REAL NUMBERS

Introduction

- There is no agreed-upon definition of the set of real numbers, thus leading to the well-known <u>continuum-paradox.</u> However, for practical purposes, we assume that the set of real numbers $\mathbb{R}$ satisfies the following 3 groups of axioms:

  1) The field axioms
  2) The order axiom
  3) The least upper bound axiom.

The field axioms

(1) Commutative axiom

$\forall x, y \in \mathbb{R}: \ x+y = y+x$

$\forall x, y \in \mathbb{R}: \ xy = yx$

(2) Associative axiom

$\forall x, y, z \in \mathbb{R}: \ (x+y)+z = x+(y+z)$

$\forall x, y, z \in \mathbb{R}: \ (xy)z = x(yz)$.

(3) Distributive axiom

$\forall x, y, z \in \mathbb{R}: x(y+z) = xy + xz$

(4) Unit element

$\forall x \in \mathbb{R}: x + 0 = 0 + x = x$

$\forall x \in \mathbb{R}: 1x = x1 = x$

(5) Inverse element

$\forall x \in \mathbb{R}: \exists y \in \mathbb{R}: x + y = y + x = 0$

$\forall x \in \mathbb{R} - \{0\}: \exists y \in \mathbb{R}: xy = yx = 1$

- The axioms 1-5 are equivalent to the statement <u>$(\mathbb{R}, +, \cdot)$ is a field</u>

<u>Uniqueness of unit elements</u>

The following statements show that the unit elements 1 and 0 from axiom 4 are unique

<u>Prop</u>: $\boxed{\forall x \in \mathbb{R}: x + y = x \Rightarrow y = 0}$

<u>Proof</u>

$0 + y = 0$ [hypothesis]

$0 + y = y$ [unit element 0]

Therefore: $y = 0$.

We can similarly show that

Prop: $\boxed{\forall x \in \mathbb{R}: \ xy = x \Rightarrow y = 1}$

<u>Uniqueness of inverse</u>

We now show that the inverse elements in axiom 5 are unique.

Prop: $$\boxed{\forall x, y_1, y_2 \in \mathbb{R}: \begin{cases} x + y_1 = 0 \\ x + y_2 = 0 \end{cases} \rightarrow y_1 = y_2}$$

Proof: Let $x, y_1, y_2 \in \mathbb{R}$ be given such that $x + y_1 = 0$ and $x + y_2 = 0$. Then

$$\begin{aligned} y_1 &= y_1 + 0 && [0 \text{ unit element}] \\ &= y_1 + (x + y_2) && [\text{hypothesis}] \\ &= (y_1 + x) + y_2 && [\text{associative}] \\ &= (x + y_1) + y_2 && [\text{commutative}] \\ &= 0 + y_2 && [\text{hypothesis}] \\ &= y_2 && [0 \text{ unit element}] \end{aligned}$$ $\square$

We can similarly show that:

Prop: $$\boxed{\forall x, y_1, y_2 \in \mathbb{R}: \begin{cases} x y_1 = 1 \\ x y_2 = 1 \end{cases} \Rightarrow y_1 = y_2}$$

- The unique inverse of x with respect to addition is denoted as $-x$.
- The unique inverse of x with respect to multiplication is denoted as $1/x$.
- We may therefore define:

$\forall x,y \in \mathbb{R} : x - y = x + (-y)$ (subtraction)

$\forall x \in \mathbb{R} : \forall y \in \mathbb{R} - \{0\} : \frac{x}{y} = x \cdot \frac{1}{y}$ (division).

A basic field property.

The following are consequences of the field axioms. Each statement is used to prove the next statement.

1 Addition Cancellation law

$$\boxed{\forall x,y,z \in \mathbb{R} : (x + z = y + z \Rightarrow x = y).}$$

Proof : Let $x,y,z \in \mathbb{R}$ be given such that $x+z = y+z$.

$$\begin{aligned} x &= x + 0 && [0 \text{ unit element}] \\ &= x + (z + (-z)) && [\text{inverse axiom}] \\ &= (x+z) + (-z) && [\text{associative}] \\ &= (y+z) + (-z) && [\text{hypothesis}] \\ &= y + (z + (-z)) && [\text{associative}] \\ &= y + 0 && [\text{inverse axiom}] \\ &= y && [0 \text{ unit element}] \end{aligned}$$

$\square$

(2) → Nullifying element

$$\boxed{\forall x \in \mathbb{R} : 0x = x0 = 0}$$

Proof

Let $x \in \mathbb{R}$ be given. Let a $y \in \mathbb{R}$ be given. Then:

$xy + x0 = x(y+0)$ [distributive]
$\quad = xy$ [0 unit element]
$\quad = xy + 0 \Rightarrow$ [0 unit element]
$\Rightarrow x0 = 0$ [addition cancellation law]

Thus $0x = x0$ [commutative]
$\quad = 0$ [previous result] ∎

(3) → Zero-factorization law

$$\boxed{\forall x, y \in \mathbb{R} : xy = 0 \Rightarrow x = 0 \lor y = 0}$$

Proof

Let $x, y \in \mathbb{R}$ be given such that $xy = 0$.
Distinguish two cases:

Case 1 : If $x = 0 \Rightarrow x = 0 \lor y = 0$.

