RELATIONS AND FUNCTIONS

▼ Cartesian Product

- An ordered-pair (x,y) is defined as an ordered collection of two elements x and y such that it satisfies the axiom:

$$\boxed{(x_1,y_1) = (x_2,y_2) \Leftrightarrow x_1 = x_2 \wedge y_1 = y_2}$$

▶ It can be shown that an ordered pair can be represented as a set, defined as

$$\boxed{(x,y) = \{x, \{x,y\}\}}$$

- We now define the <u>cartesian product</u> $A \times B$ of two sets A and B as:

$$\boxed{A \times B = \{(x,y) \mid x \in A \wedge y \in B\}}$$

It follows that

$$\boxed{(x,y) \in A \times B \Leftrightarrow x \in A \wedge y \in B}$$

- It is easy to see that
$$\emptyset \times A = A \times \emptyset = \emptyset.$$

- We also define $A^2 = A \times A$.

EXAMPLE

For $A = \{1,2\}$ and $B = \{2,3\}$

$A \times B = \{(1,2),(1,3),(2,2),(2,3)\}$

$B \times A = \{(2,1),(2,2),(3,1),(3,2)\}$

$A^2 = A \times A = \{(1,1),(1,2),(2,1),(2,2)\}$

Note that $A \times B \neq B \times A$.

EXAMPLE

Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Proof

$$
\begin{aligned}
(x,y) \in A \times (B \cup C) &\Leftrightarrow x \in A \wedge y \in B \cup C \Leftrightarrow \\
&\Leftrightarrow x \in A \wedge (y \in B \vee y \in C) \Leftrightarrow \\
&\Leftrightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \Leftrightarrow \\
&\Leftrightarrow (x,y) \in A \times B \vee (x,y) \in A \cup C \Leftrightarrow \\
&\Leftrightarrow (x,y) \in (A \times B) \cup (A \times C).
\end{aligned}
$$

EXAMPLE

Show that $C \neq \emptyset \wedge A \times C = B \times C \Rightarrow A = B$

Proof

Since $C \neq \emptyset$, there is a $y \in C$.
It follows that

$$
\begin{aligned}
x \in A &\Leftrightarrow (x,y) \in A \times C && [y \in C] \\
&\Leftrightarrow (x,y) \in B \times C && [A \times C = B \times C] \\
&\Leftrightarrow x \in B \wedge y \in C && [\text{def.}] \\
&\Leftrightarrow x \in B.
\end{aligned}
$$

EXAMPLE

Let A, B be sets with $A \neq \emptyset$ and $B \neq \emptyset$.
Show that $A \times B = B \times A \Rightarrow A = B$.

Proof

Let $y \in B$ be given, since $B \neq \emptyset$. Then,

$$
\begin{aligned}
x \in A &\Rightarrow (x,y) \in A \times B && [\text{def.}] \\
&\Rightarrow (x,y) \in B \times A && [A \times B = B \times A] \\
&\Rightarrow x \in B \wedge x \in A && [\text{def}] \\
&\Rightarrow x \in B
\end{aligned}
$$

thus $A \subseteq B$. (1)

Let $y \in A$ be given, since $A \neq \emptyset$

Then

$$
\begin{aligned}
x \in B &\Rightarrow (x, y) \in B \times A && [\text{def.}] \\
&\Rightarrow (x, y) \in A \times B && [B \times A = A \times B] \\
&\Rightarrow x \in A \wedge y \in B && [\text{def.}] \\
&\Rightarrow x \in A
\end{aligned}
$$

thus $B \subseteq A$ (2)

From (1) and (2): $A = B$.

EXERCISES

① Let $A = \{x \in \mathbb{Z} \mid 1 \leq x \leq 3\}$
$B = \{3x - 1 \mid x \in \mathbb{Z} \wedge 0 < x < 4\}$
List the elements of $A \times B$.

② Prove that for A, B, C sets
$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

③ Prove the following
a) $A \times B = \emptyset \Leftrightarrow A = \emptyset \vee B = \emptyset$
b) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
c) $(A \times B) \cap (C \times D) = \emptyset \Leftrightarrow A \cap C = \emptyset \vee B \cap D = \emptyset$.

④ Prove the following.
a) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$
b) $\{p, q\} \subseteq A \Rightarrow (A \times \{p\}) \cup (\{q\} \times A) \subseteq A \times A$

⑤ Prove the following:
a) $A \times B = B \times A \Leftrightarrow A = \emptyset \vee B = \emptyset \vee A = B$
b) $A \neq \emptyset \neq B \wedge (A \times B) \cup (B \times A) = C \times C \Rightarrow A = B = C$.

⑥ Let $\{A_a\}_{a\in I}$ and $\{B_a\}_{a\in I}$ be indexed set collections and let C be a set. Prove the following:

a) $$\left(\bigcup_{a\in I} A_a\right)\times C = \bigcup_{a\in I}(A_a\times C)$$

b) $$\left(\bigcap_{a\in I} A_a\right)\times C = \bigcap_{a\in I}(A_a\times C)$$

c) $$\bigcap_{a\in I}(A_a\times B_a) = \left(\bigcap_{a\in I} A_a\right)\times\left(\bigcap_{a\in I} B_a\right)$$

⑦ Show that for A, B sets

$$\bigcup_{S\in\mathcal{P}(A)}\left[\bigcup_{T\in\mathcal{P}(B)}\{S\times T\}\right]\subseteq\mathcal{P}(A\times B)$$

Relations

- Let A, B be two sets with $A \neq \emptyset$ and $B \neq \emptyset$. A <u>relation R from A to B</u> is any subset $R \subseteq A \times B$. The set of all such relations is denoted $\mathrm{Rel}(A, B)$. It follows that

$$\boxed{R \in \mathrm{Rel}(A, B) \Leftrightarrow R \subseteq A \times B}$$

- Let $R \in \mathrm{Rel}(A, B)$ be a relation, and let $x \in A$ and $y \in B$. We say that <u>x and y are related</u> if and only if $(x, y) \in R$. Thus,

$$\boxed{\forall x \in A : \forall y \in B : x R y \Leftrightarrow (x, y) \in R}$$

We also define:

a) The domain of R

$$\mathrm{dom}(R) = \{x \in A \mid \exists y \in B : x R y\}$$

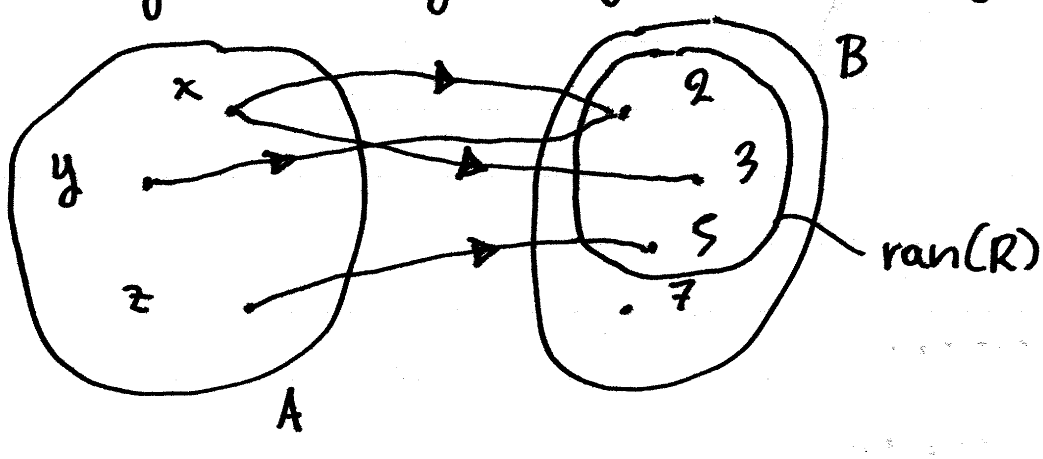
b) The range of R

$$\mathrm{ran}(R) = \{y \in B \mid \exists x \in A : x R y\}$$

EXAMPLE

Let $A = \{x, y, z\}$ and $B = \{2, 3, 5, 7\}$
Then $R = \{(x,3), (y,2), (z,5), (x,2)\}$
is a relation with $R \in Rel(A,B)$.
$dom(R) = \{x, y, z\}$
$ran(R) = \{2, 3, 5\}$
$xR2$, $xR3$, $zR5$, $yR2$ are true.

