Lecture Notes on Bussiness Calculus

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Contents

Rates of Change

- Let $x = f(t)$ be the location of an object traveling on a straight line.
At time $t + \Delta t$:

a) Displacement: $\Delta x = f(t+\Delta t) - f(t)$

b) Average velocity: $u = \dfrac{\Delta x}{\Delta t} = \dfrac{f(t+\Delta t) - f(t)}{\Delta t}$

Note that $u = u(t, \Delta t)$.

<u>example</u> : $x = t^2 + 3t \leftarrow$ Δx
$\Delta x / \Delta t$

- To define <u>instanteneous velocity</u> we want Δt to be infinitely small
This motivates the <u>Leibnitz infinitesimal</u>

<u>Leibnitz infinitesimal</u>

It is a very special number ε such that it satisfies these rules:

① It respects all the laws of algebra

e.g. $\varepsilon + \varepsilon = 2\varepsilon$

$\varepsilon / \varepsilon = 1$

$\varepsilon^2 / \varepsilon = \varepsilon$, etc.

② $\boxed{\varepsilon > 0}$

③ For any numbers a, b; with $\underline{\underline{a \neq 0}}$:

$$\boxed{a + b\varepsilon = a}$$

• Note that

$a\varepsilon + b\varepsilon^2 = \varepsilon(a + b\varepsilon) = \varepsilon \cdot a = a\varepsilon$

Similarly

$a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots + a_n\varepsilon^n = a_0$

↳ Instanteneous rate of change

Now we can define instanteneous velocity:

$$\boxed{u(t) = \frac{f(t+\varepsilon) - f(t)}{\varepsilon}}$$

More generally:

Average Velocity $\longrightarrow$ Average Rate of change

Instanteneous velocity $\longrightarrow$ Instanteneous rate of change

example : $x = t^2 + 3t \longrightarrow u(t)$?

example : cost function
$C(x) = x^3$
marginal cost function

$$MC(x) = \frac{C(x+\varepsilon) - C(x)}{\varepsilon} = \dots = 3x^2.$$

$\mapsto$ History of infinitesimal

1) Introduced by G.W. Leibnitz (16th century)
2) British Counterattack (why?)
   a) Newton: He stole it from me
   b) Bishop Berkeley: Attacked ε
3) Cauchy-Weirstrass : (17th - 18th cent)
   Calculus with limits instead of ε
4) Abraham Robinson: (20th century)
   Infinitesimal is legitimate.

Limits

- Let a, b be numbers, with $b \neq 0$

$$\boxed{\lim_{x \to a} f(x) = b \Leftrightarrow f(a+\varepsilon) = f(a-\varepsilon) = b}$$

example : $\lim_{x \to 3} x^2 = 9$

For $f(x) = x^2$

$$\left.\begin{array}{l} f(3+\varepsilon) = (3+\varepsilon)^2 = 3^2 = 9 \\ f(3-\varepsilon) = (3-\varepsilon)^2 = 3^2 = 9 \\ f(3) = 9 \end{array}\right\} \Rightarrow \lim_{x \to 3} x^2 = 9.$$

In general, if f is a polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

then

$$\lim_{x \to a} f(x) = f(a).$$

example : $f(x) = x^3 + 3x^2 - x + 1 \leftarrow \lim_{x \to 2} f(x)$.

If f, g are polynomials then

$$\boxed{g(a) \neq 0 \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}}$$

<u>example</u>: $f(x) = \dfrac{x^2+3x}{x+1} \leftarrow \lim_{x \to 3} f(x)$

If $\boxed{\lim_{x \to a} f(x) = b > 0 \Rightarrow \lim_{x \to a} \sqrt{f(x)} = \sqrt{b}}$

<u>example</u>: $f(x) = \sqrt{x^2+2x+3} \leftarrow \lim_{x \to 1} f(x).$

<u>0/0 Limits</u>

We use binomial quotient identities:

$a^2 - b^2 = (a-b)(a+b)$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

etc. , or factorization

to try to cancel the infinitesimal.

<u>examples</u>: $\lim_{x \to 2} \dfrac{x^2-4}{x-2}$; $\lim_{x \to -3} \dfrac{x^2+6x+9}{x^2+5x+6}$

Form 0/0 with $\sqrt{f(x)} - \sqrt{g(x)}$

Use the identity

$$a - b = \frac{a^2 - b^2}{a + b}$$

to eliminate the radicals and cancel the infinitesimal.

examples : $f(x) = \dfrac{\sqrt{x-1} - 2}{x-5} \leftarrow \lim_{x \to 5} f(x)$

$f(x) = \dfrac{x^2 - \sqrt{x}}{\sqrt{x} - 1} \leftarrow \lim_{x \to 1} f(x).$

Side limits and infinity

- Let a, b be numbers. ~~Then~~ with $b \neq 0$.

$$\boxed{\begin{aligned} \lim_{x \to a^+} f(x) = b &\Leftrightarrow f(a+\varepsilon) = b \\ \lim_{x \to a^-} f(x) = b &\Leftrightarrow f(a-\varepsilon) = b \end{aligned}}$$

Recall that

$$\boxed{\lim_{x \to a} f(x) = b \Leftrightarrow f(a+\varepsilon) = f(a-\varepsilon) = b}$$

It follows that

$$\boxed{\lim_{x \to a} f(x) = b \Leftrightarrow \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = b}$$

- Note that if $\displaystyle\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$,

we say that $\displaystyle\lim_{x \to a} f(x)$ <u>does not exist</u>

example : $f(x) = \dfrac{|x|}{x} \longleftarrow \displaystyle\lim_{x \to 0^+} f(x) = 1$

$$\lim_{x \to 0^-} f(x) = -1$$

$\displaystyle\lim_{x \to 0} f(x)$ does not exist

example

$$f(x) = \frac{x^2 + 2|x|}{x^2 - 2|x|} \longleftarrow \lim_{x \to 0^+} f(x) = -1$$

$$\lim_{x \to 0^-} f(x) = -1$$

$$\lim_{x \to 0} f(x) = -1$$

example

The case $b=0$

For ~~[illegible]~~ $k>0$, the number $a\varepsilon^k$ is infinitely close to 0. So we give the following definitions:

$$f(A+\varepsilon)=a\varepsilon^k \Rightarrow \lim_{x\to A^+} f(x)=0 \quad, \text{with } k>0$$

$$f(A-\varepsilon)=a\varepsilon^k \Rightarrow \lim_{x\to A^-} f(x)=0 \quad, \text{with } k>0$$

$$\lim_{x\to A^+} f(x)=\lim_{x\to A^-} f(x)=0 \Rightarrow \lim_{x\to A} f(x)=0$$

<u>example</u>: $\lim_{x\to 1^+} \left[(x-1)^2+3(x-1)(x-2)\right]$

The concept of infinity $\pm\infty$

- The symbol $\pm\infty$ does not represent a number or even an unambiguous expression involving infinitesimals.
- Note that $1/\varepsilon$, $1/\varepsilon^2$, $1/\varepsilon^3$ etc are all infinitely large. This motivates the following definitions:

$$\lim_{x\to a^+} f(x) = \begin{cases} +\infty \Leftrightarrow f(a+\varepsilon) \geq A/\varepsilon^k \text{ with } A>0, k>0 \\ -\infty \Leftrightarrow f(a+\varepsilon) \leq -A/\varepsilon^k \text{ with } A>0, k>0 \end{cases}$$

$$\lim_{x\to a^-} f(x) = \begin{cases} +\infty \Leftrightarrow f(a-\varepsilon) \geq A/\varepsilon^k \text{ with } A>0, k>0 \\ -\infty \Leftrightarrow f(a+\varepsilon) \leq -A/\varepsilon^k \text{ with } A>0, k>0 \end{cases}$$

example : $f(x) = \dfrac{1+4x}{(x-3)^3} \leftarrow \lim\limits_{x\to 3^-} f(x)$

$$f(3-\varepsilon) = \frac{1+4(3-\varepsilon)}{(3-\varepsilon-3)^3} = \frac{1+12-4\varepsilon}{(-\varepsilon)^3} = \frac{13-4\varepsilon}{-\varepsilon^3}$$

$$\simeq \frac{13}{-\varepsilon^3} = -\frac{13}{\varepsilon^3} \Rightarrow \lim_{x\to 3^-} f(x) = -\infty.$$

- We may extend definition:

$$\lim_{x\to a} f(x) = +\infty \Leftrightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = +\infty$$

$$\lim_{x\to a} f(x) = -\infty \Leftrightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = -\infty$$

<u>example</u> : $f(x) = \dfrac{x+1}{(x-2)^2} \leftarrow \lim_{x\to 2} f(x)$

$$f(2\pm\varepsilon) = \frac{2\pm\varepsilon+1}{(2\pm\varepsilon-2)^2} = \frac{3\pm\varepsilon}{(\pm\varepsilon)^2} = \frac{3}{(\pm\varepsilon)^2} = \frac{3}{\varepsilon^2}$$

$$\Rightarrow \lim_{x\to 2} f(x) = +\infty.$$

<u>example</u> : $f(x) = \dfrac{2x-1}{(x-1)^3} \leftarrow \lim_{x\to 1} f(x)$

$$f(1+\varepsilon) = \frac{2(1+\varepsilon)-1}{(1+\varepsilon-1)^3} = \frac{2+2\varepsilon-1}{\varepsilon^3} = \frac{1}{\varepsilon^3} \Rightarrow \lim_{x\to 1^+} f(x) = +\infty \quad (1)$$

$$f(1-\varepsilon) = \frac{2(1-\varepsilon)-1}{(1-\varepsilon-1)^3} = \frac{2-2\varepsilon-1}{(-\varepsilon)^3} = \frac{1}{-\varepsilon^3} \Rightarrow$$

$$\Rightarrow \lim_{x\to 1^-} f(x) = -\infty. \quad (2)$$

From (1) and (2): $\lim_{x\to 1} f(x)$ does not exist.

