Solutions to selected problems — Sets and Logic

(1c) $A = [3] \cap [2] = \{1,2,3\} \cap \{1,2\} = \{1,2\}$
$B = [2] - [6] = \{1,2\} - \{1,2,3,4,5,6\} = \emptyset$, thus
$A \cap B = A \cap \emptyset = \emptyset$
$A \cup B = A \cup \emptyset = A = \{1,2\}$
$A - B = A - \emptyset = A = \{1,2\}$
$B - A = \emptyset - A = \emptyset$.

(2f) $\mathcal{P}(\emptyset) = \{\emptyset\} \Rightarrow \mathcal{P}(\mathcal{P}(\emptyset)) = \mathcal{P}(\{\emptyset\}) =$
$= \{\emptyset, \{\emptyset\}\}$.

(3) a) T b) T c) F d) T e) F f) F
g) T h) T i) F j) F k) T l) F
m) T n) F

(4) a) $F \vee T \equiv T$
b) $T \veebar T \equiv F$
c) $F \Rightarrow T \equiv T$
d) $F \wedge T \equiv F$
e) $F \Leftrightarrow T \equiv F$

(5) a) $p \wedge q$
b) $p \veebar q$
c) $ab = 0 \Rightarrow a = 0 \vee b = 0$
d) $\bar{p}$
e) $A\hat{B}C \sim D\hat{E}F \Leftrightarrow$
$\Leftrightarrow \hat{A} = \hat{D} \wedge \hat{B} = \hat{E} \wedge \hat{C} = \hat{F}$

(6), (7), (8) Easy.
(9) similar to examples

(10) a) $\forall x \in \mathbb{R} : x = x$

b) $\exists x \in \mathbb{R} : 2x = x + 1$

c) $\forall x \in \mathbb{R} : \exists n \in \mathbb{N} : n > x$

d) $\forall x \in \mathbb{R} : \exists y \in \mathbb{C} : x - y^2 = 0$

e) $\exists x \in \mathbb{R} : \forall y \in \mathbb{R} : x + y = 0$

f) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in \mathbb{R} : (x_0 - \delta < x < x_0 + \delta \Rightarrow$
$\Rightarrow |f(x) - a| < \varepsilon$

g) $\exists b \in \mathbb{R} : \forall n \in \mathbb{N} : a_n < b$

h) $\forall \varepsilon > 0 ; \exists n_0 \in \mathbb{N} : \forall n_1, n_2 \in \mathbb{N} :$
$: (n_1 > n_0 \wedge n_2 > n_0 \Rightarrow |a_{n_1} - a_{n_2}| < \varepsilon)$

i) $\forall M > 0 : \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N} : (n > n_0 \Rightarrow a_n > M)$

(11) i) $\exists M > 0 : \forall n_0 \in \mathbb{N} : \exists n \in \mathbb{N} : (n > n_0 \wedge a_n \leq M)$

(12) a) $y \in A \Leftrightarrow \exists x \in \mathbb{Q} : 2x < 1 \wedge x^2 + 1 = y$

b) $y \in A \Leftrightarrow \exists x \in \mathbb{Z} : x$ prime number $\wedge\ 3x + 1 = y$

c) $y \in A \Leftrightarrow \exists a, b \in \mathbb{R} : \exists c \in \mathbb{Q} : a + b + c = 0 \wedge a^3 + b^3 + c^3 = y$

d) $y \in A \Leftrightarrow \exists a \in \mathbb{N} : \exists b \in \mathbb{R} : a + b > 5 \wedge a^2 - b^2 = y$

* e) $y \in A \Leftrightarrow \exists a, b \in \mathbb{R} : (a + b > 2 \vee a - b < -3) \wedge y = ab$

(13) e) $y \notin A \Leftrightarrow$
$\forall a, b \in \mathbb{R} : (a + b \leq 2 \wedge a - b \geq -3) \vee y \neq ab$

(14B) $A \cup B = A \cup C \wedge A \cap B = A \cap C \Rightarrow B = C.$

Proof.

Assume $A \cup B = A \cup C$ and $A \cap B = A \cap C$.

Case 1: Assume $x \notin A$. Then

$$\begin{aligned} x \in B &\Rightarrow x \in A \vee x \in B \\ &\Rightarrow x \in A \cup B && [\text{def}] \\ &\Rightarrow x \in A \cup C && [A \cup B = A \cup C] \\ &\Rightarrow x \in A \vee x \in C && [\text{def}] \\ &\Rightarrow x \in C. && [x \notin A] \end{aligned}$$

Similarly $x \in C \Rightarrow x \in B$.