Case 2 : If $x \neq 0 \Rightarrow \exists z \in \mathbb{R} : zx = 1$ (inverse axiom)

It follows that:

$$
\begin{aligned}
y &= 1y && [1 \text{ unit element}] \\
&= (zx)y && [\text{shown previously}] \\
&= z(xy) && [\text{associative}] \\
&= z \cdot 0 && [\text{hypotheses}] \\
&= 0 \Rightarrow && [\text{nullifying element}] \\
\Rightarrow x = 0 &\vee y = 0 && \square
\end{aligned}
$$

4 → Multiplication cancellation law

$$\boxed{\forall a \in \mathbb{R} - \{0\} : \forall x, y \in \mathbb{R} : (ax = ay \Rightarrow x = y).}$$

Proof : Let $a \in \mathbb{R} - \{0\}$ and $x, y \in \mathbb{R}$ be given such that $ax = ay$. Then:

$$
\begin{aligned}
a(x-y) &= ax - ay && [\text{distributive}] \\
&= ax - ax && [\text{hypothesis}] \\
&= ax + (-ax) && [\text{definition}] \\
&= 0 \Rightarrow && [\text{inverse axiom}] \\
\Rightarrow a = 0 \vee x - y &= 0 \Rightarrow && [\text{zero-factor law}] \\
\Rightarrow x - y &= 0 \Rightarrow && [\text{because } a \in \mathbb{R} - \{0\} \Rightarrow a \neq 0] \\
\Rightarrow x &= x + 0 && [0 \text{ unit element}] \\
&= x + [(-y) + y] = && [\text{inverse axiom}] \\
&= (x + (-y)) + y = && [\text{associative}] \\
&= (x - y) + y && [\text{definition}] \\
&= 0 + y && [\text{previous result}] \\
&= y. && [0 \text{ unit element}]. \quad \square
\end{aligned}
$$

EXERCISE

① Give detailed proofs of the following statements as requested below:

a) Use only the field axioms to show that:

$\forall x, y \in \mathbb{R} : (xy = x \Rightarrow y = 1)$

$\forall x, y_1, y_2 \in \mathbb{R} : \begin{cases} xy_1 = 1 \\ xy_2 = 1 \end{cases} \Rightarrow y_1 = y_2$

b) Use the uniqueness of the inverse and the basic field properties to show that

$\forall x, y \in \mathbb{R} : (-x)y = x(-y)$

$\forall a, x, y \in \mathbb{R} : (b + x = a \Leftrightarrow x = a - b)$

$\forall a, x \in \mathbb{R} : \forall b \in \mathbb{R}^* : bx = a \Leftrightarrow x = \dfrac{a}{b}$

c) Show, by induction, that

$\forall a_1, a_2, \dots, a_n \in \mathbb{R} : (a_1 a_2 \cdots a_n = 0 \Rightarrow a_1 = 0 \vee a_2 = 0 \vee \cdots \vee a_n = 0)$

Order axiom

(6) There is a unique subset $\mathbb{R}_+^* \subset \mathbb{R}$ such that

$$\boxed{\begin{array}{l} \forall x \in \mathbb{R} : \ x = 0 \ \underline{\vee} \ x \in \mathbb{R}_+^* \ \underline{\vee} \ -x \in \mathbb{R}_+^* \\ \forall x, y \in \mathbb{R}_+^* : \ x + y \in \mathbb{R}_+^* \ \wedge \ xy \in \mathbb{R}_+^* \end{array}}$$

- We call $\mathbb{R}_+^*$ the set of <u>positive</u> real numbers.
- <u>Definition of inequalities:</u>

$$x > y \Leftrightarrow x - y \in \mathbb{R}_+^*$$
$$x \geq y \Leftrightarrow x - y = 0 \vee x - y \in \mathbb{R}_+^*$$
$$x < y \Leftrightarrow y > x$$
$$x \leq y \Leftrightarrow y \geq x.$$

- <u>Subsets of $\mathbb{R}$:</u>

$\mathbb{R}_+^* \leftarrow$ set of positive numbers

$\mathbb{R}_+ = \mathbb{R}_+^* \cup \{0\} \leftarrow$ set of <u>nonnegative</u> numbers

$\mathbb{R}_-^* = \{x \in \mathbb{R} \mid x < 0\} \leftarrow$ set of negative numbers

$\mathbb{R}_- = \{x \in \mathbb{R} \mid x \leq 0\} \leftarrow$ set of nonpositive numbers.