The relation R can also be represented geometrically using a Venn diagram:



Each pair (a, b) gives the arrow
$a \longrightarrow b$

- A <u>relation R on A</u> is any subset of $A \times A$.
  We define
  $$Rel(A) = Rel(A, A)$$
  thus
  $$R \in Rel(A) \Leftrightarrow R \subseteq A \times A.$$

Equivalence relations

- Let $R \in Rel(A)$ be a relation on A with $A \neq 0$. We say that:

$$\boxed{\begin{array}{l} R \text{ reflexive} \Leftrightarrow \forall x \in A : xRx \\ R \text{ symmetric} \Leftrightarrow \forall x,y \in A : (xRy \Rightarrow yRx) \\ R \text{ transitive} \Leftrightarrow \forall x,y,z \in A : (xRy \land yRz \Rightarrow xRz) \end{array}}$$

and

$$\boxed{R \text{ equivalence} \Leftrightarrow \begin{cases} R \text{ reflexive} \\ R \text{ symmetric} \\ R \text{ transitive} \end{cases}}$$

EXAMPLE

Let $R \in Rel(\mathbb{Z})$ such that

$$xRy \Leftrightarrow 11x - 5y \text{ even}.$$

Show that R is an equivalence

Proof

- Reflexive:

$11x - 5x = 6x = 2(3x) \Rightarrow 11x - 5x$ even $\Rightarrow xRx$.

- Symmetric

Assume $xRy \Rightarrow 11x-5y$ even $\Rightarrow$

$\Rightarrow \exists k \in \mathbb{Z}: \quad 11x-5y=2k.$

It follows that

$$11y-5x = 11x-5y-16x+16y =$$
$$= 2k+16y-16x =$$
$$= 2(k+8y-8x) \Rightarrow$$

$\Rightarrow 11y-5x$ even $\Rightarrow yRx$.

- Transitive

Assume $xRy \wedge yRz$. Then

$xRy \Rightarrow 11x-5y$ even $\Rightarrow \exists a \in \mathbb{Z}: 11x-5y=2a \quad (1)$

$yRz \Rightarrow 11y-5z$ even $\Rightarrow \exists b \in \mathbb{Z}: 11y-5z=2b \quad (2)$

It follows that

$$11x-5z = (11x-5y)+(11y-5z)-6y =$$
$$= 2a+2b-6y = 2(a+b-3y) \Rightarrow$$

$\Rightarrow 11x-5z$ even $\Rightarrow$

$\Rightarrow \underline{xRz}$.

Thus R is an equivalence.

▼ Equivalence classes

- Let $R \in Rel(A)$ be an equivalence relation on A, and let $a \in A$ be given. We define the equivalence class $R(a)$ as

$$\boxed{R(a) = \{x \in A \mid xRa\}}$$

- The set of all possible equivalence classes ~~can~~ of R is denoted A/R. More formally, we define

$$\boxed{A/R = \bigcup_{a \in A} \{R(a)\}}$$

Properties of equivalence classes

1) $\boxed{R(a) = R(b) \Leftrightarrow aRb}$

Proof

($\Rightarrow$): Assume $R(a) = R(b)$

R equivalence $\Rightarrow aRa \Rightarrow \left.\begin{array}{l} a \in R(a) \\ R(a) = R(b) \end{array}\right\} \Rightarrow a \in R(b) \Rightarrow$

$\Rightarrow \underline{aRb}$

$(\Leftarrow)$: Assume that aRb.

If $x \in R(a) \Rightarrow \left.\begin{array}{l} xRa \\ aRb \end{array}\right\} \Rightarrow xRb \Rightarrow x \in R(b)$

thus $R(a) \subseteq R(b)$. (1)

If $x \in R(b) \Rightarrow \left.\begin{array}{l} xRb \\ aRb \Rightarrow bRa \end{array}\right\} \Rightarrow xRa \Rightarrow x \in R(a)$

thus $R(b) \subseteq R(a)$ (2)

From (1) and (2): $R(a) = R(b)$. □

2) $\boxed{R(a) \cap R(b) = \emptyset \iff a \not R b}$

<u>Proof</u>

$(\Rightarrow)$: Assume $R(a) \cap R(b) = \emptyset$.

Use proof by contradiction.

If $\left.\begin{array}{l} aRb \Rightarrow a \in R(b) \\ aRa \Rightarrow a \in R(a) \end{array}\right\} \Rightarrow a \in R(a) \cap R(b) \Rightarrow$

$\Rightarrow R(a) \cap R(b) \neq \emptyset \leftarrow$ Contradiction.

Thus $a \not R b$.

$(\Leftarrow)$: Assume $a \not R b$.

Use proof by contradiction.

If $R(a) \cap R(b) \neq \emptyset \Rightarrow \exists x \in A : x \in R(a) \cap R(b) \Rightarrow$

$\Rightarrow \left\{\begin{array}{l} x \in R(a) \\ x \in R(b) \end{array}\right. \Rightarrow \left\{\begin{array}{l} xRa \\ xRb \end{array}\right. \Rightarrow \left\{\begin{array}{l} aRx \\ xRb \end{array}\right. \Rightarrow$

$\Rightarrow aRb \leftarrow$ contradiction.

Thus $R(a) \cap R(b) = \emptyset$ □

Equivalence classes as partitions

Recall that a collection $P = \{A_a \mid a \in I\}$ of sets A_a is a partition of A if it satisfies the following properties:

a) $\forall a \in I: \ A_a \neq \emptyset$

b) $\forall a, b \in I: \ a \neq b \Rightarrow A_a \cap A_b = \emptyset$

c) $\bigcup_{a \in I} A_a = A.$

We will now argue that A/R is a partition of A, provided that R is an equivalence relation on A.
We begin by forming an index set $I(R)$ by choosing a representative element from each equivalence class. Since, by property 2, the equivalence classes are mutually disjoint, it follows that $I(R)$ will satisfy

a) $\forall a, b \in I(R): \ a \neq b \Rightarrow R(a) \neq R(b)$

b) $\bigcup_{a \in I(R)} R(a) = A.$

Thus, the set A/R reads:

$$A/R = \bigcup_{a \in A} \{R(a)\} = \{R(a) \mid a \in I(R)\}$$

We now show that

Thm: $\boxed{\left.\begin{array}{l} R \in Rel(A) \\ R \text{ equivalence} \end{array}\right\} \Rightarrow A/R \text{ partition of } A}$

Proof

We show the 3 properties of partitions:

a) Let $a \in I(R)$

$aRa \Rightarrow a \in R(a) \Rightarrow R(a) \neq \emptyset$

thus: $\forall a \in I(R): R(a) \neq \emptyset$.

b) Let $a, b \in I(R)$ with $a \neq b$.

