RELATIONS AND FUNCTIONS

Cartesian product

- An ordered pair (a,b) is defined as an ordered collection of two elements a and b such that it satisfies the axiom:
  $$(a_1,b_1)=(a_2,b_2) \Leftrightarrow a_1=a_2 \land b_1=b_2.$$
- Ordered pairs can be represented as sets:
  $$(a,b)=\{a,\{a,b\}\}$$
  Then ordered pair equality corresponds to set equality.
- Let A,B be two sets. We define the cartesian product $A\times B$ as:
  $$A\times B=\{(a,b)\mid a\in A \land b\in B\}$$
  The corresponding belonging condition is:
  $$x\in A\times B \Leftrightarrow \exists a\in A : \exists b\in B : x=(a,b).$$
  however, in practice we find it more useful to use the following statement
  $$(a,b)\in A\times B \Leftrightarrow a\in A \land b\in B.$$
- We also define $A^2=A\times A$.
- It is easy to see that
  $$\emptyset\times A=\emptyset$$
  $$A\times\emptyset=\emptyset.$$

EXAMPLES

a) For $A = \{1,2\}$ and $B = \{2,3\}$, evaluate $A \times B$, $B \times A$ and A^2.

<u>Solution</u>

$$A \times B = \{1,2\} \times \{2,3\} =$$
$$= \{(1,2), (1,3), (2,2), (2,3)\}$$
$$B \times A = \{2,3\} \times \{1,2\} =$$
$$= \{(2,1), (2,2), (3,1), (3,2)\}$$
$$A^2 = A \times A = \{1,2\} \times \{1,2\} =$$
$$= \{(1,1), (1,2), (2,1), (2,2)\}$$

b) Let A, B, C be sets. Show that
$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

<u>Solution</u>

Since,
$$(x,y) \in A \times (B \cup C) \Leftrightarrow x \in A \wedge y \in B \cup C$$
$$\Leftrightarrow x \in A \wedge (y \in B \vee y \in C)$$
$$\Leftrightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$$
$$\Leftrightarrow (x,y) \in A \times B \vee (x,y) \in A \times C$$
$$\Leftrightarrow (x,y) \in (A \times B) \cup (A \times C),$$
it follows that
$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

c) Show that; for sets A, B, C:
$(C \neq \emptyset \wedge A \times C = B \times C) \Rightarrow A = B$.

Solution

Assume that $C \neq \emptyset$ and $A \times C = B \times C$.
Since $C \neq \emptyset$, choose a $y \in C$.
Let $x \in A$ be given. Then:

$$\begin{aligned} x \in A \wedge y \in C &\Rightarrow (x,y) \in A \times C && [\text{definition}] \\ &\Rightarrow (x,y) \in B \times C && [A \times C \subseteq B \times C] \\ &\Rightarrow x \in B \wedge y \in C && [\text{definition}] \\ &\Rightarrow x \in B \end{aligned}$$

and therefore:
$(\forall x \in A : x \in B) \Rightarrow A \subseteq B$. (1)

Let $x \in B$ be given. Then

$$\begin{aligned} x \in B \wedge y \in C &\Rightarrow (x,y) \in B \times C \\ &\Rightarrow (x,y) \in A \times C \\ &\Rightarrow x \in A \wedge y \in C \\ &\Rightarrow x \in A \end{aligned}$$

and therefore
$(\forall x \in B : x \in A) \Rightarrow B \subseteq A$. (2)

From (1) and (2): $A = B$.

d) Let A, B be sets with $A \neq \emptyset$ and $B \neq \emptyset$. Show that

$A \times B = B \times A \Rightarrow A = B$.

<u>Solution</u>

Assume that $A \neq \emptyset$ and $B \neq \emptyset$ and $A \times B = B \times A$.

Let $x \in A$ be given.