Immediate consequences

The following statements are immediate consequences of the order axiom:

$$\boxed{\begin{array}{l} \forall x \in \mathbb{R}: \ x = 0 \ \veebar \ x > 0 \ \veebar \ x < 0 \\ \forall x, y \in \mathbb{R}: \ (x > 0 \land y > 0) \Rightarrow (x + y > 0 \land xy > 0) \\ \forall x, y \in \mathbb{R}: \ (x < 0 \land y < 0) \Rightarrow (x + y < 0 \land xy > 0) \end{array}}$$

The following are immediate consequences of the definition of inequalities:

$$\boxed{\begin{array}{ll} \forall x, y \in \mathbb{R}: & x > y \Leftrightarrow x - y > 0 \\ \forall x, y \in \mathbb{R}: & x \geq y \Leftrightarrow x - y \geq 0 \\ \forall x, y \in \mathbb{R}: & x < y \Leftrightarrow x - y < 0 \\ \forall x, y \in \mathbb{R}: & x \leq y \Leftrightarrow x - y \leq 0 \end{array}}$$

Properties of inequalities

The fundamental properties of inequalities are now given without proof:

1) $\forall x, y \in \mathbb{R}: \ x = y \ \veebar \ x < y \ \veebar \ x > y$
2) $\forall x, y \in \mathbb{R}: \ x, y$ same sign $\Leftrightarrow xy > 0$
   $\forall x, y \in \mathbb{R}: \ x, y$ opposite sign $\Leftrightarrow xy < 0$

3) $\forall x, y, z \in \mathbb{R}: (x > y \land y > z) \Rightarrow x > z$

Immediate consequences of properties 1-3 are:

a) $\forall x \in \mathbb{R}^*: \ x^2 > 0$
b) $\forall x \in \mathbb{R}: \ x^2 \geq 0$
c) $\forall x \in \mathbb{R}^*: \ x > 0 \Leftrightarrow \frac{1}{x} > 0$

d) $\forall x \in \mathbb{R}^*: \ x < 0 \Leftrightarrow \frac{1}{x} < 0$

e) $\forall x, y \in \mathbb{R}: \ xy > 0 \Leftrightarrow \frac{x}{y} > 0$

f) $\forall x, y \in \mathbb{R}: \ xy < 0 \Leftrightarrow \frac{x}{y} < 0$

From properties 1-3 we may then prove the following additional properties:

4) $\forall x, y, z \in \mathbb{R}: \ x > y \Rightarrow x + z > y + z$
5) $\forall a, b, x, y \in \mathbb{R}: (x > y \land a > b) \Rightarrow x + a > y + b$
6) $\forall a, x, y \in \mathbb{R}: (x > y \land a > 0) \Rightarrow ax > ay$
7) $\forall a, x, y \in \mathbb{R}: (x > y \land a < 0) \Rightarrow ax < ay$
and
* 8) $\forall a, b, x, y \in \underset{(!)}{\underline{\underline{\mathbb{R}_+^*}}} : (x > a \land y > b) \Rightarrow xy > ab$

EXERCISE

(2) Give detailed proofs for properties 1-8 of the inequalities given above. You may use already proved properties to prove the remaining properties.

Proving identities

To prove an identity involving real numbers we use the following results:

1. Newton identities

$$(x+a)(x+b) = x^2 + (a+b)x + ab$$
$$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc$$

2. Cauchy identities

$$a^2+b^2 = (a+b)^2 - 2ab$$
$$a^3+b^3 = (a+b)^3 - 3ab(a+b)$$

3. Trinomial powers

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$$
$$(a+b+c)^3 = a^3+b^3+c^3+3(a+b)(b+c)(c+a)$$

4. Binomial quotients

For $n \in \mathbb{N} - \{0\}$ with <u>n odd</u>:

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$
$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots + b^{n-1})$$

For example:

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$
$$a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$a^5-b^5=(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$$
$$a^5+b^5=(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$$

For <u>n even</u> we use $a^2-b^2=(a-b)(a+b)$.
For example:

$$a^4-b^4=(a^2-b^2)(a^2+b^2)=$$
$$=(a-b)(a+b)(a^2+b^2)$$
$$a^6-b^6=(a^3-b^3)(a^3+b^3)$$
$$=(a-b)(a^2+ab+b^2)(a+b)(a^2-ab+b^2)$$

Also note that:

$$a^4+b^4=a^4+2a^2b^2+b^4-2a^2b^2=$$
$$=(a^2+b^2)^2-2a^2b^2=$$
$$=(a^2+b^2-ab\sqrt{2})(a^2+b^2+ab\sqrt{2}).$$

5 Euler's identity

$$\boxed{\begin{array}{l} a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \\ a^3+b^3+c^3-3abc=(1/2)(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \end{array}}$$

Euler's corrollary states that

$$\boxed{a^2+b^2+c^2-ab-bc-ca=\frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]}$$

From Euler's identity we also obtain the Euler-Cauchy equivalence:

$$\boxed{a^3+b^3+c^3=3abc \iff a+b+c=0 \vee a=b=c}$$

⑥ De Moivre identity

$$\boxed{\begin{aligned} &a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2 = \\ &= (a+b+c)(a-b+c)(a+b-c)(a-b-c) \end{aligned}}$$

⑦ Lagrange identity

Using the 2x2 determinant definition

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$$

the Lagrange identity states that

$$\boxed{\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)-\left(\sum_{k=1}^{n} a_k b_k\right)^2=\sum_{k=1}^{n-1}\sum_{l=k+1}^{n}\begin{vmatrix} a_k & b_k \\ a_l & b_l \end{vmatrix}^2}$$

For $n=2$:

$$(a_1^2+a_2^2)(b_1^2+b_2^2)-(a_1b_1+a_2b_2)^2=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2$$

This is also known as the <u>Brahmagupta-Fibonacci identity</u>.

For $n=3$:

$$(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2=$$
$$=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2+\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}^2+\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}^2$$

and so on...

