MAPPINGS AND FUNCTIONS

▼ Basic Definitions

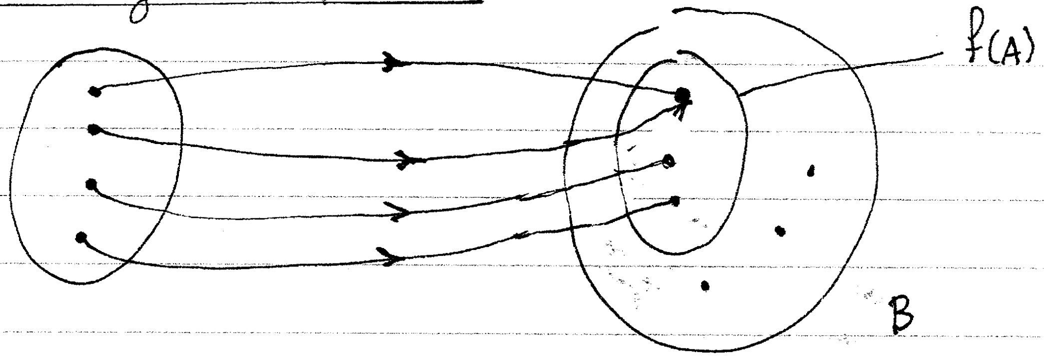
- Let A, B be two arbitrary sets. We say that f is a mapping that maps A to B (notation: $f: A \to B$) if and only if the following conditions are satisfied:

a) f is a relation $f \in Rel(A,B)$

b) $\forall x \in A : \exists y \in B : (x,y) \in f$

c) $\forall (x_1,y_1),(x_2,y_2) \in f : (x_1 = x_2 \Rightarrow y_1 = y_2)$

▶ Venn Diagram interpretation



Conditions (b) and (c) above have the following interpretations:

b) All elements of A have an outgoing arrow to some element of B

c) No element of A can have more than one outgoing arrow

Note that there are no restrictions on where the arrows go to as long as they go to some element of B.

▸ Special cases

- We denote the set of all mappings $f: A \to B$ as $Map(A,B) = \{f \in Rel(A,B) \mid f: A \to B\}$
- For $A \subseteq \mathbb{R}$ we define the set of all functions with domain A:

  $F(A) = Map(A, \mathbb{R})$.
- Also relevant are the following definitions

  $F(\mathbb{N})$ = the set of all real-valued sequences

  $Map(\mathbb{R}^n, \mathbb{R})$ = the set of all scalar fields

  $Map(\mathbb{R}^n, \mathbb{R}^n)$ = the set of all vector fields

▸ $f(x)$ notation

For every element $x \in A$, there is a unique $y \in B$ such that $(x,y) \in f$. We denote this unique y as $y = f(x)$.

EXAMPLE

For $f = \{(1,7),(2,5),(3,7)\}$, it follows that

$f(1) = 7$

$f(2) = 5$

$f(3) = 7$.

▸ $f(S)$ notation

Let $f: A \to B$ and let $S \subseteq A$. We define the image $f(S)$ of S as follows:

$$f(\mathcal{S}) = \{ f(x) \mid x \in \mathcal{S} \}$$

The belonging condition of $f(\mathcal{S})$ is given by

$$y \in f(\mathcal{S}) \Leftrightarrow \exists x \in \mathcal{S} : y = f(x)$$

We now prove the following lemma:

<u>Lemma</u>: $\boxed{(f : A \to B \land \mathcal{S} \subseteq A) \Rightarrow B \cap f(\mathcal{S}) = f(\mathcal{S})}$

<u>Proof</u>

We note that

$\underline{y \in B \cap f(\mathcal{S})} \Rightarrow y \in B \land y \in f(\mathcal{S}) \Rightarrow \underline{y \in f(\mathcal{S})}$

and therefore $B \cap f(\mathcal{S}) \subseteq f(\mathcal{S})$. (1)

Conversely, let $\underline{y \in f(\mathcal{S})} \Rightarrow \exists x \in \mathcal{S} : y = f(x)$

Since $y = f(x) \Rightarrow (x, y) \in f$ [by definition]

$\Rightarrow (x, y) \in A \times B$ [because $f \subseteq A \times B$]

$\Rightarrow x \in A \land y \in B$ [definition]

$\Rightarrow y \in B \Rightarrow \underline{y \in B \cap f(\mathcal{S})}$

and therefore $f(\mathcal{S}) \subseteq B \cap f(\mathcal{S})$ (2)

From (1) and (2): $f(\mathcal{S}) = B \cap f(\mathcal{S})$. □

▸ Domain and range of f

Let $f : A \to B$ be given. Since f is also a relation, recall that we have previously defined the <u>domain</u> and <u>range</u> of f as:

$$\mathrm{dom}(f) = \{x \in A \mid \exists y \in B : (x,y) \in f\}$$
$$\mathrm{ran}(f) = \{y \in B \mid \exists x \in A : (x,y) \in f\}$$

We will now show that:

<u>Proposition</u> : $\boxed{f : A \to B \Rightarrow (\mathrm{dom}(f) = A \wedge \mathrm{ran}(f) = f(A))}$

<u>Proof</u>

We assume that $f : A \to B$.

(a) By definition:

$\mathrm{dom}(f) = \{x \in A \mid \exists y \in B : (x,y) \in f\} \Rightarrow \mathrm{dom}(f) \subseteq A$ (1)

Sufficient to show $A \subseteq \mathrm{dom}(f)$.

Assume that $x \in A$. Since $f : A \to B \Rightarrow \exists y \in B : (x,y) \in f$. it follows that

$x \in A \wedge (\exists y \in B : (x,y) \in f) \Rightarrow$

$\Rightarrow x \in \mathrm{dom}(f)$

and therefore $A \subseteq \mathrm{dom}(f)$ (2)

From (1) and (2): $\underline{A = \mathrm{dom}(f)}$.

(b) To show $\mathrm{ran}(f) = f(A)$, we note that

$$\begin{aligned} y \in \mathrm{ran}(f) &\Leftrightarrow y \in \{z \in B \mid \exists x \in A : (x,z) \in f\} \\ &\Leftrightarrow y \in B \wedge (\exists x \in A : (x,y) \in f) \\ &\Leftrightarrow y \in B \wedge (\exists x \in A : y = f(x)) \\ &\Leftrightarrow y \in B \wedge y \in f(A) \\ &\Leftrightarrow y \in B \cap f(A). \end{aligned}$$

It follows that $\mathrm{ran}(f) = B \cap f(A) = f(A)$, using the previous lemma in the last step.

EXAMPLES

a) Let $f: A \to B$ be given and let $S \subseteq A$ and $T \subseteq A$. Show that $f(S \cup T) = f(S) \cup f(T)$.

Solution

(⇒): Let $y \in f(S \cup T)$ be given. Then

$y \in f(S \cup T) \Rightarrow \exists x \in S \cup T : f(x) = y$.

Choose $x_0 \in S \cup T$ such that $f(x_0) = y$.

Since $x_0 \in S \cup T \Rightarrow x_0 \in S \vee x_0 \in T$, we distinguish between the following cases:

Case 1 : Assume that $x_0 \in S$. Then

$$\left\{\begin{array}{l} x_0 \in S \\ f(x_0) = y \end{array}\right. \Rightarrow \exists x \in S : y = f(x) \Rightarrow y \in f(S)$$

$$\Rightarrow y \in f(S) \vee y \in f(T) \Rightarrow y \in f(S) \cup f(T).$$

Case 2 : Assume that $x_0 \in T$. Then

$$\left\{\begin{array}{l} x_0 \in T \\ f(x_0) = y \end{array}\right. \Rightarrow \exists x \in T : y = f(x) \Rightarrow y \in f(T)$$

$$\Rightarrow y \in f(S) \vee y \in f(T) \Rightarrow y \in f(S) \cup f(T).$$

In both cases we find $y \in f(S) \cup f(T)$ and therefore

$\forall y \in f(S \cup T) : y \in f(S) \cup f(T)$. (1)

(⇐) : Let $y \in f(S) \cup f(T)$ be given. Then:

$y \in f(S) \cup f(T) \Rightarrow y \in f(S) \vee y \in f(T) \Rightarrow$

$\Rightarrow (\exists x \in S : y = f(x)) \vee (\exists x \in T : y = f(x))$

We distinguish between the following two cases:

<u>Case 1</u> : Assume that $\exists x \in S : y = f(x)$.

Choose $x_0 \in S$ such that $y = f(x_0)$. Then:

$$\begin{cases} x_0 \in S \\ y = f(x_0) \end{cases} \Rightarrow \begin{cases} x_0 \in S \vee x_0 \in T \\ y = f(x_0) \end{cases} \Rightarrow \begin{cases} x_0 \in S \cup T \\ y = f(x_0) \end{cases} \Rightarrow$$

$$\Rightarrow \exists x \in S \cup T : y = f(x)$$

$$\Rightarrow y \in f(S \cup T).$$

<u>Case 2</u> : Assume that $\exists x \in T : y = f(x)$.

Choose $x_0 \in T$ such that $y = f(x_0)$. Then:

$$\begin{cases} x_0 \in T \\ y = f(x_0) \end{cases} \Rightarrow \begin{cases} x_0 \in S \vee x_0 \in T \\ y = f(x_0) \end{cases} \Rightarrow \begin{cases} x_0 \in S \cup T \\ y = f(x_0) \end{cases} \Rightarrow$$

$$\Rightarrow \exists x \in S \cup T : y = f(x)$$

$$\Rightarrow y \in f(S \cup T).$$

In both cases we find $y \in f(S \cup T)$ and therefore

$$\forall y \in f(S) \cup f(T) : y \in f(S \cup T). \qquad (2)$$

From Eq.(1) and Eq.(2):

$$\begin{cases} \forall y \in f(S \cup T) : y \in f(S) \cup f(T) \\ \forall y \in f(S) \cup f(T) : y \in f(S \cup T) \end{cases} \Rightarrow \begin{cases} f(S \cup T) \subseteq f(S) \cup f(T) \\ f(S) \cup f(T) \subseteq f(S \cup T) \end{cases}$$

$$\Rightarrow f(S \cup T) = f(S) \cup f(T).$$

b) Let $f: A \to B$ be given. Use a counterexample to explain why we cannot prove that for $S \subseteq A$ and $T \subseteq A$ we have $f(S \cap T) = f(S) \cap f(T)$.

Solution

Consider the mapping

$f = \{(a,x),(b,x),(c,y),(d,y)\}$

and define $S = \{b,c\}$ and $T = \{a,d\}$.

Then:

$$f(S \cap T) = f(\{b,c\} \cap \{a,d\}) = f(\emptyset) = \emptyset \qquad (1)$$

but

$f(b) = x \wedge f(c) = y \Rightarrow f(S) = f(\{b,c\}) = \{x,y\}$

$f(a) = x \wedge f(d) = y \Rightarrow f(T) = f(\{a,d\}) = \{x,y\}$

and therefore

$$f(S) \cap f(T) = \{x,y\} \cap \{x,y\} = \{x,y\} \qquad (2)$$

From Eq.(1) and Eq.(2):

$f(S \cap T) \neq f(S) \cap f(T)$

↳ Proof by counterexample can be very challenging. The statement $f(S \cap T) = f(S) \cap f(T)$ can be true for some choices of S, T and false for other choices of S, T. Can you find alternate choices for S, T for which the statement is true?

EXERCISES

① Let $f: A \to B$ be given, and let $S \subseteq A$ and $T \subseteq A$. Show that

a) $f(S \cap T) \subseteq f(S) \cap f(T)$

b) $f(S) - f(T) \subseteq f(S - T)$

② Find a counterexample of an $f: A \to B$ and $S \subseteq A$ and $T \subseteq A$ such that the following statements are false:

a) $f(S \cap T) = f(S) \cap f(T)$

b) $f(S) - f(T) = f(S - T)$

↳ We will later show that these statements can be proved if additional assumptions about f are introduced.

③ Let $f: A \to B$ be given and let S_a such that $\forall a \in I: S_a \subseteq A$ with I an index set. Show that

a) $f\left(\bigcup\limits_{a \in I} S_a\right) = \bigcup\limits_{a \in I} f(S_a)$

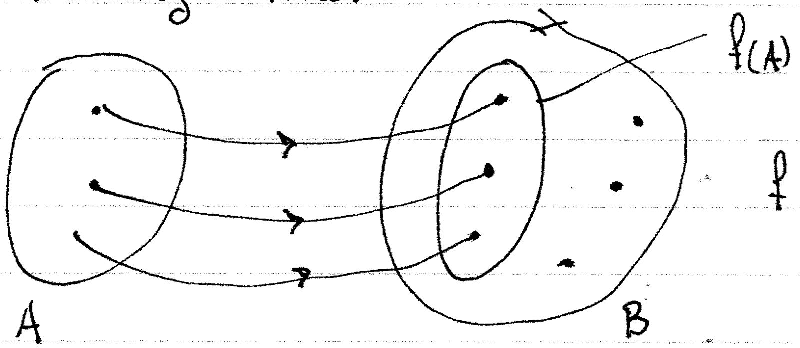
b) $f\left(\bigcap\limits_{a \in I} S_a\right) \subseteq \bigcap\limits_{a \in I} f(S_a)$

▼ One-to-one mappings/functions

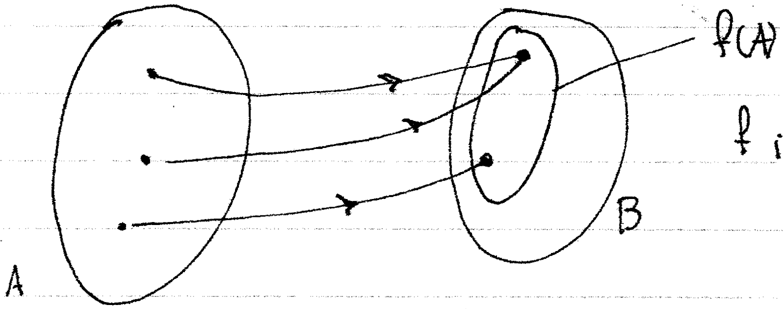
- Let $f: A \to B$ be given. We say that

$$\boxed{f \text{ one-to-one} \Leftrightarrow \forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)}$$

▶ Venn diagram interpretation: In a one-to-one mapping, every point in the range $f(A)$ receives only one incoming arrow.



f is one-to-one.



f is NOT one-to-one.