Case 2: Assume $x \in A$. Then

$$\begin{aligned} x \in B &\Rightarrow x \in A \wedge x \in B \Rightarrow \\ &\Rightarrow x \in A \cap B && [\text{def}] \\ &\Rightarrow x \in A \cap C && [A \cap B = B \cap C] \\ &\Rightarrow x \in A \wedge x \in C && [\text{def}] \\ &\Rightarrow x \in C && [T \wedge p \Rightarrow p] \end{aligned}$$

Similarly $x \in C \Rightarrow x \in B$. $\square$

(16) a) $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$

Proof

($\Rightarrow$)

Let $\underline{C \in \mathcal{P}(A) \cap \mathcal{P}(B)} \Rightarrow C \in \mathcal{P}(A) \wedge C \in \mathcal{P}(B)$

$\Rightarrow C \subseteq A \wedge C \subseteq B$.

Claim: $C \subseteq A \cap B$. To prove:

Let $x \in C \Rightarrow x \in A$, bc. $C \subseteq A$

$x \in C \Rightarrow x \in B$, bc. $C \subseteq B$

Thus $x \in A \wedge x \in B \Rightarrow x \in A \cap B$.

It follows that $C \subseteq A \cap B \Rightarrow \underline{C \in \mathcal{P}(A \cap B)}$.

($\Leftarrow$)

Let $\underline{C \in \mathcal{P}(A \cap B)} \Rightarrow C \subseteq A \cap B$.

Claim: $C \subseteq A$. To prove:

Let $x \in C \Rightarrow x \in A \cap B$ $\quad [C \subseteq A \cap B]$

$\Rightarrow x \in A \wedge x \in B$

$\Rightarrow x \in A$

thus $C \subseteq A$. Similarly we show $C \subseteq B$.

It follows that

$C \subseteq A \wedge C \subseteq B \Rightarrow C \in \mathcal{P}(A) \wedge C \in \mathcal{P}(B)$

$\Rightarrow \underline{C \in \mathcal{P}(A) \cap \mathcal{P}(B)}$

(18e) $$\left[\bigcap_{a\in I} A_a\right] \cup \left[\bigcap_{a\in I} B_a\right] = \bigcap_{a,b\in I} (A_a \cup B_b)$$

Proof

($\Rightarrow$)

Let $x \in \left[\bigcap_{a\in I} A_a\right] \cup \left[\bigcap_{a\in I} B_a\right] \Rightarrow$

$\Rightarrow x \in \bigcap_{a\in I} A_a \vee x \in \bigcap_{a\in I} B_a \Rightarrow$

$\Rightarrow [\forall a \in I : x \in A_a] \vee [\forall a \in I : x \in B_a]$

Case 1: Assume $\forall a \in I : x \in A_a$.

Sufficient to show $\forall a, b \in I : x \in A_a \cup B_b$.

Let $a, b \in I$ be given.

$x \in A_a \Rightarrow x \in A_a \cup B_b$.

thus $\forall a, b \in I : x \in A_a \cup B_b \Rightarrow x \in \bigcap_{a,b\in I} (A_a \cup B_b)$.

Case 2: Assume $\forall a \in I : x \in B_a$

Sufficient to show $\forall a, b \in I : x \in A_a \cup B_b$.

Let $a, b \in I$ be given.

$x \in B_b \Rightarrow x \in A_a \cup B_b$

thus $\forall a, b \in I : x \in A_a \cup B_b \Rightarrow x \in \bigcap_{a,b\in I} (A_a \cup B_b)$

($\Leftarrow$): Let $x \in \bigcap_{a,b\in I} (A_a \cap B_b) \Rightarrow$

$$\Rightarrow \forall a, b \in I : x \in A_a \cap B_b$$
$$\Rightarrow \forall a, b \in I : x \in A_a \wedge x \in B_b$$
$$\Rightarrow \forall a \in I : x \in A_a \Rightarrow$$
$$\Rightarrow x \in \bigcap_{a \in I} A_a \Rightarrow \underline{x \in \left[\bigcap_{a \in I} A_a\right] \cup \left[\bigcap_{a \in I} B_a\right]}$$

Thus, the conclusion follows. $\square$