$a \neq b \Rightarrow R(a) \neq R(b)$

$\Rightarrow a \not R b$ [contrapositive of prop. 1]

$\Rightarrow R(a) \cap R(b) = \emptyset$ [prop. 2]

thus: $\forall a, b \in I(R): a \neq b \Rightarrow R(a) \cap R(b) = \emptyset$.

c) By definition

$$\bigcup_{a \in I(R)} R(a) = A.$$

From (a), (b), (c): A/R partition of A. $\square$

EXAMPLE

We have previously shown that the relation $R \in Rel(\mathbb{Z})$ with

$$xRy \Leftrightarrow 11x - 5y \text{ even}$$

is an equivalence.

Find the equivalence classes of R.

Solution

Try $R(0)$:

$x \in R(0) \Leftrightarrow 11x - 5 \cdot 0$ even $\Leftrightarrow 11x$ even

$\Leftrightarrow x$ even (bc. if x odd $\Rightarrow 11x$ odd)

thus $R(0) = \{x \in \mathbb{Z} \mid x \text{ even}\}$

$= \{2x \mid x \in \mathbb{Z}\}$

Try $R(1)$:

$x \in R(1) \Leftrightarrow 11x - 5 \cdot 1$ even $\Leftrightarrow$

$\Leftrightarrow \exists \lambda \in \mathbb{Z} : 11x - 5 = 2\lambda$

$\Leftrightarrow 11x = 5 + 2\lambda = (2\lambda + 4) + 1 =$

$= 2(\lambda + 2) + 1$

$\Leftrightarrow 11x$ odd $\Leftrightarrow$

$\Leftrightarrow x$ odd (bc. if x even $\Rightarrow 11x$ even).

thus $R(1) = \{x \in \mathbb{Z} \mid x \text{ odd}\}$

$= \{2x+1 \mid x \in \mathbb{Z}$.

Since $R(0) \cup R(1) = \mathbb{Z} \Rightarrow \mathbb{Z}/R = \{R(0), R(1)\}$

and a possible $I(R) = \{0, 1\}$.

EXERCISES

(8) Write the domain and range for the following relations. Are they reflexive, symmetric, transitive?

a) $R = \{(a,a),(a,b),(a,c)\}$ on $A = \{a,b,c\}$

b) $R = \{(a,b),(b,a),(c,d),(d,c),(a,a),(b,b),$
$(c,c),(d,d)\}$ on $A = \{a,b,c,d\}$

c) $R = \{(a,b),(b,a)\}$ on $A = \{a,b\}$

d) $R = \{(a,a)\}$ on $A = \{a\}$

(9) Show that the following relations are equivalences and find the corresponding equivalence classes.

a) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow a+b$ even

b) $R \in Rel(\mathbb{N}^*)$ with $aRb \Leftrightarrow a^2+b^2$ even

c) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow 3a-7b$ even

d) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow 3 \mid a+2b$

e) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow 4 \mid a^3-b^3$

f) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow 5 \mid 2a+3b$

(10) Show that the following relations on $\mathbb{R}^* \times \mathbb{R}^*$ are equivalences:

a) $(x_1,y_1)R(x_2,y_2) \Leftrightarrow x_1y_2 - x_2y_1 = 0$

b) $(x_1,y_1)R(x_2,y_2) \Leftrightarrow \exists \lambda \in \mathbb{R}^* : x_1 = \lambda x_2 \wedge y_1 = \lambda y_2$.

(11) Write the definitions, using quantifiers, for the following statements. Here R is a relation on A.

a) R is not reflexive

b) R is not symmetric

c) R is not transitive.

(12) Let $R \in Rel(A)$. We say that

$$R \text{ circular} \Leftrightarrow \forall x, y, z \in A : (xRy \wedge yRz \Rightarrow zRx)$$

Prove that:

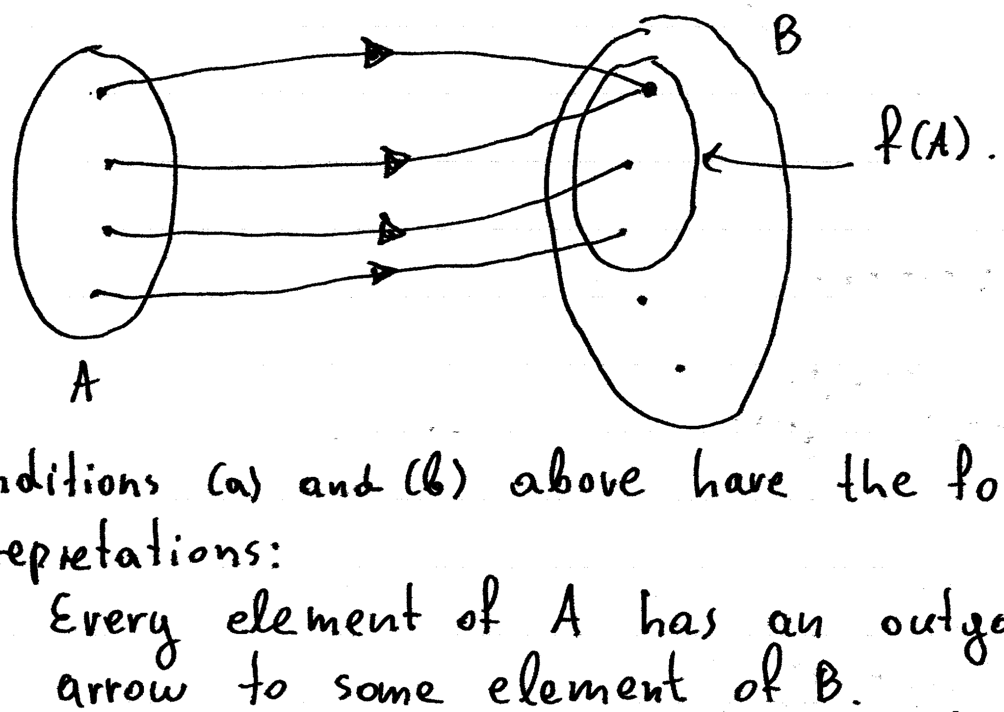
$$R \text{ equivalence} \Leftrightarrow R \text{ reflexive} \wedge R \text{ circular}.$$

(13) Let R_1, R_2 be relations on A. Show that if R_1, R_2 are equivalences then $R_1 \cap R_2$ is also an equivalence.

Mappings and Functions

- Let A, B be two sets. We say that $f : A \to B$ (f is a <u>mapping</u> that maps A to B) if and only if f is a relation $f \in Rel(A, B)$ such that it satisfies
  a) $\forall x \in A : \exists y \in B : (x, y) \in f$
  b) $\forall (x_1, y_1), (x_2, y_2) \in f : (x_1 = x_2 \Rightarrow y_1 = y_2)$.

▶ <u>Interpretation</u> : Consider a Venn diagram representation of f:



Conditions (a) and (b) above have the following intepretations:

a) Every element of A has an outgoing arrow to some element of B.

b) We do not allow any element of A to have more than one outgoing arrow (e.g. 2 arrows going to 2 distinct elements of B)

Note that it is possible that

a) B may have elements that no arrow goes to.

b) Two different elements of A can go to the same element in B.

