[2\log_{1/2}(x^2+2x) - \log_{1/2}(x^2-2x)\right]$

c) $\lim\limits_{x\to 0^+} \left[\log_{x+2}(\sin x) - \log_{x+2}(x)\right]$

d) $\lim\limits_{x\to 0^+} \left[\log_{3x+2}(\tan(2x)) - \log_{3x+2}(\tan(3x))\right]$

e) $\lim\limits_{x\to+\infty} \left[\log_x(x+\sin x) - \log_x(x)\right]$

(32) Solve the following equations with respect to x:

a) $\log_x 2 + \log_2 x = \dfrac{5}{3}$

b) $\log_x 256 = (\log_x 4)^2 + 3$

c) $\log_3 x \log_9 x = 2$

d) $2(\log_x 8)^2 + \log_x 64 + \log_x 8 = 9.$

CAL1.7: Other Inverse Functions

OTHER INVERSE FUNCTIONS

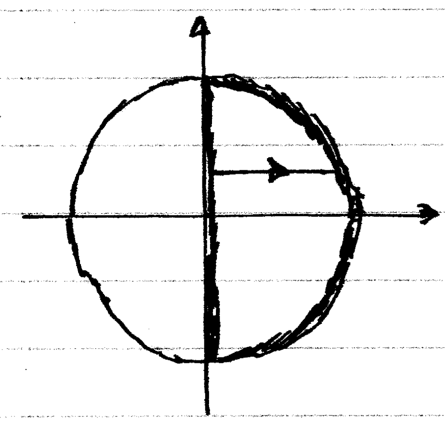
Inverse trigonometric functions

- In general, the trigonometric functions sin, cos, tan, and cot are NOT one-to-one and are NOT invertible. However, by restricting their domain into an appropriate interval, it becomes possible to define inverse trigonometric functions as follows:

▸ notation : Let $f: A \to \mathbb{R}$ be a function and let $B \subseteq A$. We define the restriction $g = f \upharpoonright B$ as follows:

$$\boxed{g = f \upharpoonright B \Leftrightarrow \begin{cases} \forall x \in B : g(x) = f(x) \\ \mathrm{dom}(g) = B \end{cases}}$$

① Inverse of sin $\longrightarrow$ $\boxed{\mathrm{Arcsin} = (\sin \upharpoonright [-\pi/2, \pi/2])^{-1}}$



$\mathrm{Arcsin}\, x = y \Leftrightarrow \begin{cases} x = \sin y \\ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \end{cases}$
$\mathrm{dom}(\mathrm{Arcsin}) = [-1, 1]$

- <u>Properties</u>

a) $\text{Arcsin} \nearrow [-1,1]$

b) Arcsin bounded at $[-1,1]$, because
$\forall x \in [-1,1] : |\text{Arcsin}(x)| \leq \pi/2$

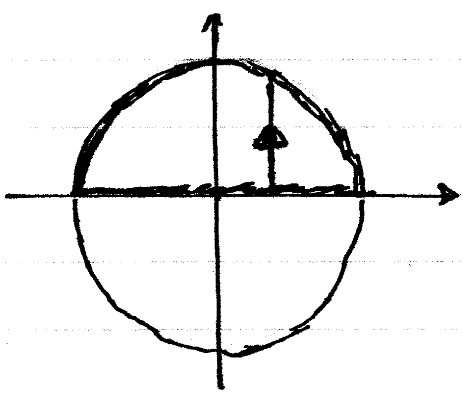
c) Arcsin is odd:
$\forall x \in [-1,1] : \text{Arcsin}(-x) = -\text{Arcsin}(x)$

d) Arcsin continuous at $[-1,1]$

* e) Arcsin differentiable at $(-1,1)$ (!!)
with

$$\boxed{\frac{d}{dx}\text{Arcsin}(x) = \frac{1}{\sqrt{1-x^2}}}$$

(2) <u>Inverse of cos</u> $\longrightarrow$ $\boxed{\text{Arccos} = (\cos \restriction [0,\pi])^{-1}}$



$\text{Arccos}\,x = y \Leftrightarrow \begin{cases} x = \cos y \\ 0 \leq y \leq \pi \end{cases}$
$\text{dom}(\text{Arccos}) = [-1,1]$

a) $\text{Arccos} \searrow [-1,1]$

b) Arccos bounded at $[-1,1]$:
$\forall x \in [-1,1] : 0 \leq \text{Arccos}(x) \leq \pi$

c) Arccos is neither even nor odd.

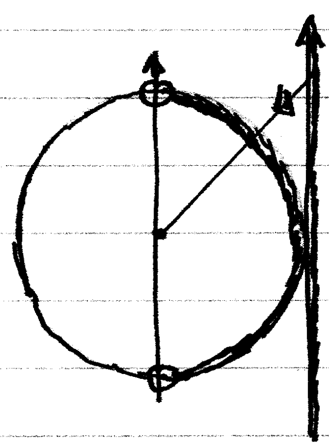
d) Arccos continuous at $[-1,1]$

e) $$\boxed{\forall x \in [-1,1] : \operatorname{Arcsin}(x) + \operatorname{Arccos}(x) = \frac{\pi}{2}}$$

f) Arccos differentiable at $(-1,1)$ with

$$\boxed{\forall x \in (-1,1) : \frac{d}{dx} \operatorname{Arccos}(x) = -\frac{1}{\sqrt{1-x^2}}}$$

③ Inverse of tan $\longrightarrow$ $\boxed{\operatorname{Arctan} = \left(\tan \restriction (-\pi/2, \pi/2)\right)^{-1}}$



$$\operatorname{Arctan}(x) = y \Leftrightarrow \begin{cases} x = \tan y \\ -\frac{\pi}{2} < y < \frac{\pi}{2} \end{cases}$$

$$\operatorname{dom}(\operatorname{Arctan}) = (-\infty, +\infty) = \mathbb{R}$$

• Properties

a) Arctan $\nearrow \mathbb{R}$

b) Arctan bounded at $\mathbb{R}$ with

$$\boxed{\forall x \in \mathbb{R} : |\operatorname{Arctan}(x)| < \frac{\pi}{2}}$$

c) Arctan is odd with $\boxed{\forall x \in \mathbb{R} : \mathrm{Arctan}(-x) = -\mathrm{Arctan}(x)}$

d) Arctan continuous at $\mathbb{R}$.

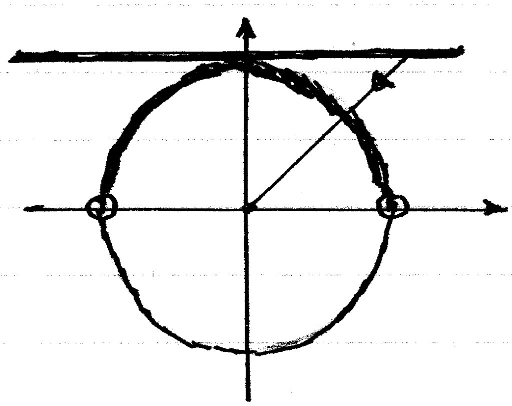
e) Limits to $\pm\infty$:

$\lim\limits_{x \to +\infty} \mathrm{Arctan}(x) = \dfrac{\pi}{2}$	$\lim\limits_{x \to -\infty} \mathrm{Arctan}(x) = -\dfrac{\pi}{2}$

f) Arctan differentiable at $\mathbb{R}$

$$\boxed{\forall x \in \mathbb{R} : \frac{d}{dx} \mathrm{Arctan}(x) = \frac{1}{1+x^2}}$$

④ Inverse of cot $\longmapsto$ $\boxed{\mathrm{Arccot} = (\cot \restriction (0,\pi))^{-1}}$



$\mathrm{Arccot}(x) = y \Leftrightarrow \begin{cases} x = \cot y \\ 0 < y < \pi \end{cases}$
$\mathrm{dom}(\mathrm{Arccot}) = \mathbb{R}$

a) Arccot $\searrow$ $\mathbb{R}$

b) Arccot bounded at $\mathbb{R}$ with

$$\boxed{\forall x \in \mathbb{R} : 0 < \mathrm{Arccot}(x) < \pi}.$$

c) Arccot is not even or odd

d)

$$\boxed{\forall x \in \mathbb{R} : \operatorname{Arctan} x + \operatorname{Arccot} x = \frac{\pi}{2}}$$

e) Limits to $\pm\infty$

$\lim\limits_{x \to +\infty} \operatorname{Arccot}(x) = 0$	$\lim\limits_{x \to -\infty} \operatorname{Arccot}(x) = \pi$

f) Arccot differentiable in $\mathbb{R}$ with

$$\boxed{\forall x \in \mathbb{R} : \frac{d}{dx} \operatorname{Arccot}(x) = \frac{-1}{1+x^2}}$$

Summary of inverse trigonometric functions

Function	Domain	Range	Monotonicity
Arcsin	$[-1,1]$	$[-\pi/2,\pi/2]$	$\nearrow$
Arccos	$[-1,1]$	$[0,\pi]$	$\searrow$
Arctan	$\mathbb{R}$	$[-\pi/2,\pi/2]$	$\nearrow$
Arccot	$\mathbb{R}$	$[0,\pi]$	$\searrow$

$$\frac{d}{dx}\operatorname{Arcsin}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\operatorname{Arctan}(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\operatorname{Arccos}(x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\operatorname{Arccot}(x) = \frac{-1}{1+x^2}$$

$$|\operatorname{Arcsin}(x)| \leq \pi/2 \ , \ \forall x \in [-1,1] \qquad |\operatorname{Arctan}(x)| < \pi/2 \ , \ \forall x \in \mathbb{R}$$

$$0 \leq \operatorname{Arccos}(x) \leq \pi \ , \ \forall x \in [-1,1] \qquad 0 < \operatorname{Arccot}(x) < \pi \ , \ \forall x \in \mathbb{R}$$

$$\lim_{x\to+\infty} \operatorname{Arctan}(x) = \frac{\pi}{2} \qquad \lim_{x\to+\infty} \operatorname{Arccot}(x) = 0$$

$$\lim_{x\to-\infty} \operatorname{Arctan}(x) = \frac{-\pi}{2} \qquad \lim_{x\to-\infty} \operatorname{Arccot}(x) = \pi$$

$$\forall x \in [-1,1]: \ \operatorname{Arcsin}(x) + \operatorname{Arccos}(x) = \pi/2$$

$$\forall x \in \mathbb{R}: \ \operatorname{Arctan}(x) + \operatorname{Arccot}(x) = \pi/2$$

Simplifying expressions with inverse trigonometric functions

To simplify such expressions we let ϑ be the value of the inverse trigonometric function and then we find the appropriate trigonometric function evaluation at ϑ using the following identities:

$\sin^2 x + \cos^2 x = 1$	
$1 + \tan^2 x = \dfrac{1}{\cos^2(x)}$	$\tan x = \dfrac{\sin x}{\cos x}$
$1 + \cot^2 x = \dfrac{1}{\sin^2 x}$	$\cot x = \dfrac{\cos x}{\sin x}$

EXAMPLE

Show $\sin(\text{Arctan}x) = \dfrac{x}{\sqrt{1+x^2}}$

Solution

Let $\vartheta = \text{Arctan}x \Rightarrow -\pi/2 < x < \pi/2 \wedge \tan\vartheta = x$.
It follows that

$$\sin^2\vartheta = 1-\cos^2\vartheta = 1-\frac{1}{1+\tan^2\vartheta} = 1-\frac{1}{1+x^2}$$

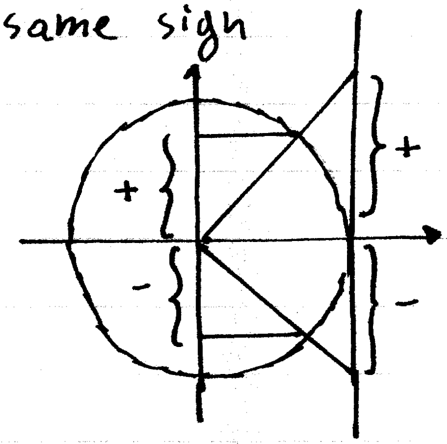
$$= \frac{(1+x^2)-1}{1+x^2} = \frac{x^2}{1+x^2} \Rightarrow \sin\vartheta = \pm\frac{x}{\sqrt{1+x^2}}$$