▶ Algebra with $\pm\infty$

- Let a be a number
  $p>0$ be a positive number
  $n<0$ be a negative number.

Then

$(+\infty)+(+\infty)=+\infty$	$a+(+\infty)=+\infty$	$\dfrac{a}{+\infty}=0$
$(-\infty)+(-\infty)=-\infty$	$a+(-\infty)=-\infty$	$\dfrac{a}{-\infty}=0$
$-(-\infty)=+\infty$	$p(+\infty)=+\infty$	
$(+\infty)(-\infty)=-\infty$	$p(-\infty)=-\infty$	
$(+\infty)(+\infty)=+\infty$	$n(+\infty)=\cancel{+\infty}\,-\infty$	
$(-\infty)(-\infty)=-\infty$	$n(-\infty)=+\infty$	

- Because $\pm\infty$ is an ambiguous symbol, the following expressions, when encountered in a limit evaluation are <u>indeterminate</u>

$$(+\infty)-(+\infty) \qquad 0\cdot(+\infty) \qquad \frac{\pm\infty}{\pm\infty}$$

$$(-\infty)-(-\infty) \qquad 0\cdot(-\infty)$$

This means that the limit could exist but we don't know yet what it equals to.

Applications

a) $\lim\limits_{x \to a^+} \dfrac{1}{x-a} = +\infty$

b) $\lim\limits_{x \to a^-} \dfrac{1}{x-a} = -\infty$

c) $\lim\limits_{x \to a} \dfrac{1}{(x-a)^{2k}} = +\infty \ , \ \forall k \in \mathbb{N} - \{0\}.$

examples

1) $f(x) = \dfrac{1-3x}{(x-2)^2} \longleftarrow \lim\limits_{x \to 2} f(x)$

2) $f(x) = \dfrac{2x+1}{2x-1} \longleftarrow \lim\limits_{x \to 1/2^+} f(x)$

3) $f(x) = \dfrac{3-4x}{(x-3)^3} \longleftarrow \lim\limits_{x \to 3} f(x)$

4) $f(x) = \dfrac{x^2+3x+2}{x^2+4x+4} \longleftarrow \lim\limits_{x \to -2^+} f(x).$

Examples with Side Limits

1) $f(x) = \dfrac{2x-1}{2-3x} \leftarrow \lim\limits_{x \to 2/3^+} f(x).$

Solution:

$$f(x) = \frac{2x-1}{2-3x} = (2x-1)\frac{1}{-3}\cdot\frac{1}{x-2/3} =$$

$$= -\frac{2x-1}{3}\,\frac{1}{x-2/3}.$$

$$\lim_{x \to 2/3^+}\left[-\frac{2x-1}{3}\right] = -\frac{2\cdot(2/3)-1}{3} = -\frac{1/3}{3} < 0 \quad (1)$$

$$\lim_{x \to 2/3^+} \frac{1}{x-2/3} = +\infty \quad (2)$$

Multiply (1) and (2): $\lim\limits_{x \to 2/3^+} f(x) = -\infty.$

2) $f(x) = \dfrac{5-2x}{(x-5)^2} \leftarrow \lim\limits_{x \to 5} f(x).$

Solution:

$$f(x) = (5-2x)\cdot\frac{1}{(x-5)^2}.$$

$$\left.\begin{aligned} &\lim_{x\to 5} (5-2x) = 5-10 < 0 \\ &\lim_{x\to 5} \frac{1}{(x-5)^2} = +\infty \end{aligned}\right\} \Rightarrow \lim_{x\to 5} f(x) = -\infty .$$

3) $f(x) = \dfrac{x+1}{(x-1)^3} \longleftarrow \lim\limits_{x\to 1} f(x)$

<u>Solution</u>

$$f(x) = (x+1)\frac{1}{(x-1)^3}$$

$$\lim_{x\to 1} (x+1) = 1+1 = 2 \qquad (1)$$

$$\lim_{x\to 1^-} \frac{1}{(x-1)^3} = (-\infty)(-\infty)(-\infty) = -\infty \qquad (2)$$

$$\lim_{x\to 1^+} \frac{1}{(x-1)^3} = (+\infty)^3 = +\infty \qquad (3)$$

From (1) and (2): $\lim\limits_{x\to 1^-} f(x) = -\infty$ (4)

From (1) and (3): $\lim\limits_{x\to 1^+} f(x) = +\infty$ (5)

From (4) and (5): $\lim\limits_{x\to 1} f(x)$ does not exist.

Limits at infinity $x \to \pm\infty$

- Let f be a function.
  Then we define:

  a) $\lim\limits_{x \to +\infty} f(x) = a \neq 0 \Leftrightarrow f(1/\varepsilon) = a$

  b) $\lim\limits_{x \to +\infty} f(x) = 0 \Leftrightarrow f(1/\varepsilon) = A\,\varepsilon^{+K}$ with $K > 0$

  c) $\lim\limits_{x \to +\infty} f(x) = +\infty \Leftrightarrow f(1/\varepsilon) \geq A\,\varepsilon^{-k}$ with $k > 0$ and $A > 0$

  d) $\lim\limits_{x \to +\infty} f(x) = -\infty \Leftrightarrow f(1/\varepsilon) \leq A\,\varepsilon^{-k}$ with $k > 0$ and $A < 0$.

example

$$f(x) = \frac{x^2 + 3x}{2x^2 - 5} \leftarrow \lim_{x \to +\infty} f(x)$$

$$f(x) = \frac{-x^3 + 5x}{3x^2 + 2} \leftarrow \lim_{x \to +\infty} f(x)$$

▸ Solution using infinitesimals

$$f(x) = \frac{x^2+3x}{2x^2-5}$$

$$f(1/\varepsilon) = \frac{(1/\varepsilon)^2+3(1/\varepsilon)}{2(1/\varepsilon)^2-5} = $$

$$= \frac{\varepsilon^2[(1/\varepsilon)^2+3(1/\varepsilon)]}{\varepsilon^2[2(1/\varepsilon)^2-5]}$$

$$= \frac{1+3\varepsilon}{2-5\varepsilon^2} \simeq \frac{1}{2} \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} f(x) = \frac{1}{2}.$$

$$f(x) = \frac{-x^3+5x}{3x^2+2}$$

$$f(1/\varepsilon) = \frac{-(1/\varepsilon)^3+5(1/\varepsilon)}{3(1/\varepsilon)^2+2} = \frac{\varepsilon^3[-(1/\varepsilon)^3+5(1/\varepsilon)]}{\varepsilon^3[3(1/\varepsilon)^2+2]}$$

$$= \frac{-1+5\varepsilon^2}{3\varepsilon+2\varepsilon^3} \simeq \frac{-1}{3\varepsilon} \Rightarrow \lim_{x\to+\infty} f(x) = -\infty.$$

- A 2nd more efficient method follows later.