▶ Negated definition

Since $\overline{p \Rightarrow q} \equiv p \wedge \overline{q}$, the negation of the above definition reads:

$$\boxed{f \text{ NOT one-to-one} \Leftrightarrow \exists x_1, x_2 \in A : (f(x_1) = f(x_2) \wedge x_1 \neq x_2)}$$

▶ Methodology

To derive statements of the form $A=B \Rightarrow C=D$ we use the following properties of real numbers

1) We can add/cancel any number to both sides of an equation:
   $\forall a,x,y \in \mathbb{R} : (x=y \Leftrightarrow a+x=a+y)$
2) We can always add or multiply two equations
   $\forall a,b,x,y \in \mathbb{R} : (a=b \land x=y \Rightarrow a+x=b+y)$
   $\forall a,b,x,y \in \mathbb{R} : (a=b \land x=y \Rightarrow ax=by)$
3) We can multiply any number to both sides of an equation:
   $\forall a,x,y \in \mathbb{R} : (x=y \Rightarrow ax=ay)$
   However the converse does not work for $a=0$.
   With the restriction $a \neq 0$ we have:
   $\forall x,y \in \mathbb{R} : \forall a \in \mathbb{R}-\{0\} : (x=y \Leftrightarrow ax=ay)$
4) We can raise both sides of an equation to any integer power:
   $\forall x,y \in \mathbb{R} : \forall n \in \mathbb{N} : (x=y \Rightarrow x^n=y^n)$
   In general, the converse does not work. However, if we require $n \neq 0$ and distinguish between odd and even powers, we have:
   $\forall x,y \in \mathbb{R} : \forall n \in \mathbb{Z} : (x^{2n+1}=y^{2n+1} \Leftrightarrow x=y)$
   $\forall x,y \in \mathbb{R} : \forall n \in \mathbb{Z}-\{0\} : (x^{2n}=y^{2n} \Leftrightarrow x=y \lor x=-y)$
5) Factored equation:
   $\forall a,b \in \mathbb{R} : (ab=0 \Leftrightarrow a=0 \lor b=0)$

EXAMPLES

a) Consider the function

$$\forall x \in \mathbb{R} - \{a\} : f(x) = \frac{x}{x-a}$$

Show that $a \neq 0 \Rightarrow f$ one-to-one.

Solution

Assume that $a \neq 0$. Let $x_1, x_2 \in \mathbb{R} - \{a\}$ be given such that $f(x_1) = f(x_2)$. Then

$$\underline{f(x_1) = f(x_2)} \Rightarrow \frac{x_1}{x_1 - a} = \frac{x_2}{x_2 - a} \Rightarrow$$

$$\Rightarrow (x_1 - a)(x_2 - a)\frac{x_1}{x_1 - a} = (x_1 - a)(x_2 - a)\frac{x_2}{x_2 - a} \Rightarrow$$

$$\Rightarrow x_1(x_2 - a) = x_2(x_1 - a) \Rightarrow x_1 x_2 - a x_1 = x_1 x_2 - a x_2$$

$$\left.\begin{array}{l} \Rightarrow -a x_1 = -a x_2 \\ \quad a \neq 0 \end{array}\right\} \Rightarrow \underline{x_1 = x_2}$$

It follows that

$$\forall x_1, x_2 \in \mathbb{R} - \{a\} : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$$

$\Rightarrow f$ one-to-one.

↳ Note that to cancel $-a$ in $-ax_1 = -ax_2$ we need the assumption $a \neq 0$, otherwise the cancellation cannot be justified.

b) Consider the function $f(x)=2x^2+6x-7, \ \forall x\in\mathbb{R}$

Show that f is not one-to-one.

Solution

Solve $f(x)=-7 \Leftrightarrow 2x^2+6x-7=-7 \Leftrightarrow 2x^2+6x=0 \Leftrightarrow$

$\Leftrightarrow 2x(x+3)=0 \Leftrightarrow 2x=0 \vee x+3=0$

$\Leftrightarrow x=0 \vee x=-3$

It follows that

$f(0)=f(-3)=-7 \wedge 0\neq -3 \Rightarrow$

$\Rightarrow \exists x_1, x_2 \in \mathbb{R} : f(x_1)=f(x_2) \wedge x_1\neq x_2$

$\Rightarrow f$ not one-to-one.

c) Let $f: A \to B$ be given. and let $S \subseteq A$ and $T \subseteq A$.

Show that

f one-to-one $\Rightarrow f(S \cap T) = f(S) \cap f(T)$.

<u>Solution</u>

Assume that f is one-to-one.

($\Rightarrow$): Let $\underline{y \in f(S \cap T)}$ be given. Then,

$y \in f(S \cap T) \Rightarrow \exists x \in S \cap T : f(x) = y$

Choose $x_0 \in S \cap T$ such that $f(x_0) = y$. It follows that

$$\begin{cases} x_0 \in S \cap T \\ f(x_0) = y \end{cases} \Rightarrow \begin{cases} x_0 \in S \wedge x_0 \in T \\ f(x_0) = y \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_0 \in S \\ f(x_0) = y \end{cases} \wedge \begin{cases} x_0 \in T \\ f(x_0) = y \end{cases} \Rightarrow$$

$$\Rightarrow (\exists x \in S : f(x) = y) \wedge (\exists x \in T : f(x) = y)$$

$$\Rightarrow y \in f(S) \wedge y \in f(T) \Rightarrow$$

$$\Rightarrow \underline{y \in f(S) \cap f(T)}.$$

($\Leftarrow$): Let $\underline{y \in f(S) \cap f(T)}$ be given. Then:

$$y \in f(S) \cap f(T) \Rightarrow y \in f(S) \wedge y \in f(T) \Rightarrow$$

$$\Rightarrow \begin{cases} \exists x \in S : f(x) = y \\ \exists x \in T : f(x) = y. \end{cases}$$

Choose $x_1 \in S$ and $x_2 \in T$ such that $f(x_1) = y$ and $f(x_2) = y$.

Then:

$$\begin{cases} f(x_1) = y = f(x_2) \\ f \text{ one-to-one} \end{cases} \Rightarrow x_1 = x_2 \in T \Rightarrow x_1 \in T.$$

and therefore:

$$\begin{cases} x_1 \in S \wedge x_1 \in T \\ f(x_1) = y \end{cases} \Rightarrow \begin{cases} x_1 \in S \cap T \\ f(x_1) = y \end{cases} \Rightarrow$$

$$\Rightarrow \exists x \in S \cap T : f(x) = y$$

$$\Rightarrow \underline{y \in f(S \cap T)}$$

From the above argument we have:

$$\begin{cases} \forall y \in f(S \cap T) : y \in f(S) \cap f(T) \\ \forall y \in f(S) \cap f(T) : y \in f(S \cap T) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f(S \cap T) \subseteq f(S) \cap f(T) \\ f(S) \cap f(T) \subseteq f(S \cap T) \end{cases} \Rightarrow$$

$$\Rightarrow f(S \cap T) = f(S) \cap f(T).$$

EXERCISES

④ Show that the following functions are one-to-one

a) $\forall x \in \mathbb{R}: f(x) = 3x^5 + 2$

b) $\forall x \in (0, +\infty): f(x) = 2x^2 + 5$

c) $\forall x \in \mathbb{R}: f(x) = ax + b$ with $a, b \in \mathbb{R} \wedge a \neq 0$

d) $\forall x \in \mathbb{R}: f(x) = (2x^3 + 1)^5$

e) $\forall x \in \mathbb{R} - \{0\}: f(x) = a/x$ with $a \in \mathbb{R} \wedge a \neq 0$

f) $\forall x \in \mathbb{R} - \{-d/c\}: f(x) = \dfrac{ax+b}{cx+d}$ with $a, b, c, d \in \mathbb{R} \wedge ad - bc \neq 0$

⑤ Show that for $\forall x \in \mathbb{R}: f(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$ and $a \neq 0$ is <u>not</u> one-to-one.

⑥ Let $f: A \to B$ be given and let $S \subseteq A$ and $T \subseteq A$. Show that

f one-to-one $\Rightarrow f(S - T) = f(S) - f(T)$.

⑦ Let $f: A \to B$ be given and let S_a be a set collection such that $\forall a \in I: S_a \subseteq A$, with I an index set. Show that

f one-to-one $\Rightarrow f\left(\bigcap_{a \in I} S_a\right) = \bigcap_{a \in I} f(S_a)$

Functions and Monotonicity

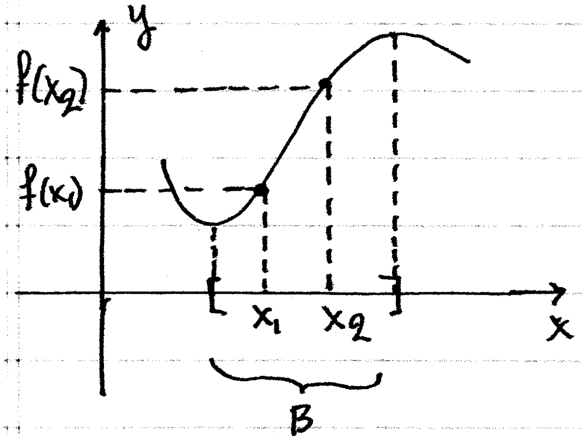
Let f be a function with $f: A \to \mathbb{R}$ and let $B \subseteq A$. We make the following definitions:

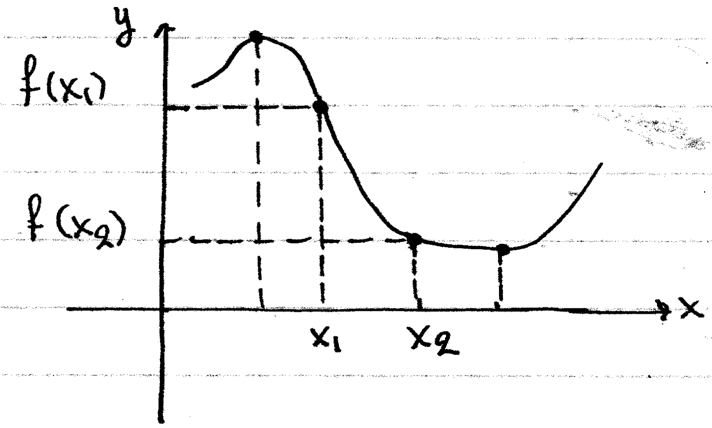
$$\boxed{\begin{array}{l} f \nearrow B \Leftrightarrow \forall x_1, x_2 \in B : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2)) \\ f \searrow B \Leftrightarrow \forall x_1, x_2 \in B : (x_1 < x_2 \Rightarrow f(x_1) > f(x_2)) \end{array}}$$

We read:

$f \nearrow B$: f is strictly increasing in B

$f \searrow B$: f is strictly decreasing in B.





Monotonicity can be determined directly from the definition with 2 methods:

1) Analytic Method
2) Synthetic Method.

In Calculus, monotonicity can also be determined using Differential Calculus.

Analytic Method

To show $f \nearrow B$ or $f \searrow B$.

- 1 Let $x_1, x_2 \in B$ be given with $x_1 < x_2$.
- 2 Calculate and <u>factor</u> $\Delta f(x_1, x_2) = f(x_2) - f(x_1)$
- 3 Determine the sign of each factor of Δf and then conclude whether $\Delta f > 0$ or $\Delta f < 0$.
- 4 Finish the argument.

EXAMPLES

a) Show that $f(x) = 3x + 5$ is strictly increasing in $\mathbb{R}$.

<u>Solution</u>

$\operatorname{dom}(f) = \mathbb{R}$.

Let $\underline{x_1, x_2 \in \mathbb{R}}$ be given with $\underline{x_1 < x_2}$.

$$\Delta f(x_1, x_2) = f(x_2) - f(x_1) = (3x_2 + 5) - (3x_1 + 5) =$$
$$= 3(x_2 - x_1)$$

Since $x_1 < x_2 \Rightarrow x_2 - x_1 > 0 \Rightarrow$

$$\Rightarrow 3(x_2 - x_1) > 0 \Rightarrow$$
$$\Rightarrow f(x_2) - f(x_1) > 0 \Rightarrow$$
$$\Rightarrow \underline{f(x_1) < f(x_2)}$$

- Thus: $\forall x_1, x_2 \in \mathbb{R}: (x_1 < x_2 \Rightarrow f(x_1) < f(x_2)) \Rightarrow f \nearrow \mathbb{R}$.

b) Show that $f(x) = \dfrac{2x}{x - 1}$ is strictly decreasing in $(1, +\infty)$.

Solution

Let $x_1, x_2 \in (1, +\infty)$ be given with $x_1 < x_2$.
Then:

$$\Delta f(x_1, x_2) = f(x_2) - f(x_1) = \frac{2x_2}{x_2 - 1} - \frac{2x_1}{x_1 - 1} =$$

$$= \frac{2x_2(x_1 - 1) - 2x_1(x_2 - 1)}{(x_1 - 1)(x_2 - 1)} =$$

$$= \frac{2x_1x_2 - 2x_2 - 2x_1x_2 + 2x_1}{(x_1 - 1)(x_2 - 1)} =$$

$$= \frac{-2x_2 + 2x_1}{(x_1 - 1)(x_2 - 1)} = \frac{2(x_1 - x_2)}{(x_1 - 1)(x_2 - 1)}$$

Since $x_1 < x_2 \Rightarrow x_1 - x_2 < 0$.

$x_1 \in (1, +\infty) \to x_1 > 1 \Rightarrow x_1 - 1 > 0$

$x_2 \in (1, +\infty) \Rightarrow x_2 > 1 \Rightarrow x_2 - 1 > 0$

therefore $\Delta f(x_1, x_2) < 0 \Rightarrow f(x_2) - f(x_1) < 0 \Rightarrow$

$\Rightarrow f(x_1) > f(x_2)$

Thus:

$\forall x_1, x_2 \in (1, +\infty) : (x_1 < x_2 \Rightarrow f(x_1) > f(x_2)) \Rightarrow$

$\Rightarrow f \searrow (1, +\infty)$.

c) Show that $f(x) = x^2 + 5x + 6$ is strictly increasing in $(-5/2, +\infty)$.

Solution

Let $\underline{x_1, x_2 \in (-5/2, +\infty)}$ be given with $\underline{x_1 < x_2}$

Then

$$\Delta f(x_1, x_2) = f(x_2) - f(x_1) = (x_2^2 + 5x_2 + 6) - (x_1^2 + 5x_1 + 6)$$

$$= (x_2^2 - x_1^2) + 5(x_2 - x_1) =$$

$$= (x_2 - x_1)(x_2 + x_1) + 5(x_2 - x_1) =$$

$$= (x_2 - x_1)(x_2 + x_1 + 5)$$

Since $x_1 < x_2 \Rightarrow x_2 - x_1 > 0 \quad (1)$

$$\left.\begin{array}{l} x_1 \in (-5/2, +\infty) \Rightarrow x_1 > -5/2 \\ x_2 \in (-5/2, +\infty) \Rightarrow x_2 > -5/2 \end{array}\right\} \Rightarrow$$

$\Rightarrow x_1 + x_2 > -5/2 - 5/2 = -5 \Rightarrow x_1 + x_2 + 5 > 0 \quad (2)$

From (1) and (2):

$\Delta f(x_1, x_2) > 0 \Rightarrow f(x_2) - f(x_1) > 0 \Rightarrow \underline{f(x_1) < f(x_2)}$

It follows that:

$\forall x_1, x_2 \in (-5/2, +\infty) : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2)) \Rightarrow$

$\Rightarrow f \uparrow (-5/2, +\infty)$

↳ For quadratics $f(x) = ax^2 + bx + c$, monotonicity changes at the axis of symmetry at $x = -b/2a$

↳ In addition to the usual properties, it is good to know the following additional properties:

1) We can add two inequalities if they have the same direction:

$$\left.\begin{array}{l} a > b \\ x > y \end{array}\right\} \Rightarrow a + x > b + y$$

2) We can multiply two inequalities if they have the same direction AND all sides are POSITIVE!

$$\left.\begin{array}{l} a>b>0 \\ x>y>0 \end{array}\right\} \Rightarrow ax>by$$

3) We can raise an inequality to a positive power if both sides of the inequality are positive

$$\left.\begin{array}{l} a>b>0 \\ p>0 \end{array}\right\} \Rightarrow a^p>b^p>0$$

e.g. $a>b>0 \Rightarrow \sqrt{a}>\sqrt{b}>0$ for $p=1/2$.