⑧ <u>Newton Binomial</u>

$(a+b)^n = \sum_{k=0}^{n} C(n,k)\, a^{n-k} b^k$
with $C(n,k) = \dfrac{n!}{k!(n-k)!}$

Here, $n! = 1\cdot 2\cdot 3\cdot \ldots \cdot n$, $0! = 1$

The coefficients $C(n,k)$ satisfy:

$$\boxed{\begin{array}{l} C(n,0)=1\ ,\ C(n,1)=n\ ,\ C(n,n-k)=C(n,k) \\ C(n,k)=C(n-1,k-1)+C(n-1,k) \end{array}}$$

For example

$(a\pm b)^2 = a^2 \pm 2ab + b^2$

$(a\pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$

$(a\pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$

$(a\pm b)^5 = a^5 \pm 5a^4b + 10a^3b^2 \pm 10a^2b^3 + 5ab^4 \pm b^5$

$(a\pm b)^6 = a^6 \pm 6a^5b + 15a^4b^2 \pm 20a^3b^3 + 15a^2b^4 \pm 6ab^5 + b^6$

and so on...

<u>Methodology</u> : To show that $A = B$ we use one of the following methods:

1) <u>Direct Method</u>

Show : $A = \dots = \dots = \dots = B.$

<u>EXAMPLE</u>

Show that $x(x+1)(x+2)(x+3)+1 = (x^2+3x+1)^2$

<u>Proof</u>

$$
\begin{aligned}
LHS &= x(x+1)(x+2)(x+3)+1 = \\
&= [x(x+3)][(x+1)(x+2)]+1 = \\
&= (x^2+3x)(x^2+3x+2)+1 \\
&= (x^2+3x)^2 + 2(x^2+3x)+1 = (x^2+3x+1)^2 = RHS \ \square
\end{aligned}
$$

2) Indirect Method

Show : $A = \cdots = \cdots = \cdots = C$ and

$B = \cdots = \cdots = \cdots = C$

It follows that $A = B$.

EXAMPLE

Show that $(x^2+y^2+xy)^2 = x^2y^2 + (x+y)^2(x^2+y^2)$

Proof

$$\begin{aligned} LHS &= (x^2+y^2+xy)^2 = \\ &= x^4+y^4+x^2y^2+2x^2y^2+2x^3y+2xy^3 \\ &= x^4+y^4+3x^2y^2+2xy(x^2+y^2) \end{aligned}$$

and

$$\begin{aligned} RHS &= x^2y^2+(x+y)^2(x^2+y^2) \\ &= x^2y^2+(x^2+2xy+y^2)(x^2+y^2) \\ &= x^2y^2+x^2(x^2+y^2)+2xy(x^2+y^2)+y^2(x^2+y^2) \\ &= x^4+y^4+3x^2y^2+2xy(x^2+y^2). \end{aligned}$$

thus the identity is proven.

3) Deductive Method

Show: $A_1 = \cdots = B_1 \Rightarrow A_2 = \cdots = B_2 \Rightarrow \cdots$
$\Rightarrow A_n = \cdots = B.$

EXAMPLE

Show that $(x^2+y^2+xy)^2 = x^2y^2 + (x+y)^2(x^2+y^2)$

Proof

Note that

$$(x^2+xy+y^2)^2 - x^2y^2 = (x^2+xy+y^2-xy)(x^2+xy+y^2+xy)$$
$$= (x^2+y^2)(x^2+2xy+y^2) =$$
$$= (x+y)^2(x^2+y^2) \Rightarrow$$
$$\Rightarrow (x^2+y^2+xy)^2 = x^2y^2 + (x+y)^2(x^2+y^2). \quad \square$$

4) Auxiliary substitution

Sometimes it helps to define auxiliary variables and use them to carry out the argument above, employing one of the above methods.

EXAMPLE

Show that

$$(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3-3(x^2-yz)(y^2-xz)\times(z^2-xy)=(x^3+y^3+z^3-3xyz)^2.$$

Proof

Define: $A=x^2-yz$, $B=y^2-zx$, $C=z^2-xy$.
It follows that

$$\text{LHS}=A^3+B^3+C^3-3ABC=$$
$$=\frac{1}{2}(A+B+C)\left[(A-B)^2+(B-C)^2+(C-A)^2\right] \quad (1)$$

Note that

$$A+B+C=(x^2-yz)+(y^2-zx)+(z^2-xy)=$$
$$=x^2+y^2+z^2-xy-yz-zx=$$
$$=\frac{1}{2}\left[(x-y)^2+(y-z)^2+(z-x)^2\right], \quad (2) \quad \text{and}$$

$$A-B=(x^2-yz)-(y^2-zx)=$$
$$=(x^2-y^2)+z(x-y)=(x-y)(x+y)+z(x-y)$$
$$=(x-y)(x+y+z).$$

By symmetry: $B-C=(y-z)(x+y+z)$
$C-A=(z-x)(x+y+z)$.