- We give a similar definition for $x \to -\infty$.

$$\lim_{x\to -\infty} f(x) = a \neq 0 \iff f(-1/\varepsilon) = a$$

$$\lim_{x\to -\infty} f(x) = 0 \iff f(-1/\varepsilon) = A\varepsilon^{+k} \text{ with } k>0$$

$$\lim_{x\to -\infty} f(x) = +\infty \iff f(-1/\varepsilon) \geq A\varepsilon^{-k} \text{ with } k>0 \text{ and } A>0$$

$$\lim_{x\to -\infty} f(x) = -\infty \iff f(-1/\varepsilon) \leq A\varepsilon^{-k} \text{ with } k>0 \text{ and } A<0$$

▸ Case 1: Monomials

- You should be able to write answer immediately. Can be justified with infinitesimals.

examples: $\lim_{x\to -\infty} (3x^3)$ $\quad \lim_{x\to -\infty} (-2x^6)$

$\lim_{x\to +\infty} \dfrac{-5}{2x^2}$ $\quad \lim_{x\to +\infty} \dfrac{10}{x\sqrt{x}}$

▶ Case 2: Polynomials

- If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
  then

$$\boxed{\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n}$$

examples

$f(x) = 2x^3 + 3x + 1 \leftarrow \lim\limits_{x \to -\infty} f(x)$

$f(x) = 3x + x^2 - x^4 + 1 \leftarrow \lim\limits_{x \to +\infty} f(x)$

$f(x) = (x^2 + 3x)(5x - 1) \leftarrow \lim\limits_{x \to -\infty} f(x)$

▶ Case 3: Rational functions

- If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
  $g(x) = b_m x^m + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$
  then

$$\boxed{\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}}$$

examples

1) $f(x) = \dfrac{x + x^3 + 1}{2x - x^2} \longleftarrow \lim\limits_{x \to -\infty} f(x)$

2) $f(x) = \dfrac{x^2 + 3x + 1}{3x^2 - 2} \longleftarrow \lim\limits_{x \to +\infty} f(x)$

3) $f(x) = \dfrac{2x^2 + 1}{x^4 - x} \longleftarrow \lim\limits_{x \to -\infty} f(x)$

V Derivatives

- Let f be a function.
  The derivative f' is defined as

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

provided that the limit exists.

- $f'(x)$ gives the instanteneous rate of change of $f(x)$ at x.
  e.g. $x = f(t)$ position of object moving on a line.
  $u = u(t) = f'(t)$ = velocity
  $a = a(t) = u'(t)$ = accelaration.

- It is possible to calculate $f'(x)$ by evaluating the limit. It is better however employ differentiation rules.

Differentiation Rules

1) Constant Rule

$f(x) = c \Rightarrow f'(x) = 0$

2) Power Rule

$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

a) For $n=0$: constant rule

b) For $n = 1/2$:

$$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$$

c) For $n = -1$

$f(x) = 1/x \Rightarrow f'(x) = -1/x^2$.

3) Scalar Multiple Rule

$f(x) = cg(x) \Rightarrow f'(x) = cg'(x)$

e.g: $f(x) = cx^n \Rightarrow f'(x) = ncx^{n-1}$.

examples : $f(x) = 3x^4$ $\qquad f(x) = \dfrac{\sqrt{x}}{\sqrt[3]{x}}$

$$f(x) = \frac{2}{3x^2}$$

$$f(x) = \frac{4}{x\sqrt{x}} \qquad f(x) = (\sqrt[5]{x})^2$$

4) Addition Rule

$$h(x) = f(x) + g(x) \Rightarrow h'(x) = f'(x) + g'(x)$$
$$h(x) = f(x) - g(x) \Rightarrow h'(x) = f'(x) - g'(x)$$

<u>examples</u>: $f(x) = 3x^2 + 2x$

$f(x) = x^5 + 3x^4 + 2x^3 + 1$

5) Product Rule

$$h(x) = f(x)g(x) \Rightarrow h'(x) = f'(x)g(x) + f(x)g'(x).$$

<u>examples</u>: $f(x) = (x^2+2x)(x^3+5x^2)$

$f(x) = (x^2+3x+1)(x^2-2x-3)$

6) Triple Product Rule

$$h(x) = f_1(x) f_2(x) f_3(x) \Rightarrow$$
$$\Rightarrow h'(x) = f_1'(x) f_2(x) f_3(x) + f_1(x) f_2'(x) f_3(x) + f_1(x) f_2(x) f_3'(x).$$

<u>example</u>: $f(x) = (x^2+1)(x^3+x)(2x+1)$

<u>example with radicals</u>

$f(x) = x^2 \sqrt{}$

7) Quotient Rule

$$h(x) = \frac{f(x)}{g(x)} \Rightarrow h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$h(x) = \frac{1}{g(x)} \Rightarrow h'(x) = -\frac{g'(x)}{[g(x)]^2}$$

<u>examples</u>

$$f(x) = \frac{3}{x^3 + 3x^2 + x + 5}$$

$$f(x) = \frac{3x+2}{3x-2}$$

$$f(x) = \frac{x^2 - x}{2x+1}$$

$$f(x) = \frac{x^3 - 4}{(2x+1)(3x-2)}$$

↳ Find derivative of denominator separately first

Chain Rule

- Let f, g be functions with derivatives f', g'. Then,

$$\boxed{h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))\, g'(x)}$$

- The chain rule is a rule that generates new differentiation rules. This is done when we choose a specific function for f but leave g arbitrary.

example : Quotient rule.

Choose $f(x) = 1/x \Rightarrow f'(x) = -1/x^2$.
Let
$$h(x) = f(g(x)) = \frac{1}{g(x)}$$
By chain rule:

$$h'(x) = f'(g(x))\, g'(x) = \left[-\frac{1}{[g(x)]^2}\right] g'(x)$$
$$= -\frac{g'(x)}{[g(x)]^2} \leftarrow \text{Quotient Rule.}$$

① Generalized Power Rule

$$\boxed{h(x) = [g(x)]^n \Rightarrow h'(x) = n[g(x)]^{n-1} g'(x)}$$

Proof

Choose $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

Let

$h(x) = f(g(x)) = [g(x)]^n \Rightarrow$

$\Rightarrow h'(x) = f'(g(x))\, g'(x) =$

$$= [n(g(x))^{n-1}]\, g'(x) = n[g(x)]^{n-1} g'(x). \quad \square$$

examples

$f(x) = (3+2x)^7$

$f(x) = (x^2+3x+1)^4$

$f(x) = (2x+1)^3 (2x-1)^2$.

↳ combine with product rule.

(2) Generalized Root Law

$$h(x) = \sqrt{g(x)} \Rightarrow h'(x) = \frac{g'(x)}{2\sqrt{g(x)}}$$

(Proof: Choose $f(x) = \sqrt{x}$ ...)

<u>examples</u>

$f(x) = \sqrt{3x+2}$
$f(x) = \sqrt{x^3+3x^2+1}$
$f(x) = 2x^2\sqrt{x+1}$

Marginal Analysis

- Let p = price of product
  x = amount of product produced
- We assume that x, p are related by a demand function:
  $$p = f(x).$$

- Total revenue: $R(x) = xp = xf(x)$
- Marginal revenue:
  $$MR(x) = R'(x) = [xf(x)]' = f(x) + xf'(x)$$

- Cost Function
  $C(x)$ = cost of producing x amount

- Marginal cost Function
  $MC(x) = C'(x)$.

- Profit Function
  $P(x) = R(x) - C(x) = xf(x) - C(x)$

- Marginal Profit Function
  $$MP(x) = P'(x) = R'(x) - C'(x)$$
  $$= f(x) + xf'(x) - C'(x).$$

<u>example</u> : Linear model:

Demand: $p = a - bx$, $a > 0, b > 0$

Cost : $C(x) = c_0 + c_1 x$, $c_0 > 0, c_1 > 0$

c_0 = overhead cost

c_1 = cost per product item

Revenue:

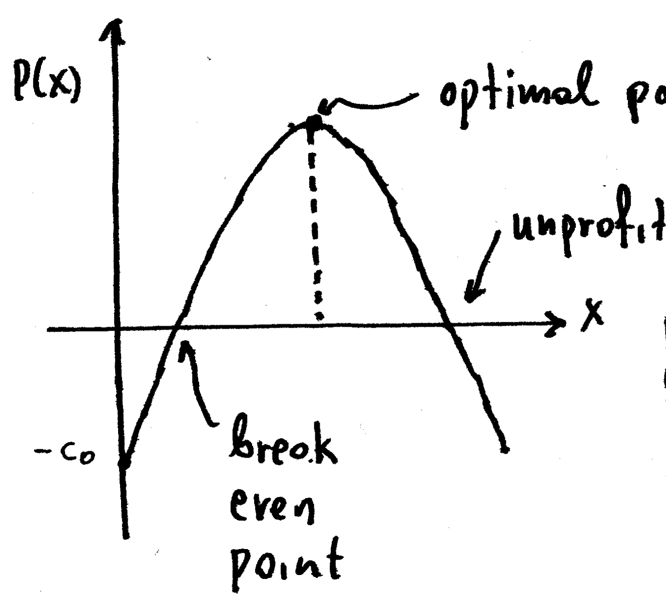
$R(x) = px = (a - bx)x = ax - bx^2$

Profit:

$$
\begin{aligned}
P(x) &= R(x) - C(x) = [ax - bx^2] - [c_0 + c_1 x] \\
&= ax - bx^2 - c_0 - c_1 x = \\
&= -bx^2 + (a - c_1)x - c_0
\end{aligned}
$$

Marginal Profit

$$
\begin{aligned}
MP(x) &= P'(x) = [-bx^2 + (a - c_1)x - c_0] \\
&= -2bx + (a - c_1)
\end{aligned}
$$



$MP(x) = 0 \Leftrightarrow -2bx + (a - c_1) = 0$

$\Leftrightarrow x = \dfrac{a - c_1}{2b}$

Need $x > 0$ thus

$a > c_1$

Average cost

- The average cost per item is given by

$$\bar{C}(x) = \frac{C(x)}{x}$$

and it is dependent on x.