4) We can raise an inequality to a negative power if both sides of the inequality are positive but then the direction of the inequality is reversed.

$$\left.\begin{array}{l} a>b>0 \\ n<0 \end{array}\right\} \Rightarrow 0<a^n<b^n$$

e.g. $a>b>0 \Rightarrow 0<\frac{1}{a}<\frac{1}{b}$ for $n=-1$.

We rely on these properties heavily for the synthetic method. We also need the following previously mentioned properties:

5) $x<y \Rightarrow x+a<y+a$

6) $\left.\begin{array}{l} x<y \\ p>0 \end{array}\right\} \Rightarrow px<py$ 7) $\left.\begin{array}{l} x<y \\ n<0 \end{array}\right\} \Rightarrow nx>ny$.

to add/multiply a constant to both sides of an inequality.

↳ Synthetic Method

To show that $f \nearrow B$ or $f \searrow B$:

- •1 Let $x_1, x_2 \in B$ be given with $x_1 < x_2$.
- •2 Use a sequence of deductions to show that

  $x_1 < x_2 \Rightarrow \dots \Rightarrow \dots \Rightarrow f(x_1) < f(x_2)$

  or

  $x_1 < x_2 \Rightarrow \dots \Rightarrow \dots \Rightarrow f(x_1) > f(x_2)$

  using the above properties of inequalities.
- •3 Wrap up the argument.

EXAMPLES

a) For $f(x) = 3 - (1-2x)^2$ show that $f \searrow (1/2, +\infty)$

Solution

Let $x_1, x_2 \in (1/2, +\infty)$ be given with $x_1 < x_2$. Then:

$$x_1 < x_2 \Rightarrow -2x_1 > -2x_2 \Rightarrow 1-2x_1 > 1-2x_2 \overset{*}{\Rightarrow}$$

$$\Rightarrow \underset{(!)}{\underline{\underline{0 < 2}}}x_1 - 1 < 2x_2 - 1 \quad [\text{because } x_1 > 1/2 \land x_2 > 1/2]$$

$$\Rightarrow (2x_1-1)^2 < (2x_2-1)^2 \overset{**}{\Rightarrow} (1-2x_1)^2 < (1-2x_2)^2$$

$$\Rightarrow -(1-2x_1) > -(1-2x_2)^2 \Rightarrow 3-(1-2x_1)^2 > 3-(1-2x_2)^2$$

$$\Rightarrow f(x_1) > f(x_2).$$

Thus: $\forall x_1, x_2 \in (1/2, +\infty) : (x_1 < x_2 \Rightarrow f(x_1) > f(x_2))$

$\Rightarrow f \searrow (1/2, +\infty)$.

* We multiply inequality with -1 to ensure that both sides are positive before going ahead and squaring it.

** Here we use $x^2 = (-x)^2$.

↳ In the above solution you should be able to identify which inequality property is used at every step.

b) For $f(x) = 3x + 1 + \sqrt{1 - x^2}$, show that $f \nearrow [-1, 0)$

Solution

Let $\underline{x_1, x_2 \in [-1, 0)}$ be given such that $\underline{x_1 < x_2}$. Then

$$x_1 < x_2 \Rightarrow 3x_1 < 3x_2 \Rightarrow 3x_1 + 1 < 3x_2 + 1 \qquad (1)$$

Also note that

$$x_1 < x_2 \Rightarrow -x_1 > -x_2 > 0 \Rightarrow (-x_1)^2 > (-x_2)^2 \Rightarrow x_1^2 > x_2^2$$

$$\Rightarrow -x_1^2 < -x_2^2 \Rightarrow 1 - x_1^2 < 1 - x_2^2 \qquad (2)$$

and

$$x_1 \in (-1, 0) \Rightarrow -1 < x_1 < 0 \Rightarrow 1 > -x_1 > 0 \Rightarrow 1 > (-x_1)^2 \Rightarrow$$

$$\Rightarrow 1 > x_1^2 \Rightarrow 1 - x_1^2 > 0 \qquad (3)$$

and similarly

$$x_2 \in (-1, 0) \Rightarrow \cdots \Rightarrow 1 - x_2^2 > 0. \qquad (4)$$

From (2), (3), (4), it follows that

$$0 < 1 - x_1^2 < 1 - x_2^2 \Rightarrow \sqrt{1 - x_1^2} < \sqrt{1 - x_2^2} \qquad (5)$$

From (1) and (5), adding the inequalities:

$3x_1+1+\sqrt{1-x_1^2} < 3x_2+1+\sqrt{1-x_2^2} \Rightarrow$

$\Rightarrow \underline{f(x_1) < f(x_2)}$

Thus $\forall x_1, x_2 \in (-1,0) : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2))$

$\Rightarrow f \nearrow (-1,0)$.

$\hookrightarrow$ Note that before we raise an inequality to any power we have to ensure/check that both sides of the inequality are positive.

Thus in the above:

$x_1 < x_2 \Rightarrow x_1^2 < x_2^2$ is WRONG

since $x_1 < 0$ and $x_2 < 0$. Be careful!!

$\hookrightarrow$ Note that it was necessary to interrupt the main line of the argument:

$x_1 < x_2 \Rightarrow \cdots \Rightarrow \sqrt{1-x_1^2} < \sqrt{1-x_2^2}$

to show that $1-x_1^2 > 0$ and $1-x_2^2 > 0$.

Note the careful use of equation labels to interrupt and restart our main argument.

c) For $f(x) = \dfrac{1}{x^2-2}$, show that $f \nearrow (-\infty, -\sqrt{2})$

Solution

Let $\underline{x_1, x_2 \in (-\infty, -\sqrt{2})}$ be given with $\underline{x_1 < x_2}$.

Then

$x_1 < x_2 \Rightarrow -x_1 > -x_2 > 0 \Rightarrow (-x_1)^2 > (-x_2)^2 \Rightarrow x_1^2 > x_2^2$

$\Rightarrow x_1^2 - 2 > x_2^2 - 2 \quad (1)$

Also note that

$x_1 \in (-\infty, -\sqrt{2}) \Rightarrow x_1 < -\sqrt{2} \Rightarrow -x_1 > \sqrt{2} \Rightarrow (-x_1)^2 > 2 \Rightarrow$

$\Rightarrow x_1^2 > 2 \Rightarrow x_1^2 - 2 > 0. \quad (2)$

and similarly $x_2 \in (-\infty, -\sqrt{2}) \Rightarrow x_2^2 - 2 > 0 \quad (3).$

From (1), (2), and (3):

$$x_1^2 - 2 > x_2^2 - 2 > 0 \Rightarrow \frac{1}{x_1^2 - 2} < \frac{1}{x_2^2 - 2} \Rightarrow \underline{f(x_1) < f(x_2)}$$

It follows that

$\forall x_1, x_2 \in (-\infty, -\sqrt{2}) : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2))$

$\Rightarrow f \nearrow (-\infty, -\sqrt{2}).$

EXERCISES

⑧ Use the analytic method to determine the monotonicity of the following functions

a) $f(x) = 3x+2$ on $\mathbb{R}$

b) $f(x) = 5-4x$ on $\mathbb{R}$

c) $f(x) = x^2-4x+5$ on $(-\infty, 2)$

d) $f(x) = \dfrac{3x+1}{x+2}$ on $(-2, +\infty)$

e) $f(x) = \dfrac{x+8}{3x+1}$ on $(-\infty, -1/3)$

f) $f(x) = (2x+5)^2-3$ on $(-\infty, -5/2)$

g) $f(x) = (x-1)(2x+1)$ on $(1, +\infty)$

⑨ Use the synthetic method to determine the monotonicity of the following functions

a) $f(x) = 5x-3$ on $\mathbb{R}$

b) $f(x) = 2-7x$ on $\mathbb{R}$

c) $f(x) = (2x+3)^2+1$ on $(0, +\infty)$

d) $f(x) = (2-5x)^3-2$ on $(0, +\infty)$

e) $f(x) = \dfrac{-2}{2x^2+3}$ on $(0, +\infty)$

f) $f(x) = \sqrt{2x-1}$ on $(1, +\infty)$

g) $f(x) = 2-3\sqrt{4-x^2}$ on $(0, 2)$

h) $f(x) = -1 + 2\sqrt{9-(x+1)^2}$ on $(-4,-1)$
i) $f(x) = 3x + 2 + \sqrt{x+1}$ on $(0,+\infty)$
j) $f(x) = (2x-1)\sqrt{2x+1}$ on $(1,+\infty)$

⑩ Let $f(x) = -1/x$, $\forall x \in (-\infty,0) \cup (0,+\infty)$

a) Show that $f \nearrow (-\infty,0)$ and $f \nearrow (0,+\infty)$.

b) Now, show that the statement $f \nearrow (-\infty,0) \cup (0,+\infty)$ is FALSE!

↳ This exercise provides a counterexample to the false conjecture

$$f \nearrow A_1 \wedge f \nearrow A_2 \Rightarrow f \nearrow A_1 \cup A_2 \longleftarrow \text{FALSE!!}$$

⑪ Consider the function

$$\forall x \in \mathbb{R} - \{-d/c\} : f(x) = \frac{ax+b}{cx+d}$$

and define $D = ad - bc$. Show that

a) $D > 0 \Rightarrow (f \nearrow (-\infty,-d/c) \wedge f \nearrow (d/c,+\infty))$

b) $D < 0 \Rightarrow (f \searrow (-\infty,-d/c) \wedge f \searrow (d/c,+\infty))$

⑫ Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ be a function. Show that

a) $f \nearrow A \Rightarrow f$ one-to-one

b) $f \searrow A \Rightarrow f$ one-to-one

(Hint: Use proof by contradiction).

▼ Algebra and properties of mappings/functions

- To properly define a mapping or function f, we have to define both the domain $\mathrm{dom}(f)$ of f and the expression $f(x)$.

▶ Equality and restriction of mappings

- Let f, g be two mappings. We say that

$$f = g \iff \begin{cases} \mathrm{dom}(f) = \mathrm{dom}(g) = A \\ \forall x \in A : f(x) = g(x) \end{cases}$$

- Let $f : A \to B$ be a mapping and let $S \subseteq A$. We define the restriction $f \upharpoonright S$ as follows:

$$\begin{cases} \mathrm{dom}(f \upharpoonright S) = S \\ \forall x \in S : (f \upharpoonright S)(x) = f(x) \end{cases}$$

▶ Algebra of functions

- Let $f \in F(A)$ and $g \in F(B)$ be two real-valued functions and let $\lambda \in \mathbb{R}$. We define $f+g$, λf, fg as follows:

$$\begin{cases} \mathrm{dom}(f+g) = \mathrm{dom}(f) \cap \mathrm{dom}(g) = A \cap B \\ \forall x \in A \cap B : (f+g)(x) = f(x) + g(x) \end{cases}$$

$$\begin{cases} \mathrm{dom}(\lambda f) = \mathrm{dom}(f) = A \\ \forall x \in A : (\lambda f)(x) = \lambda f(x) \end{cases}$$

$$\begin{cases} \mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g) = A \cap B \\ \forall x \in A \cap B : (fg)(x) = f(x) g(x) \end{cases}$$

- Note that if the domain of f, g is not given, then by default we assume the widest possible subset of $\mathbb{R}$ for which $f(x)$ can be evaluated.

▸ odd and even functions

- Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ be a function. We say that

$$f \text{ even} \iff \forall x \in A : (-x \in A \land f(-x) = f(x))$$

$$f \text{ odd} \iff \forall x \in A : (-x \in A \land f(-x) = -f(x))$$

- Note that in order for f to be even or odd, a necessary condition is that its domain A has to be symmetric around the origin, i.e. $\forall x \in A : -x \in A$.

If the domain is not symmetric, then the function can be neither even nor odd.

▸ Bounded functions

- Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ be a function and let $S \subseteq A$. We say that:

$$f \text{ upper bounded on } S \iff \exists a \in \mathbb{R} : \forall x \in S : f(x) \leq a$$

$$f \text{ lower bounded on } S \iff \exists a \in \mathbb{R} : \forall x \in S : f(x) \geq a$$

$$f \text{ bounded on } S \iff \begin{cases} f \text{ upper bounded on } S \\ f \text{ lower bounded on } S \end{cases}$$

- We will now show that

<u>Thm</u> : $f \text{ bounded on } S \iff \exists a \in (0, +\infty) : \forall x \in S : |f(x)| \leq a$.

<u>Proof</u>

$(\Rightarrow)$: Assume that <u>f Bounded on S.</u> Then.

$$f \text{ bounded on } S \Rightarrow f \text{ lower bounded on } S$$
$$\Rightarrow \exists a_1 \in \mathbb{R} : \forall x \in S : f(x) \geq a_1$$

$$f \text{ bounded on } S \Rightarrow f \text{ upper bounded on } S$$
$$\Rightarrow \exists a_2 \in \mathbb{R} : \forall x \in S : f(x) \leq a_2$$

Choose $a_1, a_2 \in \mathbb{R}$ such that $\forall x \in S : a_1 \leq f(x) \leq a_2$.

Define $a = \max\{|a_1|, |a_2|\}$.

We will show that $\forall x \in S: |f(x)| \leq a$.

Let $x \in S$ be given. Then

$f(x) \leq a_2 \leq |a_2| \leq \max\{|a_1|, |a_2|\} = a \Rightarrow f(x) \leq a$ (1)

$f(x) \geq a_1 \geq -|a_1| \geq -\max\{|a_1|, |a_2|\} = -a \Rightarrow f(x) \geq -a$ (2)

From (1) and (2):

$-a \leq f(x) \leq a \Rightarrow |f(x)| \leq a$

and therefore $\forall x \in S: |f(x)| \leq a$

We have thus shown that $\underline{\exists a \in (0, +\infty): \forall x \in S: |f(x)| \leq a}$.