It follows that

$$(A-B)^2+(B-C)^2+(C-A)^2 =$$
$$= (x-y)^2(x+y+z)^2+(y-z)^2(x+y+z)^2 +$$
$$+ (z-x)^2(x+y+z)^2 =$$
$$= (x+y+z)^2\left[(x-y)^2+(y-z)^2+(z-x)^2\right] \quad (3)$$

From (1), (2), and (3):

$$\text{LHS} = \frac{1}{2}\left[\frac{1}{2}\left[(x-y)^2+(y-z)^2+(z-x)^2\right]\right]$$
$$\times (x+y+z)^2\left[(x-y)^2+(y-z)^2+(z-x)^2\right]$$

$$= \left[\frac{1}{2}(x+y+z)\left[(x-y)^2+(y-z)^2+(z-x)^2\right]\right]^2$$
$$= (x^3+y^3+z^3-3xyz)^2 = \text{RHS}. \qquad \square$$

EXAMPLE

Show that $(x+y)^2 = 2(x^2+y^2) \Rightarrow x=y$.

Proof

$$(x+y)^2 = 2(x^2+y^2) \Rightarrow x^2+2xy+y^2 = 2x^2+2y^2 \Rightarrow$$
$$\Rightarrow 2xy = x^2+y^2 \Rightarrow x^2-2xy+y^2=0 \Rightarrow$$
$$\Rightarrow (x-y)^2=0 \Rightarrow x-y=0 \Rightarrow x=y.$$

EXERCISES

③ Prove that

a) $2a^2b(a+b) = (a^4-b^4) + 2b(a^3+b^3) - (a+b)^2(a-b)^2$

b) $(a+b)^5 - a^5 - b^5 = 5ab(a+b)(a^2+ab+b^2)$

c) $(a+b)^7 - a^7 - b^7 = 7ab(a+b)(a^2+ab+b^2)^2$

d) $(x+y)^4 + x^4 + y^4 = 2(x^2+xy+y^2)^2$

e) $64a^6 - b^{12} = (8a^3 - b^6)(2a+b^2)(4a^2-2ab^2+b^4)$

④ Prove that (using Euler's identity)

a) $(a-b)^3 + (b-c)^3 + (c-a)^3 = 3(a-b)(b-c)(c-a)$

b) $(x+y)^3 + (y+z)^3 + (z+x)^3 - 3(x+y)(y+z)(z+x) =$
$= 2(x^3+y^3+z^3-3xyz).$

c) $x^3(y-z)^3 + y^3(z-x)^3 + z^3(x-y)^3 =$
$= 3xyz(x-y)(y-z)(z-x)$

d) $(a-b)^3 - a^3 + (a+b)^3 + 3a(a-b)(a+b) = a(4a^2+3b^2)$

e) $(by+az)^3 + (bz+ax)^3 + (bx+ay)^3 -$
$- 3(by+az)(bz+ax)(bx+ay) =$
$= (a^3+b^3)(x^3+y^3+z^3-3xyz).$

⑤ Prove that (using Lagrange's identity).

a) $(a^2+b^2+c^2)(x^2+y^2) - (ax+by)^2 = (ay-bx)^2 + c^2(x^2+y^2)$

b) $(a^2+x^2+y^2)(x^2+a^2+1) - (2ax+y)^2 = (a^2-x^2)^2 +$
$+ (a-xy)^2 + (x-ay)^2.$

c) $3(a^2+b^2+c^2) - (a+b+c)^2 = (a-b)^2 + (b-c)^2 + (c-a)^2.$

5) Showing a disjunction

Some statements can be shown by using the theorem:

$$\boxed{a_1 a_2 \cdots a_n = 0 \Rightarrow a_1 = 0 \vee a_2 = 0 \vee \cdots \vee a_n = 0}$$

EXAMPLE

Show that $(a-b)^3 + (b-c)^3 + (c-a)^3 = 0$ implies that $a = b \vee b = c \vee c = a$.

Proof

Let $A = a-b$, $B = b-c$, $C = c-a$.

Then $A+B+C = (a-b)+(b-c)+(c-a) = 0 \Rightarrow$

$$\Rightarrow (a-b)^3 + (b-c)^3 + (c-a)^3 = A^3 + B^3 + C^3 = 3ABC$$

$$= 3(a-b)(b-c)(c-a) \qquad (1)$$

It follows that

$(a-b)^3 + (b-c)^3 + (c-a)^3 = 0 \Rightarrow$

$3(a-b)(b-c)(c-a) = 0 \Rightarrow$

$a-b = 0 \vee b-c = 0 \vee c-a = 0 \Rightarrow$

$a = b \vee b = c \vee c = a$. $\square$

6) Showing a conjunction

Some statements can be established by employing the following theorem:

Thm : $\boxed{\forall a, b \in \mathbb{R}: a^2 + b^2 = 0 \Rightarrow a = 0 \wedge b = 0}$

Proof

Assume $a \neq 0 \Rightarrow a^2 > 0 \Rightarrow$

$\Rightarrow b^2 = -a^2 < 0 \leftarrow$ contradiction

because $b^2 \geq 0$. Thus $a = 0$.

Similarly, we show that $b = 0$.