Since $-\pi/2 \leq \vartheta \leq \pi/2 \Rightarrow$

$\Rightarrow$ $\sin\vartheta$, $\tan\vartheta$ have the same sign (see graph)

$\Rightarrow$ $\sin\vartheta$, x have the same sign

$\Rightarrow$ $\sin\vartheta = \dfrac{x}{\sqrt{1+x^2}}$



Differentiating inverse trigonometric functions

From the chain rule, we have:

$\dfrac{d}{dx}\,\text{Arcsin}(f(x)) = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$\dfrac{d}{dx}\,\text{Arccos}(f(x)) = \dfrac{-f'(x)}{\sqrt{1-[f(x)]^2}}$
$\dfrac{d}{dx}\,\text{Arctan}(f(x)) = \dfrac{f'(x)}{1+[f(x)]^2}$
$\dfrac{d}{dx}\,\text{Arccot}(f(x)) = \dfrac{-f'(x)}{1+[f(x)]^2}$

EXERCISES

(1) Find the default domain for the following functions:

a) $f(x) = \text{Arctan}\left(\sqrt{x^2+x-2}\right)$

b) $f(x) = \text{Arcsin}(3x-2)$

c) $f(x) = \text{Arctan}(e^x)$

d) $f(x) = \text{Arcsin}(e^x)$

e) $f(x) = \text{Arccos}\left(\dfrac{x+1}{2x-1}\right)$

f) $f(x) = \text{Arccos}(\ln x)$

g) $f(x) = \text{Arccos}(x^2+x-2)$

(2) Show that

a) $\cos(\text{Arctan}\,x) = \dfrac{1}{\sqrt{1+x^2}}$

b) $\sin(\text{Arctan}\,x) = \dfrac{x}{\sqrt{1+x^2}}$

c) $\tan(\text{Arccos}\,x) = \dfrac{\sqrt{1-x^2}}{x}$

d) $\tan(\text{Arcsin}\,x) = \dfrac{x}{\sqrt{1-x^2}}$

e) $\sin(\text{Arccos}\,x) = \sqrt{1-x^2}$

f) $\cos(2\,\text{Arctan}\,x) = \dfrac{1-x^2}{1+x^2}$

g) $\sin(2\,\text{Arctan}\,x) = \dfrac{2x}{1+x^2}$

h) $\sin(2\,\text{Arccos}\,x) = 2x\sqrt{1-x^2}$

(3) Evaluate the derivatives of the following functions

a) $f(x) = (x^2+3x+1)\,\text{Arctan}\,x$

b) $f(x) = (1-x^2)\,\text{Arcsin}\,x$

c) $f(x) = (1+x^2)\,\text{Arccot}\,x$

d) $f(x) = \sqrt{1-x^2}\,\text{Arccos}\,x$

④ Similarly for the following functions

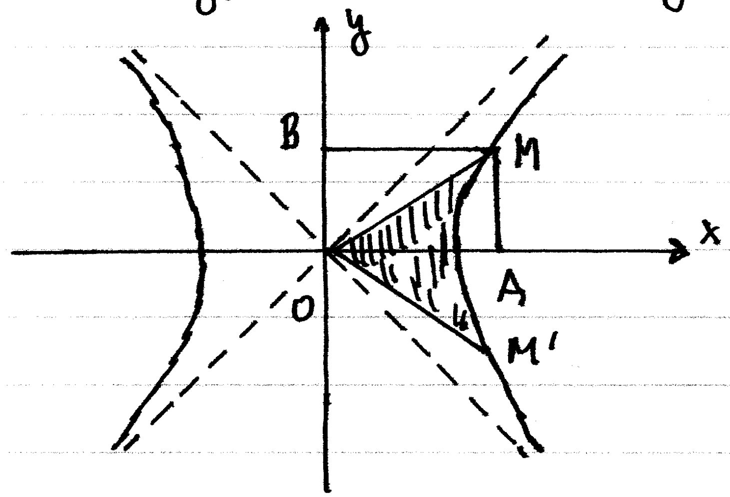
a) $f(x) = \text{Arcsin}(1-2x)$
b) $f(x) = \text{Arctan}(\sin\vartheta)$
c) $f(x) = \ln(\text{Arcsin}x)$
d) $f(x) = \text{Arctan}(\ln x)$
e) $f(x) = \text{Arcsin}\left(\sqrt{1-x^2}\right)$
f) $f(x) = \text{Arctan}\left(\sqrt{1-x^2}\right)$
g) $f(x) = \text{Arccos}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$

⑤ Evaluate the following limits.

a) $\lim\limits_{x \to +\infty} \text{Arcsin}\left(\dfrac{x^2}{x^2+x+1}\right)$
b) $\lim\limits_{x \to -\infty} \text{Arctan}\left(\dfrac{x^2\sqrt{3}}{(x+3)^2}\right)$
c) $\lim\limits_{x \to 3^-} \text{Arctan}\left(\dfrac{x^2+1}{x^2-9}\right)$
d) $\lim\limits_{x \to 1^+} \text{Arccot}\left(\dfrac{1-2x^2}{x^2+4x+3}\right)$
e) $\lim\limits_{x \to -\infty} \text{Arctan}\left((x+2)^2(x-3)^3\right)$
f) $\lim\limits_{x \to 0^+} \text{Arctan}(\ln(\sin x))$

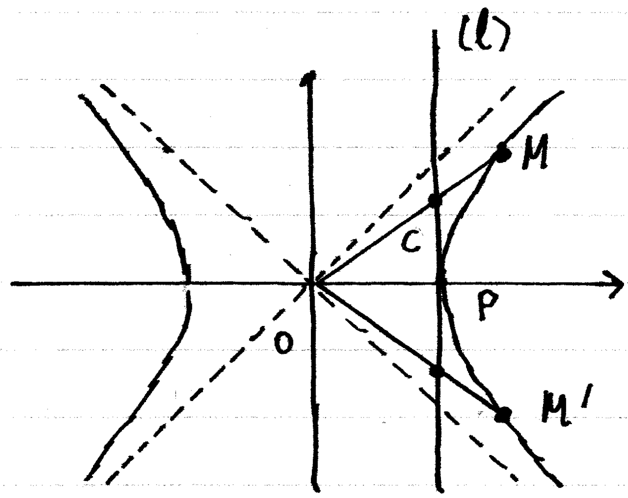
Hyperbolic functions

- They are defined geometrically in terms of the hyperbola (c): $x^2 - y^2 = 1$



x = the area between the hyperbola and OM and its mirror image OM'. Project M to A and B. For $x<0$ we use the left branch of the hyperbola. Thus we define

$$\boxed{\begin{aligned} \sinh(x) &= \overline{OB} \\ \cosh(x) &= \overline{OA} \end{aligned}}$$



To define tanh, for $x>0$, consider the line $(\ell): x = 1$. Let C be the point where OM intersects (ℓ). For $x<0$ we use the left branch, the line $(\ell): x=-1$ and place the point M below the x-axis. Then we define

$$\boxed{\tanh(x) = \overline{PC}}$$

with P the point where (ℓ) intersects x-axis.

Algebraic properties

Function	Domain	Range	Parity
$\sinh(x) = \dfrac{e^x - e^{-x}}{2}$	$\mathbb{R}$	$\mathbb{R}$	odd
$\cosh(x) = \dfrac{e^x + e^{-x}}{2}$	$\mathbb{R}$	$[1, +\infty)$	even
$\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	$\mathbb{R}$	$(-1, 1)$	odd

Variation Table

x		0	
$\sinh(x)$	$-\infty$ ↗		↗ $+\infty$
$\cosh(x)$	$+\infty$ ↘		↗ $+\infty$
$\tanh(x)$	-1 ↗		↗ 1

Derivatives

$\dfrac{d}{dx} \sinh(x) = \cosh(x)$
$\dfrac{d}{dx} \cosh(x) = \sinh(x)$
$\dfrac{d}{dx} \tanh(x) = 1/(\cosh x)^2 = 1 - \tanh^2 x$

Identities

$\sinh(-x) = -\sinh(x)$ $\cosh(-x) = \cosh(x)$ $\cosh^2 x - \sinh^2 x = 1$ $\dfrac{1}{\cosh^2 x} = 1 - \tanh^2 x$	$\sinh(x+y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x)$ $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

Inverse hyperbolic functions

Function	Domain	Range
$\sinh^{-1}(x) = \ln(x+\sqrt{x^2+1})$	$\mathbb{R}$	$\mathbb{R}$
$\cosh^{-1}(x) = \ln(x+\sqrt{x^2-1})$	$[1,+\infty)$	$[0,+\infty)$ (!!)
$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$(-1,1)$	$\mathbb{R}$

- Note that cosh is not "1-1" (because cosh is even), thus to define $\cosh^{-1}$ we restrict the domain of cosh to $[0,+\infty)$ before inverting. Thus

$$\cosh^{-1} = (\cosh \restriction [0,+\infty))^{-1}.$$

Derivatives of inverse trigonometric functions

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}, \quad \forall x \in \mathbb{R}$$

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}, \quad \underline{\underline{\forall x \in (-1,+\infty)}} \text{ (!!)}$$

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}, \quad \forall x \in (-1,1)$$

- Note that $\cosh^{-1}$ is not differentiable over its entire domain (not differentiable at $x=-1$)

EXERCISES

⑥ Prove, by definition, that

a) $\sinh(x+y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x)$

b) $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

c) $\cosh^2 x - \sinh^2 x = 1$.

Then use these identities to show that

d) $\sinh(x+y) + \sinh(x-y) = 2\sinh(x)\cosh(x)$

e) $\cosh(x+y) + \cosh(x-y) = 2\cosh(x)\cosh(y)$

f) $\cosh(x+y) - \cosh(x-y) = 2\sinh(x)\sinh(y)$

g) $\sinh(x+y)\sinh(x-y) = \sinh^2 x - \sinh^2 y$

h) $\cosh(x+y)\cosh(x-y) = \cosh^2 x + \sinh^2 y$

⑦ Show that:

a) $\cosh(2x) = \cosh^4 x - \sinh^4 x$

b) $\sinh^2(x)[1 - \tanh^2(x)] = \tanh^2(x)$

c) $(\cosh x + \sinh x)^n = \cosh(nx) + \sinh(nx)$

⑧ Use the properties of logarithms to show that

a) $\sinh^{-1}(x) + \sinh^{-1}(-x) = 0$

b) $\tanh^{-1}(x) + \tanh^{-1}(-x) = 0$

⑨ Prove the results given for the derivatives of the inverse hyperbolic functions $\sinh^{-1}(x)$, $\cosh^{-1}(x)$, $\tanh^{-1}(x)$.

(10) Show that

a) $\lim\limits_{x \to -1^+} \tanh^{-1}(x) = -\infty$

b) $\lim\limits_{x \to 1^-} \tanh^{-1}(x) = +\infty$

c) $\lim\limits_{x \to +\infty} \cosh^{-1}(x) = +\infty$

d) $\lim\limits_{x \to +\infty} \sinh^{-1}(x) = +\infty$

e) $\lim\limits_{x \to +\infty} [\cosh^{-1}(x) - \sinh^{-1}(x)] = 0$

f) $\lim\limits_{x \to -1^+} \sinh^{-1}(\tanh^{-1}(x)) = -\infty$

(11) Use the identity $e^{ix} = \cos x + i \sin x$ to show that

a) $\sin x = -i \sinh(ix)$

b) $\cos x = \cosh(ix)$

c) $\tan x = \tanh(ix)$

(Here: $i = \sqrt{-1}$).

↳ This exercise establishes the relationship between trigonometric functions and hyperbolic functions.

De L'Hospital's theorem

- The De L'Hospital theorem provides an additional method for resolving indeterminate forms $0/0$ and ∞/∞.

- **Thm**: Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ be two functions that satisfy the following conditions:

1) f, g differentiable at $N(x_0, \delta)$
2) $g'(x) \neq 0, \forall x \in N(x_0, \delta)$
3) $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \vee \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \pm\infty$
4) $\lim_{x \to x_0} \dfrac{f'(x)}{g'(x)} = \ell \in \mathbb{R} \vee \lim_{x \to x_0} \dfrac{f'(x)}{g'(x)} = \pm\infty$

Then, it follows that

$$\boxed{\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

↳ In using this theorem, we show condition (4) retroactively by successfully evaluating the limit. However, if application of De L'Hospital gives a limit that does not exist, then this shows that applying De L'Hospital is not properly justified, and the limit may or may not exist.