($\Leftarrow$): Assume that $\underline{\exists a \in (0, +\infty): \forall x \in S: |f(x)| \leq a}$

Let $x \in S$ be given. Then $f(x) \leq |f(x)| \leq a$ and $f(x) \geq -|f(x)| \geq -a$. It follows that

$$\begin{cases} \forall x \in S: f(x) \leq a \\ \forall x \in S: f(x) \geq -a \end{cases} \Rightarrow \begin{cases} f \text{ upper bounded on } S \\ f \text{ lower bounded on } S \end{cases} \Rightarrow$$

$$\Rightarrow \underline{f \text{ bounded on } S.} \quad \square$$

↳ In arguments involving absolute values, we use the following properties:

$\forall a \in \mathbb{R}: -|a| \leq a \leq |a|$

$\forall a, b \in \mathbb{R}: |a+b| \leq |a| + |b|$

$\forall a, b \in \mathbb{R}: |a-b| \leq |a| + |b|$

$\forall a, b \in \mathbb{R}: |ab| = |a||b|$

$\forall a \in \mathbb{R}: \forall b \in \mathbb{R} - \{0\}: \left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

EXAMPLES

a) Given the functions $f_1, f_2 \in F(A)$ and $g_1, g_2 \in F(B)$ show that

$$f_1 = f_2 \land g_1 = g_2 \Rightarrow f_1 + g_1 = f_2 + g_2$$

Solution

Assume that $f_1 = f_2 \land g_1 = g_2$. Then

$$\left.\begin{array}{l} \mathrm{dom}(f_1+g_1) = \mathrm{dom}(f_1) \cap \mathrm{dom}(g_1) = A \cap B \\ \mathrm{dom}(f_2+g_2) = \mathrm{dom}(f_2) \cap \mathrm{dom}(g_2) = A \cap B \end{array}\right\} \Rightarrow$$

$$\Rightarrow \mathrm{dom}(f_1+g_1) = \mathrm{dom}(f_2+g_2) \qquad (1)$$

We will show: $\forall x \in A \cap B : (f_1+g_1)(x) = (f_2+g_2)(x)$.

Let $x \in A \cap B$ be given. We note that:

$$f_1 = f_2 \Rightarrow f_1(x) = f_2(x) \qquad (2)$$

$$g_1 = g_2 \Rightarrow g_1(x) = g_2(x) \qquad (3)$$

and therefore:

$$\begin{aligned} (f_1+g_1)(x) &= f_1(x) + g_1(x) && [\text{definition}] \\ &= f_2(x) + g_2(x) && [\text{eq. } (2),(3)] \\ &= (f_2+g_2)(x) && [\text{definition}] \end{aligned}$$

It follows that $\forall x \in A \cap B : (f_1+g_1)(x) = (f_2+g_2)(x)$ (4)

From (1) and (4): $f_1 + g_1 = f_2 + g_2$.

→ To show that two functions are equal, we have to show that

a) They have the same domain

b) They have the same formula.

b) Let A, B be two sets with $A \cap B \neq \emptyset$. Show that:

$\forall a, b \in \mathbb{R} : \forall f \in \mathcal{F}(A) : \forall g \in \mathcal{F}(B) : (af)(bg) = (ab)(fg)$

Solution

Let $a, b \in \mathbb{R}$ and $f \in \mathcal{F}(A)$ and $g \in \mathcal{F}(B)$ be given. Then

$$\operatorname{dom}((af)(bg)) = \operatorname{dom}(af) \cap \operatorname{dom}(bg) = \operatorname{dom}(f) \cap \operatorname{dom}(g) = A \cap B \quad (1)$$

and

$$\operatorname{dom}((ab)(fg)) = \operatorname{dom}(fg) = \operatorname{dom}(f) \cap \operatorname{dom}(g) = A \cap B \quad (2)$$

From (1) and (2): $\operatorname{dom}((af)(bg)) = \operatorname{dom}((ab)(fg))$ (3).

Let $x \in A \cap B$ be given. Then

$$[(af)(bg)](x) = (af)(x) \cdot (bg)(x) = af(x)\, bg(x) = ab\, f(x) g(x) = ab\,(fg)(x) = [(ab)(fg)](x).$$

and therefore $\forall x \in A \cap B : [(af)(bg)](x) = [(ab)(fg)](x)$. (4)

From (3) and (4): $(af)(bg) = (ab)(fg)$

and it follows that

$\forall a, b \in \mathbb{R} : \forall f \in \mathcal{F}(A) : \forall g \in \mathcal{F}(B) : (af)(bg) = (ab)(fg)$.

c) Let f, g be two functions. Show that

f even $\wedge$ g odd $\Rightarrow$ fg odd

Solution

Assume that f even $\wedge$ g odd.

Define $A = \mathrm{dom}(f)$ and $B = \mathrm{dom}(g)$.

f even $\Rightarrow \forall x \in A : (-x \in A \wedge f(-x) = f(x))$ (1)

g odd $\Rightarrow \forall x \in B : (-x \in B \wedge g(-x) = -g(x))$. (2)

Note that $\mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g) = A \cap B$.

Let $\underline{x \in A \cap B}$ be given. Then:

$$\begin{aligned} x \in A \cap B &\Rightarrow x \in A \wedge x \in B && [\text{definition}] \\ &\Rightarrow -x \in A \wedge -x \in B && [\text{from } (1), (2)] \\ &\Rightarrow \underline{-x \in A \cap B} \end{aligned}$$

and

$$\begin{aligned} \underline{(fg)(-x)} &= f(-x)g(-x) = f(x)[-g(x)] = -f(x)g(x) \\ &= \underline{-(fg)(x)}. \end{aligned}$$

It follows that

$\forall x \in A \cap B : (-x \in A \cap B \wedge (fg)(-x) = -(fg)(x))$

$\Rightarrow fg$ odd.

d) Define $\forall x \in \mathbb{R}: f(x) = 2\sin x\,(\cos(2x) + \cos(3x))$

Show that f bounded in $\mathbb{R}$.

Solution

Let $x \in \mathbb{R}$ be given. Then

$$|f(x)| = |2\sin x\,[\cos(2x) + \cos(3x)]| =$$
$$= 2|\sin x| \cdot |\cos(2x) + \cos(3x)|$$
$$\leqslant 2|\cos(2x) + \cos(3x)| \leqslant 2(|\cos(2x)| + |\cos(3x)|)$$
$$\leqslant 2(1+1) = 2 \cdot 2 = 4 \Rightarrow |f(x)| \leqslant 4.$$

It follows that

$\forall x \in \mathbb{R}: |f(x)| \leqslant 4 \Rightarrow f$ bounded in $\mathbb{R}$.

e) Let $f, g \in F(\mathbb{R})$ be two functions, both bounded on $\mathbb{R}$.

Define h as:

$\forall x \in \mathbb{R}: h(x) = f(x)(2 + \cos x) - g(x)(1 - \sin x)^3$

Show that h is bounded in $\mathbb{R}$.

Solution

f bounded on $\mathbb{R} \Rightarrow \exists a \in (0, +\infty): \forall x \in \mathbb{R}: |f(x)| \leqslant a$

g bounded on $\mathbb{R} \Rightarrow \exists b \in (0, +\infty): \forall x \in \mathbb{R}: |g(x)| \leqslant b$

Choose $a, b \in (0, +\infty)$ such that $\forall x \in \mathbb{R}: (|f(x)| \leqslant a \wedge |g(x)| \leqslant b)$.

Let $x \in \mathbb{R}$ be given. Then:

$$|h(x)| = |f(x)(2 + \cos x) - g(x)(1 - \sin x)^3|$$
$$\leqslant |f(x)(2 + \cos x)| + |g(x)(1 - \sin x)^3|$$
$$= |f(x)||2 + \cos x| + |g(x)|(|1 - \sin x|)^3$$
$$\leqslant a|2 + \cos x| + b|1 - \sin x|^3$$
$$\leqslant a(2 + |\cos x|) + b(1 + |\sin x|)^3$$

$$\leqslant a(2+1) + b(1+1)^3 = 3a + 8b.$$

and therefore

$\forall x \in \mathbb{R} : (|h(x)| \leqslant 3a + 8b) \Rightarrow$

$\Rightarrow h$ bounded at $\mathbb{R}$.

↳ In addition to properties of absolute values, we also use:

$\forall x \in \mathbb{R} : |\sin x| \leqslant 1$

$\forall x \in \mathbb{R} : |\cos x| \leqslant 1.$

f) Let $f, g \in F(\mathbb{R})$ be two functions that are upper bounded on $\mathbb{R}$. Show that $f+g$ are upper bounded on $\mathbb{R}$.

<u>Solution</u>

f upper bounded on $\mathbb{R} \Rightarrow \exists a \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) \leqslant a$ (1)

g upper bounded on $\mathbb{R} \Rightarrow \exists b \in \mathbb{R} : \forall x \in \mathbb{R} : g(x) \leq b$ (2)

Let $x \in \mathbb{R}$ be given. Then:

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) && [\text{definition}]\\ &\leqslant a + g(x) && [\text{via eq.}(1)]\\ &\leqslant a + b && [\text{via eq.}(2)]\end{aligned}$$

and therefore

$\forall x \in \mathbb{R} : (f+g)(x) \leqslant a + b$

$\Rightarrow f+g$ upper bounded on $\mathbb{R}$.

g) Let $f, g \in F(\mathbb{R})$ be two functions. Show that
$f \uparrow \mathbb{R} \wedge g \uparrow \mathbb{R} \Rightarrow f+g \uparrow$

Solution

1st method : Let $x_1, x_2 \in \mathbb{R}$ be given with $x_1 < x_2$. Then

$f \uparrow \mathbb{R} \Rightarrow f(x_1) < f(x_2)$ (1)

$g \uparrow \mathbb{R} \Rightarrow g(x_1) < g(x_2)$ (2)

From (1) and (2):

$f(x_1) + g(x_1) < f(x_2) + g(x_2) \Rightarrow$

$\Rightarrow (f+g)(x_1) < (f+g)(x_2)$

It follows that

$\forall x_1, x_2 \in \mathbb{R} : (x_1 < x_2 \Rightarrow (f+g)(x_1) < (f+g)(x_2)) \Rightarrow$

$\Rightarrow f+g \uparrow \mathbb{R}$.

2nd method : Let $x_1, x_2 \in \mathbb{R}$ be given with $x_1 < x_2$. Then

$$\begin{aligned}\Delta(x_1, x_2) &\equiv (f+g)(x_2) - (f+g)(x_1) \\ &= [f(x_2) + g(x_2)] - [f(x_1) + g(x_1)] \\ &= [f(x_2) - f(x_1)] + [g(x_2) - g(x_1)] > 0\end{aligned}$$

because:

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \Rightarrow f(x_2) - f(x_1) > 0$

$x_1 < x_2 \Rightarrow g(x_1) < g(x_2) \Rightarrow g(x_2) - g(x_1) > 0$

It follows that $(f+g)(x_1) < (f+g)(x_2)$, and therefore:

$\forall x_1, x_2 \in \mathbb{R} : (x_1 < x_2 \Rightarrow (f+g)(x_1) < (f+g)(x_2))$

$\Rightarrow f+g \uparrow \mathbb{R}$.

EXERCISES

⑬ Let A, B be two sets with $A \cap B \neq \emptyset$. Show that

a) $\forall f \in F(A) : \forall g \in F(B) : (-f)(-g) = fg$

b) $\forall a, b \in \mathbb{R} : \forall f \in F(A) : \forall g, h \in F(B) : (ag + bh)f = a(fg) + b(fh)$

c) $\forall f, g \in F(A) : \forall h \in F(B) : (f = g \Rightarrow f + h = g + h)$

d) $\forall f, g \in F(A) : \forall h \in F(B) : (f = g \Rightarrow fh = gh)$

⑭ Let $f, g \in F(\mathbb{R})$ be two functions. Show that:

a) f even $\wedge$ g even $\Rightarrow f+g$ even

b) f even $\wedge$ g even $\Rightarrow fg$ even

c) f odd $\wedge$ g odd $\Rightarrow f+g$ odd

d) f odd $\wedge$ g odd $\Rightarrow fg$ even

e) f odd $\wedge$ $f \nearrow [0, +\infty) \Rightarrow f \nearrow \mathbb{R}$
   (Hint: use proof by cases)

f) f even $\wedge$ $f \nearrow [0, +\infty) \Rightarrow f \searrow (-\infty, 0)$

⑮ Let $f : A \to \mathbb{R}$ be a function. Show that

a) $f \nearrow A \Rightarrow f$ one-to-one

b) $f \searrow A \Rightarrow f$ one-to-one

c) f even $\Rightarrow f$ not one-to-one

⑯ Show that the following functions are bounded in $\mathbb{R}$.

a) $\forall x \in \mathbb{R} : f(x) = \sin x (\cos x + \sin x)$

b) $\forall x \in \mathbb{R} : f(x) = (1 - \sin x)^2 \cos x + \sin x$

c) $\forall x \in \mathbb{R}: f(x) = (1-\cos x)(1-\sin x) + \sin x$

⑰ Let $f, g \in F(\mathbb{R})$ be two functions bounded in $\mathbb{R}$. Show that $h \in F(\mathbb{R})$, defined as follows, is also bounded in $\mathbb{R}$.

a) $\forall x \in \mathbb{R}: h(x) = f(x)\, g(x) \cos x$

b) $\forall x \in \mathbb{R}: h(x) = f(x)\,(1+\sin x) + g(x) \cos^2 x$

c) $\forall x \in \mathbb{R}: h(x) = \sin(f(x)) + g(x) \cos(g(x))$

d) $\forall x \in \mathbb{R}: h(x) = f(g(x))\,[\sin x + g(x) \cos(\sin x)]$

⑱ Let $f \in F(\mathbb{R})$ be defined as:

$\forall x \in \mathbb{R}: f(x) = ax^2 + bx + c$

Show that $g = f \restriction [-1,1]$ is bounded on $[-1,1]$.

⑲ Let $f \in F(\mathbb{R})$ be a general polynomial function defined by:

$$\forall x \in \mathbb{R}: f(x) = \sum_{K=1}^{n} a_K x^k$$

and let $a, b \in \mathbb{R}$ be given with $a < b$. Show that $g = f \restriction [a,b]$ is bounded in $[a,b]$.

(Hint: For $x \in [a,b]$, first show that $|x| \leq \max\{|a|, |b|\}$.

⑳ Let $f, g, h \in F(\mathbb{R})$ be three functions. Show that

a) $\begin{cases} h = f + 3g \\ f, g \text{ lower bounded on } \mathbb{R} \end{cases} \Rightarrow h$ lower bounded on $\mathbb{R}$.

b) $\begin{cases} h = 2f - 5g \\ f \text{ upper bounded on } \mathbb{R} \\ g \text{ lower bounded on } \mathbb{R} \end{cases} \Rightarrow h$ upper bounded on $\mathbb{R}$.

c) $\begin{cases} h = fg \\ f \text{ upper bounded on } \mathbb{R} \\ \forall x \in \mathbb{R} : 0 < g(x) < 1 \end{cases} \Rightarrow h$ upper bounded on $\mathbb{R}$.

d) $\begin{cases} h = fg \\ f \text{ lower bounded on } \mathbb{R} \\ \forall x \in \mathbb{R} : 0 < g(x) < 2 \end{cases} \Rightarrow h$ lower bounded on $\mathbb{R}$.