- We see that every element $x \in A$ is mapped to a unique element in B.

  a) We denote that unique element as $f(x)$

  b) For $S \subseteq A$ we define

  $$f(S) = \{f(x) \mid x \in S\} \leftarrow \underline{\text{image}} \text{ of } S.$$

or: $y \in f(S) \Leftrightarrow \exists x \in S : f(x) = y$

↳ <u>Domain and range</u>

Recall that since $f: A \to B$ is also a relation, it has a domain and range defined as

$$\text{dom}(f) = \{x \in A \mid \exists y \in B : (x,y) \in f\}$$

$$\text{ran}(f) = \{y \in B \mid \exists x \in B : (x,y) \in f\}.$$

We will now show that

- $$\boxed{f: A \longrightarrow B \Rightarrow \text{dom}(f) = A \wedge \text{ran}(f) = f(A)}$$

Proof

$$x \in \mathrm{dom}(f) \Leftrightarrow x \in A \wedge (\exists y \in B : (x,y) \in f)$$
$$\Leftrightarrow x \in A$$

because $\forall x \in A : \exists y \in B : (x,y) \in f$ by definition.

Thus $\mathrm{dom}(f) = A$.

Similarly

$$y \in \mathrm{ran}(f) \Leftrightarrow y \in B \wedge (\exists x \in A : (x,y) \in f)$$
$$\Leftrightarrow y \in B \wedge (\exists x \in A : y = f(x))$$
$$\Leftrightarrow y \in B \wedge y \in \{f(x) \mid x \in A\}$$

thus

$$\mathrm{ran}(f) = B \cap \{f(x) \mid x \in A\} = B \cap f(A) = f(A),$$

because $f(A) \subseteq B$. $\square$

- Thus A is the domain of $f : A \to B$ and $f(A)$ is the range. B is the codomain of f.

One-to-one and onto mappings

Let $f : A \to B$ be a mapping. We say that

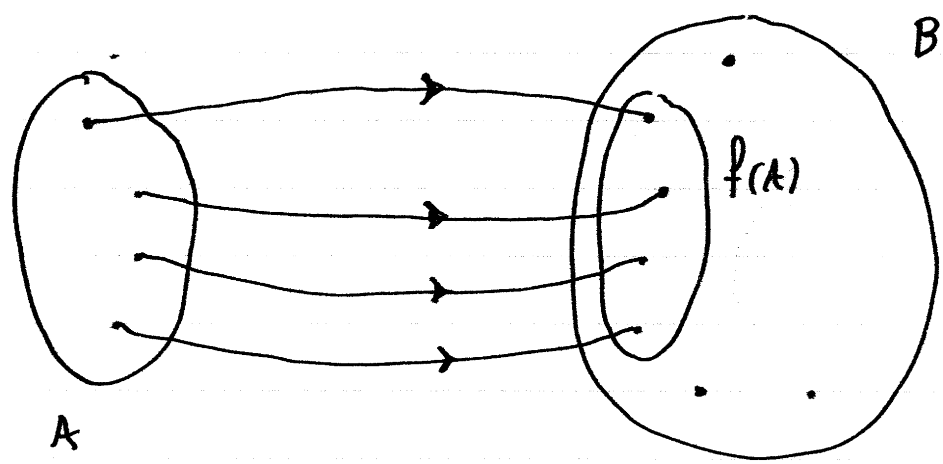
$$\boxed{\begin{array}{l} f \text{ "1-1"} \Leftrightarrow \forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2) \\ f \text{ onto} \Leftrightarrow f(A) = B \end{array}}$$

Note that in general $f(A) \subseteq B$

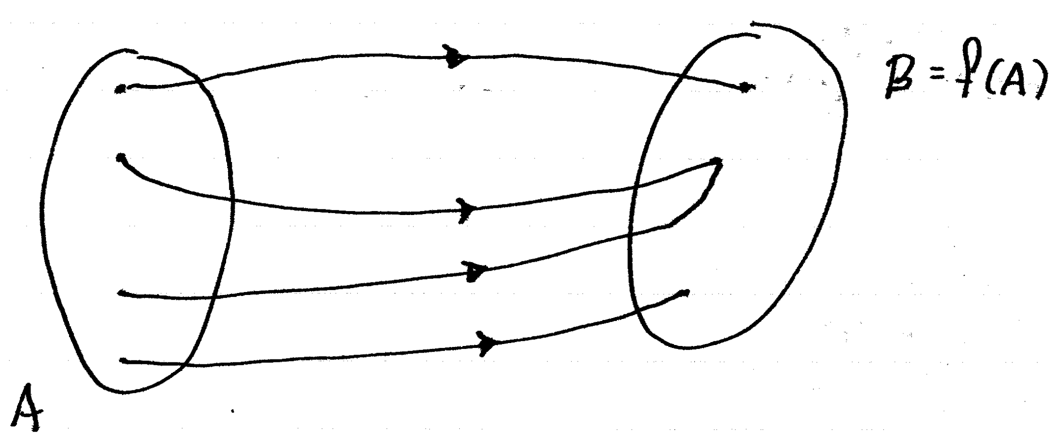
▸ Interpretation

a) In a "1-1" mapping $f: A \to B$ we do not allow two arrows from two elements of A to map into the same element in B

b) In an "onto" mapping $f: A \to B$, every element of B has an incoming arrow from some element of A.



A "1-1" mapping.



An "onto" mapping.

We also say that

$$\boxed{f \text{ bijective} \Leftrightarrow f \text{ "1-1"} \wedge f \text{ "onto"}}$$

Equality and restriction

Let f, g be two mappings. We say that

$$f = g \Leftrightarrow \begin{cases} \mathrm{dom}(f) = \mathrm{dom}(g) = A \\ \forall x \in A : f(x) = g(x) \end{cases} \quad \text{(equality)}$$

$$f = g|S \Leftrightarrow \begin{cases} \mathrm{dom}(f) = S \subseteq \mathrm{dom}(g) \\ \forall x \in S : f(x) = g(x) \end{cases} \quad \text{(restriction)}$$

- In general to define a mapping we need a statement that defines the domain of the mapping and a statement that defines $f(x)$.

EXERCISES

(14) Which of the following relations are mappings?

a) $f = \{(1,3),(2,2),(3,5)\}$
b) $f = \{(2,4),(4,3),(3,2),(2,5)\}$
c) $f = \{(3,1)\}$
d) $f = \{(1,1),(2,1),(3,1)\}$

(15) Let $f: A \to B$ be a mapping and let $S \subseteq A$ and $T \subseteq A$. Prove that
a) $f(S \cup T) = f(S) \cup f(T)$
b) $f(S \cap T) \subseteq f(S) \cap f(T)$
c) $f(S) - f(T) \subseteq f(S - T)$
If we furthermore assume that f is "1-1" prove that:
d) $f(S \cap T) = f(S) \cap f(T)$
e) $f(S - T) = f(S) - f(T)$

(16) Let $f: A_1 \to B_1$ and $g: A_2 \to B_2$ be mappings and define a mapping
$h: A_1 \times A_2 \longrightarrow B_1 \times B_2$
with
$\forall (x_1, x_2) \in A_1 \times A_2 : h(x_1, x_2) = (f(x_1), g(x_2))$
Prove that
h bijection $\Leftrightarrow$ f bijection $\land$ g bijection.

(17) Let $f: A \times B \longrightarrow A$ be a mapping such that
$\forall (x_1, x_2) \in A \times B : f(x_1, x_2) = x_1$
Prove that:
a) f onto
b) $B = \{a\} \Rightarrow f$ "1-1"
c) $B = \{a, b\} \Rightarrow f$ not "1-1".