Since $B \neq \emptyset$, choose a $y \in B$. Then

$$\begin{aligned} x \in A \wedge y \in B &\Rightarrow (x, y) \in A \times B \\ &\Rightarrow (x, y) \in B \times A \quad [\text{via } A \times B \subseteq B \times A] \\ &\Rightarrow x \in B \wedge y \in A \\ &\Rightarrow x \in B. \end{aligned}$$

and therefore:

$(\forall x \in A : x \in B) \Rightarrow A \subseteq B$. (1)

Let $x \in B$ be given.

Since $A \neq \emptyset$, choose a $y \in A$. Then

$$\begin{aligned} x \in B \wedge y \in A &\Rightarrow (x, y) \in B \times A \\ &\Rightarrow (x, y) \in A \times B \quad [\text{via } B \times A \subseteq A \times B] \\ &\Rightarrow x \in A \wedge y \in B \\ &\Rightarrow x \in A. \end{aligned}$$

and therefore

$(\forall x \in B : x \in A) \Rightarrow B \subseteq A$. (2)

From (1) and (2): $A = B$.

e) Let $\{A_a\}_{a\in I}$, $\{B_a\}_{a\in I}$ be indexed set collections and let G be a set. Show that

$$C\times\left[\bigcup_{a\in I}(A_a-B_a)\right]\subseteq\bigcup_{a\in I}\left[(C\times A_a)-(C\times B_a)\right]$$

Solution

Since

$$(x,y)\in G\times\left[\bigcup_{a\in I}(A_a-B_a)\right]\Rightarrow$$

$$\Rightarrow x\in G\land y\in\bigcup_{a\in I}(A_a-B_a)\Rightarrow$$

$$\Rightarrow x\in G\land\exists a\in I:\ y\in A_a-B_a$$

$$\Rightarrow x\in G\land\exists a\in I:\ (y\in A_a\land y\notin B_a)$$

$$\Rightarrow\exists a\in I:\ (x\in G\land y\in A_a\land y\notin B_a)$$

$$\Rightarrow\exists a\in I:\ \left[(x\in G\land y\in A_a)\land\underline{\underline{(x\notin G\lor y\notin B_a)}}\right]\qquad(!!!)$$

$$\Rightarrow\exists a\in I:\ \left((x,y)\in G\times A_a\land\overline{(x\in C\land y\in B_a)}\right)$$

$$\Rightarrow\exists a\in I:\ \left((x,y)\in G\times A_a\land(x,y)\notin G\times B_a\right)$$

$$\Rightarrow\exists a\in I:\ (x,y)\in(G\times A_a)-(G\times B_a)$$

$$\Rightarrow(x,y)\in\bigcup_{a\in I}\left[(G\times A_a)-(G\times B_a)\right]$$

it follows that:

$$C\times\left[\bigcup_{a\in I}(A_a-B_a)\right]\subseteq\bigcup_{a\in I}\left[(G\times A_a)-(G\times B_a)\right]$$

↳ Note that the (!!) step is valid but cannot be reversed.

EXERCISES

① Let $A = \{x \in \mathbb{Z} \mid 1 \leq x \leq 3\}$
$B = \{3x - 1 \mid x \in \mathbb{Z} \wedge 0 < x < 4\}$
List the elements of $A \times B$.

② Prove that for A, B, C sets
$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

③ Prove the following

a) $A \times B = \emptyset \Leftrightarrow A = \emptyset \vee B = \emptyset$

b) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

c) $(A \times B) \cap (C \times D) = \emptyset \Leftrightarrow A \cap C = \emptyset \vee B \cap D = \emptyset$.

④ Prove the following.

a) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$

b) $\{p, q\} \subseteq A \Rightarrow (A \times \{p\}) \cup (\{q\} \times A) \subseteq A \times A$

⑤ Prove the following:

a) $A \times B = B \times A \Leftrightarrow A = \emptyset \vee B = \emptyset \vee A = B$

b) $A \neq \emptyset \neq B \wedge (A \times B) \cup (B \times A) = C \times C \Rightarrow A = B = C$.