EXAMPLES

• $\boxed{0/0 \text{ Form}}$

a) $$\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}=\left(\frac{0}{0}\right)\overset{\text{L'Hospital}}{=}\lim_{x\to 1}\frac{[x-1-\ln x]'}{[(x-1)\ln x]'}=$$

$$=\lim_{x\to 1}\frac{1-0-\frac{1}{x}}{(x-1)'\ln x+(x-1)(\ln x)'}=$$

$$=\lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}}=\left(\frac{0}{0}\right)\underset{\text{L'Hospital.}}{=}$$

$$=\lim_{x\to 1}\frac{0-\left(-\frac{1}{x^2}\right)}{\frac{1}{x}+\frac{(x-1)'x-(x-1)(x)'}{x^2}}=$$

$$=\lim_{x\to 1}\frac{\frac{1}{x^2}}{\frac{1}{x}+\frac{x-(x-1)}{x^2}}=\lim_{x\to 1}\frac{\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}}=$$

$$=\frac{1}{1+1}=\frac{1}{2}$$

b) $$\lim_{x\to 0} \frac{2\sin x - \sin(2x)}{x - \sin x} = \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{2\cos x - 2\cos(2x)}{1-\cos x}$$

$$= \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{\sin x} = \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-2\cos x + 8\cos 2x}{\cos x} = \frac{-2+8}{1} = 6.$$

↳ We apply De L'Hospital 3 times!

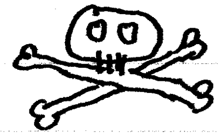
- $\boxed{\text{Form } \infty/\infty}$

c) $$\lim_{x\to +\infty} \frac{\ln(1+e^x)}{x+1} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to +\infty} \frac{\frac{(1+e^x)'}{1+e^x}}{1} =$$

$$= \lim_{x\to +\infty} \frac{e^x}{e^x+1} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to +\infty} \frac{e^x}{e^x} = 1.$$

- <u>Failure of De L'Hospital:</u>

d) $$\lim_{x\to \infty} \frac{x+\sin x}{x} = \left(\frac{\infty}{\infty}\right) \overset{\uparrow}{=} \lim_{x\to \infty} \frac{1+\cos x}{1} =$$



(wrong step)
(condition 4 fails)

$$= \lim_{x\to \infty} (1+\cos x) \longleftarrow \text{does not exist}$$

However: $f(x) = \dfrac{x+\sin x}{x} = 1 + \dfrac{\sin x}{x}$ and since

$$\left.\begin{array}{l}\sin x \text{ bounded at } \mathbb{R} \\ \lim\limits_{x\to+\infty}\frac{1}{x}=0\end{array}\right\} \Rightarrow \lim_{x\to+\infty}\frac{\sin x}{x}=0 \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(1+\frac{\sin x}{x}\right) = 1+0 = 1.$$

so the limit does in fact exist!

- <u>Application</u>: $\boxed{\lim\limits_{x\to+\infty}[f(x)+f'(x)] = \ell \Rightarrow \begin{cases}\lim\limits_{x\to+\infty} f(x)=\ell \\ \lim\limits_{x\to+\infty} f'(x)=0\end{cases}}$

<u>Proof</u>

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{e^x f(x)}{e^x} = \lim_{x\to+\infty}\frac{(e^x)' f(x) + e^x f'(x)}{(e^x)'} =$$

$$= \lim_{x\to+\infty}\frac{e^x[f(x)+f'(x)]}{e^x} =$$

$$= \lim_{x\to+\infty}[f(x)+f'(x)] = \ell.$$

Thus: $\lim\limits_{x\to+\infty} f'(x) = \lim\limits_{x\to+\infty}[(f(x)+f'(x)) - f(x)] =$

$$= \lim_{x\to+\infty}(f(x)+f'(x)) - \lim_{x\to+\infty} f(x) =$$

$$= \ell - \ell = 0. \quad \square$$

Other indeterminent forms

Other indeterminent forms can be reduced to to the forms 0/0 and ∞/∞.

- $\boxed{\text{Form } \infty - \infty}$: Simplify and reduce to 0/0 or ∞/∞.

a) $$\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=(\infty-\infty)=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}=\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^4}\,\frac{x^2}{\sin^2 x}\right)=$$

$$=\lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^4}\right)\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2=$$

$$=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4}\underset{*}{=}\lim_{x\to 0}\frac{2x-2\sin x\cos x}{4x^3}=$$

$$=\lim_{x\to 0}\frac{2x-\sin 2x}{4x^3}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2-2\cos 2x}{12x^2}=$$

$$=\lim_{x\to 0}\frac{1-\cos 2x}{6x^2}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sin 2x}{6x}=\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{2\cos 2x}{6}=\frac{2\cos 0}{6}=\frac{1}{3}.$$

As can be seen from this example, De L'Hospital's theorem supplements our previous techniques for evaluating trigonometric limits but does <u>not</u> render them obsolete!

• $\boxed{\text{Form } 0\cdot\infty}$ ⟶ It occurs for a function of the form $f(x) = g_1(x) g_2(x)$ with $\lim\limits_{x\to\sigma} g_1(x) = 0$ and $\lim\limits_{x\to\sigma} g_2(x) = \pm\infty$

We use the following technique:

$$f(x) = g_1(x) g_2(x) = \frac{g_1(x)}{1/g_2(x)} = \left(\frac{0}{0}\right)$$

and apply De L'Hospital. Or: $f(x) = \dfrac{g_2(x)}{1/g_1(x)} = \left(\dfrac{\infty}{\infty}\right)$

EXAMPLE

6) $$\lim_{x\to 0^+} (\tan x \cdot \ln x) = (0\cdot(-\infty)) = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\tan x}} = \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{\tan^2 x}(\tan x)'} =$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{\tan^2 x}\frac{1}{\cos^2 x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}}$$

$$= \lim_{x\to 0^+} \frac{-\sin^2 x}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} (-\sin x)$$

$$= 1\cdot(-0) = 0.$$

• $\boxed{\text{Form } 0^0, \infty^0, 1^\infty}$ → Occurs from functions of the form $f(x) = a(x)^{b(x)}$.

We use the following technique:

$$\lim_{x\to\sigma} f(x) = \lim_{x\to\sigma} a(x)^{b(x)} = \lim_{x\to\sigma} \exp(b(x)\ln a(x))$$

$$= \exp\left(\lim_{x\to\sigma} b(x)\ln a(x)\right) = \cdots$$

The resulting limit is $0/0$, ∞/∞, $0\cdot\infty$.

EXAMPLE

a) $$\lim_{x\to 0}\left(\tan\frac{x}{2}\right)^{1/\ln x} = (0^0) = \lim_{x\to 0}\exp\left(\frac{1}{\ln x}\ln\left(\tan\frac{x}{2}\right)\right)$$

$$= \exp\left[\lim_{x\to 0}\frac{\ln(\tan(x/2))}{\ln x}\right] = \left(\frac{\infty}{\infty}\right)$$

$$= \exp\left[\lim_{x\to 0}\frac{\frac{1}{\tan(x/2)}\left(\tan(x/2)\right)'}{1/x}\right] =$$

$$= \exp\left[\lim_{x\to 0}\frac{\frac{1}{\tan(x/2)}\,\frac{1}{\cos^2(x/2)}\,\frac{1}{2}}{1/x}\right]$$

$$= \exp\left[\lim_{x\to 0}\frac{x}{2\sin(x/2)\cos(x/2)}\right] =$$

$$= \exp\left[\lim_{x\to 0} \frac{x/2}{\sin(x/2)} \lim_{x\to 0} \frac{1}{\cos(x/2)}\right] =$$

$$= \exp\left[1 \cdot \frac{1}{\cos 0}\right] = e^1 = e.$$

8) $$\lim_{x\to+\infty} \left(\frac{x+2}{x+1}\right)^x = \lim_{x\to+\infty} \exp\left[x \ln\left(\frac{x+2}{x+1}\right)\right] =$$

$$= \exp\left[\lim_{x\to+\infty} x \ln\left(\frac{x+2}{x+1}\right)\right] =$$

$$= \exp\left[\lim_{x\to+\infty} \frac{\ln\left(\frac{x+2}{x+1}\right)}{1/x}\right] = \left(\frac{0}{0}\right)$$

$$= \exp\left[\lim_{x\to+\infty} \frac{\frac{x+1}{x+2}\left(\frac{x+2}{x+1}\right)'}{-1/x^2}\right] =$$

$$= \exp\left[\lim_{x\to+\infty} \left(-x^2 \frac{x+1}{x+2} \frac{(x+2)'(x+1)-(x+2)(x+1)'}{(x+1)^2}\right)\right]$$

$$= \exp\left[\lim_{x\to+\infty} \left(-x^2 \frac{x+1}{x+2} \frac{(x+1)-(x+2)}{(x+1)^2}\right)\right] =$$

$$= \exp\left[\lim_{x\to+\infty} \left(-x^2 \frac{x+1}{x+2} \frac{-1}{(x+1)^2}\right)\right] =$$

$$= \exp\left[\lim_{x \to +\infty}\left(\frac{x^2}{(x+1)(x+2)}\right)\right] = \exp\left[\lim_{x \to +\infty}\frac{x^2}{x^2}\right]$$

$$= \exp(1) = e.$$

EXERCISES

(12) Evaluate the following limits:

1) $\lim\limits_{x\to\pi/2} \dfrac{\cos x}{2x-\pi}$

2) $\lim\limits_{x\to 0} \dfrac{\cos 3x - 1}{x^2}$

3) $\lim\limits_{x\to 0} \dfrac{e^x+e^{-x}-2\cos x}{1-\cos x}$

4) $\lim\limits_{x\to 0} \dfrac{1-\cos x-\ln(\cos x)}{x^2}$

5) $\lim\limits_{x\to +\infty} \dfrac{\ln^2 x}{x}$

6) $\lim\limits_{x\to 0} \dfrac{\cos x-\cos 2x}{x^2}$

7) $\lim\limits_{x\to +\infty} \dfrac{e^x}{x+\ln x}$

8) $\lim\limits_{x\to 0^+} \left(\dfrac{1}{1-\cos x}-\dfrac{1}{x}\right)$

9) $\lim\limits_{x\to 0^+} \left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)$

10) $\lim\limits_{x\to 1^+} \left(\dfrac{1}{x-1}-\dfrac{1}{\ln x}\right)$

11) $\lim\limits_{x\to 0^+} x^x$

12) $\lim\limits_{x\to +\infty} (\ln x)^{1/x}$

13) $\lim\limits_{x\to 0^+} x^{\sin x}$

14) $\lim\limits_{x\to +\infty} \left[x^2 \ln\left(\cos\left(\dfrac{\pi}{x}\right)\right)\right]$

15) $\lim\limits_{x\to 0} \left(\dfrac{3e^x-e^{-x}}{2}\right)^{1/x}$

16) $\lim\limits_{x\to 0} (\sin x \cdot \ln x)$

17) $\lim\limits_{x\to 0^+} (\sin x)^{\sin x}$

18) $\lim\limits_{x\to 0^+} (\sin x)^x$

19) $\lim\limits_{x\to 1^+} \left(\dfrac{1}{x-1}\right)^{x-1}$

20) $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2}-\cot^2 x\right)$

21) $\lim\limits_{x\to 0^+} (1+x^2)^{\cot x}$

22) $\lim\limits_{x\to +\infty} \left(1+\dfrac{3}{x}\right)^x$

(13) Consider the limit $\lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin x}$

a) Show that the first 3 conditions of the De L'Hospital theorem are satisfied but that the 4th condition is violated. (Hint: Use without proof the statement that $\lim_{x\to\pm\infty} \cos x$ does not exist)

b) Use other methods to show that

$$\lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin x} = 0$$

(14) Let f be a function differentiable in $(-a, a)$ with $a>0$ and $f(0) = f'(0) = 1$ and $\forall x \in (-a, a): f(x) > 0$. Show that

$$\lim_{x\to 0} [f(x)]^{1/x} = e$$

(15) Let f be a function that is twice differentiable in $\mathbb{R}$ with f' continuous in $\mathbb{R}$ and $f(0) = f'(0) = 0$. Let g be defined as

$$g(x) = \begin{cases} \frac{f(x)}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that

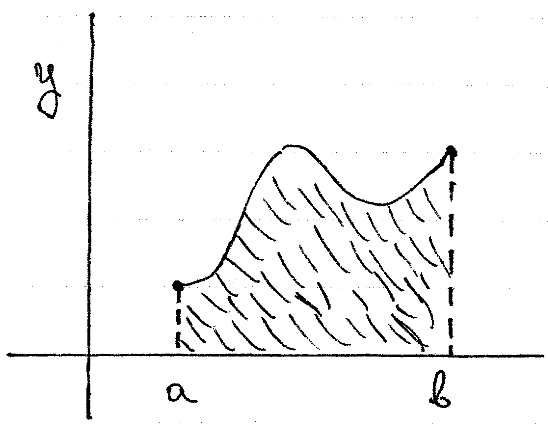
a) g differentiable in $\mathbb{R}$

b) g' continuous in $\mathbb{R}$.