(18) Write, using quantifier notation, the definitions for the following statements

a) $f: A \longrightarrow B$ not one-to-one
b) $f: A \longrightarrow B$ not onto

Functions

- A <u>function f</u> is a mapping $f: A \longrightarrow \mathbb{R}$ with $A \neq \emptyset$ and $A \subseteq \mathbb{R}$.

$$\boxed{f \text{ function} \Leftrightarrow f: A \longrightarrow \mathbb{R} \wedge A \neq \emptyset \wedge A \subseteq \mathbb{R}}$$

Properties of functions

1) <u>Even / odd</u>

f even $\Leftrightarrow \forall x \in A: (-x \in A \wedge f(-x) = f(x))$

f odd $\Leftrightarrow \forall x \in A: (-x \in A \wedge f(-x) = -f(x))$

2) <u>Monotonicity.</u>

Let $B \subseteq A$.

$f \nearrow B \Leftrightarrow \forall x_1, x_2 \in A: (x_1 < x_2 \Rightarrow f(x_1) < f(x_2))$

$f \searrow B \Leftrightarrow \forall x_1, x_2 \in A: (x_1 < x_2 \Rightarrow f(x_1) > f(x_2))$

Terminology: $f \nearrow B \longrightarrow f$ strictly increasing in B.

$f \searrow B \longrightarrow f$ strictly decreasing in B.

3) <u>Boundedness</u>

Let $B \subseteq A$.

f Bounded at $B \Leftrightarrow \exists \lambda \in \mathbb{R}^*_+ : \forall x \in B : |f(x)| \leq \lambda$

Operations with functions

1) Addition

$$h = f+g \iff \begin{cases} \mathrm{dom}(h) = \mathrm{dom}(f) \cap \mathrm{dom}(g) \\ \forall x \in \mathrm{dom}(h) : h(x) = f(x) + g(x) \end{cases}$$

2) Multiplication

$$h = fg \iff \begin{cases} \mathrm{dom}(h) = \mathrm{dom}(f) \cap \mathrm{dom}(g) \\ \forall x \in \mathrm{dom}(h) : h(x) = f(x)g(x) \end{cases}$$

3) Scalar multiplication

$$h = \lambda f \iff \begin{cases} \mathrm{dom}(h) = \mathrm{dom}(f) \\ \forall x \in \mathrm{dom}(h) : h(x) = \lambda f(x). \end{cases}$$

EXAMPLES

1) Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be two functions with $f \nearrow A$ and $g \nearrow A$.
Show that $f+g \nearrow A$.

Proof

$\vdots$

Sufficient to show that
$\forall x_1, x_2 \in A : (x_1 < x_2 \Rightarrow (f+g)(x_1) < (f+g)(x_2))$.
Let $\underline{x_1, x_2 \in A}$ be given with $\underline{x_1 < x_2}$.
$f \nearrow A \Rightarrow f(x_1) < f(x_2)$ (1)
$g \nearrow A \Rightarrow g(x_1) < g(x_2)$ (2)
From (1) and (2)
$f(x_1) + g(x_1) < f(x_2) + g(x_2) \Rightarrow$
$\Rightarrow \underline{(f+g)(x_1) < (f+g)(x_2)}$

Thus we get $(f+g) \nearrow A$. □

2) Let $f(x) = ax^2 + bx + c$ with $a > 0$. Show that $f \nearrow (-b/(2a), +\infty)$.
<u>Proof</u>

Let $\underline{x_1, x_2 \in (-b/(2a), +\infty)}$ be given with $\underline{x_1 < x_2}$.
Then
$f(x_1) - f(x_2) = (ax_1^2 + bx_1 + c) - (ax_2^2 + bx_2 + c) =$
$= a(x_1^2 - x_2^2) + b(x_1 - x_2) =$
$= a(x_1 - x_2)(x_1 + x_2) + b(x_1 - x_2) =$
$= (x_1 - x_2)[a(x_1 + x_2) + b]$ (1).
Sufficient to show that $f(x_1) - f(x_2) < 0$.
Since
$x_1, x_2 \in (-b/2a, +\infty) \Rightarrow$
$x_1 > -b/2a \wedge x_2 > -b/2a \Rightarrow$

$$\Rightarrow x_1+x_2 > -b/a \underset{a>0}{\Rightarrow} a(x_1+x_2) > -b \Rightarrow$$

$$\Rightarrow a(x_1+x_2)+b > 0 \qquad (2)$$

Since $x_1 < x_2 \Rightarrow x_1 - x_2 < 0 \underset{(2)}{\Rightarrow}$

$$\Rightarrow (x_1-x_2)[a(x_1+x_2)+b] < 0 \Rightarrow$$

$$\Rightarrow f(x_1) - f(x_2) < 0 \Rightarrow \underline{f(x_1) < f(x_2)}.$$

It follows that

$\forall x_1, x_2 \in (-b/2a, +\infty) : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2)) \Rightarrow$

$\Rightarrow f \nearrow (-b/2a, +\infty).$

3) Let $f(x) = \dfrac{x}{x-c}$ with $f : \mathbb{R} - \{c\} \longrightarrow \mathbb{R}$.

Show that f is one-to-one, if $c \neq 0$.

<u>Proof</u>

Let $\underline{x_1, x_2 \in \mathbb{R} - \{c\}}$ be given.

Sufficient to show that

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Calculate:

$$f(x_1) - f(x_2) = \frac{x_1}{x_1 - c} - \frac{x_2}{x_2 - c} =$$

$$= \frac{x_1(x_2-c) - x_2(x_1-c)}{(x_1-c)(x_2-c)} =$$

$$= \frac{x_1x_2 - cx_1 - x_1x_2 + c\,x_2}{(x_1-c)(x_2-c)} =$$

$$= \frac{c(x_2-x_1)}{(x_1-c)(x_2-c)} \qquad (1)$$

It follows that

$\underline{f(x_1) = f(x_2)} \Rightarrow f(x_1) - f(x_2) = 0 \Rightarrow$

$$\Rightarrow \frac{c(x_2-x_1)}{(x_1-c)(x_2-c)} = 0 \Rightarrow$$

$$\Rightarrow c(x_2 - x_1) = 0 \Rightarrow$$

$$\Rightarrow c = 0 \lor x_2 - x_1 = 0 \Rightarrow \qquad [\text{use } c \neq 0]$$

$$\Rightarrow x_1 - x_2 = 0 \Rightarrow \underline{x_1 = x_2}.$$

Thus

$\forall x_1, x_2 \in \mathbb{R} - \{c\} : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2) \Rightarrow$

$\Rightarrow f$ one-to-one.

↳ For boundedness proofs we use the following results from college algebra and precalculus:

$$\boxed{\begin{array}{l} \forall a, b \in \mathbb{R} : \ |a \pm b| \leq |a| + |b| \\ \forall a, b \in \mathbb{R} : \ |ab| = |a||b| \\ \forall a \in \mathbb{R} : \forall b \in \mathbb{R} - \{0\} : \ \left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|} \\ \forall x \in \mathbb{R} : \ |\sin x| \leq 1 \land |\cos x| \leq 1 \end{array}}$$

4) Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be two functions bounded at $\mathbb{R}$. Show that

$$h(x) = f(x)(2+\cos x) - g(x)(1-\sin x)^2, \ \forall x\in\mathbb{R}$$

is also bounded in $\mathbb{R}$.