⑥ Let $\{A_a\}_{a\in I}$ and $\{B_a\}_{a\in I}$ be indexed set collections and let C be a set. Prove the following:

a) $$\left(\bigcup_{a\in I} A_a\right)\times C = \bigcup_{a\in I}(A_a\times C)$$

b) $$\left(\bigcap_{a\in I} A_a\right)\times C = \bigcap_{a\in I}(A_a\times C)$$

c) $$\bigcap_{a\in I}(A_a\times B_a) = \left(\bigcap_{a\in I} A_a\right)\times\left(\bigcap_{a\in I} B_a\right)$$

⑦ Show that for A, B sets

$$\bigcup_{S\in\mathcal{P}(A)}\left[\bigcup_{T\in\mathcal{P}(B)}\{S\times T\}\right]\subseteq\mathcal{P}(A\times B)$$

V Relations

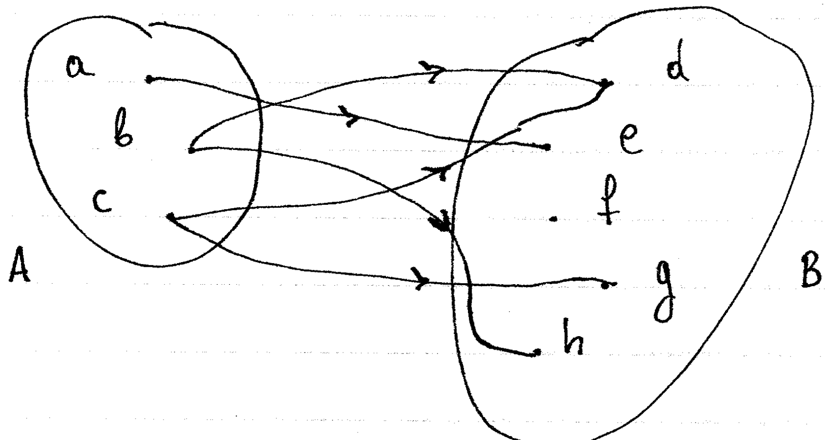
- Let A, B be two sets with $A \neq \emptyset$ and $B \neq \emptyset$. We define the set of all relations from A to B via the following belonging condition:
$$R \in Rel(A,B) \Leftrightarrow R \subseteq A \times B$$
- If $R \in Rel(A,B)$, we say that R is a <u>relation</u> from A to B.
- Let $R \in Rel(A,B)$ be a relation and let $x \in A$ and $y \in B$. Then we define the statements xRy and $x\not{R}y$ as follows:
$$\forall x \in A : \forall y \in B : (xRy \Leftrightarrow (x,y) \in R)$$
$$\forall x \in A : \forall y \in B : (x\not{R}y \Leftrightarrow (x,y) \notin R)$$
We say that:
xRy : x is related with y via relation R.
$x\not{R}y$: x is NOT related with y via relation R.

EXAMPLE

Let $A = \{a,b,c\}$ and $B = \{d,e,f,g,h\}$. Then
$R = \{(a,e),(b,d),(c,g),(b,h),(c,d)\}$
is a relation from A to B (i.e. $R \in Rel(A,B)$). Then

$(a,e) \in R \Rightarrow aRe$ $\qquad$ $(b,h) \in R \Rightarrow bRh$
$(b,d) \in R \Rightarrow bRd$ $\qquad$ $(c,d) \in R \Rightarrow cRd$
$(c,g) \in R \Rightarrow cRg$

↳ The relation R can be represented geometrically using a <u>Venn diagram</u>, as follows:



Each ordered pair (x,y) is represented by an arrow from x to y.