Thus $a = 0 \wedge b = 0$ $\square$

By induction, we generalize the theorem to:

$$\boxed{a_1^2 + a_2^2 + \cdots + a_n^2 = 0 \Rightarrow a_1 = 0 \wedge a_2 = 0 \wedge \cdots \wedge a_n = 0}$$

EXAMPLE

Show that

$$(a^2+b^2+c^2)^2 - (ab+bc+ca)^2 = 0 \Rightarrow$$

$$\Rightarrow a^2 = bc \wedge b^2 = ca \wedge c^2 = ab.$$

Proof

Note that, using Lagrange's identity:

$$\begin{aligned}
LHS &= (a^2+b^2+c^2)^2-(ab+bc+ca)^2 = \\
&= (a^2+b^2+c^2)(b^2+c^2+a^2)-(ab+bc+ca)^2 \underset{*}{=} \\
&= \begin{vmatrix} a & b \\ b & c \end{vmatrix}^2 + \begin{vmatrix} b & c \\ c & a \end{vmatrix}^2 + \begin{vmatrix} a & b \\ c & a \end{vmatrix}^2 = \\
&= (ac-b^2)^2+(ab-c^2)^2+(a^2-bc)^2=0 \Rightarrow
\end{aligned}$$

$$\Rightarrow ac-b^2=0 \wedge ab-c^2=0 \wedge a^2-bc=0 \Rightarrow$$
$$\Rightarrow a^2=bc \wedge b^2=ca \wedge c^2=ab \qquad \square$$

EXERCISES

⑥ Prove that

a) $a^4 - b^4 = (a-b)^3(a+b) \Rightarrow a=0 \vee b=0 \vee a=b \vee a=-b$

b) $(a+b)^3 = a^3+b^3 \wedge a \neq -b \Rightarrow a=0 \vee b=0$

c) $(x+y+z)(xy+yz+zx) = xyz \Rightarrow x=-y \vee y=-z \vee z=-x$

d) $(x-a)^2(b-c)+(x-b)^2(c-a)+(x-c)^2(a-b)=0 \Rightarrow$

$\Rightarrow a=b \vee b=c \vee c=a$

e) $x(y+z)^2+y(z+x)^2+z(x+y)^2 = 4xyz \Rightarrow$

$\Rightarrow x+y=0 \vee y+z=0 \vee z+x=0$.

f) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{1}{a+b+c} \Rightarrow$

$\Rightarrow a+b=0 \vee b+c=0 \vee c+a=0$

⑦ Prove that

a) $3(a^2+b^2+c^2) = (a+b+c)^2 \Rightarrow a=b=c$

b) $(x+y+z)^2+(x-y)^2+(y-z)^2+(z-x)^2=0 \Rightarrow$

$\Rightarrow x=0 \wedge y=0 \wedge z=0$

c) $(a^2+b^2+c^2)^2-(ab+bc+ca)^2=0 \Rightarrow$

$\Rightarrow a^2=bc \wedge b^2=ac \wedge c^2=ab$

d) $(a+b)^3+2(a^3+b^3)=0 \Rightarrow a=-b \vee (a=0 \wedge b=0)$

e) $(x+y+z)^2 = 3(xy+yz+zx) \Rightarrow x=y=z$

f) $(ax+by)^2+(ay-bx)^2+c^2(x^2+y^2)=0 \Rightarrow$

$\Rightarrow a=b=c=0 \vee x=y=0$.

(8) Let $a, b, c, d \in \mathbb{R} - \{0\}$. Show that

a) $a^4+b^4+c^4+d^4-4abcd = (a^2+b^2-c^2-d^2)^2 + 2(ad-bc)^2$

b) $a^4+b^4+c^4+d^4 = 4abcd \Rightarrow a=b=c=d$.

(Hint: Use $a^4+b^4 = (a^2+b^2)^2 - 2a^2b^2$ and later the Lagrange identity to show (a))

(9) Let $a, b, c \in \mathbb{R}$ such that

$a+b+c = x \wedge ab+bc+ca = y$.

Prove that

$$(a+b)(b+c)(c+a) + abc = \frac{1}{3}\left[x^3 - x(x^2+y)\right]$$

(Hint: Investigate what $a^3+b^3+c^3-3abc$ is doing first)

(10) Show that

a) $ab+bc+ca = 0 \Rightarrow (a^2+b^2+c^2)^3 = (a^3+b^3+c^3-3abc)^2$.

b) $a+b+c = 1 \wedge a^2+b^2+c^2 = 1 \Rightarrow a^3+b^3+c^3-3abc = 1$.

Proving inequalities

The objective is to prove a statement of the form

$$C \gtrless D \Rightarrow A \gtrless B$$

Such arguments are based on the properties of inequalities. The following theorems are also useful:

1) $xy > 0 \wedge x < y \Rightarrow \frac{1}{x} > \frac{1}{y}$

   $xy < 0 \wedge x < y \Rightarrow \frac{1}{x} < \frac{1}{y}$

2) $a > b > 0 \Rightarrow a^2 > b^2$

   $0 > a > b \Rightarrow a^2 < b^2$

3) $\forall a, b \in \mathbb{R}: a^2 + b^2 \geq 2ab$

4) $\forall a, b, c \in \mathbb{R}: a^2 + b^2 + c^2 \geq ab + bc + ca$

5) $\forall a, b \in \mathbb{R}: a^2 + b^2 \leq 0 \Rightarrow a = b = 0$

6) $a > 0 \Rightarrow a + \frac{1}{a} \geq 2$

7) Schwarz Inequality

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq$$
$$\geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

Methodology

To construct a proof of an inequality, we use one of the following methods, listed by order of preference.