<u>Proof</u>

f bounded at $\mathbb{R} \Rightarrow \exists\, a\in\mathbb{R}^*_+ : \forall x\in\mathbb{R} : |f(x)| \leq a$

g bounded at $\mathbb{R} \Rightarrow \exists\, b\in\mathbb{R}^*_+ : \forall x\in\mathbb{R} : |g(x)| \leq a.$

It follows that

$$\begin{aligned}
|h(x)| &= |f(x)(2+\cos x) - g(x)(1-\sin x)^2| \leq \\
&\leq |f(x)(2+\cos x)| + |-g(x)(1-\sin x)^2| \\
&= |f(x)|\cdot|2+\cos x| + |g(x)|\cdot(|1-\sin x|)^2 \\
&\leq a|2+\cos x| + b(|1-\sin x|)^2 \leq \\
&\leq a(2+|\cos x|) + b(1+|\sin x|)^2 \leq \\
&\leq a(2+1) + b(1+1)^2 = \\
&= 3a+4b, \quad \forall x\in\mathbb{R} \Rightarrow h \text{ bounded at } \mathbb{R}.
\end{aligned}$$

- Note that the validity of the proof above is strongly dependent on knowing that $a>0$ and $b>0$.

EXERCISES

(19) Let $f: A \longrightarrow \mathbb{R}$ be a function. Use quantifier notation to write the definitions for the following statements

a) f not even

b) f not odd

c) f not strictly increasing in $S \subseteq A$

d) f not strictly decreasing in $S \subseteq A$

e) f not bounded at $S \subseteq A$.

(20) Let $f = \{(1,3), (2,4), (3,1), (4,7)\}$ and
$g = \{(2,0), (3,2), (4,1), (5,3)\}$

Define the functions

a) $h = f + g$ b) $h = fg$ c) $h = 3f + g$.

(21) Let f, g, h be functions with $f: A \to \mathbb{R}$, $g: A \to \mathbb{R}$, and $h: B \to \mathbb{R}$. Show that

a) $f = g \Rightarrow f + h = g + h$

b) $f = g \Rightarrow fh = gh$

c) $\forall a, b \in \mathbb{R}: (af)(bg) = (ab)(fg)$.

(22) Let f, g be functions with $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$. Show that

a) $(f \text{ even} \wedge g \text{ even}) \Rightarrow (f+g \text{ even} \wedge fg \text{ even})$

b) $(f \text{ odd} \wedge g \text{ odd}) \Rightarrow (f+g \text{ odd} \wedge fg \text{ even})$

c) $(f \text{ odd} \wedge f \nearrow [0,+\infty)) \Rightarrow f \nearrow \mathbb{R}$

d) $(f \text{ even} \wedge f \nearrow (0,+\infty)) \Rightarrow f \searrow (-\infty,0)$.

(23) Show that the following functions are one-to-one

a) $f(x) = ax+b$ with $a \neq 0$

b) $f(x) = a/x$ with $a \neq 0$

c) $f(x) = \dfrac{ax+b}{cx+d}$ with $D = ad-bc \neq 0$

(24) Let $f: A \to \mathbb{R}$. Show that

a) $f \nearrow A \Rightarrow f$ one-to-one

b) $f \searrow A \Rightarrow f$ one-to-one

c) f even $\Rightarrow f$ not one-to-one

(25) Show that $f(x) = ax^2+bx+c$ with $a \neq 0$ is not one-to-one.
(Hint: Distinguish two cases: $b \neq 0$ and $b = 0$.)

(26) Let $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = ax^3+b$. Show that

a) $a > 0 \Rightarrow f \nearrow \mathbb{R}$

b) $a < 0 \Rightarrow f \searrow \mathbb{R}$.

(27) Let $f:\mathbb{R}\to\mathbb{R}$ with $f(x)=ax^2+bx+c$. Prove that
$a<0 \Rightarrow f\searrow[-b/2a,+\infty) \wedge f\nearrow(-\infty,-b/2a]$

(28) Let $f(x)=-1/x$ with $f:\mathbb{R}-\{0\}\to\mathbb{R}$.
Prove that

a) $f\nearrow(-\infty,0)$

b) $f\nearrow(0,+\infty)$

c) f <u>NOT</u> $\nearrow(-\infty,0)\cup(0,+\infty)$ (!!)

(29) Let $f:\mathbb{R}-\{-d/c\}\to\mathbb{R}$ with $f(x)=\dfrac{ax+b}{cx+d}$
Show that

a) $D=ad-bc>0 \Rightarrow f\nearrow(-\infty,-d/c) \wedge f\nearrow(-d/c,+\infty)$

b) $D=ad-bc<0 \Rightarrow f\searrow(-\infty,-d/c)\wedge f\searrow(-d/c,+\infty)$

(30) Let $f:A\to\mathbb{R}$ and $g:A\to\mathbb{R}$ be two functions bounded at B with $B\subseteq A$. Show that $f+g$ and fg are also bounded at B.

(31) Show that the following functions are bounded in $\mathbb{R}$.

a) $f(x)=\sin x\cdot(\cos x+\sin x)$ with $f:\mathbb{R}\to\mathbb{R}$

b) $f(x)=(1-\sin x)^2\cos x+\sin x$ with $f:\mathbb{R}\to\mathbb{R}$

c) $f(x)=(1-\cos x)(1-\sin x)+\sin x$ with $f:\mathbb{R}\to\mathbb{R}$.

(32) Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be bounded at $\mathbb{R}$. Show that $h:\mathbb{R}\to\mathbb{R}$, defined as follows, is also bounded in $\mathbb{R}$.

a) $h(x) = f(x)g(x)\cos x$

b) $h(x) = f(x)(1+\sin x) + g(x)\cos^2 x$

c) $h(x) = \sin(f(x)) + g(x)\cos(g(x))$

d) $h(x) = f(g(x))(\sin x + g(x)\cos(\sin x))$

(33) Let $f:[-1,3]\longrightarrow\mathbb{R}$ with $f(x) = ax^2+bx+c$. Show that f is bounded at $[-1,3]$.

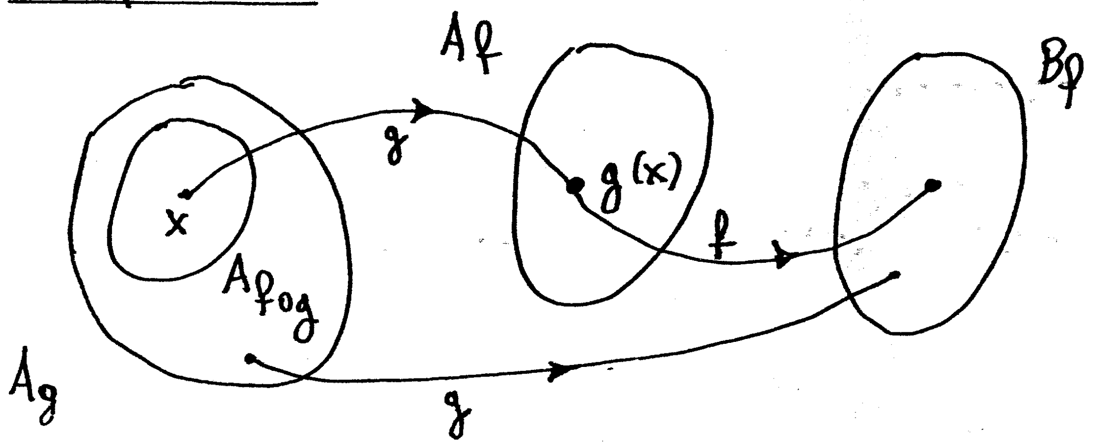
Mapping composition

- Let $f : A_f \to B_f$ and $g : A_g \to B_g$ be two mappings. We define the composition $h = f \circ g$ as follows:

$$\boxed{\begin{array}{l} f\circ g : A_{f\circ g} \longrightarrow B_f \text{ with} \\ A_{f\circ g} = \{x \in A_g \mid g(x) \in A_f\} \\ \forall x \in A_{f\circ g} : (f\circ g)(x) = f(g(x)) \end{array}}$$

We may similarly define $g \circ f$.