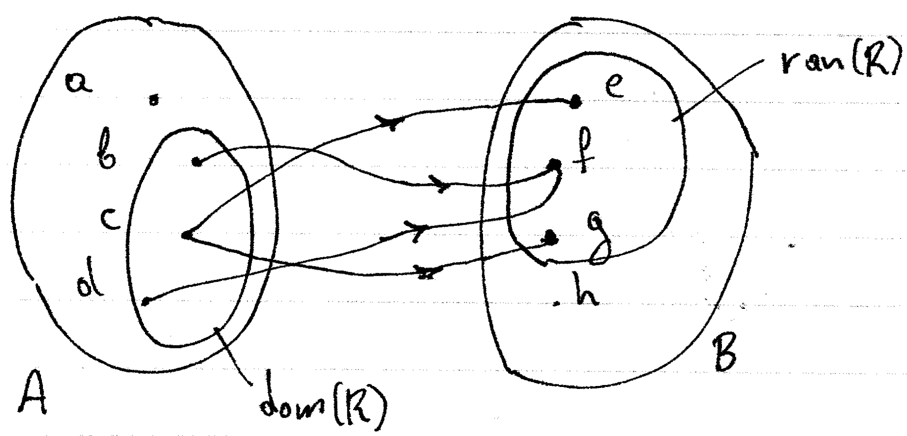
↳ Domain and range of a relation

- Let $R \in Rel(A,B)$ be a relation from A to B. We define the domain $dom(R)$ and range $ran(R)$ of R as:
$$dom(R) = \{x \in A \mid \exists y \in B : xRy\} \subseteq A$$
$$ran(R) = \{y \in B \mid \exists x \in A : xRy\} \subseteq B$$
- $dom(R)$ contains all the elements of A that are related with some element of B. In terms of Venn diagrams, $dom(R)$ has all the elements of A that have an outgoing arrow.
- $ran(R)$ contains all the elements of B that are related with some element of A. In terms of Venn diagrams,

ran(R) has all the elements of B that have an incoming arrow.

EXAMPLE

For $A=\{a,b,c,d\}$ and $B=\{e,f,g,h\}$, let $R\in Rel(A,B)$ be a relation from A to B with
$R=\{(b,f),(c,e),(d,f),(c,g)\}$.
Then: $dom(R)=\{b,c,d\}$ and
$ran(R)=\{e,f,g\}$



Relations on A

We define $Rel(A)=Rel(A,A)$. Then:

$$R\in Rel(A)\Leftrightarrow R\subseteq A\times A$$

and we say that R is <u>a relation on A</u>.

Equivalence relations

- Let $R \in Rel(A)$ be a relation on A with $A \neq \emptyset$.
We say that
  R reflexive $\Leftrightarrow \forall x \in A : xRx$
  R symmetric $\Leftrightarrow \forall x,y \in A : (xRy \Rightarrow yRx)$
  R transitive $\Leftrightarrow \forall x,y,z \in A : ((xRy \wedge yRz) \Rightarrow xRz)$
and:

$$R \text{ equivalence} \Leftrightarrow \begin{cases} R \text{ reflexive} \\ R \text{ symmetric} \\ R \text{ transitive} \end{cases}$$

EXAMPLE

a) Let $R \in Rel(\mathbb{Z})$ such that
$$xRy \Leftrightarrow 11x - 5y \text{ even.}$$
Show that R is an equivalence.

Proof

- Reflexive

Let $x \in \mathbb{Z}$ be given. Then:
$11x - 5x = 6x = 2(3x)$
and therefore, for $\mu = 3x$:
$$(\exists \mu \in \mathbb{Z} : 11x - 5x = 2\mu) \Rightarrow 11x - 5x \text{ even}$$
$$\Rightarrow xRx$$
It follows that: $(\forall x \in \mathbb{Z} : xRx) \Rightarrow R$ reflexive. (1)

• Symmetric

Let $x, y \in \mathbb{Z}$ be given. Assume that xRy. Then

$xRy \Rightarrow 11x - 5y$ even

$\Rightarrow \exists k \in \mathbb{Z}: 11x - 5y = 2k$

Choose $k \in \mathbb{Z}$ such that $11x - 5y = 2k$. Then, we have:

$11y - 5x = -5y + 16y + 11x - 16x = (11x - 5y) + (16y - 16x)$

$= 2k + (16y - 16x) = 2(k + 8y - 8x)$

Then, for $\mu = k + 8y - 8x \in \mathbb{Z}$:

$(\exists \mu \in \mathbb{Z}: 11y - 5x = 2\mu) \Rightarrow 11y - 5x$ even

$\Rightarrow yRx$

It follows that

$(\forall x, y \in \mathbb{Z}: (xRy \Rightarrow yRx)) \Rightarrow R$ symmetric. (2)

• Transitive

Let $x, y, z \in \mathbb{Z}$ be given. Assume that $xRy \land yRz$. Then

$xRy \Rightarrow 11x - 5y$ even

$\Rightarrow \exists k \in \mathbb{Z}: 11x - 5y = 2k$

$yRz \Rightarrow 11y - 5z$ even

$\Rightarrow \exists \lambda \in \mathbb{Z}: 11y - 5z = 2\lambda$

Choose $k, \lambda \in \mathbb{Z}$ such that $11x - 5y = 2k$ and $11y - 5z = 2\lambda$. Then,

$11x - 5z = (11x - 5y) + (11y - 5z) - 6y$

$= 2k + 2\lambda - 6y = 2(k + \lambda - 3y)$

Then, for $\mu = k + \lambda - 3y \in \mathbb{Z}$:

$(\exists \mu \in \mathbb{Z}: 11x - 5z = 2\mu) \Rightarrow 11x - 5z$ even

$\Rightarrow xRz$

It follows that: $(\forall x, y, z \in \mathbb{Z}: ((xRy \land yRz) \Rightarrow xRz)) \Rightarrow$

$\Rightarrow R$ transitive. (3)

From (1), (2), (3):

$$\begin{cases} R \text{ reflexive} \\ R \text{ symmetric} \\ R \text{ transitive} \end{cases} \Rightarrow R \text{ equivalence.}$$

b) Let $R \in Rel(A)$ be a relation on A. Show that

R reflexive $\Rightarrow dom(R) = A$

<u>Proof</u>

Assume that R reflexive. Since

$dom(R) = \{x \in A \mid \exists y \in A : xRy\} \Rightarrow \underline{dom(R) \subseteq A}$ (1)

$\Rightarrow$ Sufficient to show that $A \subseteq dom(R)$.

Let $\underline{x \in A}$ be given. Then:

R reflexive $\Rightarrow xRx$

$\Rightarrow \exists y \in A : xRy$ (for $y = x$)

It follows that since:

$$\begin{cases} x \in A \\ \exists y \in A : xRy \end{cases} \Rightarrow \underline{x \in dom(R)}.$$

and therefore

$(\forall x \in A : x \in dom(R)) \Rightarrow \underline{A \subseteq dom(R)}$ (2)

From (1) and (2):

$$\begin{cases} dom(R) \subseteq A \\ A \subseteq dom(R) \end{cases} \Rightarrow dom(R) = A.$$

$\hookrightarrow$ For $R \in Rel(A, B)$ note the belonging conditions:

$$x \in dom(R) \Leftrightarrow x \in A \wedge (\exists y \in B : xRy)$$
$$y \in ran(R) \Leftrightarrow y \in B \wedge (\exists x \in A : xRy)$$

These belonging conditions are used in the proofs above.

c) Let $R \in Rel(A)$. We define

$$R \text{ circular} \Leftrightarrow \forall x,y,z \in A : ((xRy \wedge yRz) \Rightarrow zRx)$$

Show that:

$$(R \text{ transitive} \wedge R \text{ symmetric}) \Rightarrow R \text{ circular}.$$

Proof

Assume that R transitive $\wedge$ R symmetric.