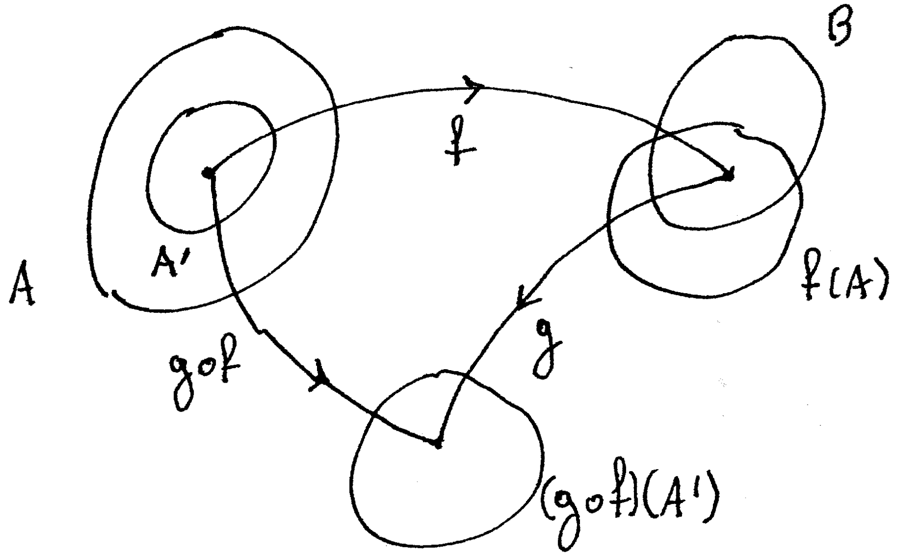
▼ Function composition

- Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be two functions. We assume that $f(A) \cap B \neq \emptyset$. Let A' be the subset of A whose elements are mapped by f into the intersection $f(A) \cap B$. Thus A' is given by

$$A' = \{x \in A \mid f(x) \in B\}.$$

We may therefore define the function $g \circ f : A' \to \mathbb{R}$ as follows:

$$\boxed{\begin{array}{l} \mathrm{dom}(g \circ f) = \{x \in \mathrm{dom}(f) \mid f(x) \in \mathrm{dom}(g)\} = A' \\ \forall x \in A' : \ (g \circ f)(x) = g(f(x)) \end{array}}$$



- We note that the belonging condition for $g \circ f$ is

$$\boxed{x \in \mathrm{dom}(g \circ f) \Leftrightarrow \begin{cases} x \in \mathrm{dom}(f) \\ f(x) \in \mathrm{dom}(g) \end{cases}}$$

↳ Properties of mapping composition

- Let f, g, h be 3 mappings. Then

$$\boxed{(f\circ g)\circ h = f\circ(g\circ h)} \quad \text{(associative)}$$

Proof

First we establish that the domains are equal.

$$x \in \operatorname{dom}((f\circ g)\circ h) \Leftrightarrow$$
$$\Leftrightarrow x\in\operatorname{dom}(h) \wedge h(x) \in \operatorname{dom}(f\circ g)$$
$$\Leftrightarrow x\in\operatorname{dom}(h) \wedge [h(x)\in\operatorname{dom}(g) \wedge g(h(x))\in\operatorname{dom}(f)]$$
$$\Leftrightarrow (x\in\operatorname{dom}(h)\wedge h(x)\in\operatorname{dom}(g)) \wedge (g\circ h)(x)\in\operatorname{dom}(f)$$
$$\Leftrightarrow x\in\operatorname{dom}(g\circ h) \wedge (g\circ h)(x)\in\operatorname{dom}(f)$$
$$\Leftrightarrow x\in\operatorname{dom}(f\circ(g\circ h)).$$

therefore,

$$\operatorname{dom}((f\circ g)\circ h) = \operatorname{dom}(f\circ(g\circ h)) = A$$

Let $x\in A$ be given. Then

$$\left.\begin{array}{l} [(f\circ g)\circ h](x) = (f\circ g)(h(x)) = f(g(h(x))) \\ {[f\circ(g\circ h)](x)} = f((g\circ h)(x)) = f(g(h(x))) \end{array}\right\} \Rightarrow$$

$$\Rightarrow [(f\circ g)\circ h](x) = [f\circ(g\circ h)](x), \forall x\in A.$$

It follows that $(f\circ g)\circ h = f\circ(g\circ h)$. ☐

- In general, it is usually <u>not true</u> that $f \circ g = g \circ f$, although exceptions are possible for specific choices of f, g.

- Let f, g be two mappings. Then

$$\boxed{\begin{cases} f \text{ one-to-one} \\ g \text{ one-to-one} \end{cases} \Rightarrow f \circ g \text{ one-to-one.}}$$

<u>Proof</u>

Let $A = \mathrm{dom}(f \circ g)$. Let $x_1, x_2 \in A$ be given such that $(f \circ g)(x_1) = (f \circ g)(x_2)$. Then,

$$\begin{aligned} (f \circ g)(x_1) = (f \circ g)(x_2) &\Rightarrow f(g(x_1)) = f(g(x_2)) && [\text{definition}] \\ &\Rightarrow g(x_1) = g(x_2) && [f \text{ one-to-one}] \\ &\Rightarrow x_1 = x_2 && [g \text{ one-to-one}] \end{aligned}$$

and it follows that

$$\forall x_1, x_2 \in A : ((f \circ g)(x_1) = (f \circ g)(x_2) \Rightarrow x_1 = x_2)$$

$\Rightarrow f \circ g$ one-to-one.

Methodology

- 1 To define a function f, we have to define both the expression f(x) and the domain dom(f) of f.
- 2 When the domain of a function is not given, the implied domain is the widest possible subset of $\mathbb{R}$ for which the function formula f(x) can be evaluated. To derive the belonging condition of the domain, we note that

a) We cannot DIVIDE BY ZERO

b) We cannot take the SQUARE ROOT OF A NEGATIVE NUMBER.

- 3 To find the domain of fog:
  a) First we find dom(f) and dom(g)
  b) The belonging condition of dom (fog) is given by

$$x \in \mathrm{dom}(f \circ g) \Leftrightarrow \begin{cases} x \in \mathrm{dom}(g) \\ g(x) \in \mathrm{dom}(f) \end{cases} \Leftrightarrow \cdots$$

EXAMPLE

a) Given $f(x)=\sqrt{1-x}$ and $g(x)=1-3x$, define the functions $h_1=f\circ g$ and $h_2=g\circ f$.

Solution

- Domain of f

Require $1-x\geqslant 0 \Leftrightarrow x\leqslant 1 \Leftrightarrow x\in(-\infty,1]$.

It follows that $\operatorname{dom}(f)=(-\infty,1]$.

- Domain of g.

There are no requirements, therefore $\operatorname{dom}(g)=\mathbb{R}$.

- Definition of $h_1=f\circ g$.

$$x\in\operatorname{dom}(f\circ g)\Leftrightarrow\begin{cases}x\in\operatorname{dom}(g)\\ g(x)\in\operatorname{dom}(f)\end{cases}\Leftrightarrow\begin{cases}x\in\mathbb{R}\\ (1-3x)\in(-\infty,1]\end{cases}\Leftrightarrow$$

$$\Leftrightarrow (1-3x)\in(-\infty,1]\Leftrightarrow 1-3x\leqslant 1\Leftrightarrow -3x\leqslant 0\Leftrightarrow$$

$$\Leftrightarrow x\geqslant 0\Leftrightarrow x\in[0,+\infty).$$

and therefore $\operatorname{dom}(f\circ g)=[0,+\infty)$.

$$\forall x\in[0,+\infty):(f\circ g)(x)=f(g(x))=f(1-3x)=\sqrt{1-(1-3x)}$$
$$=\sqrt{1-1+3x}=\sqrt{3x}$$

thus $\forall x\in[0,+\infty):(f\circ g)(x)=\sqrt{3x}$.

- Definition of $h_2=g\circ f$.

$$x\in\operatorname{dom}(g\circ f)\Leftrightarrow\begin{cases}x\in\operatorname{dom}(f)\\ f(x)\in\operatorname{dom}(g)\end{cases}\Leftrightarrow\begin{cases}x\in(-\infty,1]\\ \sqrt{1-x}\in\mathbb{R}\end{cases}\Leftrightarrow$$

$$\Leftrightarrow x\in(-\infty,1]$$

and therefore: $\operatorname{dom}(g\circ f)=(-\infty,1]$.

$$\forall x\in(-\infty,1]:(g\circ f)(x)=g(f(x))=g(\sqrt{1-x})=1-3\sqrt{1-x}$$

b) Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be two functions.
Show that: $f\searrow\mathbb{R} \land g\nearrow\mathbb{R} \Rightarrow f\circ g\searrow\mathbb{R}$.

Solution

Assume that $f\searrow\mathbb{R}$ and $g\nearrow\mathbb{R}$.
Since $\operatorname{dom}(f)=\mathbb{R}$ and $\operatorname{dom}(g)=\mathbb{R}$, it follows that

$$\begin{aligned}\operatorname{dom}(f\circ g) &= \{x\in\operatorname{dom}(g) \mid g(x)\in\operatorname{dom}(f)\} = \\ &= \{x\in\mathbb{R} \mid g(x)\in\mathbb{R}\} = \mathbb{R}.\end{aligned}$$

Let $x_1,x_2\in\mathbb{R}$ be given with $x_1<x_2$. Then

$$\begin{aligned} x_1<x_2 &\Rightarrow g(x_1)<g(x_2) && [g\nearrow\mathbb{R}] \\ &\Rightarrow f(g(x_1))>f(g(x_2)) && [f\searrow\mathbb{R}] \\ &\Rightarrow (f\circ g)(x_1)>(f\circ g)(x_2) && [\text{definition}] \end{aligned}$$

and therefore:

$$\forall x_1,x_2\in\mathbb{R}: (x_1<x_2 \Rightarrow (f\circ g)(x_1)>(f\circ g)(x_2))$$
$$\Rightarrow f\circ g\searrow\mathbb{R}$$

c) Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be two functions.
Show that f even $\land$ g odd $\Rightarrow f\circ g$ even.

Solution

Assume that f even and g odd. Since $\operatorname{dom}(f)=\mathbb{R}$ and $\operatorname{dom}(g)=\mathbb{R}$, it follows that

$$\begin{aligned}\operatorname{dom}(f\circ g) &= \{x\in\operatorname{dom}(g) \mid g(x)\in\operatorname{dom}(f)\} = \\ &= \{x\in\mathbb{R} \mid g(x)\in\mathbb{R}\} = \mathbb{R}\end{aligned}$$

which is symmetric: $\forall x\in\mathbb{R}: -x\in\mathbb{R}$.
Let $x\in\mathbb{R}$ be given. Then:

$$
\begin{aligned}
(f\circ g)(-x) &= f(g(-x)) && [\text{definition}]\\
&= f(-g(x)) && [g \text{ odd}]\\
&= f(g(x)) && [f \text{ even}]\\
&= (f\circ g)(x) && [\text{definition}]
\end{aligned}
$$

and therefore $\forall x \in \mathbb{R} : (f\circ g)(-x) = (f\circ g)(x)$. (2)

From (1) and (2):

$\forall x \in \mathbb{R} : (-x \in \mathbb{R} \wedge (f\circ g)(-x) = (f\circ g)(x))$

$\Rightarrow f\circ g$ even.

d) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that

f odd $\wedge$ f bounded on $[0, +\infty) \Rightarrow f$ bounded on $\mathbb{R}$.

Solution

Assume that f odd and f bounded on $[0, +\infty)$. Since:

f bounded on $[0, +\infty) \Rightarrow \exists a \in (0, +\infty) : \forall x \in [0, +\infty) : |f(x)| \leq a$. (1)

Let $x \in \mathbb{R}$ be given. We distinguish the following cases:

Case 1 : If $x \in [0, +\infty)$, then from (1): $|f(x)| \leq a$.

Case 2 : If $x \in (-\infty, 0)$, then

$$
\begin{aligned}
|f(x)| &= |-f(x)| = \\
&= |f(-x)| && [f \text{ odd}]\\
&\leq a && [\text{eq.}(1) \text{ and } -x \in [0, +\infty)]
\end{aligned}
$$

It follows that

$(\forall x \in \mathbb{R} : |f(x)| \leq a) \Rightarrow f$ bounded on $\mathbb{R}$.

EXERCISES

㉑ Define the functions $f \circ g$ and $g \circ f$ for f, g given by:

a) $f(x) = 3x+2$, $g(x) = x^2+5x+3$

b) $f(x) = x^2+1$, $g(x) = \sqrt{3-x}$

c) $f(x) = \sqrt{4-x^2}$, $g(x) = \sqrt{1-x^2}$

d) $f(x) = \dfrac{x+2}{x-1}$, $g(x) = \dfrac{2x-1}{x+3}$

㉒ Let $f, g, h \in F(\mathbb{R})$ be three functions. Show that $f = g \Rightarrow f \circ h = g \circ h$.

㉓ Let $f, g \in F(\mathbb{R})$ be two functions. Show that

a) f even $\wedge$ g even $\Rightarrow$ $f \circ g$ even

b) f odd $\wedge$ g odd $\Rightarrow$ $f \circ g$ odd

c) f even $\wedge$ g odd $\Rightarrow$ $f \circ g$ even

d) $f \nearrow \mathbb{R} \wedge g \nearrow \mathbb{R} \Rightarrow f \circ g \nearrow \mathbb{R}$

e) $f \nearrow \mathbb{R} \wedge g \searrow \mathbb{R} \Rightarrow f \circ g \searrow \mathbb{R}$

f) $f \searrow \mathbb{R} \wedge g \searrow \mathbb{R} \Rightarrow g \circ f \nearrow \mathbb{R}$

g) f odd $\wedge$ $f \nearrow [0, +\infty) \Rightarrow f \nearrow \mathbb{R}$

h) f even $\wedge$ $f \nearrow (0, +\infty) \Rightarrow f \searrow (-\infty, 0)$.

i) f even $\wedge$ f bounded on $[0, +\infty) \Rightarrow f$ bounded on $\mathbb{R}$

Inverse mappings

- Let $f : A \to B$ and $g : B \to A$ be two mappings. We say that

$$\boxed{\begin{aligned} &g \text{ left inverse of } f \Leftrightarrow \forall x \in A : (g \circ f)(x) = x \\ &g \text{ right inverse of } f \Leftrightarrow \forall x \in B : (f \circ g)(x) = x \end{aligned}}$$

- These definitions can be abbreviated if written in terms of the identity mapping $id[A] : A \to A$ defined as: $\forall x \in A : id[A](x) = x$.

  Then, it follows that for $f : A \to B$ and $g : B \to A$

  g left inverse of $f \Leftrightarrow g \circ f = id[A]$

  g right inverse of $f \Leftrightarrow f \circ g = id[B]$

- We note that in general:

$$f \circ id[S] = f \upharpoonright S$$
$$id[S] \circ f = f \upharpoonright \{x \in dom(f) \mid f(x) \in S\}$$

  To eliminate the need for restrictions, for $f : A \to B$ we have:

$$f \circ id[A] = f$$
$$id[f(A)] \circ f = f$$

Criteria for existence of left/right inverse

Let $f : A \to B$ be a mapping. Recall that we defined 1-1 mappings as follows:

$\boxed{f \text{ one-to-one} \Leftrightarrow \forall x_1, x_2 \in A : (\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2)}$

We now introduce the following definitions:

$$\boxed{\begin{array}{l} f \text{ onto} \Leftrightarrow f(A) = B \\ f \text{ bijection} \Leftrightarrow f \text{ onto} \wedge f \text{ one-to-one} \end{array}}$$

We will now show that

Thm : Let $f : A \to B$ be a mapping. Then:

a) f has a left inverge $g : B \to A \Leftrightarrow f$ one-to-one

b) f has a right inverse $g : B \to A \Leftrightarrow f$ onto

Proof

a) ($\Rightarrow$): Assume that f has a left inverse $g : B \to A$. Let $x_1, x_2 \in A$ be given with $f(x_1) = f(x_2)$. Then:

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow g(f(x_1)) = gf(x_2)) \\ &\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2) \quad [\text{Definition}] \\ &\Rightarrow id[A](x_1) = id[A](x_2) \quad [g \text{ left inverse of } f] \\ &\Rightarrow x_1 = x_2 \quad [\text{Definition}] \end{aligned}$$

It follows that

$\forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$

$\Rightarrow f$ one-to-one.