- interpretation



- Note that the domain of $f\circ g$ involves two restrictions:

$$\boxed{x \in \mathrm{dom}(f\circ g) \Leftrightarrow x \in \mathrm{dom}(g) \wedge g(x) \in \mathrm{dom}(f)}$$

Properties of mapping composition

- Let f, g, h be 3 mappings. Then

$$\boxed{(f \circ g) \circ h = f \circ (g \circ h)} \quad \text{(associative)}$$

Proof

First we establish that the domains are equal.

$x \in \mathrm{dom}((f \circ g) \circ h) \Leftrightarrow$

$\Leftrightarrow x \in \mathrm{dom}(h) \wedge h(x) \in \mathrm{dom}(f \circ g)$

$\Leftrightarrow x \in \mathrm{dom}(h) \wedge [h(x) \in \mathrm{dom}(g) \wedge g(h(x)) \in \mathrm{dom}(f)]$

$\Leftrightarrow (x \in \mathrm{dom}(h) \wedge h(x) \in \mathrm{dom}(g)) \wedge (g \circ h)(x) \in \mathrm{dom}(f)$

$\Leftrightarrow x \in \mathrm{dom}(g \circ h) \wedge (g \circ h)(x) \in \mathrm{dom}(f)$

$\Leftrightarrow x \in \mathrm{dom}(f \circ (g \circ h))$.

therefore,

$$\mathrm{dom}((f \circ g) \circ h) = \mathrm{dom}(f \circ (g \circ h)) = A$$

Let $x \in A$ be given. Then

$$\left.\begin{aligned} [(f \circ g) \circ h](x) &= (f \circ g)(h(x)) = f(g(h(x))) \\ [f \circ (g \circ h)](x) &= f((g \circ h)(x)) = f(g(h(x))) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow [(f \circ g) \circ h](x) = [f \circ (g \circ h)](x), \forall x \in A.$$

It follows that $(f \circ g) \circ h = f \circ (g \circ h)$. $\square$

- In general, it is usually not true that $f \circ g = g \circ f$, although exceptions are possible for specific choices of f, g.

- Let f, g be two mappings.

$$\boxed{f \text{ "1-1"} \land g \text{ "1-1"} \Rightarrow f \circ g \text{ "1-1"}}$$

Proof

Suffic

Let $A = \mathrm{dom}(f \circ g)$. Sufficient to show that

$$\forall x_1, x_2 \in A : (f \circ g)(x_1) = (f \circ g)(x_2) \Rightarrow x_1 = x_2. \quad (1)$$

Let $x_1, x_2 \in A$ be given. Then

$$\begin{aligned} (f \circ g)(x_1) = (f \circ g)(x_2) &\Rightarrow \\ &\Rightarrow f(g(x_1)) = f(g(x_2)) && [\text{def.}] \\ &\Rightarrow g(x_1) = g(x_2) && [f \text{ "1-1"}] \\ &\Rightarrow x_1 = x_2 && [g \text{ "1-1"}] \end{aligned}$$

Thus we have shown $(1) \Rightarrow f \circ g$ "1-1" $\square$.

- We do not have a similarly broad result for the onto property. However, it can be shown that if

$$\boxed{\left\{\begin{array}{ll} f: B \to G & \text{onto} \\ g: A \to B & \text{onto}\end{array}\right. \Rightarrow \left\{\begin{array}{l} f\circ g: A \to G \\ f\circ g \ \text{onto}\end{array}\right.}$$

<u>Proof</u>

$$\begin{aligned}
g \text{ onto} &\Rightarrow g(A) = B \Rightarrow \\
&\Rightarrow \forall x \in A : g(x) \in B \\
&\Rightarrow \mathrm{dom}(f\circ g) = \{x \in A \mid g(x) \in B\} = A \qquad (1) \\
&\Rightarrow f\circ g : A \to G
\end{aligned}$$

Furthermore

$$\begin{aligned}
(f\circ g)(\mathrm{dom}(f\circ g)) &= (f\circ g)(A) && [\text{use } (1)] \\
&= f(g(A)) && [\text{def.}] \\
&= f(B) && [g \text{ onto}] \\
&= C && [f \text{ onto}] \quad \square
\end{aligned}$$

EXAMPLES

1) Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ with $f \searrow \mathbb{R}$ and $g \nearrow \mathbb{R}$. Show that $f\circ g \searrow \mathbb{R}$.

Proof

Let $x_1, x_2 \in \mathbb{R}$ be given with $x_1 < x_2$.
Sufficient to show that $(f\circ g)(x_1) > (f\circ g)(x_2)$.
Since

$$
\begin{aligned}
x_1 < x_2 &\Rightarrow g(x_1) < g(x_2) && [g \nearrow \mathbb{R}] \\
&\Rightarrow f(g(x_1)) > f(g(x_2)) && [f \searrow \mathbb{R}] \\
&\Rightarrow \underline{(f\circ g)(x_1) > (f\circ g)(x_2)}.
\end{aligned}
$$

Thus $f\circ g \searrow \mathbb{R}$.

2) Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ with f and g odd. Show that $f\circ g$ odd.

Proof

Let $x\in\mathbb{R}$ be given.

$$
\begin{aligned}
(f\circ g)(-x) &= f(g(-x)) \\
&= f(-g(x)) && [g \text{ odd}] \\
&= -f(g(x)) && [f \text{ odd}] \\
&= -(f\circ g)(x), \quad \forall x\in\mathbb{R} \Rightarrow f\circ g \text{ odd}.
\end{aligned}
$$

EXERCISES

(34) Let $f = \{(1,3),(2,4),(3,1),(4,2)\}$ and
$g = \{(2,4),(3,1),(4,2)\}$.
Define $f \circ g$ and $g \circ f$.

(35) Let $f = \{(1,2),(3,2),(2,4),(4,4)\}$ and
$g = \{(1,3),(2,1),(3,5),(4,2)\}$
Define $f \circ g$ and $g \circ f$.

(36) Let f, g, h be three functions. Show that
$$f = g \Rightarrow f \circ h = g \circ h$$

(37) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Show that

a) f even $\land$ g even $\Rightarrow$ $f \circ g$ even

b) f odd $\land$ g odd $\Rightarrow$ $f \circ g$ odd

c) f even $\land$ g odd $\Rightarrow$ $f \circ g$ even

d) $f \nearrow \mathbb{R} \land g \nearrow \mathbb{R} \Rightarrow f \circ g \nearrow \mathbb{R}$

e) $f \nearrow \mathbb{R} \land g \searrow \mathbb{R} \Rightarrow f \circ g \searrow \mathbb{R}$

f) $f \searrow \mathbb{R} \land g \searrow \mathbb{R} \Rightarrow g \circ f \nearrow \mathbb{R}$.