Let $x, y, z \in A$ be given and assume that $xRy \wedge yRz$. Then

$$\left\{\begin{array}{l} xRy \\ yRz \end{array}\right. \Rightarrow xRz \quad [R \text{ is transitive}]$$
$$\Rightarrow zRx \quad [R \text{ is symmetric}]$$

From the above, it follows that

$$\forall x,y,z \in A : ((xRy \wedge yRz) \Rightarrow zRx)$$
$$\Rightarrow R \text{ circular}.$$

EXERCISES

(8) Show that the following relations are equivalences:

a) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow a+b$ even

b) $R \in Rel(\mathbb{N}^*)$ with $aRb \Leftrightarrow a^2+b^2$ even

c) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow 3a-7b$ even

d) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow 3 \mid (a+2b)$

e) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow 4 \mid (a^3-b^3)$

f) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow 5 \mid (2a+3b)$

(9) Show that the following relations on $\mathbb{R}^* \times \mathbb{R}^*$ are equivalences:

a) $(x_1,y_1) R (x_2,y_2) \Leftrightarrow x_1y_2 - x_2y_1 = 0$

b) $(x_1,y_1) R (x_2,y_2) \Leftrightarrow \exists \lambda \in \mathbb{R}^* : \begin{cases} x_1 = \lambda x_2 \\ y_1 = \lambda y_2 \end{cases}$

(Recall that $\mathbb{R}^* = \mathbb{R} - \{0\}$).

(10) Let $R \in Rel(A)$ be a relation on A. Show that

a) R reflexive $\Rightarrow ran(R) = A$

b) R symmetric $\Rightarrow dom(R) = ran(R)$

c) (R circular $\land$ R symmetric) $\Rightarrow$ R transitive

d) R equivalence $\Leftrightarrow$ (R reflexive $\land$ R circular)

$\hookrightarrow$ We use the definition:

R circular $\Leftrightarrow \forall x,y,z \in A : ((xRy \land yRz) \Rightarrow zRx)$

(11) Let $R \in Rel(A)$ be a relation on A. Write the definitions, using quantifiers for the following statements:

a) R is not reflexive

b) R is not symmetric

c) R is not transitive.

Equivalence classes

- Let $R \in Rel(A)$ be an equivalence relation on A, and let $a \in A$. We define the equivalence class $R(a)$ as:

  $$R(a) = \{x \in A \mid x R a\}$$

  The belonging condition of $R(a)$ is given by:

  $$x \in R(a) \Leftrightarrow x \in A \land x R a$$

- The set of all possible equivalence classes of R is denoted as A/R:

  $$A/R = \{R(a) \mid a \in A\}$$

Properties of equivalence classes

- Let $R \in Rel(A)$ be an equivalence relation. Then

$$\boxed{\begin{array}{l} 1)\ \forall a, b \in A: R(a) = R(b) \Leftrightarrow aRb \\ 2)\ \forall a, b \in A: R(a) \cap R(b) = \emptyset \Leftrightarrow a \not R b \end{array}}$$

Proof of (1)

Let $a, b \in A$ be given.

($\Rightarrow$): Assume that $R(a) = R(b)$. Then.

$$\begin{aligned} R \text{ equivalence} &\Rightarrow R \text{ reflexive} && [\text{definition}] \\ &\Rightarrow aRa && [\text{definition}] \\ &\Rightarrow a \in R(a) && [\text{belonging condition}] \\ &\Rightarrow a \in R(b) && [\text{hypothesis } R(a) = R(b)] \\ &\Rightarrow \underline{aRb} && [\text{belonging condition}]. \end{aligned}$$

$(\Leftarrow)$: Assume that aRb. Let $x \in R(a)$ be given. Then

$$
\begin{aligned}
x \in R(a) &\Rightarrow xRa && [\text{belonging condition}] \\
&\Rightarrow xRb && [aRb \wedge R \text{ transitive}] \\
&\Rightarrow x \in R(b) && [\text{belonging condition}]
\end{aligned}
$$

It follows that $(\forall x \in R(a) : x \in R(b)) \Rightarrow \underline{R(a) \subseteq R(b)}$. (1)

Let $x \in R(b)$ be given. Then

$$
\begin{aligned}
x \in R(b) &\Rightarrow xRb && [\text{belonging condition}] \\
&\Rightarrow bRx && [R \text{ symmetric}] \\
&\Rightarrow aRx && [aRb \wedge R \text{ transitive}] \\
&\Rightarrow xRa && [R \text{ symmetric}] \\
&\Rightarrow x \in R(a) && [\text{belonging condition}]
\end{aligned}
$$