CAL1.8: Introduction to integrals

INTEGRAL EQUATIONS

▼ Riemann integral definition



The problem is to calculate the area A between the x-axis, the lines $(l_1): x = a$ and $(l_2): x = b$, and the curve $(c): y = f(x)$. The solution of the problem, according to Riemann, is as follows:

•1 Divide the interval $[a, b]$ to n equal intervals $[x_{k-1}, x_k]$ with

$$x_k = a + (b-a)(k/n) \ , \ \forall k \in [n] \cup \{0\}$$

with $[n] = \{1, 2, \ldots, n\}$.

•2 Let m_k and M_k be the minimum and maximum value of f in the interval $[x_{k-1}, x_k]$, defined as:

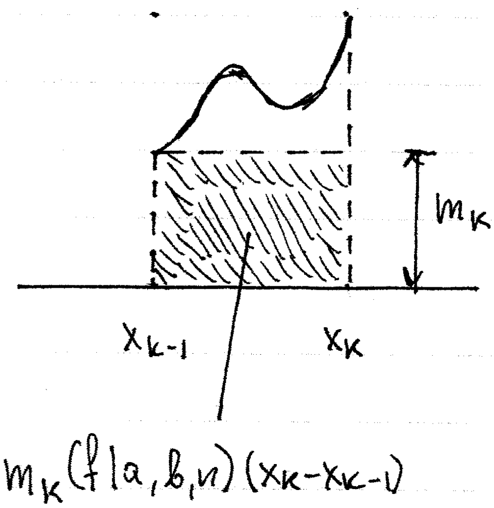
$$m_k(f \mid a, b, n) = \min_{x \in [x_{k-1}, x_k]} f(x) \ , \ \forall k \in [n]$$

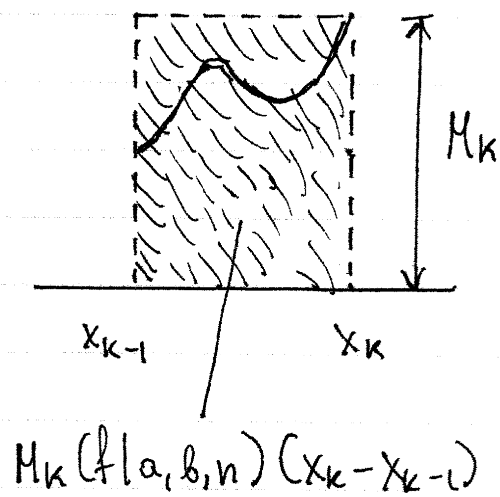
$$M_k(f \mid a, b, n) = \max_{x \in [x_{k-1}, x_k]} f(x) \ , \ \forall k \in [n]$$

•3 We form the Riemann sums $L_n(f \mid a, b)$ and $U_n(f \mid a, b)$ given by:

$$L_n(f\mid a,b) = \sum_{k=1}^{n} m_k(f\mid a,b,n)(x_k - x_{k-1})$$

$$U_n(f\mid a,b) = \sum_{k=1}^{n} M_k(f\mid a,b,n)(x_k - x_{k-1})$$





Obviously, the actual area A will satisfy:
$$\forall n\in\mathbb{N}-\{0\}: L_n(f\mid a,b) \leq A \leq U_n(f\mid a,b)$$

• 4 We prove that
$$\lim_{n\in\mathbb{N}} L_n(f\mid a,b) = \lim_{n\in\mathbb{N}} U_n(f\mid a,b) = A$$

$\longmapsto$ <u>Riemann integrability</u>

• Let $f: A\to\mathbb{R}$ be a function with $[a,b]\subseteq A$. We say that

$$\boxed{\begin{array}{l} f \text{ integrable} \iff \exists A \in \mathbb{R} : \lim\limits_{n \in \mathbb{N}^*} L_n(f \mid a, b) = \lim\limits_{n \in \mathbb{N}^*} U_n(f \mid a, b) = A \\ \text{at } [a, b] \end{array}}$$

- A very important result of Riemann's theory, which is difficult to prove, is that

$$\boxed{f \text{ continuous at } [a,b] \Rightarrow f \text{ integrable at } [a,b]}$$

→ <u>Integral notation</u>

If f is a function that is integrable at $[a,b]$, then we define:

$\int_a^b f(x)\,dx = \lim\limits_{n \in \mathbb{N}^*} L_n(f \mid a, b) = \lim\limits_{n \in \mathbb{N}^*} U_n(f \mid a, b)$
$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
$\int_a^a f(x)\,dx = 0$

▼ Integral evaluation via approximation sequence

- Let $f: A \to \mathbb{R}$ be a function with $[a,b] \subseteq A$ and assume that f is integrable at $[a,b]$. Then:

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\in\mathbb{N}^*} \left[\frac{b-a}{n} \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right) \right]}$$

- Note that the existence of the limit above does not imply that f is integrable at $[a,b]$. We need to know ahead of time that f is integrable at $[a,b]$ to apply the above statement.
- Evaluating integrals via approximation sequences requires use of the following basic sums:

$$\boxed{\begin{aligned} S_1(n) &= \sum_{k=1}^{n} k = 1+2+\cdots+n = \frac{n(n+1)}{2} \\ S_2(n) &= \sum_{k=1}^{n} k^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6} \\ S_3(n) &= \sum_{k=1}^{n} k^3 = 1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4} = [S_1(n)]^2 \end{aligned}}$$

- Proof of $S_1(n)$

We note that $(x+1)^2 = x^2+2x+1$, $\forall x \in \mathbb{R}$

For $x=1$: $\quad 2^2 = 1^2 + 2\cdot 1 + 1$

For $x=2$: $\quad 3^2 = 2^2 + 2\cdot 2 + 1$

For $x=3$: $\quad 4^2 = 3^2 + 2\cdot 3 + 1$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

For $x=n$: $\quad (n+1)^2 = n^2 + 2n + 1.$

Adding the above equations gives:

$$\sum_{k=2}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + n \Rightarrow$$

$$\Rightarrow (n+1)^2 + \sum_{k=2}^{n} k^2 = 1^2 + \sum_{k=2}^{n} k^2 + 2S_1(n) + n \Rightarrow$$

$$\Rightarrow (n+1)^2 = 1 + 2S_1(n) + n \Rightarrow$$

$$\Rightarrow 2S_1(n) = (n+1)^2 - n - 1 = (n+1)^2 - (n+1) =$$

$$= (n+1)\left[(n+1) - 1\right] = n(n+1) \Rightarrow$$

$$\Rightarrow S_1(n) = \frac{n(n+1)}{2}$$

↳ The expressions for $S_2(n), S_3(n)$ can be derived similarly using

$(x+1)^3 = x^3 + 3x^2 + 3x + 1$

$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$

etc...

↳ To evaluate the resulting sequence limits, we use the following statement:

$$\lim_{x\to+\infty} f(n) = a \Rightarrow \lim_{n\in\mathbb{N}^*} f(n) = a$$

EXAMPLE

Evaluate via approximation sequences the integral; and show:

$$I = \int_0^a x^2 dx = \frac{a^3}{3}$$

Solution

Define $\forall x \in \mathbb{R} : f(x) = x^2$. Since:

f continuous at $[0,a] \Rightarrow f$ integrable at $[0,a] \Rightarrow$

$$\Rightarrow I = \int_0^a x^2 dx = \lim_{n\in\mathbb{N}^*} \left[\frac{a-0}{n} \sum_{k=1}^{n} f\left(0 + k\frac{a-0}{n}\right)\right] =$$

$$= \lim_{n\in\mathbb{N}^*} \left[\frac{a}{n} \sum_{k=1}^{n} f\left(\frac{ka}{n}\right)\right] = \lim_{n\in\mathbb{N}^*} \left[\frac{a}{n} \sum_{k=1}^{n} \left(\frac{ka}{n}\right)^2\right]$$

$$= \lim_{n\in\mathbb{N}^*} \left[\left(\frac{a}{n}\right)^3 \sum_{k=1}^{n} k^2\right] = \lim_{n\in\mathbb{N}^*} \left[\left(\frac{a}{n}\right)^3 S_2(n)\right] =$$

$$= \lim_{n\in\mathbb{N}^*} \left[\frac{a^3}{n^3} \frac{n(n+1)(2n+1)}{6}\right] = \frac{a^3}{6} \lim_{n\in\mathbb{N}^*} \left[\frac{(n+1)(2n+1)}{n^2}\right]$$

$$= \frac{a^3}{6} \lim_{x\to+\infty} \left(\frac{(x+1)(2x+1)}{x^2}\right) = \frac{a^3}{6} \lim_{x\to+\infty} \frac{2x^2}{x^2} =$$

$$= \frac{a^3}{6} \cdot 2 = \frac{a^3}{3}.$$

$\hookrightarrow$ This method for solving the corresponding geometric problem (the area under a parabola) was initially discovered more than 2000 years ago in Ancient Greece by Archimedes.

Integral of the exponential function

- The integrals of some exponential functions can also be evaluated directly from approximation sequences as follows: First we show that:

$$\boxed{\forall a \in \mathbb{R}-\{1\}: \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}}$$

Proof

Let $a \in \mathbb{R}-\{1\}$ be given. Then

$$(1-a)\sum_{k=0}^{n} a^k = \sum_{k=0}^{n}(a^k - a^{k+1}) = \sum_{k=0}^{n} a^k - \sum_{k=0}^{n} a^{k+1} =$$

$$= \sum_{k=0}^{n} a^k - \sum_{k=1}^{n+1} a^k =$$

$$= a^0 + \sum_{k=1}^{n} a^k - \sum_{k=1}^{n} a^k - a^{n+1} =$$

$$= 1 - a^{n+1} \Rightarrow$$

$$\Rightarrow \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \qquad \square$$

EXAMPLE

Use apporoximation sequences to show that

$$\int_0^a e^x \, dx = e^a - 1, \ \forall a \in (0, +\infty)$$

Solution

Define $\forall x \in \mathbb{R}: f(x) = e^x$. Then

f continuous at $[0,a] \Rightarrow f$ integrable at $[0,a] \Rightarrow$

$$\Rightarrow I = \int_0^a e^x dx = \lim_{n\in\mathbb{N}^*}\left[\frac{a-0}{n}\sum_{k=1}^{n} f\left(0 + k\frac{a-0}{n}\right)\right]$$

$$= \lim_{n\in\mathbb{N}^*}\left[\frac{a}{n}\sum_{k=1}^{n} f\left(\frac{ka}{n}\right)\right] = \lim_{n\in\mathbb{N}^*}\left[\frac{a}{n}\sum_{k=1}^{n} e^{ka/n}\right] =$$

$$= \lim_{n\in\mathbb{N}^*}\left[\frac{a}{n}\sum_{k=1}^{n} (e^{a/n})^k\right]$$

$$= \lim_{n\in\mathbb{N}^*}\left[\frac{a}{n}\left(\sum_{k=0}^{n} (e^{a/n})^k\right) - \frac{a}{n}\right] =$$

$$= \lim_{n\in\mathbb{N}^*}\left[\frac{a}{n}\sum_{k=0}^{n} (e^{a/n})^k\right] = \lim_{n\in\mathbb{N}^*}\left[\frac{a}{n}\,\frac{1-e^{a(n+1)/n}}{1-e^{a/n}}\right]$$