($\Leftarrow$): Assume that f is one-to-one.

▸ Definition of $g : B \to A$

Let $y \in f(A)$ be given. Since $y \in f(A) \Rightarrow \exists x \in A : f(x) = y$ we choose an $x \in A$ and define $h(y) = x$ such that $f(x) = y$. Consequently, we may define a mapping $h : f(A) \to A$ such that

$$\forall y \in f(A): \ f(h(y)) = y \qquad (1)$$

We now define $g: B \to A$ as:

$$\forall y \in B: \ g(y) = \begin{cases} h(y) & , \text{ if } y \in f(A) \\ y & , \text{ if } y \in B - f(A) \end{cases}$$

- Analysis: We now show g left inverse of f.

Let $x \in A$ be given. Define $y = f(x) \in f(A)$ and $x_0 = g(y)$. Note that it is not yet obvious that $x_0 = x$. Since

$$y \in f(A) \Rightarrow f(h(y)) = y \qquad [\text{Eq. } (1)]$$
$$= f(x) \qquad [\text{Definition}]$$

and therefore,

$$\begin{cases} f(h(y)) = f(x) \\ f \text{ one-to-one} \end{cases} \Rightarrow h(y) = x \qquad (2)$$

consequently,

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && [\text{Definition}] \\ &= g(y) && [\text{Definition}] \\ &= h(y) && [\text{because } y \in f(A)] \\ &= x && [\text{Eq. } (2)] \end{aligned}$$

It follows that

$$\forall x \in A: (g \circ f)(x) = x$$
$$\Rightarrow g \text{ left inverse of } f.$$

b) $(\Rightarrow)$: Assume that $g: B \to A$ is a right inverse of f. By definition, we know that $f(A) \subseteq B$. We claim that $B \subseteq f(A)$. Let $y \in B$ be given. Define $x = g(y) \in A$. Then:

$$\begin{aligned} f(x) &= f(g(y)) && [\text{because } x = g(y)] \\ &= (f \circ g)(y) && [\text{definition}] \\ &= id[B](y) && [g \text{ right inverse of } f] \\ &= y && [\text{definition}] \end{aligned}$$

It follows that

$(\exists x \in A : y = f(x)) \Rightarrow y \in f(A).$

and therefore $\forall y \in B : y \in f(A) \Rightarrow B \subseteq f(A)$

Since $\left\{ \begin{array}{l} f(A) \subseteq B \\ B \subseteq f(A) \end{array} \right. \Rightarrow f(A) = B \Rightarrow f$ onto.

$(\Leftarrow)$: Assume that f onto.

- Definition of $g : B \to A$

f onto $\Rightarrow f(A) = B \Rightarrow B \subseteq f(A) \Rightarrow \forall y \in B : y \in f(A)$

$\Rightarrow \forall y \in B : \exists x \in A : f(x) = y \quad (1)$

Let $y \in B$ be given. From (1), we choose an $x \in A$ such that $f(x) = y$, and define $g(y) = x$.

It follows that we have thus defined a $g : B \to A$ such that $\forall y \in B : (g(y) = x \Rightarrow f(x) = y)$

- Analysis

Let $y \in B$ be given. Define $x = g(y)$. Then $f(x) = y$.

It follows that

$$\begin{aligned} (f \circ g)(y) &= f(g(y)) && [\text{definition}] \\ &= f(x) && [\text{because } x = g(y)] \\ &= y && [\text{because } f(x) = y] \end{aligned}$$

and therefore

$(\forall y \in B : (f \circ g)(y) = y) \Rightarrow g$ right inverse of f. □

From the proof of this theorem we see that the left and right inverse do not have to be unique. However we will show that when both exist, they have to be equal to each other.

Prop: Let $f: A \to B$ be a mapping. Then

$$\boxed{\left\{\begin{array}{l} g_1 \text{ left inverse of } f \\ g_2 \text{ right inverse of } f \end{array}\right. \Rightarrow g_1 = g_2}$$

Proof

Assume that g_1 left inverse of f and g_2 right inverse of f.

g_1 left inverse of $f \Rightarrow g_1 \circ f = id[A]$ (1)

g_2 right inverse of $f \Rightarrow f \circ g_2 = id[B]$. (2)

It follows that

$$\begin{aligned} g_1 &= g_1 \circ id[B] && [\text{identity mapping}] \\ &= g_1 \circ (f \circ g_2) && [\text{eq. (2)}] \\ &= (g_1 \circ f) \circ g_2 && [\text{associative property}] \\ &= id[A] \circ g_2 && [\text{eq. (1)}] \\ &= g_2 && [\text{identity mapping}] \end{aligned}$$

and therefore $g_1 = g_2$ □

Definition of inverse mapping

- Let $f : A \to B$ be a mapping. We say that

$$\boxed{g \text{ inverse of } f \Leftrightarrow \begin{cases} g \text{ left inverse of } f \\ g \text{ right inverse of } f \end{cases}}$$

Equivalently, the definition can be rewritten as

$$\boxed{g \text{ inverse of } f \Leftrightarrow \begin{cases} g \circ f = \mathrm{id}[A] \\ f \circ g = \mathrm{id}[B] \end{cases} \Leftrightarrow \begin{cases} \forall x \in A : (g \circ f)(x) = x \\ \forall x \in B : (f \circ g)(x) = x \end{cases}}$$

Existence of inverse of mapping

<u>Thm</u> : Let $f : A \to B$ be a mapping. Then

$$\boxed{\exists\, g \in \mathrm{Map}(B,A) : g \text{ inverse of } f \Leftrightarrow f \text{ bijection}}$$

<u>Proof</u>

<u>($\Rightarrow$)</u>: Assume that f has an inverse $g : B \to A$. Then

$$g \text{ inverse of } f \Rightarrow \begin{cases} g \text{ left inverse of } f \\ g \text{ right inverse of } f \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f \text{ one-to-one} \\ f \text{ onto} \end{cases} \Rightarrow$$

$$\Rightarrow f \text{ bijection.}$$

($\Leftarrow$) : Assume that f is a bijection. Then

f bijection $\Rightarrow f$ one-to-one $\Rightarrow$

$\Rightarrow \exists g_1 \in Map(B,A) : g_1$ left inverse of f. (1)

f bijection $\Rightarrow f$ onto $\Rightarrow$

$\Rightarrow \exists g_2 \in Map(B,A) : g_2$ right inverse of f (2)

Choose $g_1, g_2 \in Map(B,A)$ such that g_1 left inverse and g_2 right inv. of f.

$$\left\{\begin{array}{l} g_1 \text{ left inverse of } f \\ g_2 \text{ right inverse of } f \end{array}\right. \Rightarrow g_1 = g_2 .$$

Define $g \equiv g_1 = g_2$. Then

$$\left\{\begin{array}{l} g \text{ left inverse of } f \\ g \text{ right inverse of } f \end{array}\right. \Rightarrow g \text{ inverse of } f. \quad \square$$

↳ Uniqueness of inverse mapping

Thm: Let $f: A \to B$ be a mapping. Then, we have:

$$\left\{\begin{array}{l} g_1 \text{ inverse of } A \\ g_2 \text{ inverse of } A \end{array}\right. \Rightarrow g_1 = g_2$$

Proof

Assume that $g_1 : B \to A$ and $g_2 : B \to A$ are inverses of f. Then, we have:

$$\begin{aligned} g_1 &= id[A] \circ g_1 && [A \text{ codomain of } g_1] \\ &= (g_2 \circ f) \circ g_1 && [g_2 \text{ left inverse of } f] \\ &= g_2 \circ (f \circ g_1) && [\text{associative property}] \\ &= g_2 \circ id[B] && [g_1 \text{ right inverse of } f] \\ &= g_2 && [B \text{ domain of } g_2] \end{aligned}$$

and therefore $g_1 = g_2$.

↳ Notation: If $f: A \to B$ is a bijection, then according to the previous two results, there is a unique function g which is the inverse of f. We denote the unique inverse of f as $f^{-1} = g$.

↦ Equivalent characterization of inverse mapping

Thm: Let $f : A \to B$ and $g : B \to A$ be two mappings. Then:

$$\boxed{g = f^{-1} \Leftrightarrow \forall x \in A : \forall y \in B : (y = f(x) \Leftrightarrow x = g(y))}$$

Proof

($\Rightarrow$): Assume that $g = f^{-1}$. Let $x \in A$ and $y \in B$ be given. We will show that $y = f(x) \Leftrightarrow x = g(y)$.

- To show $y = f(x) \Rightarrow x = g(y)$:

Assume that $y = f(x)$. Then

$$\begin{array}{ll} x = id[A](x) = & [\text{Definition of } id] \\ \quad = (g \circ f)(x) = & [g \text{ left inverse of } f] \\ \quad = g(f(x)) = & [\text{Definition}] \\ \quad = g(y) & [\text{Hypothesis } y = f(x)] \end{array}$$

- To show $x = g(y) \Rightarrow y = f(x)$:

Assume that $x = g(y)$. Then

$$\begin{array}{ll} y = id[B](y) = & [\text{Definition of } id] \\ \quad = (f \circ g)(y) = & [g \text{ right inverse of } f] \\ \quad = f(g(y)) = & [\text{Definition}] \\ \quad = f(x). & [\text{Hypothesis } x = g(y)] \end{array}$$

It follows that $\forall x \in A : \forall y \in B : (y = f(x) \Leftrightarrow x = g(y))$.

($\Leftarrow$): Assume that $\forall x \in A : \forall y \in B : (y = f(x) \Leftrightarrow x = g(y))$

We will show that $f \circ g = id[B]$ and $g \circ f = id[A]$.

Let $x \in A$ be given. Define $y = f(x)$. Then, by hypothesis, we have $x = g(y)$, and

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && [\text{Definition}] \\ &= g(y) && [\text{Definition } y = f(x)] \\ &= x && [\text{because } x = g(y)] \end{aligned}$$

It follows that $\forall x \in A : (g \circ f)(x) = x$ (1)

Let $y \in B$ be given. Define $x = g(y)$. By hypothesis, it follows that $y = f(x)$ and

$$\begin{aligned} (f \circ g)(y) &= f(g(y)) && [\text{Definition}] \\ &= f(x) && [\text{because } x = g(y)] \\ &= y && [\text{because } y = f(x)] \end{aligned}$$

It follows that $\forall y \in B : (f \circ g)(y) = x$ (2)

From (1) and (2)

$$\begin{cases} g \circ f = id[A] \\ f \circ g = id[B] \end{cases} \Rightarrow \begin{cases} g \text{ left inverse of } f \\ g \text{ right inverse of } f \end{cases} \Rightarrow$$

$$\Rightarrow g = f^{-1}. \qquad \square$$

EXAMPLES

a) Let $f: A \to B$ be a mapping. Show that if:

$$\left\{\begin{array}{l} f \text{ odd} \\ g \text{ inverse of } f \end{array}\right. \Rightarrow g \text{ odd.}$$

Solution

Assume that f odd and $g: B \to A$ is an inverse of f.
It is sufficient to show that $\forall y \in B : (-y \in B \wedge g(-y) = -g(y))$.
Let $\underline{y \in B}$ be given.

- Proof that $-y \in B$.

We first note that:

f has an inverse $\Rightarrow f$ bijection $\Rightarrow f$ onto $\Rightarrow f(A) = B$.

$$\begin{aligned} \text{Since } \quad y \in B &\Rightarrow y \in f(A) && [\text{because } f(A) = B] \\ &\Rightarrow \exists x \in A : f(x) = y && [\text{definition of } f(A)] \end{aligned}$$

We note that $x \in A \wedge f \text{ odd} \Rightarrow -x \in A$.

We may therefore evaluate:

$$\begin{aligned} f(-x) &= -f(x) && [f \text{ odd}] \\ &= -y \Rightarrow && [\text{because } f(x) = y] \\ \Rightarrow \exists x' \in A : f(x') = -y &\Rightarrow -y \in f(A) && [\text{Definition}] \\ &\Rightarrow \underline{-y \in B}. && [\text{because } f(A) = B] \end{aligned}$$

We also note that

f bijection $\Rightarrow f$ one-to-one.

and

$$\begin{aligned} f(g(-y)) &= (f \circ g)(-y) = && [\text{Definition}] \\ &= -y = && [g \text{ right inverse on } f] \end{aligned}$$

$= -(f \circ g)(y) =$ [g right inverse of f]

$= -f(g(y)) =$ [definition]

$= f(-g(y))$ [f odd]

and therefore:

$$\left.\begin{cases} f \text{ one-to-one} \\ f(g(-y)) = f(-g(y)) \end{cases}\right\} \Rightarrow \underline{g(-y) = -g(y).}$$

We have thus shown that

$\forall y \in B: (-y \in B \land g(-y) = -g(y))$

$\Rightarrow g$ odd.

b) Let $f: A \to B$ be a bijection with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$.

Show that: $f \nearrow A \Rightarrow f^{-1} \nearrow B$.

<u>Solution</u>

Assume that $f \nearrow A$. Let $\underline{y_1, y_2 \in B}$ be given with $\underline{y_1 < y_2.}$

To derive a contradiction, assume that $f^{-1}(y_1) \geq f^{-1}(y_2)$. Then:

$f^{-1}(y_1) \geq f^{-1}(y_2) \Rightarrow f(f^{-1}(y_1)) \geq f(f^{-1}(y_2))$ [$f \nearrow A$]

$\Rightarrow (f \circ f^{-1})(y_1) \geq (f \circ f^{-1})(y_2))$ [definition]

$\Rightarrow y_1 \geq y_2$ (1) [f^{-1} right inverse]

Eq. (1) contradicts the hypothesis $y_1 < y_2$. It follows that $\underline{f^{-1}(y_1) < f^{-1}(y_2)},$ and therefore

$\forall y_1, y_2 \in B: (y_1 < y_2 \Rightarrow f^{-1}(y_1) < f^{-1}(y_2))$

$\Rightarrow f^{-1} \nearrow B.$

EXERCISES

㉔ Study the preceding proofs on inverse mappings, and learn how to reproduce them, for the following statements:

a) f has a left inverse $\Leftrightarrow$ f one-to-one

b) f has a right inverse $\Leftrightarrow$ f onto

c) $\left.\begin{cases} g_1 \text{ left inverse of } f \\ g_2 \text{ right inverse of } f \end{cases}\right. \Rightarrow g_1 = g_2$

d) f has an inverse $\Leftrightarrow$ f bijection

e) $\left.\begin{cases} f \text{ bijection} \\ g_1 \text{ inverse of } f \\ g_2 \text{ inverse of } f \end{cases}\right. \Rightarrow g_1 = g_2$

f) $g = f^{-1} \Leftrightarrow \forall x \in A : \forall y \in B : (y = f(x) \Leftrightarrow g(y) = x)$

㉕ Let $f : A \to B$ be a bijection with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. Show that $f \searrow A \Rightarrow f^{-1} \searrow B$.