It follows that:

$$(\forall x \in R(b) : x \in R(a)) \Rightarrow \underline{R(b) \subseteq R(a)} \qquad (2)$$

From (1) and (2):

$$\begin{cases} R(a) \subseteq R(b) \\ R(b) \subseteq R(a) \end{cases} \Rightarrow \underline{R(a) = R(b)}.$$

From the above argument it follows that

$\forall a, b \in A : (R(a) = R(b) \Leftrightarrow aRb)$.

<u>Proof of (2)</u>

Let $a, b \in A$ be given.

$(\Rightarrow)$: Assume that $R(a) \cap R(b) = \emptyset$. To show that $a \not R b$, assume that aRb. Then:

$aRb \Rightarrow a \in R(b)$ (1)

and

R equivalence $\rightarrow R$ reflexive $\Rightarrow aRa$

$\Rightarrow a \in R(a) \qquad (2)$

From (1) and (2):

$a \in R(a) \wedge a \in R(b) \Rightarrow a \in R(a) \cap R(b)$

$\Rightarrow R(a) \cap R(b) \neq \emptyset \leftarrow$ Contradiction

because by hypothesis: $R(a) \cap R(b) = \emptyset$.

It follows that $\underline{a \not R b}$.

$\underline{(\Leftarrow)}$: Assume that $\underline{a \not R b}$. To show that $R(a) \cap R(b) = \emptyset$, assume that $R(a) \cap R(b) \neq \emptyset$. We may therefore choose some $x \in R(a) \cap R(b)$. Then:

$$x \in R(a) \cap R(b) \Rightarrow \begin{cases} x \in R(a) \\ x \in R(b) \end{cases} \Rightarrow \begin{cases} xRa \\ xRb \end{cases} \Rightarrow \begin{cases} aRx \\ xRb \end{cases}$$

$\Rightarrow aRb \leftarrow$ Contradiction,

because by hypothesis $a \not R b$.

It follows that $R(a) \cap R(b) = \emptyset$.

From the above argument it follows that

$\forall a, b \in A : (R(a) \cap R(b) = \emptyset \Leftrightarrow a \not R b)$.

EXAMPLE

We have previously shown that the relation $R \in Rel(\mathbb{Z})$ defined as:

$$xRy \Leftrightarrow 11x - 5y \text{ even}$$

is an equivalence. Find the equivalence classes of R.

For this type of problem it is useful to know and use the following previously proven statements:

$$\boxed{\begin{array}{l} \forall a,b \in \mathbb{Z}: \ ab \text{ even} \Leftrightarrow (a \text{ even} \vee b \text{ even}) \\ \forall a,b \in \mathbb{Z}: \ ab \text{ odd} \Leftrightarrow (a \text{ odd} \wedge b \text{ odd}) \end{array}}$$

Solution

Try $R(0) = \{x \in \mathbb{Z} \mid xR0\}$.

$$x \in R(0) \Leftrightarrow xR0 \Leftrightarrow 11x - 5\cdot 0 \text{ even} \Leftrightarrow 11x \text{ even}$$

$$\Leftrightarrow 11 \text{ even} \vee x \text{ even}$$

$$\Leftrightarrow x \text{ even}$$

and therefore $R(0) = \{x \in \mathbb{Z} \mid x \text{ even}\}$.