- The marginal average cost per item is:

$$\overline{MC}(x) = [\bar{C}(x)]' = \left[\frac{C(x)}{x}\right]' =$$

$$= \frac{C'(x)\,x - C(x)\,(x)'}{x^2} =$$

$$= \frac{MC(x)\cdot x - C(x)}{x^2}$$

thus we showed that

$$\boxed{\overline{MC}(x) = \frac{x\,MC(x) - C(x)}{x^2}}$$

<u>example</u>: $C(x) = \dfrac{2x+1}{x+3}$

Want $\bar{C}(x)$, $\overline{MC}(x)$.

Emploee model

- Let x = amount of product produced
  p = sale price per unit
  N = number of employees.
  Want profit as a function of N.

- Assume demand curve:
  $p = f(x)$
- Assume production curve
  $x = g(N)$

- Revenue Model:

$$R(N) = px = f(x)\,x = f(g(N))\,g(N)$$

$$MR(N) = R'(N) = [f(g(N))\,g(N)]' =$$

$$= [f(g(N))]'\,g(N) + f(g(N))\,[g(N)]'$$

$$= f'(g(N))\,g'(N)\,g(N) + f(g(N))\,g'(N) =$$

$$= [f'(g(N))\,g(N) + f(g(N))]\,g'(N).$$

- Cost model

$$C(N) = c_0 + c_1 X + c_2 N = $$
$$= c_0 + c_1 g(N) + c_2 N$$

with c_0 = overhead cost
c_1 = cost of raw materials
c_2 = cost of employees

$$MC(N) = C'(N) = c_1 g'(N) + c_2$$

<u>example</u> : $\begin{cases} p = 100/x \\ x = 5\sqrt{n} \end{cases}$ $\quad c_0 = 2$
$c_1 = 3$, $c_2 = 5$

Derivative of exponential function

- The main result is

$$\boxed{f(x) = \exp(x) = e^x \Rightarrow f'(x) = \exp(x) = e^x}$$

examples : $f(x) = (x^2+3x)e^x$

$$f(x) = \frac{e^x+1}{e^x-1}$$

$$f(x) = (e^x+x^2)^3$$

- From the chain rule we get the more general result:

$$\boxed{f(x) = \exp(g(x)) \Rightarrow f'(x) = g'(x)\exp(g(x))}$$

examples : $f(x) = e^{-x^2}$

$$f(x) = \exp(x-2x^3)$$

$$f(x) = (x+1)\exp(-x^3)$$

- For more general power function:

$$\boxed{f(x) = a^x \Rightarrow f'(x) = a^x \ln a}$$

<u>examples</u> : $f(x) = (x^3+x)\,3^x$

$f(x) = \dfrac{2^x}{3^x+1}$

- From the chain rule:

$$\boxed{f(x) = a^{g(x)} \Rightarrow f'(x) = g'(x)\,a^{g(x)} \ln a}$$

<u>examples</u> : $f(x) = x^2\,5^{x-x^3}$

Derivative of Logarithmic Function

- The main result is

$$\boxed{\begin{array}{ll} f(x) = \ln x \Rightarrow f'(x) = \dfrac{1}{x} & \text{for } x > 0 \\ f(x) = \ln|x| \Rightarrow f'(x) = \dfrac{1}{x} & \text{for } x \neq 0 \end{array}}$$

examples : $f(x) = (x^3 + 5x^2)\ln x$

$f(x) = \ln(4x^6)$

$f(x) = \dfrac{\ln|x|}{1 + \ln|x|}$ $\qquad f(x) = e^x \ln x$

- From the chain rule we get the generalized rule that

$$\boxed{f(x) = \ln(g(x)) \Rightarrow f'(x) = \frac{g'(x)}{g(x)}}$$

example : $f(x) = \ln(x^3 + 3x^2 + x)$ $\qquad f(x) = \ln(\ln x)$

$f(x) = \ln\left[\dfrac{x^2+1}{x^2-1}\right]$

$f(x) = \ln\sqrt{x^3 + 4x}$, $f(x) = \ln(e^x + 1)$

- For the decimal logarithm:

$$f(x) = \log x \Rightarrow f'(x) = \frac{1}{x \ln 10} \quad , \text{ if } x > 0$$

$$f(x) = \log |x| \Rightarrow f'(x) = \frac{1}{x \ln 10} \quad , \text{ if } x \neq 0$$

$$f(x) = \log [g(x)] \Rightarrow f'(x) = \frac{g'(x)}{g(x) \ln 10}$$

<u>examples</u> :

$f(x) = (x^2 + x) \log x$

$f(x) = \log (3x^4)$

$f(x) = \log (x^3 + 3x + 1)$

Review of equations

1) The linear equation $ax+b=0$

a) If $a\neq 0$, there is a unique solution

$$x=-b/a$$

b) If $a=0$ and $b\neq 0$ then solutions do not exist.

c) If $a=0$ and $b=0$ then every number is a solution.

2) The quadratic equation $ax^2+bx+c=0$

First calculate the discriminant

$$\Delta=b^2-4ac$$

Then

a) If $\Delta>0 \Rightarrow$ Two solutions $x_{1,2}=\dfrac{-b\pm\sqrt{\Delta}}{2a}$

b) If $\Delta=0 \Rightarrow$ One solution $x=-\dfrac{b}{2a}$

c) If $\Delta<0 \Rightarrow$ No REAL solution.

Sign Charts Review

We use sign charts to determine the intervals where a function is positive or negative.

1) For $f(x) = ax + b$ with $a \neq 0$.

x		$-b/a$	
$f(x)$	$-$	0	$+$

if $a > 0$

x		$-b/a$	
$f(x)$	$+$	0	$-$

if $a < 0$

- Note that $f(x) = (ax+b)^{2n}$ is always POSITIVE and $f(x) = (ax+b)^{2n+1}$ has the SAME SIGN as $f(x) = ax+b$.
- For product of linear factors we build a sign chart for every factor and for the function itself.

<u>example</u>: $f(x) = (2x+1)(x-2)^3(x+1)^2$

- Find zeroes: $-1/2,\ 2,\ -1$
- Sort zeroes: $-1,\ -1/2,\ 2$
- Make sign chart:

x		-1		$-1/2$		2	
$2x+1$	$-$		$-$	0	$+$		$+$
$x-2$	$-$		$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$		$+$
$f(x)$	$-$	0	$+$	0	$-$	0	$+$

- 1 Place factors/zeroes on sign chart
- 2 Place the 0 zero markers in the correct locations
- 3 Enter signs for each factor.
- 4 Multiply signs to obtain sign for factor.

<u>example</u>: $f(x) = (2-x)(x+1)^2(x+3)^3$

Zeroes: $2, -1, -3$

Sort: $-3, -1, 2$

x		-3		-1		2	
$2-x$	$+$		$+$		$+$	0	$-$
$(x+1)^2$	$+$		$+$	0	$+$		$+$
$(x+3)^3$	$-$	0	$+$		$+$		$+$
$f(x)$	$-$	0	$+$	0	$+$	0	$-$

← Always positive

← Same sign as without the power 3.

2) <u>For $f(x) = ax^2 + bx + c$</u> with $a \neq 0$

The sign depends on a and $\Delta = b^2 - 4ac$

a) If $\Delta > 0 \Leftrightarrow f(x)$ has the same sign as a outside of the two zeroes x_1, x_2 and the opposite sign between the zeroes

$a > 0$

x		x_1		x_2	
$f(x)$	$+$	0	$-$	0	$+$

$a < 0$

x		x_1		x_2	
$f(x)$	$-$	0	$+$	0	$-$

b) If $\Delta = 0 \Leftrightarrow f(x)$ has the same sign as a when $x \neq -b/2a$

$a > 0$

x		$-b/2a$	
$f(x)$	$+$	0	$+$

$a < 0$

x		$-b/2a$	
$f(x)$	$-$	0	$-$

c) If $\Delta < 0 \Leftrightarrow f(x)$ has the same sign as a for all $x \in \mathbb{R}$.