We note that:

$$\lim_{x\to+\infty}\frac{a(x+1)}{x} = \lim_{x\to+\infty}\frac{ax}{x} = a \Rightarrow \lim_{x\to+\infty}\exp\left(\frac{a(x+1)}{x}\right) = e^a$$

$$\Rightarrow \lim_{x\to+\infty}\left[1-\exp\left(\frac{a(x+1)}{x}\right)\right] = 1-e^a \qquad (1)$$

and for $y = a/x$, we have $x\to+\infty \Rightarrow y\to 0^+$ with $y\neq 0$ and therefore:

$$\lim_{x\to+\infty}\left[\frac{a}{x}\,\frac{1}{1-e^{a/x}}\right] = \lim_{y\to 0^+}\left[\frac{y}{1-e^y}\right] = \lim_{y\to 0^+}\frac{1}{-e^y}$$

$$= \frac{1}{-e^0} = -1. \qquad (2)$$

From Eq. (1) and Eq. (2) it follows that:

$$I = \lim_{x\to+\infty}\left[\frac{a}{x}\,\frac{1-\exp\left(\frac{a(x+1)}{x}\right)}{1-\exp\left(\frac{a}{x}\right)}\right] =$$

$$= \lim_{x\to+\infty}\left[\frac{a}{x}\,\frac{1}{1-\exp(a/x)}\right]\lim_{x\to+\infty}\left[1-\exp\left(\frac{a(x+1)}{x}\right)\right]$$

$$= (-1)(1-e^{a}) = e^{a}-1 .$$

EXERCISES

① Prove the results for the following basic sums:

a) $S_2(n) = \sum_{k=1}^{n} k^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

b) $S_3(n) = \sum_{k=1}^{n} k^3 = 1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4} = [S_1(n)]^2$

(Hint: Use $(x+1)^3 = x^3+3x^2+3x+1$
$(x+1)^4 = x^4+4x^3+6x^2+4x+1$)

② Use approximation sequences to show that

a) $\int_a^b c\,dx = c(b-a)$

b) $\int_a^b x\,dx = \frac{b^2-a^2}{2}$

c) $\int_a^b x^2 dx = \frac{b^3-a^3}{3}$

d) $\int_a^b x^3 dx = \frac{b^4-a^4}{4}$

e) $\int_a^b e^x dx = e^b - e^a$

f) $\int_a^b 2^x dx = \frac{2^b-2^a}{\ln 2}$

▼ Properties of integrals

① Linearity

Let f, g be integrable at $[a,b]$. Then:

$$\boxed{\begin{aligned} &\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \\ &\forall \lambda \in \mathbb{R}: \int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx \end{aligned}}$$

By induction, these properties give; in general:

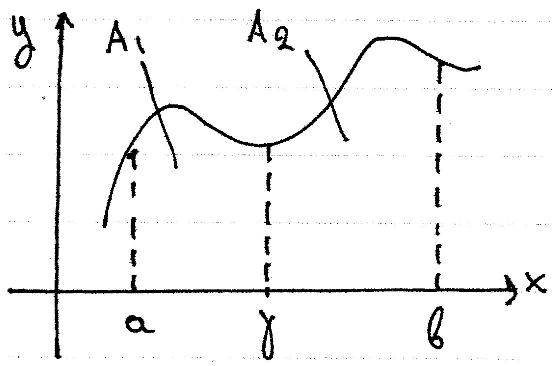
$$\boxed{\left.\begin{aligned} &\forall k \in [n]: f_k \text{ integrable at } [a,b] \\ &\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R} \end{aligned}\right\} \Rightarrow \int_a^b \left[\sum_{k=1}^n \lambda_k f_k(x)\right] dx = \sum_{k=1}^n \lambda_k \int_a^b f_k(x)\,dx}$$

↳ These properties can be easily proved using approximation sequences.

② Chasles theorem

$$\boxed{\left.\begin{aligned} &f \text{ integrable at } I \\ &I \text{ interval} \\ &a, b, \gamma \in I \end{aligned}\right\} \Rightarrow \int_a^b f(x)\,dx = \int_a^\gamma f(x)\,dx + \int_\gamma^b f(x)\,dx}$$

▸ geometric interpratation



Let

A_1 = area between a, γ

A_2 = area between γ, b

A = area between a, b

Then it is easy to see that

$A = A_1 + A_2$.

Although proving the Chasles theorem geometrically is easy, proving it directly from the Riemann definition requires a lot of more effort.

▸ generalization: By induction, Chasles theorem can be generalized to give:

$$\boxed{\left.\begin{array}{l} f \text{ integrable at } I \\ I \text{ interval} \\ a, b, \gamma_1, \dots, \gamma_n \in I \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx = \int_a^{\gamma_1} f(x)\,dx + \sum_{k=1}^{n-1} \int_{\gamma_k}^{\gamma_{k+1}} f(x)\,dx + \int_{\gamma_n}^{b} f(x)\,dx}$$

③ Integral bounding

$$\boxed{\left.\begin{array}{l} f \text{ integrable at } [a,b] \\ \forall x \in [a,b] : f(x) \geq m \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx \geq m(b-a)}$$

$$\boxed{\left.\begin{array}{l} f \text{ integrable at } [a,b] \\ \forall x \in [a,b] : f(x) \leq m \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx \leq m(b-a)}$$

Proof

With no loss of generality, assume that f integrable at $[a,b]$ and that $\forall x \in [a,b]: f(x) \geq m$. Then

$$\forall x \in [a,b]: f(x) \geq m \Rightarrow$$

$$\Rightarrow \forall k \in [n]: f\left(a + k\frac{b-a}{n}\right) \geq m$$

$$\Rightarrow \forall k \in [n]: \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) \geq mn$$

$$\Rightarrow \quad \frac{b-a}{n} \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right) \geq mn\,\frac{b-a}{n} = m(b-a)$$

$$\Rightarrow I = \int_a^b f(x)\,dx = \lim_{n \in \mathbb{N}^*} \left[\frac{b-a}{n} \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)\right]$$

$$\geq m(b-a) \Rightarrow$$

$$\Rightarrow \int_a^b f(x)\,dx \geq m(b-a). \qquad \square$$

$\mapsto$ For $m=0$, the above statement gives:

$$\left.\begin{array}{l} f \text{ integrable at } [a,b] \\ \forall x \in [a,b]: f(x) \geq 0 \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx \geq 0$$

and an immediate corollary is that

$$\boxed{\left.\begin{array}{l} f,g \text{ integrable at } [a,b] \\ \forall x \in [a,b]: f(x) \geq g(x) \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx}$$

Note that the above results require $a<b$!!

(4) Integral Mean Value Theorem

$$\left.\begin{array}{l} f \text{ continuous at } [a,b] \\ g \text{ integrable at } [a,b] \\ \forall x\in[a,b]: g(x)\geqslant 0 \end{array}\right\} \Rightarrow \exists \xi\in[a,b]: \int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$$

Proof

From the extremum value theorem:

f continuous at $[a,b] \Rightarrow$

$\Rightarrow \exists x_1, x_2 \in [a,b]: \forall x\in[a,b]: f(x_1)\leqslant f(x)\leqslant f(x_2)$

Since $\forall x\in[a,b]: g(x)\geqslant 0$, it follows that

$\forall x\in[a,b]: f(x_1)g(x)\leqslant f(x)g(x)\leqslant f(x_2)g(x)$

$$\Rightarrow \int_a^b f(x_1)g(x)\,dx \leqslant \int_a^b f(x)g(x)\,dx \leqslant \int_a^b f(x_2)g(x)\,dx$$

$$\Rightarrow f(x_1)\int_a^b g(x)dx \leqslant \int_a^b f(x)g(x)dx \leqslant f(x_2)\int_a^b g(x)dx \quad (1)$$

We also note that:

$$\forall x\in[a,b]: g(x)\geqslant 0 \Rightarrow \int_a^b g(x)dx \geqslant 0$$

and we may therefore distinguish between the following cases:

Case 1: Assume that $\int_a^b g(x)dx = 0$. Then, from Eq. (1):

$$0\leqslant \int_a^b g(x)dx \leqslant 0 \Rightarrow \int_a^b g(x)dx = 0$$

$$0 \leqslant \int_a^b f(x)g(x)dx \leqslant 0 \Rightarrow$$

$$\Rightarrow \forall \xi \in [a,b]: \int_a^b f(x)g(x)\,dx = 0 = f(\xi)\cdot 0 = f(\xi)\int_a^b g(x)\,dx$$

$$\Rightarrow \exists \xi \in [a,b]: \int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)\,dx.$$

<u>Case 2</u>: Assume that $\int_a^b g(x)dx > 0$.

Let us define $A = \dfrac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)\,dx}$. From Eq. (1).

$f(x_1) \leqslant A \leqslant f(x_2)$. Assume with no loss of generality that $x_1 < x_2$

Furthermore, from the intermediate value theorem:

$$\left.\begin{array}{l} f \text{ continuous at } [a,b] \\ x_1, x_2 \in [a,b] \end{array}\right\} \Rightarrow [f(x_1), f(x_2)] \subseteq f([x_1, x_2])$$

and it follows that

$$\begin{aligned} f(x_1) \leqslant A \leqslant f(x_2) &\Rightarrow A \in [f(x_1), f(x_2)] \\ &\Rightarrow A \in f([x_1, x_2]) \\ &\Rightarrow \exists \xi \in [x_1, x_2] : f(\xi) = A \\ &\Rightarrow \exists \xi \in [a,b]: f(\xi) = A \\ &\Rightarrow \exists \xi \in [a,b]: \int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx \quad \square \end{aligned}$$

↳ For $\forall x \in [a,b]: g(x) = 1$, the integral mean-value theorem gives:

$$\boxed{f \text{ continuous at } [a,b] \Rightarrow \int_a^b f(x)dx = f(\xi)(b-a)}$$

(5) Absolute value of integral

$$\left.\begin{array}{l} f \text{ integrable at } [a,b] \\ |f| \text{ integrable at } [a,b] \end{array}\right\} \Rightarrow \left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx$$

Proof

Let us define:

$$A_n = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)$$

$$B_n = \frac{b-a}{n} \sum_{k=1}^{n} \left|f\left(a + k\frac{b-a}{n}\right)\right|$$

Then

$$\left\{\begin{array}{l} f \text{ integrable at } [a,b] \\ |f| \text{ integrable at } [a,b] \end{array}\right. \Rightarrow \lim_{n\in\mathbb{N}^*} A_n = \int_a^b f(x)\,dx \wedge \lim_{n\in\mathbb{N}^*} B_n = \int_a^b |f(x)|\,dx$$

We also note that

$$\forall n\in\mathbb{N}^*: |A_n| = \left|\frac{b-a}{n} \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)\right| =$$

$$= \left|\frac{b-a}{n}\right| \left|\sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)\right| =$$

$$= \frac{b-a}{n} \left|\sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)\right|$$

$$\leqslant \frac{b-a}{n} \sum_{k=1}^{n} \left|f\left(a+k\frac{b-a}{n}\right)\right| = B_n \Rightarrow$$

$$\Rightarrow \forall n\in\mathbb{N}^*: |A_n| \leqslant B_n \Rightarrow \forall n\in\mathbb{N}^*: -B_n \leqslant A_n \leqslant B_n \Rightarrow$$

$$\Rightarrow -\lim_{n\in\mathbb{N}^*} B_n \leqslant \lim_{n\in\mathbb{N}^*} A_n \leqslant \lim_{n\in\mathbb{N}^*} B_n \Rightarrow \left|\lim_{n\in\mathbb{N}^*} A_n\right| \leqslant \lim_{n\in\mathbb{N}^*} B_n$$

$$\Rightarrow \left| \int_a^b f(x)dx \right| = \left| \lim_{n\in\mathbb{N}^*} A_n \right| \leqslant \lim_{n\in\mathbb{N}^*} B_n = \int_a^b |f(x)|dx$$

$$\Rightarrow \left| \int_a^b f(x)dx \right| \leqslant \int_a^b |f(x)|dx \qquad \square$$

▸ Summary: Properties of integrals

① $(\forall k \in [n] : f_k \text{ integrable at } [a,b])$
$$\Rightarrow \forall \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R} : \int_a^b \left[\sum_{k=1}^{n} \lambda_k f_k(x) \right] dx = \sum_{k=1}^{n} \lambda_k \int_a^b f_k(x)\,dx$$