㉖ Let $f : B \to C$ and $g : A \to B$ be bijections. Show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

㉗ Let $f : B \to C$ and $g : A \to B$ be two mappings. Show that for $S \subseteq A$, we have $(f \circ g)(S) = f(g(S))$

↳ This statement was used in the proof that the inverse mapping is unique. Prove it!

㉘ Let $f : A \to B$ be a mapping. Show that
$S \subseteq T \subseteq A \Longrightarrow f(S) \subseteq f(T)$.

㉙ Let $f : B \to C$ and $g : A \to B$ be two mappings. Show that

$$\left\{ \begin{array}{l} f \circ g \text{ onto} \\ g \text{ not onto} \end{array} \right. \Longrightarrow f \text{ not one-to-one.}$$

(Hint: Exercise 28 can help shorten the proof for this very challenging problem).

▼ Cardinality

- Given two finite sets A, B, if there is a bijection $f: A \to B$ then A and B have to have the same number of elements. Cantor proposed extending his observation to infinite sets according to the following definitions:

Def: Let A, B be two sets. We say that

$$A \sim B \Leftrightarrow \exists f \in \mathrm{Map}(A, B) : f \text{ bijection}$$

- The statement $A \sim B$ reads "A, B are equipotent", or "A and B have the same cardinality".
- Recall the definition

  $[n] = \{x \in \mathbb{N}^* \mid x \leq n\} = \{1, 2, 3, \ldots, n\}$

  Based on that, we introduce the following characterizations:

$$
\begin{aligned}
&A \text{ finite set} \Leftrightarrow A = \emptyset \vee (\exists n \in \mathbb{N}^* : A \sim [n]) \\
&A \text{ infinite set} \Leftrightarrow A \text{ not finite set} \\
&\qquad\qquad\qquad \Leftrightarrow A \neq \emptyset \wedge (\forall n \in \mathbb{N}^* : \overline{A \sim [n]}) \\
&A \text{ countable set} \Leftrightarrow \exists B \in \mathcal{P}(\mathbb{N}) : A \sim B \\
&A \text{ countably infinite} \Leftrightarrow A \sim \mathbb{N} \\
&A \text{ uncountable} \Leftrightarrow A \text{ not countable}
\end{aligned}
$$

- A relative comparison of sets in terms of cardinality is:

$$\text{finite} \leq \underbrace{\text{countable} \leq \text{countably infinite} < \text{uncountable}}_{\text{infinite}}$$

It should be stressed that since $\emptyset, \mathbb{N} \in \mathcal{P}(\mathbb{N})$ and
$\forall n \in \mathbb{N}^* : [n] \in \mathcal{P}(\mathbb{N})$
it follows that

$$A \text{ finite} \Rightarrow A \text{ countable}$$
$$A \text{ countably infinite} \Rightarrow A \text{ countable}$$

However, the converse statements do not hold.

▸ Interpretation: A countably infinite set contains an infinite number of elements, however the existence of some bijection $f : A \to \mathbb{N}$ allows us to enumerate each element of A by assigning it to a unique natural number from $\mathbb{N}$.

▸ $\mathbb{Z}$ and $\mathbb{Q}$ are countable

Recall that

$$\mathbb{Z} = \mathbb{N} \cup \{-x \mid x \in \mathbb{N}^*\} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$$
$$\mathbb{Q} = \{(a/b) \mid a, b \in \mathbb{Z} \wedge b \neq 0\}$$

with $\mathbb{Z}$ the set of integers and $\mathbb{Q}$ the set of rational numbers. The remarkable insight of Cantor is that even though $\mathbb{Z}$ and $\mathbb{Q}$ contain "more numbers" than $\mathbb{N}$, in the sense that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$, from the standpoint of cardinality, we can show that $\mathbb{Z} \sim \mathbb{N}$ and $\mathbb{Q} \sim \mathbb{N}$. Equivalently, we can show that

$$\begin{cases} \mathbb{Z} \text{ countably infinite} \\ \mathbb{Q} \text{ countably infinite} \end{cases}$$

▸ $\mathbb{R}$ is uncountable

With some additional theory we can show that the set $\mathbb{R}$ of all real numbers satisfies the following statements:

$$\begin{cases} \mathbb{R} \text{ is uncountable} \\ \mathbb{R} \sim \mathcal{P}(\mathbb{N}) \end{cases}$$

↱ Proof of $\mathbb{Z} \sim \mathbb{N}$ ($\mathbb{Z}$ is countably infinite)

We define the mapping $f : \mathbb{Z} \to \mathbb{N}$ such that

$$\forall x \in \mathbb{Z} : f(x) = \begin{cases} 2x-1 & , \text{if } x > 0 \\ -2x & , \text{if } x \leq 0 \end{cases}$$

and note that

$$f = \{(0,0), (1,1), (-1,2), (2,3), (-2,4), (3,5), (-3,6), \dots\}$$

which indicates that f is a bijection. To prove that, we show that f is one-to-one and that f is onto.

- <u>one-to-one</u> : Sufficient to show that

$$\forall x_1, x_2 \in \mathbb{Z} : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$$

Let <u>$x_1, x_2 \in \mathbb{Z}$</u> be given and assume that <u>$f(x_1) = f(x_2)$</u>. We distinguish between the following cases.

<u>Case 1</u> : Assume that $f(x_1) = -2x_1$ and $f(x_2) = -2x_2$. Then,

$$f(x_1) = f(x_2) \Rightarrow -2x_1 = -2x_2 \Rightarrow x_1 = x_2.$$

<u>Case 2</u>: Assume that $f(x_1) = 2x_1 - 1$ and $f(x_2) = 2x_2 - 1$. Then

$$f(x_1) = f(x_2) \Rightarrow 2x_1 - 1 = 2x_2 - 1 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$$

<u>Case 3</u> : Assume that $f(x_1) = 2x_1 - 1$ and $f(x_2) = -2x_2$. Then

$$f(x_1) = f(x_2) \Rightarrow 2x_1 - 1 = -2x_2 \Rightarrow 2x_1 + 2x_2 = 1 \Rightarrow$$
$$\Rightarrow 2(x_1 + x_2) = 1 \Rightarrow x_1 + x_2 = 1/2$$

This is a contradiction, because

$$x_1, x_2 \in \mathbb{Z} \Rightarrow x_1 + x_2 \in \mathbb{Z} \Rightarrow x_1 + x_2 \neq 1/2$$

therefore case 3 does not materialize.

From the above cases we conclude that <u>$x_1 = x_2$</u> and therefore:

$$\forall x_1, x_2 \in \mathbb{Z} : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2) \qquad (1)$$

- <u>Onto:</u> Sufficient to show that $\forall y \in \mathbb{N} : \exists x \in \mathbb{Z} : f(x) = y$.

Let $\underline{y \in \mathbb{N}}$ be given. From the division theorem we have:

$\exists k \in \mathbb{N} : (y = 2k \vee y = 2k+1)$

Choose a $k \in \mathbb{N}$ such that $y = 2k \vee y = 2k+1$ and distinguish between the following cases.

<u>Case 1</u>: Assume that $y = 2k$. Then:

$$k \in \mathbb{N} \Rightarrow k \geqslant 0 \Rightarrow -k \leqslant 0 \Rightarrow f(-k) = -2(-k) = 2k = y \Rightarrow$$
$$\Rightarrow \underline{\exists x \in \mathbb{Z} : f(x) = y} \quad (\text{for } x = -k)$$

<u>Case 2</u>: Assume that $y = 2k+1$. Then:

$$k \in \mathbb{N} \Rightarrow k \geqslant 0 \Rightarrow k+1 > 0 \Rightarrow$$
$$\Rightarrow f(k+1) = 2(k+1) - 1 = 2k+2-1 = 2k+1 = y \Rightarrow$$
$$\Rightarrow \underline{\exists x \in \mathbb{Z} : f(x) = y.} \quad (\text{for } x = k+1)$$

From the above argument, in all cases, we find that

$$(\forall y \in \mathbb{N} : \exists x \in \mathbb{Z} : f(x) = y) \Rightarrow \forall y \in \mathbb{N} : y \in f(\mathbb{Z}) \Rightarrow$$
$$\Rightarrow \mathbb{N} \subseteq f(\mathbb{Z}) \Rightarrow$$
$$\Rightarrow f(\mathbb{Z}) = \mathbb{N} \cancel{\Rightarrow} \qquad (2)$$

From Eq. (1) and Eq. (2).

$$\begin{cases} \forall x_1, x_2 \in \mathbb{Z} : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2) \\ f(\mathbb{Z}) = \mathbb{N} \end{cases} \Rightarrow$$
$$\Rightarrow \begin{cases} f \text{ one-to-one} \\ f \text{ onto} \end{cases} \Rightarrow f : \mathbb{Z} \to \mathbb{N} \text{ bijection}$$
$$\Rightarrow \mathbb{Z} \sim \mathbb{N} \Rightarrow \mathbb{Z} \text{ countably infinite.}$$

Sketch of proof that $\mathbb{Q} \sim \mathbb{N}$

A bijection $f: \mathbb{Q} \to \mathbb{N}$ can be constructed via the process of diagonalization, originally proposed by Cantor. We will explain this process and the overall argument informally, for the sake of clarity. We sequence the rational numbers using the diagonalizing pattern shown in the table below, making sure to skip any numbers previously encounterd in an equivalent fractional representation:

	0	1	2	3	4	...
1	0/1 →	1/1	2/1	3/1	4/1	...
2	0/2	1/2	2/2	3/2	4/2	...
3	0/3	1/3	2/3	3/3	4/3	...
4	0/4	1/4	2/4	3/4	4/4	...
5	0/5	...	...	...	...	
⋮	⋮					

Consequently, we sequence the rational numbers of $\mathbb{Q}$ as follows:

$\underline{0/1}$, $\underline{1/1}$, $0/2$, $\underline{2/1}$, $\underline{1/2}$, $0/3$, $\underline{3/1}$, $2/2$, $\underline{1/3}$, $0/4$, $\underline{4/1}$, $\underline{3/2}$, $\underline{2/3}$, $\underline{1/4}$, $0/5$, etc.

where we have underlined all rational numbers that appear for the first time and thus are not being skipped. We can thus define a bijection $f: \mathbb{Z} \to \mathbb{Q}$

with the initial assignments:

$f(0) = 0/1 = 0$ $\quad f(4) = 3/1$ $\quad f(8) = 2/3$

$f(1) = 1/1 = 1$ $\quad f(5) = 1/3$ $\quad f(9) = 1/4$

$f(2) = 2/1 = 2$ $\quad f(6) = 4/1$

$f(3) = 1/2$ $\quad f(7) = 3/2$ $\quad$ etc.

The algorithm for generating this bijection is as follows:

```
for $a = 0, 1, 2, 3, 4, \dots$
    for $b = 0, 1, 2, \dots, a$
        if it has not occured previously then add
        the number $(a-b)/(b+1)$ to the sequence.
    end for
end for.
```

To account for negative rational numbers, we extend the definition by the algorithm above as follows:

$\forall x \in \mathbb{N}^* : f(-x) = -f(x)$

and that completes the bijection $f: \mathbb{Z} \to \mathbb{Q}$. Skipping numbers that occured previously ensures that f is one-to-one. It is also clear that any rational number will be reached by this algorithm with a finite numbers of steps, which ensures that f is onto. Thus, it follows that

$$
\begin{aligned}
f: \mathbb{Z} \to \mathbb{Q} \text{ bijection} &\Rightarrow \mathbb{Q} \sim \mathbb{Z} \quad [\text{definition}] \\
&\Rightarrow \mathbb{Q} \sim \mathbb{N} \quad [\text{via } \mathbb{Z} \sim \mathbb{N}] \\
&\Rightarrow \mathbb{Q} \text{ countable} \qquad \square
\end{aligned}
$$

EXAMPLE – APPLICATION

↱ The following problem is also a necessary first step towards proving that $\mathbb{R}$ is uncountable.

Show that $\boxed{\mathbb{R} \sim (0,1)}$

Solution

Define $\forall x \in \mathbb{R}: f(x) = (1/2) + (1/\pi)\,\text{Arctan}(x)$.

We will show that $f: \mathbb{R} \to (0,1)$ is a bijection.

- Onto : Sufficient to show $\begin{cases} \forall y \in f(\mathbb{R}): y \in (0,1) \\ \forall y \in (0,1): y \in f(\mathbb{R}). \end{cases}$

($\Rightarrow$): Let $y \in f(\mathbb{R})$ be given. Then

$y \in f(\mathbb{R}) \Rightarrow \exists x \in \mathbb{R}: f(x) = y$.

Choose $x_0 \in \mathbb{R}$ such that $f(x_0) = y$. Then,

$$-\pi/2 < \text{Arctan}(x_0) < \pi/2 \Rightarrow$$
$$\Rightarrow -1/2 < (1/\pi)\,\text{Arctan}(x_0) < 1/2 \Rightarrow$$
$$\Rightarrow 0 < (1/2) + (1/\pi)\,\text{Arctan}(x_0) < 1 \Rightarrow$$
$$\Rightarrow 0 < f(x_0) < 1 \Rightarrow 0 < y < 1 \Rightarrow \underline{y \in (0,1)}$$

It follows that $\forall y \in f(\mathbb{R}): y \in (0,1)$. (1)

($\Leftarrow$): Let $y \in (0,1)$ be given. Then, we note that

$$f(x) = y \Leftrightarrow (1/2) + (1/\pi)\,\text{Arctan}(x) = y \Leftrightarrow$$
$$\Leftrightarrow (1/\pi)\,\text{Arctan}(x) = y - 1/2$$
$$\Leftrightarrow \text{Arctan}(x) = \pi(y - 1/2) \qquad (2)$$

and also that

$$y \in (0,1) \Rightarrow 0 < y < 1 \Rightarrow -1/2 < y - 1/2 < 1/2 \Rightarrow$$
$$\Rightarrow -\pi/2 < \pi(y - 1/2) < \pi/2 \Rightarrow \tan \text{ is defined at } \pi(y-1/2).$$

Now we can define $x_0 = \tan(\pi(y-1/2))$ and conclude that

$$\mathrm{Arctan}(x_0) = \mathrm{Arctan}(\tan(\pi(y-1/2))) = \pi(y-1/2) \overset{(2)}{\Longrightarrow}$$

$$\Rightarrow f(x_0) = y \Rightarrow \exists\, x \in \mathbb{R} : f(x) = y \Rightarrow$$

$$\Rightarrow \underline{y \in f(\mathbb{R})}$$

and therefore,

$$\forall y \in (0,1) : y \in f(\mathbb{R}) \qquad (3)$$

From Eq.(2) and Eq.(3):

$$\begin{cases} \forall y \in f(\mathbb{R}) : y \in (0,1) \\ \forall y \in (0,1) : y \in f(\mathbb{R}) \end{cases} \Rightarrow \begin{cases} f(\mathbb{R}) \subseteq (0,1) \\ (0,1) \subseteq f(\mathbb{R}) \end{cases} \Rightarrow f(\mathbb{R}) = (0,1)$$

$$\Rightarrow f \text{ onto.} \qquad (4)$$

- <u>One-to-one</u>

Let $\underline{x_1, x_2 \in \mathbb{R}}$ be given and assume that $\underline{f(x_1) = f(x_2)}$. Then,

$$f(x_1) = f(x_2) \Rightarrow (1/2) + (1/\pi)\mathrm{Arctan}(x_1) = (1/2) + (1/\pi)\mathrm{Arctan}(x_2) \Rightarrow$$

$$\Rightarrow (1/\pi)\mathrm{Arctan}(x_1) = (1/\pi)\mathrm{Arctan}(x_2) \Rightarrow$$

$$\Rightarrow \mathrm{Arctan}(x_1) = \mathrm{Arctan}(x_2) \Rightarrow$$

$$\Rightarrow \tan(\mathrm{Arctan}(x_1)) = \tan(\mathrm{Arctan}(x_2))$$

$$\Rightarrow \underline{x_1 = x_2}$$

and therefore, we have

$$\forall x_1, x_2 \in \mathbb{R} : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$$

$$\Rightarrow f \text{ one-to-one} \qquad (5)$$

From Eq.(4) and Eq.(5):

$$\begin{cases} f \text{ onto} \\ f \text{ one-to-one} \end{cases} \Rightarrow f : \mathbb{R} \to (0,1) \text{ bijection} \Rightarrow \mathbb{R} \sim (0,1).$$

EXERCISES

⑧ Learn the proofs for the following statements

a) $\mathbb{Z}$ is countable

b) $\mathbb{Q}$ is countable

c) $\mathbb{R} \sim (0,1)$

⑨ Let A, B be two sets. Show that

A countable $\wedge$ B countable $\Rightarrow A \cup B$ countable.