$a > 0$

x	
$f(x)$	$+$

$a < 0$

x	
$f(x)$	$-$

$\hookrightarrow$ We see that $f(x)$ always has the same sign as a EXCEPT when $\Delta > 0$ (two zeroes x_1, x_2) and x is BETWEEN the two zeroes.

- For n integer, $n>0$:

a) $f(x)=(ax^2+bx+c)^{2n}$ is ALWAYS positive

b) $f(x)=(ax^2+bx+c)^{2n+1}$ has the same sign as $g(x)=ax^2+bx+c$.

<u>example</u> : $f(x)=(x+1)(x^2+x-3)$.

$$\left.\begin{aligned} &\text{For } g(x)=x^2+x-3\\ &\Delta = 1^2-4\cdot 1\cdot(-3)=\\ &=1+12=13\end{aligned}\right\} \Rightarrow x_{1,2}=\frac{-1\pm\sqrt{13}}{2}=\begin{cases}\dfrac{-1-\sqrt{13}}{2}\\[2ex] \dfrac{-1+\sqrt{13}}{2}\end{cases}$$

Zeroes: $\dfrac{-1-\sqrt{13}}{2}$, -1, $\dfrac{-1+\sqrt{13}}{2}$

x		$(-1-\sqrt{13})/2$		-1		$(-1+\sqrt{13})/2$	
$x+1$	$-$		$-$	0	$+$		$+$
x^2+x-3	$+$	0	$-$		$-$	0	$+$
$f(x)$	$-$	0	$+$	0	$-$	0	$+$

- To compare the zeroes, argue as follows:

$$\underline{\underline{\frac{-1-\sqrt{13}}{2}}} < \frac{-1-\sqrt{9}}{2} = \frac{-1-3}{2} = -2 < \underline{\underline{-1}}$$

$$\underline{\underline{-1}} < -\frac{1}{2} < \underline{\underline{\frac{-1+\sqrt{13}}{2}}}$$

- If you have a fraction, then the zeroes of the denominator are points of singularity of the function:

example : $f(x) = \dfrac{x-1}{x^2+5x+6}$

▶ For $\left.\begin{aligned} g(x) &= x^2+5x+6 \\ \Delta &= 5^2-4\cdot1\cdot6 = \\ &= 25-24 = 1 \end{aligned}\right\} \Rightarrow x_{1,2} = \dfrac{-5 \pm 1}{2} = \begin{cases} -2 \\ -3 \end{cases}$

Zeroes : $-3, -2, +1$

x		-3		-2		1	
$x-1$	$-$		$-$		$-$	0	$+$
x^2+5x+6	$+$	0	$-$	0	$+$		$+$
$f(x)$	$-$	$\nexists$	$+$	$\nexists$	$-$	0	$+$

singular points.

- The expression $\exp(f(x)) = e^{f(x)} > 0$ is ALWAYS positive and NEVER zero

<u>example</u> : $f(x) = e^{x}(x-2)^2(x+1)^3$

Zeroes: $-1, 2$

x		-1		2	
e^x	$+$		$+$		$+$
$(x-2)^2$	$+$		$+$	0	$+$
$(x+1)^3$	$-$	0	$+$		$+$
$f(x)$	$-$	0	$+$	0	$+$

Monotonicity of a function

- The sign of the derivative $f'(x)$ of a function $f(x)$ can be used to determine the intervals where $f(x)$ is increasing or decreasing.

- Methodology : Monotonicity.

- 1 Calculate $f'(x)$
- 2 Factor $f'(x)$
- 3 Construct a sign chart for $f'(x)$ with an additional entry for $f(x)$.
- 4 $f(x)$ is increasing when $f'(x) > 0$
  $f(x)$ is decreasing when $f'(x) < 0$
- 5 f has a local max when f' changes from $+$ to $-$
  f has a local min when f' changes from $-$ to $+$
  However, singular points cannot be local min or local max.

example : $f(x) = (x-1)^2(x+2)^3$

- $_1$ $Af = \mathbb{R}$.
- $_2$ $f'(x) = \dots = (x-1)(x+2)^2(5x+1)$
  with $Af' = \mathbb{R}$.

x		-2		$-1/5$		1	
$x-1$	$-$		$-$		$-$	0	$+$
$(x+2)^2$	$+$	0	$+$		$+$		$+$
$(5x+1)$	$-$		$-$	0	$+$		$+$
f'	$+$	0	$+$	0	$-$	0	$+$
f	$\nearrow$		$\nearrow$	max	$\searrow$	min	$\nearrow$

$f \nearrow$ at $(-\infty,-2)$, $(-2,-1/5)$, $(1,+\infty)$
$f \searrow$ at $(-1/5,1)$
max at $x=-1/5$
min at $x=1$
▶ $x=-2$ is not a min or max!

<u>example</u> : $f(x) = x^2 e^{4x} + 3$

$$f'(x) = [x^2 e^{4x}]' + 0 = (x^2)' e^{4x} + x^2 (e^{4x})' =$$
$$= 2x e^{4x} + x^2 e^{4x} \cdot (4x)' =$$
$$= 2x e^{4x} + 4x^{\cancel{3}2} e^{4x} =$$
$$= 2x e^{4x} (1 + 2x). \leftarrow \text{zeroes: } -1/2 \,, 0$$

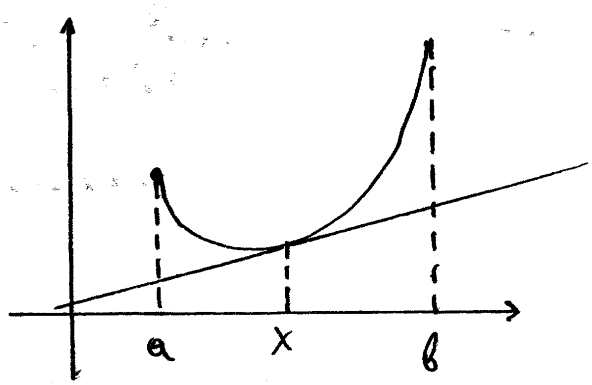
x		$-1/2$		0	
$2x$	$-$		$-$	0	$+$
e^{4x}	$+$		$+$		$+$
$1+2x$	$-$	0	$+$		$+$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$
		max		min	

$f \nearrow (-\infty, -1/2)$
$f \searrow (-1/2, 0)$
$f \nearrow (0, +\infty)$
local max at $x = -1/2$
local min at $x = 0$.

Concavity

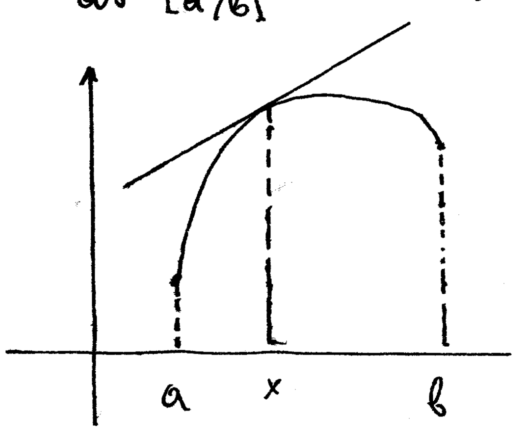
- Let f be a function. We say that f

a) f is concave up at $[a,b]$ $\Leftrightarrow$ The graph of f is ABOVE every tangent line at $x \in [a,b]$

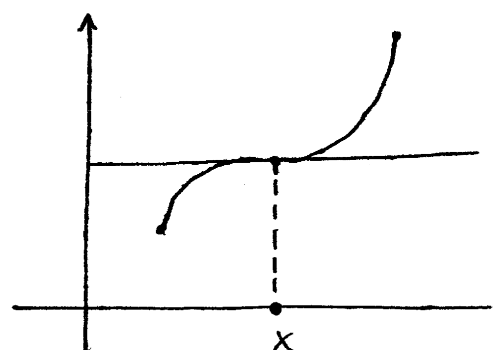


"concave up"

b) f is concave down at $[a,b]$ $\Leftrightarrow$ The graph of f is BELLOW every tangent line at $x \in [a,b]$



"concave down"



"inflection point"

c) An inflection point x is a point where the function's concavity changes.