② f integrable at I, I interval, $a, b, \gamma_1, \ldots, \gamma_n \in I$
$$\Rightarrow \int_a^b f(x)\,dx = \int_a^{\gamma_1} f(x)\,dx + \sum_{k=1}^{n-1} \int_{\gamma_k}^{\gamma_{k+1}} f(x)\,dx + \int_{\gamma_n}^{b} f(x)\,dx$$

③ f integrable at $[a,b]$, $\forall x \in [a,b] : f(x) \geq m$
$$\Rightarrow \int_a^b f(x)\,dx \geq m(b-a)$$

f integrable at $[a,b]$, $\forall x \in [a,b] : f(x) \leq m$
$$\Rightarrow \int_a^b f(x)\,dx \leq m(b-a)$$

↳ f, g integrable at $[a,b]$, $\forall x \in [a,b] : f(x) \geq g(x)$
$$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

④ f continuous at $[a,b]$, g integrable at $[a,b]$, $\forall x \in [a,b] : g(x) \geq 0$
$$\Rightarrow \exists \xi \in [a,b] : \int_a^b f(x) g(x)\,dx = f(\xi) \int_a^b g(x)\,dx$$

↳ f continuous at $[a,b] \Rightarrow \exists \xi \in [a,b] : \int_a^b f(x)\,dx = f(\xi)(b-a)$

⑤ f integrable at $[a,b]$, $|f|$ integrable at $[a,b]$
$$\Rightarrow \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

EXAMPLES

a) Let f be continuous on an interval I and let $a,b,c,d \in I$. Show that:

$$\int_a^b f(x)dx \int_c^d f(x)dx + \int_a^c f(x)dx \int_d^b f(x)dx + \int_a^d f(x)dx \int_b^c f(x)dx = 0$$

Solution

$$I = \int_a^b f(x)dx \int_c^d f(x)dx + \int_a^c f(x)dx \int_d^b f(x)dx + \int_a^d f(x)dx \int_b^c f(x)dx =$$

$$= \int_a^b f(x)dx \int_c^d f(x)dx + \left[\int_a^b f(x)dx + \int_b^c f(x)dx\right] \int_d^b f(x)dx + \int_a^d f(x)dx \int_b^c f(x)dx$$

$$= \int_a^b f(x)dx \left[\int_c^d f(x)dx + \int_d^b f(x)dx\right] + \int_b^c f(x)dx \left[\int_d^b f(x)dx + \int_a^d f(x)dx\right]$$

$$= \int_a^b f(x)dx \int_c^b f(x)dx + \int_b^c f(x)dx \int_a^b f(x)dx =$$

$$= -\int_a^b f(x)dx \int_b^c f(x)dx + \int_a^b f(x)dx \int_b^c f(x)dx = 0.$$

→ The following example is based on the integral properties and the following result, which can be proved directly from approximating sequences:

$$\int_a^b c\,dx = c(b-a).$$

b) Evaluate $I = \int_0^2 \frac{(2x^2+1)dx}{x^2+2} - 3\int_2^0 \frac{dx}{x^2+2}$

Solution

$$I = \int_0^2 \frac{2x^2+1}{x^2+2}dx - 3\int_2^0 \frac{dx}{x^2+2} =$$
$$= \int_0^2 \frac{2x^2+1}{x^2+2}dx + 3\int_0^2 \frac{dx}{x^2+2} =$$
$$= \int_0^2 \left[\frac{2x^2+1}{x^2+2} + \frac{3}{x^2+2}\right]dx$$
$$= \int_0^2 \frac{2x^2+1+3}{x^2+2}dx = \int_0^2 \frac{2x^2+4}{x^2+2}dx =$$
$$= \int_0^2 \frac{2(x^2+2)}{x^2+2}dx = \int_0^2 2dx = 2\cdot 2 = 4.$$

c) Let f be a function that is continuous on $[a,b]$ such that $\forall x \in [a,b]: f(x) > 0$.
Show that $\int_a^b f(x)dx > 0$.

Solution

From the integral mean-value theorem:
f continuous at $[a,b] \Rightarrow \exists \xi \in [a,b]: \int_a^b f(x)dx = f(\xi)(b-a)$
Since:
$$(\forall x \in [a,b]: f(x) > 0) \Rightarrow f(\xi) > 0 \Rightarrow f(\xi)(b-a) > 0 \Rightarrow$$
$$\Rightarrow \int_a^b f(x)dx > 0 \qquad \square$$

d) Show that $\lim\limits_{a\to+\infty}\displaystyle\int_a^{a+1}\frac{dx}{x^2+1}=0$

Solution

Define $\forall x\in\mathbb{R}: f(x)=\dfrac{1}{x^2+1}\Rightarrow$

$\Rightarrow \forall x\in\mathbb{R}: f'(x)=\dfrac{-(x^2+1)'}{(x^2+1)^2}=\dfrac{-2x}{(x^2+1)^2}$

x		0	
$-2x$	$+$	0	$-$
$(x^2+1)^2$	$+$		$-$
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$		$\searrow$

It follows that

$f\searrow(0,+\infty)\Rightarrow\forall a\in(0,+\infty):\forall x\in[a,a+1]: f(a)\geq f(x)\geq f(a+1)$

$\Rightarrow\forall a\in(0,+\infty): f(a)[(a+1)-a]\geq\displaystyle\int_a^{a+1}f(x)dx\geq f(a+1)[(a+1)-a]$

$\Rightarrow\forall a\in(0,+\infty): f(a)\geq\displaystyle\int_a^{a+1}f(x)dx\geq f(a+1)\qquad(1).$

We also note that

$\lim\limits_{a\to+\infty}f(a)=\lim\limits_{a\to+\infty}\dfrac{1}{a^2+1}=\lim\limits_{a\to+\infty}\dfrac{1}{a^2}=0\qquad(2)$

$\lim\limits_{a\to+\infty}f(a+1)=\lim\limits_{a\to+\infty}\dfrac{1}{(a+1)^2+1}=\lim\limits_{a\to+\infty}\dfrac{1}{a^2}=0\qquad(3)$

From Eq.(1) and Eq.(2) and Eq.(3): $\lim\limits_{a\to+\infty}\displaystyle\int_a^{a+1}f(x)dx=0.$

e) Show that

$$\lim_{x\to+\infty}\left[\frac{1}{x^2}\int_0^x \cos t\,[\sin(2t)+\operatorname{Arctan}(3t)]\,dt\right]=0$$

Solution

Define:

$$\forall x\in\mathbb{R}:\ f(x)=\frac{1}{x^2}\int_0^x \cos t\,[\sin(2t)+\operatorname{Arctan}(3t)]\,dt$$

and note that

$$\begin{aligned}
\forall x\in\mathbb{R}:\ |f(x)| &= \left|\frac{1}{x^2}\int_0^x \cos t\,[\sin(2t)+\operatorname{Arctan}(3t)]\,dt\right| \\
&= \left|\frac{1}{x^2}\right|\left|\int_0^x \cos t\,[\sin(2t)+\operatorname{Arctan}(3t)]\,dt\right| \\
&= \frac{1}{x^2}\left|\int_0^x \cos t\,[\sin(2t)+\operatorname{Arctan}(3t)]\,dt\right| \\
&\leqslant \frac{1}{x^2}\int_0^x |\cos t\,(\sin(2t)+\operatorname{Arctan}(3t))|\,dt \\
&= \frac{1}{x^2}\int_0^x |\cos t|\cdot|\sin(2t)+\operatorname{Arctan}(3t)|\,dt \\
&\leqslant \frac{1}{x^2}\int_0^x |\sin(2t)+\operatorname{Arctan}(3t)|\,dt \\
&\leqslant \frac{1}{x^2}\int_0^x (|\sin(2t)|+|\operatorname{Arctan}(3t)|)\,dt \\
&\leqslant \frac{1}{x^2}\int_0^x (1+\pi/2)\,dt = \frac{1+\pi/2}{x^2}\int_0^x dt
\end{aligned}$$

$$= \frac{1+\pi/2}{x^2}(x-0) = \frac{\pi+2}{2x} \Rightarrow$$

$$\Rightarrow \forall x \in \mathbb{R} : |f(x)| \leqslant \frac{\pi+2}{2x} \quad (1).$$

and

$$\lim_{x\to+\infty} \frac{\pi+2}{x} = (\pi+2)\lim_{x\to+\infty} \frac{1}{x} = (\pi+2)\cdot 0 = 0 \quad (2).$$

From Eq.(1) and Eq.(2):

$$\left\{ \begin{array}{l} \forall x \in \mathbb{R} : |f(x)| \leqslant \dfrac{\pi+2}{2x} \\ \lim\limits_{x\to+\infty} \dfrac{\pi+2}{2x} = 0 \end{array} \right. \Rightarrow \lim_{x\to+\infty} f(x) = 0 .$$

↳ We have used the bounds:

$\forall x \in \mathbb{R} : |\sin x| \leqslant 1$

$\forall x \in \mathbb{R} : |\cos x| \leqslant 1$

$\forall x \in \mathbb{R} : |\operatorname{Arctan} x| \leqslant \pi/2$

combined with the theorem:

$$\left. \begin{array}{l} \forall x \in N(\sigma,\delta) : |f(x)| \leqslant g(x) \\ \lim\limits_{x\to\sigma} g(x) = 0 \end{array} \right\} \Rightarrow \lim_{x\to\sigma} f(x) = 0 .$$

EXERCISES

(3) Show the following statements:

a) $\forall a, b \in (0, +\infty)$: $\displaystyle\int_a^b \frac{dx}{x^3+1} + \int_a^b \frac{x^3\,dx}{x^3+1} = b - a$

b) $\displaystyle\int_1^2 \frac{dx}{x^2+4x} + \int_2^3 \frac{dx}{x^2+4x} + \int_3^1 \frac{dx}{x^2+4x} = 0$

c) Let f be integrable on an interval I and let $a, b, c, d \in I$. Then:

$$\int_a^b f(x)\,dx + \int_c^d f(x)\,dx = \int_c^b f(x)\,dx + \int_a^d f(x)\,dx$$

$$\int_a^b f(x)\,dx \int_c^d f(x)\,dx - \int_a^c f(x)\,dx \int_b^d f(x)\,dx = \int_a^b f(x)\,dx \int_c^b f(x)\,dx$$

d) $\left\{\begin{array}{l} f \text{ continuous at } [a,b] \\ \forall x \in [a,b] : f(x) < 0 \end{array}\right. \Rightarrow \displaystyle\int_a^b f(x)\,dx < 0$

e) $\left\{\begin{array}{l} f \text{ continuous at } [a,b] \\ \displaystyle\int_a^b f(x)\,dx > 0 \end{array}\right. \Rightarrow \exists \xi \in [a,b] : f(\xi) > 0$

f) $\forall a, b \in \mathbb{R}$: $0 < a < b \Rightarrow \displaystyle\frac{b-a}{b^2} \leq \int_a^b \frac{dx}{x^2} \leq \frac{b-a}{a^2}$

g) $0 \leq \displaystyle\int_1^e \frac{\ln x}{x}\,dx \leq \frac{e-1}{e}$

h) $0 \leq \displaystyle\int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx \leq \frac{e^2-1}{e^2+1}$

i) $0 \leqslant \int_{\pi/6}^{\pi/2} \frac{1-\sin x}{\sin x}\, dx \leqslant \frac{\pi}{3}$

j) $\lim_{x\to+\infty} \int_{x-1}^{x+1} \frac{dt}{\sqrt{t^2+1}} = 0$

k) $\lim_{x\to+\infty} \int_{x^2}^{x^2+1} \operatorname{Arctan}(t)\, dt = 0$

l) $\lim_{x\to+\infty} \int_{x}^{2x} \cos x \sin(1/t)\, dt = 0$ (Hint: Use zero-bounded theorem).

m) $\lim_{x\to+\infty} \left[\frac{1}{x^3} \int_0^{x^2} (\sin t \sin(2t) + \cos^3(3t)]\, dt \right] = 0$

n) $\lim_{x\to+\infty} \left[\frac{1}{2x+1} \int_0^{\sqrt{x}} \cos(2t)(1+\operatorname{Arctan}(t))\, dt \right] = 0$

o) $\lim_{x\to+\infty} \left[\frac{1}{x^2+3x} \int_0^{x} \frac{3+\sin t}{2+\cos t}\, dt \right] = 0$