1) <u>Equivalence method</u>:

- Show:

  $A \gtrless B \Leftrightarrow A - B \gtrless 0 \Leftrightarrow \cdots \Leftrightarrow$ <u>Obvious Inequality</u>

- Then, prove the Obvious Inequality using the condition $C \gtrless D$

2) <u>Deductive method</u>

- Show: $C \gtrless D \Rightarrow \cdots \Rightarrow \cdots \Rightarrow A \gtrless B$

3) <u>Direct method</u>

- Establish necessary lemmas from the condition $C \gtrless D$.
- Use lemmas to show

  $A \gtrless \cdots = \cdots \gtrless \cdots = \cdots \gtrless \cdots B$

  (sequence of equalities and inequalities where ALL inequalities point in the <u>same direction</u>).

Obvious Inequalities

Let $p, p_1, p_2, \dots, p_n \in \mathbb{R}_+^*$ (positive reals)
$n, n_1, n_2, \dots, n_m \in \mathbb{R}_-^*$ (negative reals)

We consider the following as obvious inequalities thus not requiring an explicit proof, only a "because statement"

1) $a^2 \geq 0$, because it is a perfect square
2) $a^2+b^2 \geq 0$, because it is a sum of two squares
3) $a^2+p > 0$, because it is the sum of a square and a positive real.
4) $p_1+p_2+\dots+p_n > 0$, because it is the sum of positive reals
   $n_1+n_2+\dots+n_n < 0$, because it is the sum of negative reals
5) $p_1 p_2 \cdots p_n > 0$, because it is a product of positive reals
   $p_1 n_1 < 0$, because it is a product of reals with opposite sign
   $n_1 n_2 > 0$, because it is a product of negative reals.

6) The quadratic $f(x) = ax^2+bx+c$ with $\Delta = b^2-4ac < 0$ has the same sign as $\underline{\underline{a}}$ for all $x \in \mathbb{R}$.

EXAMPLES

Show that if $xy > 0$ and $x \neq y$, then $x^4 + y^4 > x^3y + xy^3$.

<u>Proof</u> (by equivalence method)

Note that

$$x^4 + y^4 > x^3y + xy^3 \Leftrightarrow x^4 - x^3y + y^4 - xy^3 > 0 \Leftrightarrow$$
$$\Leftrightarrow x^3(x-y) - y^3(x-y) > 0 \Leftrightarrow$$
$$\Leftrightarrow (x^3 - y^3)(x-y) > 0 \Leftrightarrow$$
$$\Leftrightarrow (x-y)(x^2+xy+y^2)(x-y) > 0$$
$$\Leftrightarrow (x-y)^2(x^2+xy+y^2) > 0. \qquad (1)$$

Since $x \neq y \Rightarrow x - y \neq 0 \Rightarrow (x-y)^2 > 0 \qquad (2)$

and $x^2 + xy + y^2 \geqslant 2xy + xy = 3xy \underset{\substack{\uparrow \\ xy>0}}{>} 0 \qquad (3)$

From (2) and (3), (1) is true, thus the statement we want to show follows $\square$

b) Show that if $a > b > 0$, then

$$a^7 - \frac{1}{a} > b^7 - \frac{1}{b}$$

<u>Proof</u> (Deductive)

Since $a > b > 0 \Rightarrow a^7 > b^7$ (1)

Also $a > b > 0 \Rightarrow \frac{1}{a} < \frac{1}{b} \Rightarrow -\frac{1}{a} > -\frac{1}{b}$ (2)

Add (1) and (2): $a^7 - \frac{1}{a} > b^7 - \frac{1}{b}$ $\square$

c) Show that if $a > b + c$ and $b > 0$ and $c > 0$ then $a^4 + b^4 + c^4 > 2(a^2b^2 + b^2c^2 + c^2a^2)$.

<u>Proof</u> (Mixed: Direct / Deductive)

$a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 + c^2a^2) =$

$= a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2 =$

$= (a+b+c)(a-b+c)(a+b-c)(a-b-c).$ (1)

Since $a > b + c \Rightarrow a - b - c > 0$ (2)

$a + b - c > a - b - c$ [$b > 0$]

> 0 [(2)]

$\Rightarrow a + b - c > 0$ (3)

$a - b + c > a - b - c$ [$c > 0$]

> 0 [(2)]

$\Rightarrow a - b + c > 0$ (4)

$a+b+c > a+b-c \qquad [c>0]$

$\qquad\quad > 0 \qquad\qquad [(3)]$

$\Rightarrow a+b+c > 0 \qquad (5)$

From $(2),(3),(4),(5)$:

$(a+b+c)(a-b+c)(a+b-c)(a-b-c) > 0 \Rightarrow$

$\Rightarrow a^4+b^4+c^4-2(a^2b^2+b^2c^2+c^2a^2) > 0 \Rightarrow$

$\Rightarrow a^4+b^4+c^4 > 2(a^2b^2+b^2c^2+c^2a^2)$. $\square$

d) Show that $\forall a,b,c \in \mathbb{R}: 2[(a-b)^2+(b-c)^2] \geq (a-c)^2$

<u>Proof</u>: (Direct Method).