- The concavity of $f(x)$ depends on the sign of the <u>second derivative</u> $f''(x)$ which is defined as

$$f''(x) = [f'(x)]'$$

- <u>Methodology</u> : To determine concavity

1. Calculate and factor $f'(x)$ and then $f''(x)$
2. Make a sign chart for $f''(x)$ with an additional entry for $f(x)$.
3. f is concave up when $f''(x) > 0$
   f is concave down when $f''(x) < 0$
4. Inflection points are located at the <u>zeroes</u> of $f''(x)$ <u>where the sign changes.</u>

- <u>Methodology</u>: Curve Analysis

1. First make a monotonicity chart
2. Then make a concavity chart.
3. Merge the two charts into a curve analysis (variation) chart consisting of
   a) The zeroes of both f', f'' charts
   b) The entries f', f'', f
   c) Label f as: , , ,

<u>example</u>: $f(x) = (x^2+1)e^{-x}$

$$f'(x) = \cdots = -(x-1)^2 e^{-x}$$
$$f''(x) = \cdots = (x-3)(x-1)e^{-x}$$

Monotonicity:

x		1	
$-(x-1)^2$	$-$	0	$-$
e^{-x}	$+$		$+$
$f'(x)$	$-$	0	$-$
$f(x)$	$\searrow$		$\searrow$

Concavity:

x		1		3	
$x-3$	$-$		$-$	0	$+$
$x-1$	$-$	0	$+$		$+$
e^{-x}	$+$		$+$		$+$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$		$\cap$		$\cup$
		infl.		infl.	

Curve Analysis

x		1		3	
$f'(x)$	$-$	0	$-$		$-$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\searrow$		$\searrow$		$\searrow$

No local min or max
Inflection points at
$x=1$ and $x=3$.

<u>example</u> : $f(x) = \dfrac{x^3}{x^2-1} \longleftarrow \begin{cases} \text{Monotonicity} \\ \text{Convexity} \\ \text{Variation} \end{cases}$

$$f'(x) = \dots = \frac{x^2(x^2-3)}{(x-1)^2(x+1)^2}$$

$$f''(x) = \dots = \frac{2x(x^2+3)}{(x-1)^3(x+1)^3}$$

- Monotonicity

x		$-\sqrt{3}$		-1		0		1		$+\sqrt{3}$	
x^2	+		+		+	0	+		+		+
x^2-3	+	0	−		−		−		−	0	+
$(x-1)^2$	+		+		+		+	0	+		+
$(x+1)^2$	+		+	0	+		+		+		+
$f'(x)$	+	0	−	$\Vert$	−	0	−	$\Vert$	−	0	+
$f(x)$	↗		↘		↘		↘		↘		↗

- Convexity

x		-1		0		1	
$2x$	−		−	0	+		+
x^2+3	+		+		+		+
$(x-1)^3$	−		−		−	0	+
$(x+1)^3$	−	0	+		+		+
f''	−	$\Vert$	+	0	−	$\Vert$	+
f	$\cap$		$\cup$		$\cap$		$\cup$

- Variation Table

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$+$	0	$-$	$\parallel$	$-$	0	$-$	$\parallel$	$-$	0	$+$
f''	$-$		$-$	$\parallel$	$+$	0	$-$	$\parallel$	$+$		$+$
f	$\nearrow$		$\searrow$	$\parallel$	$\searrow$		$\searrow$	$\parallel$	$\searrow$		$\nearrow$
		max		vertical asymptote		infl.		vertical asymptote		min	

Optimization Applications

- Recall that

$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

Optimization in $[a,b]$

- The problem is to optimize a function $f(x)$ in an interval $[a,b]$.
- Let $x_0 \in [a,b]$. We say that
  a) x_0 is absolute max of f in $[a,b]$ iff
  $$a \le x \le b \Rightarrow f(x) \le f(x_0)$$
  b) x_0 is absolute min of f in $[a,b]$ iff
  $$a \le x \le b \Rightarrow f(x) \ge f(x_0)$$

- Methodology:

- 1 Solve the equation
  $$f'(x)$$
  and accept only the solutions that lie in the interval $[a,b]$.
- 2 The critical points of the problem are the accepted solutions AND the endpoints $x=a$ and $x=b$.

- 3 Evaluate the function f at the critical points
- 4 By the <u>extremum value theorem</u> the critical point where f has the <u>largest value</u> is the <u>absolute ~~local~~ max</u> and the critical point where f has the <u>smallest value</u> is the <u>absolute min</u>.
- 5 If you need to identify if the remaining critical points are <u>local min</u> or <u>local max</u>, you must build a sign chart and include the interval constraint in the sign chart.

<u>examples</u>

a) $f(x) = \dfrac{1-x}{3+x}$ at $x \in [0,3]$

b) $f(x) = \dfrac{x}{x^2+2}$ at $x \in [-1,4]$

<u>Optimization in (a,b)</u>

<u>Methodology:</u> Same as in optimization in $[a,b]$ above. However we DO NOT include $x=a$ or $x=b$ as critical points.

AND a table of signs may be needed to confirm if a point is min or max, IF you only have ONE critical point.

examples

a) $f(x) = x^3 - 3x + 1$ at $x \in (0, 2)$

b) $f(x) = xe^{2-x^2}$ at $x \in (0, +\infty)$

Application Problems

To solve applied min/max problems we work as follows:

- 1 Read the problem very carefully to understand it.
- 2 Read the problem again and draw diagrams where you identify and label all the relevant quantities.
- 3 Translate the problem to mathematical statements that show how the relevant quantities are related to each other.
- 4 Recast the question of the problem as an optimization problem using the mathematical statements from step 3. This requires you to define the function that must be optimized and to identify its domain.

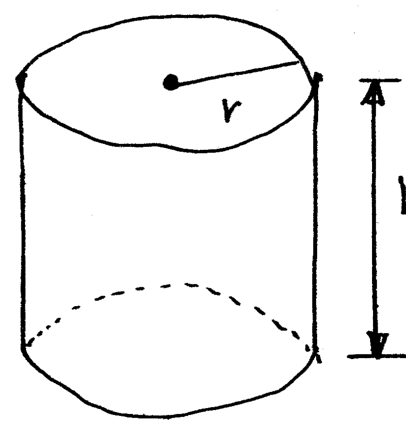
• 5 Solve the optimization problem.

examples

a) A cylindrical can with volume $V = 58 \text{ in}^3$ must be at least 1in high and 2in in diameter. What dimensions use the least amount of material?

Solution

• Diagram



• Volume $V = \pi r^2 h$
• Material used:

$$A = A_{top} + A_{bottom} + A_{around} =$$

$$= \pi r^2 + \pi r^2 + 2\pi r h$$
$$= 2\pi r^2 + 2\pi r h$$

• Constraints $r \geq 1$ and $h \geq 1$.
• Want to minimize A with $V = 58$ fixed.

- Formulation

$$V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2}$$

thus

$$A(r) = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2} =$$

$$= 2\pi r^2 + \frac{2V}{r} = \frac{2\pi r^3 + 2V}{r}$$

Also $h \geq 1 \Leftrightarrow \dfrac{V}{\pi r^2} \geq 1 \Leftrightarrow \pi r^2 \leq V \Leftrightarrow$

$$\Leftrightarrow r^2 \leq \frac{V}{\pi} \Leftrightarrow r \leq \sqrt{\frac{V}{\pi}}$$

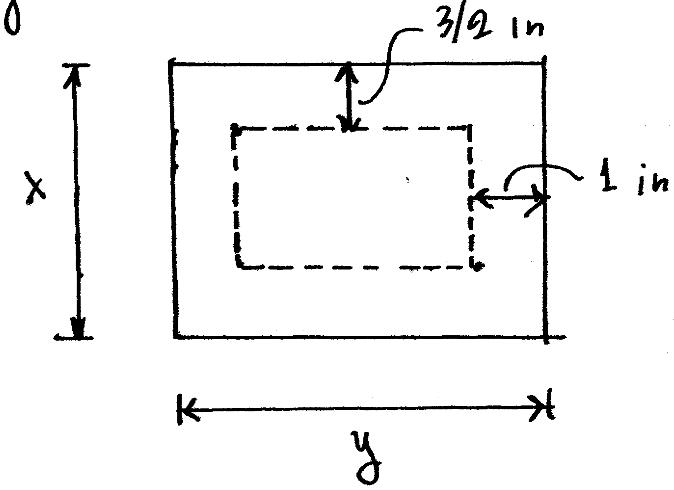
Thus: $1 \leq r \leq \sqrt{\dfrac{V}{\pi}}$.

Optimization problem:

Minimize : $\boxed{\begin{array}{l} A(r) = \dfrac{2\pi r^3 + 2V}{r} \\ \text{with } r \in \left[1, \sqrt{\dfrac{V}{\pi}}\right] \end{array}}$

6) A book has 36 in^2 of printed matter with 1 in side margin and 3/2 in top-bottom margin. Find the page dimensions using the minimum amount of paper.