▼ Fundamental theorem of calculus

Before proving the fundamental theorem of calculus we have to use the definition of the integral to prove directly that

$$\int_a^b c\,dx = c(b-a)\ ,\ \forall a,b,c \in \mathbb{R}.$$

We also use the property that

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

① Fundamental Theorem. I

$$\boxed{\left.\begin{array}{l} f \text{ continuous at } [a,b] \\ F(x) = \displaystyle\int_a^x f(t)dt\ ,\ \forall x \in [a,b] \end{array}\right\} \Rightarrow \left\{\begin{array}{l} F \text{ differentiable in } [a,b] \\ F'(x) = f(x)\ ,\ \forall x \in [a,b] \end{array}\right.}$$

Proof

Let $x_0 \in [a,b]$ be given. It is sufficient to show that

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$$

Let $\varepsilon > 0$ be given. Since f continuous at $[a,b] \Rightarrow$
$\Rightarrow \exists \delta > 0 : \forall t \in (x_0-\delta, x_0+\delta) : |f(t) - f(x_0)| < \varepsilon$. (1)
Let $x \in (x_0-\delta, x_0+\delta)$ be given. It follows that:

$$\frac{F(x)-F(x_0)}{x-x_0} - f(x_0) = \frac{1}{x-x_0}\left[\int_a^x f(t)dt - \int_a^{x_0} f(t)dt\right] - f(x_0)$$
$$= \frac{1}{x-x_0}\int_{x_0}^x f(t)\,dt - \frac{f(x_0)}{x-x_0}\int_{x_0}^x dt =$$
$$= \frac{1}{x-x_0}\int_{x_0}^x f(t)dt - \frac{1}{x-x_0}\int_{x_0}^x f(x_0)dt =$$
$$= \frac{1}{x-x_0}\left[\int_{x_0}^x f(t)dt - \int_{x_0}^x f(x_0)dt\right] =$$
$$= \frac{1}{x-x_0}\int_{x_0}^x [f(t) - f(x_0)]\,dt \Rightarrow$$

$$\Rightarrow \left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| = \left|\frac{1}{x-x_0}\int_{x_0}^x [f(t)-f(x_0)]\,dt\right| =$$
$$= \frac{1}{|x-x_0|}\left|\int_{x_0}^x [f(t)-f(x_0)]dt\right| \leqslant$$
$$\leqslant \frac{1}{|x-x_0|}\int_{x_0}^x |f(t)-f(x_0)|\,dt \leqslant$$
$$\leqslant \frac{1}{|x-x_0|}\int_{x_0}^x \varepsilon\,dt = \frac{\varepsilon}{|x-x_0|}\cdot(x-x_0) \leqslant$$
$$\leqslant \frac{\varepsilon}{|x-x_0|}|x-x_0| = \varepsilon,\ \forall x \in (x_0-\delta, x_0+\delta)$$

We have thus shown that

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in (x_0 - \delta, x_0 + \delta) :$$
$$: \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon \Rightarrow$$

$$\Rightarrow \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0) \Rightarrow$$

$\Rightarrow$ F differentiable at x_0 with $F'(x_0) = f(x_0)$. □

↳ In Leibnitz notation the theorem reads:

$$\boxed{\frac{d}{dx} \int_c^x f(t)\,dt = f(x)}$$

↳ Combining the fundamental theorem of calculus with the chain rule gives the following more general differentiation rule.

$$\boxed{\frac{d}{dx} \int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))\,b'(x) - f(a(x))\,a'(x)}$$

EXERCISES

(4) Evaluate and simplify the derivatives of the following functions

a) $f(x) = \int_1^x (t^2 \ln t + e^{-t})\,dt$

b) $f(x) = \int_x^3 t^2 e^t (t-1)^3\,dt$

c) $f(x) = \int_0^{1/x} t \ln t\,dt$

d) $f(x) = \int_{x^2-1}^{x^2+1} \frac{t^2-1}{t^2+1}\,dt$

e) $f(x) = \int_{\ln x}^{\ln(x^2)} t e^{-t}\,dt$

(5) Show that

a) $$\frac{d}{dx}\int_{\cos x}^{\sin x} \operatorname{Arcsin}(t)\,dt = x(\cos x - \sin x) + \frac{\pi}{2}\sin x$$

b) $$\frac{d}{dx}\int_0^{\cos x} \operatorname{Arccos}(2t^2-1)\,dt = -2x\sin x$$

c) $$\frac{d}{dx}\int_0^{\tan x} \operatorname{Arctan}\left(\frac{2t}{1+t^2}\right)dt = 2x(1+\tan^2 x)$$

(6) Show that the function $f(x)=\int_0^{\sin x} \text{Arccos}(t)\,dt$ has a local minimum at $x=0$.

(7) Analyze the function

$$f(x)=\int_x^{1-x} \frac{dt}{1+t^2}$$

with respect to monotonicity and concavity. Locate the local minimum and maximum points and the inflection points.

(8) Use the fundamental theorem of calculus and L'Hospital's theorem to evaluate the following limits

a) $\lim\limits_{x\to 0} \dfrac{1}{x}\displaystyle\int_0^{x} \cosh(t)\,dt$

b) $\lim\limits_{x\to 0} \dfrac{1+\cos x}{1-\cos x}\displaystyle\int_0^{x} \tan(3t)\,dt$

c) $\lim\limits_{x\to 0} \dfrac{\sin x}{x^2}\displaystyle\int_1^{e^x} \ln(t)\,dt$

d) $\lim\limits_{x\to e} \dfrac{1}{\ln(\ln x)}\displaystyle\int_1^{\ln x} e^{2t}(t+1)\,dt$

e) $\lim\limits_{x\to 0} \dfrac{\tan x}{x^3}\displaystyle\int_0^{x} dt \int_1^{\cos t} ds\, \text{Arcsin}(s)$

② Fundamental theorem of calculus II

$$\boxed{\left.\begin{array}{l} F \text{ differentiable at } [a,b] \\ F'(x) = f(x),\ \forall x \in [a,b] \\ f \text{ continuous at } [a,b] \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)}$$

Proof

$$\forall x \in [a,b] : \frac{d}{dx}\int_a^x f(t)\,dt = f(x) = \frac{dF(x)}{dx} \Rightarrow$$

$$\Rightarrow \exists\, c \in \mathbb{R} : \int_a^x f(t)\,dt = F(x) + c \quad , \ \forall x \in [a,b]$$

For $x = a$:

$$F(a) + c = \int_a^a f(t)\,dt = 0 \Rightarrow c = -F(a)$$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a). \quad \square$$

Equivalently

$$\boxed{\int_a^b f'(x)\,dx = f(b) - f(a)}$$

↳ The FTC II motivates the definition of the indefinite integral:

$$\boxed{\int f'(x)\,dx = f(x) + c}$$

↳ <u>Integration formulas</u>

$$1)\ \int x^a\,dx = \begin{cases} \dfrac{x^{a+1}}{a+1} + c & , \text{ if } a \neq -1 \\ \ln|x| + c & , \text{ if } a = -1 \end{cases}$$

▸ Special cases:

$$a)\ \int dx = x + c \qquad b)\ \int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + c$$

$$2)\ \int \sin x\,dx = -\cos x + c \qquad 6)\ \int a^x\,dx = \frac{a^x}{\ln a} + c, \text{ if } a \neq 1$$

$$3)\ \int \cos x\,dx = \sin x + c \qquad \hookrightarrow \int e^x\,dx = e^x + c$$

$$4)\ \int \frac{dx}{\cos^2 x} = \tan x + c \qquad 7)\ \int \frac{dx}{1+x^2} = \text{Arctan}\,x + c$$

$$5)\ \int \frac{dx}{\sin^2 x} = -\cot x + c \qquad 8)\ \int \frac{dx}{\sqrt{1-x^2}} = \text{Arcsin}\,x + c$$

EXAMPLES

1) $I = \int_1^2 \frac{x+1}{x^3} dx = \int_1^2 (x^{-2} + x^{-3}) dx =$

$$= \int_1^2 x^{-2} dx + \int_1^2 x^{-3} dx = \left[\frac{x^{-1}}{-1}\right]_1^2 + \left[\frac{x^{-2}}{-2}\right]_1^2 =$$

$$= \left[-\frac{1}{x}\right]_1^2 + \left[-\frac{1}{2x^2}\right]_1^2 =$$

$$= \left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right) + \left(-\frac{1}{2 \cdot 2^2}\right) - \left(\frac{-1}{2 \cdot 1^2}\right) =$$

$$= -\frac{1}{2} + 1 - \frac{1}{8} + \frac{1}{2} = 1 - \frac{1}{8} = \frac{7}{8}$$

2) $I = \int_0^{\pi/4} \frac{1 + \cos^3 x}{\cos^2 x} dx = \int_0^{\pi/4} \frac{dx}{\cos^2 x} + \int_0^{\pi/4} \cos x \, dx$

$$= \Big[\tan x\Big]_0^{\pi/4} + \Big[\sin x\Big]_0^{\pi/4} =$$

$$= \tan\left(\frac{\pi}{4}\right) - \tan 0 + \sin\left(\frac{\pi}{4}\right) - \sin 0 =$$

$$= 1 + \frac{\sqrt{2}}{2} = \frac{2 + \sqrt{2}}{2}$$

3) $I = \int_{-1}^{1} \frac{dx}{x^3} \longrightarrow f(x) = 1/x^3$ NOT continuous at $[-1, 1]$ thus we cannot apply the FTC II !!

EXERCISES

(9) Evaluate the following integrals:

a) $I = \int_{-1}^{1} x^3 dx$ b) $I = \int_{-2}^{0} (x + e^x) dx$ c) $I = \int_{2}^{1} \sqrt{x}\, dx$

d) $I = \int_{1}^{2} \frac{dx}{\sqrt{x}}$ e) $I = \int_{2}^{3} x\sqrt{x}\, dx$ f) $I = \int_{0}^{\pi/4} \sin x\, dx$

g) $I = \int_{-\pi/6}^{\pi/6} \frac{dx}{\cos^2 x}$ h) $I = \int_{\pi/6}^{\pi/3} \frac{dx}{\sin^2 x}$

i) $I = \int_{0}^{2} (3^x + \sqrt{x}) dx$ j) $I = \int_{-\sqrt{3}}^{\sqrt{3}/3} \frac{dx}{1+x^2}$

k) $I = \int_{1/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$ l) $I = \int_{0}^{2} (3^x + 4^x) dx$

(10) Evaluate the following integrals

a) $I = \int_{1}^{2} x(x+2)^2 dx$ b) $I = \int_{1}^{3} x^2(\sqrt{x} + 1) dx$

c) $I = \int_{-\pi/3}^{2\pi/3} \sin(x/2)\cos(x/2)\, dx$ d) $I = \int_{0}^{1} \frac{x^2+2}{x^2+1} dx$

e) $I = \int_0^{1/2} \frac{\sqrt{1-x^2}}{(1+x)(1-x)}\, dx$

f) $I = \int_0^{\pi/4} (2\cos^2(x/2) - 1)\, dx$ g) $I = \int_0^3 \frac{2^x}{e^x}\, dx$

h) $I = \int_1^2 2^x (3^x - 5^x)\, dx$ i) $I = \int_0^1 \frac{3^x + 4^x}{5^x}\, dx$

Method of substitution

The method of substitution is based on the following theorem:

Thm:

$$\left.\begin{array}{l} g \text{ differentiable at } [a,b] \\ g' \text{ continuous at } [a,b] \\ f \text{ continuous at } g([a,b]) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy$$

Remark: If we let $y = g(x)$ then the substitution theorem implies the formal relationship:

$$dy = g'(x)\,dx\,.$$

Proof

Let $F(x) = \int_{g(a)}^{x} f(t)\,dt$. It follows that $F'(x) = f(x)$.

and thus:

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_a^b F'(g(x))\,g'(x)\,dx =$$

$$= \int_a^b [F(g(x))]'\,dx =$$

$$= F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} F'(y)\,dy =$$

$$= \int_{g(a)}^{g(b)} f(y)\,dy \qquad \square$$

Immediate consequences

1) $\displaystyle\int \sin(ax)\,dx = \frac{-\cos(ax)}{a} + c$

2) $\displaystyle\int \cos(ax)\,dx = \frac{\sin(ax)}{a} + c$

3) $\displaystyle\int a^{bx}\,dx = \frac{a^{bx}}{b\ln a} + c$

$\quad\rightarrow \displaystyle\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$

EXAMPLES

1) $I = \int_0^{\pi/2} 2^{\cos 3x} \sin(3x)\,dx$

Let $y = g(x) = \cos(3x) \Rightarrow \begin{cases} dy = -3\sin(3x)\,dx \\ g(0) = \cos(0) = 1 \\ g(\pi/2) = \cos(3\pi/2) = 0 \end{cases} \Rightarrow$

$$\Rightarrow I = \int_1^0 2^y (-1/3)\,dy = (-1/3)\int_1^0 2^y\,dy =$$

$$= -\frac{1}{3}\left[\frac{2^y}{\ln 2}\right]_1^0 = -\frac{1}{3}\,\frac{2^0 - 2^1}{\ln 2} =$$

$$= -\frac{1}{3}\,\frac{1-2}{\ln 2} = \frac{1}{3\ln 2}$$

2) $I = \int (2x-1)e^{x^2-x}\,dx$

Let $y = x^2 - x \Rightarrow dy = (2x-1)\,dx \Rightarrow$

$$\Rightarrow I = \int e^y\,dy = e^y + c \underset{\uparrow}{=} e^{x^2-x} + c.$$

backsubstitution.