(38) Let $f : B \to C$ and $g : A \to B$ be two functions.
Show that:
$f \circ g$ onto $\land$ g not onto $\Rightarrow$ f not one-to-one

(39) Let $f: A \to A$ be a mapping and let $i_A : A \to A$ be defined as

$$\forall x \in A : i_A(x) = x.$$

Show that

$$f \circ f = i_A \Rightarrow f \text{ is bijective}.$$

[Hint: You can use the result of the previous question]

(40) Let $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = ax + b$. Show that

$$\mathcal{S} = \{(a,b) \in \mathbb{R} \times \mathbb{R} \mid f \circ f = i\}$$
$$= \{(1,0)\} \cup \{(-1,x) \mid x \in \mathbb{R}\}$$

Here $i: \mathbb{R} \to \mathbb{R}$ with $\forall x \in \mathbb{R} : i(x) = x$.

(41) Investigate the set $\mathcal{S}$ in question (40) for the function $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = ax^2 + bx + c$.

▼ Inverse mapping

- Let $f : A \to B$ be a mapping. We say that

a) g is left-inverse of $f \Leftrightarrow$
$$\Leftrightarrow \forall x \in dom(g \circ f) : (g \circ f)(x) = x$$

b) g is right-inverse of $f \Leftrightarrow$
$$\Leftrightarrow \forall x \in dom(f \circ g) : (f \circ g)(x) = x$$

c) g is inverse of f if and only if g is both left-inverse and right-inverse of f. (notation: $g = f^{-1}$)

Formally:

$$\boxed{g = f^{-1} \Leftrightarrow \begin{cases} dom(g) = ran(f) \\ g \text{ left-inverse of } f \\ g \text{ right-inverse of } f \end{cases}}$$

- These definitions can be abbreviated using the identity mapping i:

$$\boxed{\forall x \in dom(i) : i(x) = x}$$

Then:

$$\boxed{g = f^{-1} \Leftrightarrow f \circ g = g \circ f = i}$$

Existence of inverse mapping

Let $f : A \to B$ be a mapping.

1) $\boxed{f \text{ "1-1"} \Rightarrow f \text{ has a left inverse } g : B \to A}$

<u>Proof</u>

- <u>Construction</u> of $g : B \to A$

  Let $y \in B$. If $y \in f(A) \Rightarrow \exists\, x \in A : f(x) = y$
  Use this x to define $g(y) = x$.
  If $y \in B - f(A)$, define $g(y) = y$.

- <u>Left-inverse</u> : We show that g is a left inverse.

Let $x \in A$ be given. Define $y = f(x)$ and $x_0 = g(y)$ (it is not obvious yet that g will return to the original x). By construction of g we only know that $f(x_0) = y$, since $y \in f(A)$.
It follows that

$$f(x_0) = y = f(x) \Rightarrow x = x_0 \quad [f \text{ "1-1"}]$$

$$\Rightarrow g(y) = x$$

$$\Rightarrow (g \circ f)(x) = g(f(x)) = g(y) = x, \ \forall x \in A$$

$$\Rightarrow g \text{ left-inverse of } f \quad \square$$

27) $\boxed{f \text{ onto} \Rightarrow f \text{ has a right inverse } g: B \to A}$

Proof

- Construction of $g: B \to A$

Since f onto $\Rightarrow f(A) = B \Rightarrow$

$\Rightarrow \forall y \in B : \exists x \in A : f(x) = y$.

Let $y \in B$ be given. Choose some $x \in A$ such that $f(x) = y$ and thus define the mapping

$g : y \in B \longrightarrow x \in A$.

- Right inverse : We show that g is a right inverse. Let $y \in B$ be given. Define $x = g(y)$.

By construction of g we know that $f(x) = y$.

It follows that

$(f \circ g)(y) = f(g(y)) = f(x) = y, \; \forall y \in B \Rightarrow$

$\Rightarrow g$ right inverse of f. □

It is easier to show that the converse statements of the two results above are also true: (see homework):

f has left inverse $\Rightarrow f$ "1-1"

f has right inverse $\Rightarrow f$ onto.

Thus:

$$\boxed{f \text{ has an inverse} \Leftrightarrow \begin{cases} f \text{ "1-1"} \\ f \text{ onto} \end{cases}}$$

EXAMPLES

1) Let $f: A \to B$ be a bijection with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. If $f \nearrow A$, show that $f^{-1} \nearrow B$.

Proof

Let $y_1, y_2 \in B$ be given with $y_1 < y_2$.
Define $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$.
Sufficient to show that $x_1 < x_2$.
Assume that $x_1 \geq x_2$. Then,

$$x_1 \geq x_2 \Rightarrow f(x_1) \geq f(x_2) \quad [f \nearrow A]$$
$$\Rightarrow f(f^{-1}(y_1)) \geq f(f^{-1}(y_2)) \quad [\text{def.}]$$
$$\Rightarrow y_1 \geq y_2 \leftarrow \text{contradiction}$$

It follows that $x_1 < x_2 \Rightarrow f^{-1}(y_1) < f^{-1}(y_2)$
Thus we showed
$\forall y_1, y_2 \in B: (y_1 < y_2 \Rightarrow f^{-1}(y_1) < f^{-1}(y_2)) \Rightarrow$

$\Rightarrow f^{-1} \nearrow B$. $\square$

*2) Let $f: A \to B$ be a bijection. Show that the inverse mapping of f is unique

Proof

Assume that there are two inverse mappings $g_1 : B \longrightarrow A$ and $g_2 : B \longrightarrow A$. It is sufficient to show that $g_1 = g_2$.

By definition:

$$\forall x \in A : \; g_1(f(x)) = x \qquad (1)$$

$$\forall x \in A : \; g_2(f(x)) = x \qquad (2)$$

From (1) and (2):

$$((g_1 - g_2) \circ f)(x) = g_1(f(x)) - g_2(f(x)) = x - x = 0, \quad \forall x \in A \Rightarrow$$

$$\Rightarrow ((g_1 - g_2) \circ f)(A) = \{0\} \qquad (3)$$

From (3):

$$(g_1 - g_2)(B) = (g_1 - g_2)(f(A)) \qquad [f \text{ onto}]$$

$$= ((g_1 - g_2) \circ f)(A)$$

$$= \{0\} \Rightarrow \qquad [\text{from } (3)]$$

$$\Rightarrow \forall y \in B : \; (g_1 - g_2)(y) = g_1(y) - g_2(y) = 0 \Rightarrow$$

$$\Rightarrow \forall y \in B : \; g_1(y) = g_2(y) \Rightarrow$$

$\Rightarrow g_1 = g_2 \Rightarrow$ the inverse is unique. $\square$

EXERCISES

• (42) Let $f: A \to B$ be a mapping. Show that f has an inverse $\Rightarrow f$ bijection

Combined with the theory discussed in lecture this establishes that

$$\boxed{f \text{ has an inverse} \Leftrightarrow f \text{ bijection}}$$

(43) Let $f: \mathbb{R} \to \mathbb{R}$. be a bijection. Show that f odd $\Rightarrow f^{-1}$ odd

(44) Let $f: [0,1] \to \mathbb{R}$ be a bijection. Show that f^{-1} is bounded.

(45) Let $f: A \to B$ be a bijection with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. Show that $f \searrow A \Rightarrow f^{-1} \searrow B$.

(46) Let $f: C \to D$, $g: B \to C$, $h: A \to B$ be bijections. Show that

a) $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

b) $(f \circ g \circ h)^{-1} = h^{-1} \circ g^{-1} \circ f^{-1}$

[Hint: Use (a) and associative property to show (b)]