- Diagram



- Amount of paper used: $A = xy$.
- Area of printed matter:

$$36 = (x - 2\cdot(3/2))(y - 2\cdot 1) =$$
$$= (x-3)(y-2) \qquad (1)$$

- Solve for y:

$$(1) \Leftrightarrow y - 2 = \frac{36}{x-3} \Leftrightarrow y = 2 + \frac{36}{x-3} =$$
$$= \frac{2(x-3)+36}{x-3} = \frac{2x-6+36}{x-3} = \frac{2x+30}{x-3}$$

consequently:

$$A = xy = x \cdot \frac{2x+30}{x-3} = \frac{x(2x+30)}{x-3}$$

Constraints:

$$\begin{cases} x-3 > 0 \\ y-2 > 0 \end{cases} \Leftrightarrow \begin{cases} x > 3 \\ y > 2. \end{cases}$$

For $y > 2 \Leftrightarrow \dfrac{2x+30}{\underbrace{x-3}_{+}} > 2 \Leftrightarrow 2x+30 > 2(x-3)$

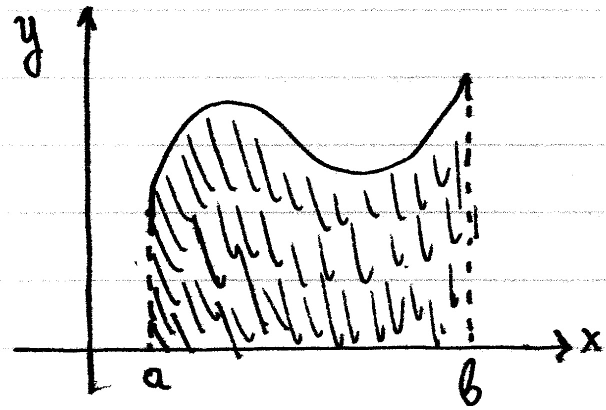
$\Leftrightarrow 2x+30 > 2x-6 \Leftrightarrow 30 > -6 \leftarrow$ always true.

- Formulation:

Minimize $\boxed{A(x) = \dfrac{x(2x+30)}{x-3} \quad \text{with } x > 3.}$

Integral Calculus

Definition of the Riemann integral



The problem is to calculate the area A between the x-axis, the lines $(l_1): x = a$ and $(l_2): x = b$ and the curve $(c): y = f(x)$.

The solution of the problem, according to Riemann is as follows:

- 1 Divide the interval $[a, b]$ to n equal intervals $[x_{k-1}, x_k]$ with
$$x_k = a + (b-a)(k/n), \quad \forall k \in [n]$$
with $[n] = \{0, 1, 2, \ldots, n\}$.

- 2 Let m_k and M_k be the min and max value of f in the interval $[x_{k-1}, x_k]$:

$$m_k = \min_{x \in [x_{k-1}, x_k]} f(x)$$

$$M_k = \max_{x \in [x_{k-1}, x_k]} f(x)$$

• 3 We form the Riemann sums

$$L_n = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$$

$$U_n = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$$

Obviously the area A will satisfy

$$\forall n \in \mathbb{N} : L_n \leqslant A \leqslant U_n \quad (1)$$

• 4 We prove that $\lim L_n = \lim U_n = \ell$ which combined with (1) implies that

$$\boxed{\lim L_n = \lim U_n = A}$$

If the limits $\lim L_n$ and $\lim U_n$ converge and coincide, we say that

f integrable at $[a,b]$

and write

$$\boxed{\lim L_n = \lim U_n = \int_a^b f(x)\,dx}$$

This definition assumes that $a<b$. For convenience we generalize by defining:

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

From the definition it follows that the integral can be calculated as the limit of the following sequence:

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty}\left[\frac{b-a}{n}\sum_{k=1}^{n} f\left(a+k\,\frac{b-a}{n}\right)\right]$$

Basic Sums

$$S_1(n) = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$S_2(n) = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$S_3(n) = 1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4} = [S_1(n)]^2.$$

example : $\int_0^a x^2\,dx = \frac{a^3}{3}$.

Properties of the integral

① f continuous at $[a,b] \Rightarrow f$ integrable at $[a,b]$

② Let f, g integrable at $[a,b]$
Then

a) $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

b) $\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx \quad , \forall \lambda \in \mathbb{R}$

c) $\gamma \in [a,b] \Rightarrow \int_a^b f(x)\,dx = \int_a^\gamma f(x)\,dx + \int_\gamma^b f(x)\,dx$

③ Let f integrable at $[a,b]$.

a) $(\forall x \in [a,b] : f(x) \geq 0) \Rightarrow \int_a^b f(x)\,dx \geq 0$

b) $(\forall x \in [a,b] : f(x) \leq g(x)) \Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

c) $\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$

Fundamental theorem of calculus.

- Let f be a function.
  If $F'(x) = f(x)$ then we say that F is the antiderivative of f.

- If F, G are both antiderivatives of f then there is a $c \in \mathbb{R}$ such that
  $$F(x) = G(x) + c$$

- Fundamental theorem of calculus:

  If F is the antiderivative of f then
  $$\boxed{\int_a^b f(x)dx = F(b) - F(a)}$$

  Thus, to evaluate a definite integral of f it is sufficient to find the antiderivative of f.

- This motivates the definition of the indefinite integral.
  $$\int f(x)dx = F(x) + c \quad \text{with } F'(x) = f(x).$$

Integration formulas

$$1)\ \int x^a\,dx = \begin{cases} \dfrac{x^{a+1}}{a+1} + c & , \text{ if } a \neq -1 \\ \ln|x| + c & , \text{ if } a = -1 \end{cases}$$

- Special cases

$$a)\ \int dx = x + c \qquad (a = 0)$$

$$b)\ \int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + c \qquad (a = -1/2)$$

<u>examples</u>

$$1)\ I = \int_1^2 (2x+1)(x-1)\,dx$$

$$2)\ I = \int \frac{x^2+1}{\sqrt{x}}\,dx$$

$$3)\ I = \int \frac{x\sqrt{x}}{\sqrt[3]{x}}\,dx$$

$$4)\ I = \int_1^3 \frac{(x+1)^2}{x}\,dx$$

2) $\displaystyle\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$, for $a \neq 0$.

- For $a = 0 \Rightarrow e^{ax} = e^{0} = 1 \Rightarrow \int e^{ax}\,dx = \int dx =$

$= x + c.$

<u>examples</u>

$$I = \int_0^2 e^{3x}\,dx$$

$$I = \int_0^2 e^{-x}\left(1 + e^{x}\right)dx$$

$$I = \int \frac{\left(e^{x}+1\right)^2}{e^{x}}\,dx$$

$$I = \int \frac{x e^{-x} + 1}{x}\,dx$$

Method of substitution

The method of substitution is based on the identity:

$$\boxed{\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(y)\, dy}$$

which is derived from the chain rule and the fundamental theorem of calculus.

Proof

Let F be the antiderivative of f. Then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_a^b F'(g(x))\, g'(x)\, dx =$$

$$= \int_a^b [F(g(x))]'\, dx =$$

$$= F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} F(y)\, dy.$$

<u>Methodology</u>: Definite Integrals $\int_a^b f(x)\,dx$

- 1 Identify the required substitution
$$y = g(x)$$
on a case by case basis
- 2 Calculate the differential
$$dy = g'(x)\,dx$$
- 3 Calculate the new limits of integration $g(a)$ and $g(b)$.

- 4 Change the integral in terms of y and change the limits of integration.

<u>example</u>: $I = \int_1^2 \sqrt{2x+3}\,dx$

Let $y = 2x+3 = g(x) \Rightarrow \begin{cases} dy = 2dx \Rightarrow dx = (1/2)\,dy \\ g(1) = 2\cdot 1 + 3 = 5 \\ g(2) = 2\cdot 2 + 3 = 7 \end{cases}$

$$\Rightarrow I = \int_5^7 \sqrt{y}\,(1/2)\,dy = \int_5^7 y^{1/2}\cdot(1/2)\,dy =$$

$$= \left[\frac{y^{3/2}}{3/2}(1/2)\right]_5^7 = \left[\frac{y\sqrt{y}}{3}\right]_5^7 =$$

$$= \frac{7\sqrt{7} - 5\sqrt{5}}{3}$$

Methodology: Indefinite Integrals $I = \int f(x)\,dx$

- 1 Identify the required substitution
$$y = g(x)$$
- 2 Calculate the differential
$$dy = g'(x)\,dx$$
- 3 Rewrite the integral in terms of y and then perform the integral.
- 4 For indefinite integrals: you obtain an answer in terms of an auxiliary variable. You must rewrite the final answer in terms of x. $\leftarrow$ Backsubstitution

example: $I = \int 3x e^{x^2}\,dx$

Let $y = x^2 \Rightarrow dy = 2x\,dx \Rightarrow 3x\,dx = (3/2)\,dy \Rightarrow$

$$\Rightarrow I = \int e^y (3/2)\,dy = (3/2)e^y + c \underset{\uparrow}{=}$$

Backsubstitution

$$= (3/2)e^{x^2} + c$$

Substitution Forms

1) Form $I = \int [f(x)]^a f'(x)\,dx$

- let $y = f(x)$

examples: a) $I = \int_0^1 (x^2+1)^5\, x\,dx$

b) $I = \int x^3 \sqrt{x^4+1}\,dx$

- Linear substitutions $y = ax+b$ always work provided they simplify the integral

c) $I = \int_0^2 2x\,(3x+1)^7\,dx$

2) Form $I = \int e^{f(x)} f'(x)\,dx$

- Let $y = f(x)$

examples: a) $I = \int_0^2 x^2 e^{-x^3}\,dx$

b) $I = \int \dfrac{3x}{e^{x^2}}\,dx$

3) Form $I = \int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + c$

▶ Or, let $y = f(x)$.