↳ With indefinite integrals we work similarly as with definite integrals but it is necessary to use backsubstitution to return to the original variable.

Methodology

1) $$\boxed{I = \int \frac{f'(x)}{f(x)}\,dx} \longrightarrow \text{Let } y = f(x)$$

EXAMPLE

$$I = \int_0^2 \frac{2x}{x^2+1}\,dx$$

Let $y = g(x) = x^2+1 \Rightarrow \begin{cases} dy = 2x\,dx \\ g(0) = 0^2+1 = 1 \\ g(2) = 2^2+1 = 5 \end{cases} \Rightarrow$

$$\Rightarrow I = \int_1^5 \frac{dy}{y} = \Big[\ln|y|\Big]_1^5 = \ln 5 - \ln 1 = \ln 5$$

2) $$\boxed{I = \int \frac{f'(x)}{\sqrt{f(x)}}\,dx} \longrightarrow \text{Let } y = f(x).$$

EXAMPLE

$$I = \int_0^{\pi/8} \frac{\sin 2x}{\sqrt{1+\cos 2x}}\,dx$$

$$\text{Let } y = g(x) = 1+\cos 2x \Rightarrow \begin{cases} dy = -2\sin 2x\,dx \\ g(0) = 1+\cos 0 = 1+1 = 2 \\ g(\pi/8) = 1+\cos(\pi/4) = \\ \quad = 1+\sqrt{2}/2 \end{cases}$$

$$\Rightarrow I = -\frac{1}{2}\int_0^{\pi/8} \frac{-2\sin 2x}{\sqrt{1+\cos 2x}}\,dx =$$

$$= \frac{-1}{2}\int_2^{1+\sqrt{2}/2} \frac{dy}{\sqrt{y}} = \frac{-1}{2}\Big[2\sqrt{y}\Big]_2^{1+\sqrt{2}/2} =$$

$$= \frac{-1}{2}\left[2\sqrt{1+\frac{\sqrt{2}}{2}} - 2\sqrt{2}\right] =$$

$$= -\sqrt{1+\frac{\sqrt{2}}{2}} + \sqrt{2}$$

3) $\boxed{I = \int f(ax+b)\,dx} \longrightarrow$ Let $y = ax+b$

EXAMPLE

$$I = \int_0^1 \frac{dx}{(2x+1)^4}$$

$$\text{Let } y = g(x) = 2x+1 \Rightarrow \begin{cases} dy = 2dx \Rightarrow dx = (1/2)\,dy \\ g(0) = 2\cdot 0+1 = 1 \\ g(2) = 2\cdot 1+1 = 3 \end{cases} \Rightarrow$$

$$\Rightarrow I = \int_1^3 \frac{(1/2)\,dy}{y^4} = \frac{1}{2}\left[\frac{y^{-3}}{-3}\right]_1^3 = \frac{1}{2}\left[\frac{-1}{3y^3}\right]_1^3 =$$

$$= \frac{1}{2}\cdot\left(\frac{-1}{3}\right)\left(\frac{1}{3^3} - \frac{1}{1^3}\right) =$$

$$= \frac{-1}{6}\left(\frac{1}{27} - 1\right) = \frac{-1}{6}\left(-\frac{26}{27}\right) =$$

$$= \frac{13}{3 \cdot 27} = \frac{13}{81}$$

4) $\boxed{I = \int F(x, \sqrt{ax+b})\, dx}$

$\to$ Let $y = \sqrt{ax+b} \Leftrightarrow y^2 = ax+b \Leftrightarrow ax = y^2 - b$

$\Leftrightarrow x = \dfrac{y^2 - b}{a}$

It follows that $dx = \dfrac{2}{a} y\, dy$ and therefore

$$I = \int F\left(\frac{y^2-b}{a}, y\right) \frac{2}{a} y\, dy$$

The new integral does not have radicals.

EXAMPLE

$$I = \int_1^2 x\sqrt{3x+2}\, dx$$

Let $y = \sqrt{3x+2} = g(x) \Rightarrow \begin{cases} g(1) = \sqrt{3+2} = \sqrt{5} \\ g(2) = \sqrt{6+2} = \sqrt{8} \end{cases}$

and since

$$y=\sqrt{3x+2} \Leftrightarrow y^2=3x+2 \Leftrightarrow 3x=y^2-2 \Leftrightarrow$$
$$\Leftrightarrow x=\frac{y^2-2}{3}$$

we have $dx=(1/3)2y\,dy=(2/3)y\,dy$.
Thus

$$I=\int_{\sqrt{5}}^{\sqrt{8}}\frac{y^2-2}{3}\,y\cdot(2/3)y\,dy=\frac{2}{9}\int_{\sqrt{5}}^{\sqrt{8}}y^2(y^2-2)\,dy=$$
$$=\frac{2}{9}\int_{\sqrt{5}}^{\sqrt{8}}(y^4-2y^2)\,dy=\frac{2}{9}\left[\frac{y^5}{5}-\frac{2y^3}{3}\right]_{\sqrt{5}}^{\sqrt{8}}=$$

$$=\frac{2}{9}\left[\frac{(\sqrt{8})^5-(\sqrt{5})^5}{5}-\frac{2(\sqrt{8})^3-2(\sqrt{5})^3}{3}\right]$$
$$=\frac{2}{9}\left[\frac{8^2\sqrt{8}-5^2\sqrt{5}}{5}-\frac{2\cdot8\sqrt{8}-2\cdot5\sqrt{5}}{3}\right]$$
$$=\frac{2}{9}\left[\frac{64\sqrt{8}-25\sqrt{5}}{5}-\frac{16\sqrt{8}-10\sqrt{5}}{3}\right]$$
$$=\frac{2}{9}\left[\left(\frac{64}{5}-\frac{16}{3}\right)\sqrt{8}+\left(\frac{10}{3}-\frac{25}{5}\right)\sqrt{5}\right]$$
$$=\dots=\frac{224}{135}\sqrt{8}-\frac{10}{27}\sqrt{5}=\frac{448}{135}\sqrt{2}-\frac{10}{27}\sqrt{5}.$$

In some problems human creativity is needed to "see" the correct substitution

EXAMPLE

$$I = \int x^5 \sqrt{1-x^2}\, dx$$

Let $y = 1 - x^2 \Rightarrow dy = -2x\,dx$ and since $x^2 = 1 - y^2 \Rightarrow x^4 = (1-y^2)^2$.
It follows that

$$I = \int (1-y^2)^2 \sqrt{y} \cdot (-1/2)\, dy =$$

$$= -\frac{1}{2} \int \sqrt{y}\,(1 - 2y^2 + y^4)\, dy =$$

$$= -\frac{1}{2} \int (\sqrt{y} - 2y^2\sqrt{y} + y^4\sqrt{y})\, dy =$$

$$= -\frac{1}{2} \left[\frac{y\sqrt{y}}{3/2} - 2\,\frac{y^3\sqrt{y}}{7/2} + \frac{y^5\sqrt{y}}{11/2} \right] + c$$

$$= -\frac{y\sqrt{y}}{3} + \frac{y^3\sqrt{y}}{7} - \frac{y^5\sqrt{y}}{11} + c$$

$$= -y\sqrt{y} \left[\frac{1}{3} - \frac{y^2}{7} + \frac{y^4}{11} \right] + c$$

$$= -y\sqrt{y} \cdot \frac{77 - 33y^2 + 21y^4}{232} + c$$

$$= -(1-x^2)\sqrt{1-x^2}\left[\frac{77-33(1-x^2)^2+21(1-x^2)^4}{232}\right]+c$$

$$= \frac{-1}{232}(1-x^2)(77-33(1-x^2)^2+21(1-x^2)^4)\sqrt{1-x^2}+c$$

EXERCISES

(11) Evaluate the following integrals.

a) $\displaystyle\int_0^{-10} e^{x+5}\,dx$

b) $\displaystyle\int_0^{1} \frac{3x^2}{\sqrt{x^3+1}}\,dx$

c) $\displaystyle\int_{-1}^{0} (2x+3)e^{x^2+3x+2}\,dx$

d) $\displaystyle\int_0^{1} (5x-3)^5\,dx$

e) $\displaystyle\int_1^{2} \sqrt[3]{x+1}\,dx$

f) $\displaystyle\int_0^{1} \frac{dx}{3x+1}$

g) $\displaystyle\int_0^{\pi/2} \cos(3x-\pi/2)\,dx$

h) $\displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{\cos^2(-4x)}$

i) $\displaystyle\int_0^{e-1} \sin(x+1)\,dx$

j) $\displaystyle\int_0^{\sqrt[3]{2}} x^2 e^{x^3}\,dx$

k) $\displaystyle\int_0^{\sqrt{\pi}/2} x\sin(4x^2)\,dx$

l) $\displaystyle\int_{-1}^{1} \frac{2x-1}{x^2-x+1}\,dx$

m) $\displaystyle\int_0^{\pi} \cos^2 x \sin x\, 2^{\cos^3 x}\,dx$

n) $\displaystyle\int_1^{e} \frac{e^{\ln x}}{x}\,dx$

o) $\displaystyle\int_0^{\pi/4} \frac{e^{\tan x}}{\cos^2 x}\,dx.$

(12) Evaluate the following integrals.

a) $\displaystyle\int_0^{\sqrt{3}} \frac{2x+1}{x^2+1}\,dx$

b) $\displaystyle\int \frac{\tan x}{\cos^2 x}\,dx$

c) $\displaystyle\int e^{\cos x} \sin x\,dx$

d) $\displaystyle\int_0^{3/4} \frac{\sin\sqrt{1-x}}{\sqrt{1-x}}\,dx$

e) $\displaystyle\int \frac{\sin(\ln(4x^2))}{x}\,dx$

f) $\displaystyle\int \frac{3e^{2x}}{\sqrt{1-e^{2x}}}\,dx$

g) $\displaystyle\int_0^{\pi/4} \frac{\sin x}{1+\cos^2 x}\,dx$

h) $\displaystyle\int_0^{\pi/6} 2^{\cos x} \cdot \sin x\,dx$

i) $\displaystyle\int_0^{\pi/3} \tan x\,dx$

j) $\displaystyle\int \frac{x^2\cos(x^3-2)}{\sin^2(x^3-2)}\,dx$

k) $\displaystyle\int_0^{\sqrt{3}/6} \frac{\exp(\operatorname{Arctan}(2x))}{1+4x^2}\,dx$

(13) Consider the integrals

$$I_1 = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx\ ,\quad I_2 = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx$$

Show that $I_1 = I_2 = \pi/4$.

(14) Evaluate the following integrals

a) $\int_0^1 x\sqrt{x+1}\,dx$

d) $\int \frac{dx}{x-\sqrt{x}}$

b) $\int_0^{5/3} \frac{x\,dx}{\sqrt{3x+4}}$

e) $\int_1^{5/2} x^2\sqrt{2x-1}\,dx$

c) $\int \frac{x^2+3x}{\sqrt{x+4}}$

f) $\int_0^2 \frac{x}{1+\sqrt{x+1}}\,dx$

References

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Lecture notes by Pistofides are available for download at

http://www.math.utpa.edu/lf/OGS/pistofides.html