$2[(a-b)^2+(b-c)^2] = (1^2+1^2)[(a-b)^2+(b-c)^2]$

$\geq [1\cdot(a-b)+1\cdot(b-c)]^2 = (a-b+b-c)^2 =$

$= (a-c)^2 \Rightarrow$

$\Rightarrow 2[(a-b)^2+(b-c)^2] \geq (a-c)^2$. $\square$

EXERCISES

(11) Prove that:

a) $\forall a \in \mathbb{R} : 3(a^4+a^2+1) \geq (a^2+a+1)^2$

b) $\forall a_1, a_2, x_1, x_2 \in \mathbb{R} : (a_1x_1+a_2x_2)^2 - (a_1^2+a_2^2)(x_1^2+x_2^2) \leq 0$

c) $\begin{cases} a, b, c \in \mathbb{R}_+^* \\ a \neq b \neq c \neq a \end{cases} \Rightarrow a^3+b^3+c^3 > 3abc$

d) $a, b, c \in \mathbb{R}_+^* \Rightarrow (a+b+c)^3 > 3(a+b)(b+c)(c+a)$

e) a, b have same sign $\Rightarrow (1+a)(1+b) > 1+a+b$

f) a, b have opposite sign $\Rightarrow \frac{a}{b} + \frac{b}{a} \leq -2$

g) $a, b \in \mathbb{R}_+^* \Rightarrow ab \leq \left(\frac{a+b}{2}\right)^2$

h) $a, b, c \in \mathbb{R}_+^* \Rightarrow a^2(b+c) + b^2(c+a) + c^2(a+b) \geq 6abc$

i) $b \geq c \Rightarrow 2a^2 + b^2 + c^2 - 2a(b+c) \geq c - b$

j) $(x+y-z)^2 + (x-y+z)^2 + (y+z-x)^2 \geq xy+yz+zx$

k) $(ab+bc+ca)^2 \geq 3abc(a+b+c)$

l) $2a^4+1 \geq 2a^3+a^2$.

(12) Prove that

a) $a \geq b > 0 \Rightarrow a^3 - b^3 \geq (a-b)^3$

b) $a > 0 \wedge b > 0 \Rightarrow a^3+b^3 \geq a^2b + ab^2$

c) $a, b, c \in \mathbb{R}_+^* \wedge abc = 1 \Rightarrow (1+a)(1+b)(1+c) \geq 8$

d) $ab > 0 \wedge a \neq b \Rightarrow \left(\frac{2ab}{a+b}\right)^2 < ab < \left(\frac{a+b}{2}\right)^2$

e) $ab > 0 \wedge a \neq b \Rightarrow (a^2-b^2)^2 > (a-b)^4$.

Inequalities by induction

- It is possible to prove certain types of inequalities by method of induction

EXAMPLE

Show that $n \in \mathbb{N} \wedge n \geq 3 \Rightarrow 3^n > 3n+1$.

Proof

For $n=3$: $\left.\begin{array}{l} 3^n = 27 \\ 3n+1 = 3\cdot 3+1 = 9+1 = 10 \end{array}\right\} \Rightarrow 3^n > 3n+1$

Assume that for $n=k$: $3^k > 3k+1$

Sufficient to show that $3^{k+1} > 3(k+1)+1$

We have:

$$3^{k+1} = 3\cdot 3^k > 3[3k+1] = 9k+3 \underset{k \geq 3}{\geq} 8k+6$$
$$> 8k+4 > 3k+4 =$$
$$= 3(k+1)+1$$

Thus $\forall n \in \mathbb{N} : (n \geq 3 \Rightarrow 3^n > 3n+1)$.

EXERCISES

(13) Show by induction the following inequalities:

a) $\forall n \in \mathbb{N} : (n \geq 4 \Rightarrow (3/2)^n > n+1)$
b) $\forall n \in \mathbb{N} : 2^n > n$
c) $\forall n \in \mathbb{N} - \{0\} : 2^{n+2} > 2n+5$
d) $\forall n \in \mathbb{N} : (n \geq 2 \Rightarrow 5^n > 5n+2)$
e) $\forall n \in \mathbb{N} : (n \geq 4 \Rightarrow 3^{n-1} > n^2)$
f) $\forall n \in \mathbb{N} : (n \geq 10 \Rightarrow 2^n > n^3)$
g) $\forall n \in \mathbb{N} : (n \geq 4 \Rightarrow n! > 2^n)$

(14) Prove the Bernoulli inequality
a) $x > -1 \Rightarrow \forall n \in \mathbb{N} : (1+x)^n \geq 1+nx$

(15) Prove that
$$\forall n \in \mathbb{N} - \{0\} : \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$$

(16) Let $a_1, a_2, \ldots, a_n \in \mathbb{R}_+^*$. Prove that:

a) $a_1 a_2 \cdots a_n = 1 \Rightarrow a_1 + a_2 + \cdots + a_n \geq n$
b) $a_1 + a_2 + \cdots + a_n \leq 1/2 \Rightarrow (1-a_1)(1-a_2)\cdots(1-a_n) \geq \frac{1}{2}$