e.g: a) $I = \int_1^3 \frac{2x+5}{x^2+5x+12} \, dx$

b) $I = \int_1^4 \frac{dx}{x \ln x}$

c) $I = \int_1^2 \frac{e^x}{e^x+1} \, dx$

4) Form $I = \int \frac{f(\ln x)}{x} \, dx$

▶ Let $y = \ln x$

e.g. : a) $I = \int \frac{(\ln x)^2 + 1}{x \ln x} \, dx$

b) $I = \int_1^3 \frac{x + \ln(3x^2)}{x} \, dx$

5) Form $I = \int f(e^x)\, e^x \, dx$

▶ Let $y = e^x$

<u>example</u> : $I = \int_0^2 \dfrac{(3e^{2x}+1)\, e^x}{e^{3x}+e^x} \, dx$

↳ Backsubstitution

We apply this method to integrals of the form

$$I = \int f(x, \sqrt[n]{ax+b})\,dx \qquad \text{OR}$$

$$I = \int f\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$$

The idea is to employ the substitution theorem in reverse.

Methodology

- 1 Let $y = \sqrt[n]{ax+b}$ $\left(\text{or } y = \sqrt[n]{\frac{ax+b}{cx+d}}\right)$
- 2 Solve for x: $x = g(y)$.
- 3 Calculate $dx = g'(y)\,dy$.
- 4 If the integral is <u>definite</u>, compute the new limits of integration.
- 5 Rewrite the integral in terms of y and proceed to evaluate it.

• 6 If the integral is <u>indefinite</u> rewrite the final answer in terms of x.

<u>examples</u>

a) $I = \int 2x\sqrt{3x-2}\,dx$

b) $I = \int_0^2 x^2\sqrt{x+2}\,dx$

c) $I = \int \frac{x-1}{\sqrt{2x+1}}\,dx$

▼ Application : Consumer / Producer surplus

- Consider a product with

  Demand curve : $p = D(q)$

  Supply curve : $p = \$(q)$

  with p = price and q = quantity.

- The equilibrium price p_0 occurs at quantity q_0 that satisfies

  $$D(q_0) = \$(q_0)$$

  with $p_0 = D(q_0) = \$(q_0)$.

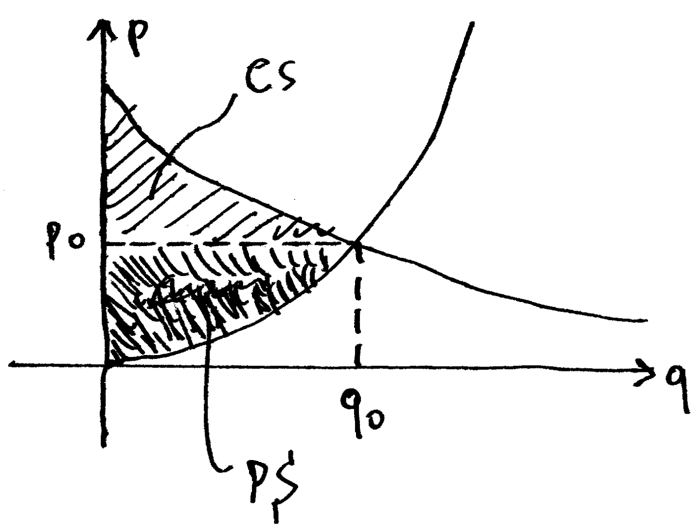
- Define:

1) Consumer Surplus

$$C\$ = \int_0^{q_0} [D(q) - p_0]\, dq$$

2) Producer Surplus

$$P\$ = \int_0^{q_0} [p_0 - \$(q)]\, dq.$$



- Interpretation

a) The consumer surplus represents how much more than the equilibrium price p_0 would some consumers be willing to pay. Thus consumer surplus measures the efficiency of the market with respect to the interest of the consumer.

b) The producer surplus represents how much less than the equilibrium price p_0 would some producers be willing to charge. Thus producer surplus measures the efficiency of the market with respect to the interest of the producer.

example : Find CS and PS when

$$\begin{cases} S(q) = 2q^2 \\ D(q) = 6 - 4q \end{cases}$$

example : Find CS and PS when

$$\begin{cases} S(q) = e^{q/2} - 1 \\ D(q) = 199 - e^{q/2} \end{cases}$$

Multivariate Calculus

Functions of several variables

- A function of two independent variables ~~is~~ f is a <u>rule</u> (mapping) that <u>maps</u> a pair (x,y) of two independent variables onto a <u>dependent variable</u> $z = f(x,y)$ which is unique for each choice of (x,y).

Partial Derivatives

- Let $z = f(x,y)$. The partial derivatives of f with respect to x and y are defined as

$$f_x(x,y) = \frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$f_y(x,y) = \frac{\partial f(x,y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

- <u>Method:</u> It is possible to use the rules of differentiation to evaluate <u>$\partial f / \partial x$</u> if we <u>treat y as a constant.</u>

Likewise, we may evaluate $\partial f/\partial y$ if we treat x as a constant.

examples

1) $f(x,y) = \ln(x^4 + xy) \leftarrow \begin{array}{l} \partial f/\partial x \\ \partial f/\partial y \end{array}$

2) $f(x,y) = x^2 e^{x+y} \leftarrow \begin{array}{l} \partial f/\partial x \\ \partial f/\partial y \end{array}$

Second order partial derivatives

- Let $z = f(x,y)$. The second order partial derivatives are defined in terms of the previously defined partial derivatives as follows:

$$f_{xx}(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial x}\right)$$

$$f_{yy}(x,y) = \frac{\partial^2 f(x,y)}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial y}\right)$$

$$f_{xy}(x,y) = \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right)$$

$$f_{yx}(x,y) = \frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right).$$

- Under the condition that f_{xy} and f_{yx} are continuous, we have

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x}$$

<u>example</u>: For $f(x,y) = 9xy^2 - x^3 - y^3$
find
f_{xx}, f_{yy}, f_{xy}

▼ Extrema of Multivariate Functions

- Let $f(x,y)$ be a function and consider a point (x_0,y_0) in its domain. We say that

a) f has local maximum at (x_0,y_0) iff there is a circular region B centered around (x_0,y_0) such that

$$(x_0,y_0) \in B \Rightarrow f(x,y) \leq f(x_0,y_0)$$

b) f has a local minimum at (x_0,y_0) iff there is a circular region B centered around (x_0,y_0) such that

$$(x_0,y) \in B \Rightarrow f(x,y) \geq f(x_0,y_0).$$

- If $z=f(x,y)$ has a local min or max at (a,b) then, provided the derivatives exist,

$$\begin{cases} \dfrac{\partial f(a,b)}{\partial x} = 0 \\[2ex] \dfrac{\partial f(a,b)}{\partial y} = 0 \end{cases} \qquad (1)$$

All points that satisfy (1) are called <u>critical points.</u>

- A saddle point is a critical point which is not a local max or a local min.

- Let $z = f(x,y)$ be a function. Its <u>Jacobian</u> is defined as

$$M = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left[\frac{\partial^2 f}{\partial x \partial y} \right]^2$$

If (a,b) is a critical point of f then

a) $M(a,b) > 0$ and $f_{xx}(a,b) < 0 \Rightarrow$
$\Rightarrow (a,b)$ local max.

b) $M(a,b) > 0$ and $f_{xx}(a,b) > 0 \Rightarrow$
$\Rightarrow (a,b)$ local min

c) $M(a,b) < 0 \Rightarrow (a,b)$ saddle point.

- <u>Remark</u>: For the case $M(a,b) = 0$ we do not know if (a,b) is local min or max or saddle point. In this case, an alternative method is to reduce the problem to a 1-variable problem.

<u>examples</u>

a) $f(x,y) = x^2 + y^2 - 2x - 6y + 14$ (local min at $(1,3)$)

b) $f(x,y) = x^3 + y^3 - 3xy$ (local min at $(1,1)$ saddle point at $(0,0)$)

c) $f(x,y) = e^{x^2+y^2}$. (local min at $(0,0)$)