Try $R(1) = \{x \in \mathbb{Z} \mid xR1\}$. Let $x \in \mathbb{Z}$ be given.

$$x \in R(1) \Leftrightarrow xR1 \Leftrightarrow 11x - 5\cdot 1 \text{ even} \Leftrightarrow$$

$$\Leftrightarrow \exists k \in \mathbb{Z}: 11x - 5 = 2k$$

$$\Leftrightarrow \exists k \in \mathbb{Z}: 11x = 2k + 5 = 2k + 4 + 1 = 2(k+2) + 1$$

$$\Leftrightarrow \exists \lambda \in \mathbb{Z}: 11x = 2\lambda + 1 \quad (\text{for } \lambda = k+2)$$

$$\Leftrightarrow 11x \text{ odd} \Leftrightarrow 11 \text{ odd} \wedge x \text{ odd} \Leftrightarrow x \text{ odd}$$

and therefore

$$R(1) = \{x \in \mathbb{Z} \mid x \text{ odd}\}.$$

Since $R(0) \cup R(1) = \mathbb{Z}$, it follows that we have all equivalence classes and therefore

$$A/R = \{R(0), R(1)\}$$

EXERCISE

(12) The following relations were previously shown to be equivalences. Find the corresponding equivalence classes.

a) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow a+b$ even

b) $R \in Rel(\mathbb{N}^*)$ with $aRb \Leftrightarrow a^2+b^2$ even

c) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow 3a-7b$ even

d) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow 3 \mid a+2b$

Methodology for writing proofs

Proving implications

(1) To prove $\boxed{p \Rightarrow q}$

- **Direct Method**

  Assume p is true.

  [Prove q]

- **Contrapositive Method**

  We will show that $\bar{q} \Rightarrow \bar{p}$

  Assume $\bar{q}$ is true.

  [Prove $\bar{p}$]

  It follows that $p \Rightarrow q$

- **Contradiction Method**

  Assume p is true.

  To derive a contradiction, assume $\bar{q}$.

  [Prove r, using $p \wedge \bar{q}$]

  [Prove $\bar{r}$] ← Contradiction.

  It follows that q is true.

(2) To prove $\boxed{p \Leftrightarrow q}$

($\Rightarrow$): Assume p is true
[Prove q]

($\Leftarrow$): Assume q is true
[Prove p]

Proofs involving sets

Let A, B be two sets.

(1) To prove $\boxed{A \subseteq B}$

[We prove $x \in A \Rightarrow x \in B$]

(2) To prove $\boxed{A = B}$

[We prove $x \in A \Rightarrow x \in B$]
It follows that $A \subseteq B$ (1)
[We prove $x \in B \Rightarrow x \in A$]
It follows that $B \subseteq A$ (2)
From (1) and (2): $A = B$.

- For proofs involving sets, we recall that

$$x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$$
$$x \in A \cup B \Leftrightarrow x \in A \vee x \in B$$
$$x \in A - B \Leftrightarrow x \in A \wedge x \notin B$$
$$x \in \{x \in A \mid p(x)\} \Leftrightarrow x \in A \wedge p(x)$$
$$x \in \{\varphi(x) \mid x \in A \wedge p(x)\} \Leftrightarrow \exists y \in A : (\varphi(y) = x \wedge p(y))$$

Proofs involving identities

Let a, b be two expressions.
To prove $a = b$.

- Direct Method

$$a = \cdots = \cdots = \\ = \cdots = \cdots = b$$

- Indirect Method

$a = \cdots = \cdots = c$ (1)
$b = \cdots = \cdots = c$ (2)
From (1) and (2): $a = b$.

Proofs involving quantified statements

(1) To prove $\boxed{\forall x \in A : p(x)}$

Let $x \in A$ be given.
[Prove $p(x)$]
It follows that $\forall x \in A : p(x)$.

(2) To prove $\boxed{\exists x \in A : p(x)}$

- 1st method

[Define an $x \in A$]
[Prove that $p(x)$ is true]
It follows that $\exists x \in A : p(x)$

(Note that x can be indirectly defined by deducing a statement of the form $\exists x \in B : r(x)$ via a theorem or by constructing it from other variables that have been indirectly defined via existential statements)

▷ <u>2nd method</u>

$p(x) \Leftrightarrow \cdots \Leftrightarrow \cdots \Leftrightarrow x \in \$$

Choose an $x \in \$$. Show that $x \in A \wedge p(x)$.

It follows that $\exists x \in A : p(x)$.