⑩ Let A_a with $a \in \mathbb{N}$ be a set collection. Show that:

a) $(\forall a \in \mathbb{N} : A_a \text{ finite}) \Rightarrow \bigcup_{a \in \mathbb{N}} A_a$ countable

b) Use part (a) to show that

$(\forall a \in \mathbb{N} : A_a \sim \mathbb{N}) \Rightarrow \bigcup_{a \in \mathbb{N}} A_a \sim \mathbb{N}$

⑪ Given 3 sets A, B, C show that the set equivalence satisfies the reflexive, symmetric, and transitive properties.

a) $A \sim A$

b) $A \sim B \rightarrow B \sim A$

c) $A \sim B \wedge B \sim C \Rightarrow A \sim C$

⑫ Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$ and consider the intervals

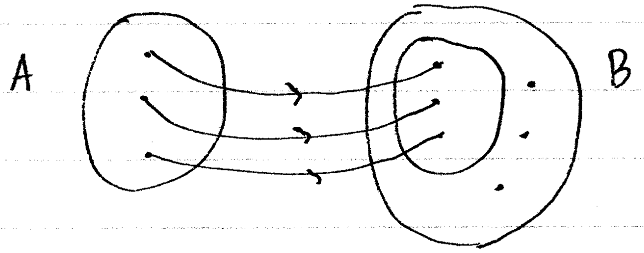
$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

$[c,d] = \{x \in \mathbb{R} \mid c \leq x \leq d\}$

Construct a bijection to show that $[a,b] \sim [c,d]$.

▼ Cardinality inequalities

If we can define a mapping $f: A \to B$ which is one-to-one but not necessarily onto, then from an intuitive standpoint the only conclusion that can be drawn is that either A, B are of "equal cardinality" or "B has greater cardinality than A", as illustrated by the following figure:



$$\begin{cases} f \text{ one-to-one} \\ f \text{ not onto} \end{cases}$$

Consequently, we propose the following definitions:

$$\boxed{\begin{aligned} A \leqslant B &\iff \exists f \in \mathrm{Map}(A,B): f \text{ one-to-one} \\ A < B &\iff A \leqslant B \wedge A \not\sim B \end{aligned}}$$

Note that it is easy to show that:

$$A \sim B \wedge B \sim C \Rightarrow A \sim C$$
$$A \leqslant B \wedge B \leqslant C \Rightarrow A \leqslant C$$
$$A \subseteq B \Rightarrow A \leqslant B$$

which are left as homework problems. Starting from Cantor, the following two major theorems will be used to show that $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ uncountable.

① → Cantor's theorem

Thm : For any set A, $A < \mathcal{P}(A)$

Proof

Define $f_0 : A \to \mathcal{P}(A)$ such that $\forall x \in A : f_0(x) = \{x\}$. Then:

$$\forall x_1, x_2 \in A : (\{x_1\} = \{x_2\} \Rightarrow x_1 = x_2) \Rightarrow$$
$$\Rightarrow \forall x_1, x_2 \in A : (f_0(x_1) = f_0(x_2) \Rightarrow x_1 = x_2)$$
$$\Rightarrow f_0 \text{ one-to-one} \Rightarrow$$
$$\Rightarrow \exists f \in \mathrm{Map}(A, \mathcal{P}(A)) : f \text{ one-to-one} \quad (\text{for } f = f_0)$$
$$\Rightarrow A \preccurlyeq \mathcal{P}(A). \quad (1)$$

To show that $A \nsim \mathcal{P}(A)$, assume that $A \sim \mathcal{P}(A)$. Then

$A \sim \mathcal{P}(A) \Rightarrow \exists f \in \mathrm{Map}(A, \mathcal{P}(A)) : f$ bijection

Choose an $f \in \mathrm{Map}(A, \mathcal{P}(A))$ such that $f : A \to \mathcal{P}(A)$ is a bijection. We define a set of "bad elements"

$$B = \{x \in A \mid x \notin f(x)\} \subseteq A \Rightarrow B \in \mathcal{P}(A).$$

and note that

$$f \text{ bijection} \Rightarrow f \text{ onto} \Rightarrow f(A) = \mathcal{P}(A) \Rightarrow \mathcal{P}(A) \subseteq f(A)$$
$$\Rightarrow \forall y \in \mathcal{P}(A) : y \in f(A) \Rightarrow$$
$$\Rightarrow \forall y \in \mathcal{P}(A) : \exists x \in A : f(x) = y$$

Let $y = B$ and choose a $b \in A$ such that $f(B) = B$.

We distinguish between the following cases.

Case 1 : Assume that $b \in B$. Then

$$b \in B \Rightarrow b \in \{x \in A \mid x \notin f(x)\} \Rightarrow$$
$$\Rightarrow b \in A \wedge b \notin f(b) \Rightarrow b \notin f(b) \Rightarrow b \notin B$$

which is a contradiction, therefore case 1 does not materialize.

Case 2 : Assume that $b \notin B$. We also now, by definition, that $b \in A$, and therefore:

$$\begin{cases} b \in A \\ b \notin B \end{cases} \Rightarrow \begin{cases} b \in A \\ b \notin f(b) \end{cases} \Rightarrow b \in \{x \in A \mid x \notin f(x)\} \Rightarrow$$

$$\Rightarrow b \in B$$

which is also a contradiction.

Since none of the possible cases materialize, it follows that $A \not\sim \mathcal{P}(A)$. (2)

From Eq.(1) and Eq.(2):

$$\begin{cases} A \not\sim \mathcal{P}(A) \\ A \leqslant \mathcal{P}(A) \end{cases} \Rightarrow A < \mathcal{P}(A).$$

② → Schroeder - Bernstein theorem

<u>Thm</u> : Let A, B be two sets. Then
$$A \leq B \wedge B \leq A \Rightarrow A \sim B$$

<u>Proof</u>

Assume that $A \leq B$ and $B \leq A$. Then

$$\begin{cases} A \leq B \\ B \leq A \end{cases} \Rightarrow \begin{cases} \exists f \in Map(A,B) : f \text{ one-to-one} \\ \exists g \in Map(B,A) : g \text{ one-to-one} \end{cases} \quad (1)$$

Choose $f \in Map(A,B)$ and $g \in Map(B,A)$ such that f, g are one-to-one.

Define $C_0 = A - g(B)$ and distinguish between the following two cases.

<u>Case 1</u> : Assume that $C_0 = \emptyset$. By construction, we have
$g \in Map(B,A) \Rightarrow g(B) \subseteq A$.

We will show that $A \subseteq g(B)$. (2)

Let $x \in A$ be given. To show that $x \in g(B)$, assume that $x \notin g(B)$ in order to derive a contradiction. It follows that

$$\begin{cases} x \in A \\ x \notin g(B) \end{cases} \Rightarrow x \in A - g(B) \Rightarrow x \in C_0 \Rightarrow x \in \emptyset$$

which is a contradiction. We conclude that <u>$x \in g(B)$</u>

We have thus shown that

$$\forall x \in A : x \in g(B) \Rightarrow A \subseteq g(B) \quad (3)$$

From Eq.(1), Eq.(2), Eq.(3) we conclude that:

$$\begin{cases} A\subseteq g(B) \wedge g(B)\subseteq A \\ g \text{ one-to-one}\end{cases} \Rightarrow \begin{cases} g(B)=A \\ g \text{ one-to-one}\end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} g \text{ onto} \\ g \text{ one-to-one}\end{cases} \Rightarrow g: B\to A \text{ bijection}$$

$$\Rightarrow B\sim A \Rightarrow \underline{A\sim B}.$$

Case 2 : Assume that $C_0\neq\emptyset$. Then we define by recursion

$$\forall n\in\mathbb{N}: C_{n+1} = g(f(C_n)) = g(\{f(x)\mid x\in C_n\}) = \\ = \{g(f(x))\mid x\in C_n\}$$

We construct the needed bijection $h: A\to B$ by the following definition:

$$\forall x\in A: h(x) = \begin{cases} f(x) & , \text{ if } \exists n\in\mathbb{N}: x\in C_n \\ g^{-1}(x) & , \text{ if } \forall n\in\mathbb{N}: x\notin C_n\end{cases}$$

Since we do not know if g is a bijection, we need to prore that $A-\bigcup_{n\in\mathbb{N}} C_n \subseteq g(B)$ to ensure that $g^{-1}(x)$ has a unique evaluation.

To show the claim, let $x\in A-\bigcup_{n\in\mathbb{N}} C_n$ be given. Then:

$$x\in A-\bigcup_{n\in\mathbb{N}} C_n \Rightarrow x\in A \wedge x\notin \bigcup_{n\in\mathbb{N}} C_n \Rightarrow x\notin \bigcup_{n\in\mathbb{N}} C_n \Rightarrow$$

$$\Rightarrow \overline{\exists n\in\mathbb{N}: x\in C_n} \Rightarrow$$

$$\Rightarrow \forall n\in\mathbb{N}: x\notin C_n \Rightarrow x\notin C_0.$$

To show that $x\in g(B)$, assume that $x\notin g(B)$. Then

$$\begin{cases} x\in A \\ x\notin g(B)\end{cases} \Rightarrow x\in A-g(B) \Rightarrow x\in C_0$$

which is a contradiction, since we previously showed that $x\notin C_0$.

We conclude that

$$\forall x \in A - \bigcup_{n \in \mathbb{N}} C_n : x \in g(B) \Rightarrow A - \bigcup_{n \in \mathbb{N}} C_n \subseteq g(B)$$

which proves the claim.

- We will show that <u>h is one-to-one.</u>

Let <u>$x_1, x_2 \in A$</u> be given and assume that <u>$h(x_1) = h(x_2)$.</u>
We distinguish between the following subcases.

<u>Case A</u>: Assume that $\begin{cases} \exists n \in \mathbb{N} : x_1 \in C_n \\ \exists n \in \mathbb{N} : x_2 \in C_n \end{cases}$

Then $h(x_1) = h(x_2) \Rightarrow f(x_1) = f(x_2)$ [definition of h]
$\Rightarrow \underline{x_1 = x_2}$ [f one-to-one]

<u>Case B</u>: Assume that $\begin{cases} \forall n \in \mathbb{N} : x_1 \notin C_n \\ \forall n \in \mathbb{N} : x_2 \notin C_n \end{cases}$. Then,

$$\begin{aligned} h(x_1) = h(x_2) &\Rightarrow g^{-1}(x_1) = g^{-2}(x_2) \Rightarrow \text{[definition of h]} \\ &\Rightarrow g(g^{-1}(x_1)) = g(g^{-1}(x_2)) \Rightarrow \\ &\Rightarrow \underline{x_1 = x_2}. \end{aligned}$$

<u>Case C</u>: Assume that $\begin{cases} \exists n \in \mathbb{N} : x_1 \in C_n \\ \forall n \in \mathbb{N} : x_2 \notin C_n \end{cases}$

Choose $n_0 \in \mathbb{N}$ such that $x_1 \in C_{n_0}$. We note that

$$\begin{cases} x_2 \in A \\ \forall n \in \mathbb{N} : x_2 \notin C_n \end{cases} \Rightarrow x_2 \in A - \bigcup_{n \in \mathbb{N}} C_n \Rightarrow g^{-1}(x_2) \text{ is defined}$$

and therefore:

$$\begin{aligned} x_2 &= g(g^{-1}(x_2)) \\ &= g(h(x_2)) && \text{[Definition of } h(x) \text{ - 2nd case]} \\ &= g(h(x_1)) && \text{[Hypothesis } h(x_1) = h(x_2)\text{]} \\ &= g(f(x_1)) && \text{[Definition of } h(x) \text{ - 1st case]} \end{aligned}$$

$\Rightarrow \exists x \in C_{n_0} : g(f(x)) = x_2 \Rightarrow$

$\Rightarrow x_2 \in \{g(f(x)) \mid x \in C_{n_0}\}$

$\Rightarrow x_2 \in g(f(C_{n_0}))$

$\Rightarrow x_2 \in C_{n_0+1}$

This is a contradiction because

$(\forall n \in \mathbb{N} : x_2 \notin C_n) \Rightarrow x_2 \notin C_{n_0+1}$

therefore Case C does not materialize. In all of the above cases we conclude that $x_1 = x_2$ and therefore:

$\forall x_1, x_2 \in A : (h(x_1) = h(x_2) \Rightarrow x_1 = x_2)$

$\Rightarrow h$ one-to-one. (4)

•2 We will show that $h(A) = B$.

By definition, we have $h(A) \subseteq B$, so it is sufficient to show that $\forall y \in B : y \in h(A)$. Let $\underline{y \in B}$ be given. We distinguish between the following cases.

Case 1 : Assume that $\exists n \in \mathbb{N} : y \in f(C_n)$.

Choose $n_0 \in \mathbb{N}$ such that $y \in f(C_{n_0})$. Since

$h(C_{n_0}) = \{h(x) \mid x \in C_{n_0}\}$

$= \{f(x) \mid x \in C_{n_0}\}$ [Definition of $h(x)$ – 1st case]

$= f(C_{n_0})$

it follows that

$y \in f(C_{n_0}) \Rightarrow y \in h(C_{n_0})$ [Because $h(C_{n_0}) = f(C_{n_0})$]

$\Rightarrow \underline{y \in h(A)}$ [because $C_{n_0} \subseteq A$]

Case 2 : Assume that $\forall n \in \mathbb{N} : y \notin f(C_n)$.

We claim that $\forall n \in \mathbb{N} : g(y) \notin C_n$.

To show the claim, we note that:

$$\begin{aligned}\forall n\in\mathbb{N}: y\notin f(C_n) &\Rightarrow \forall n\in\mathbb{N}: g(y)\notin g(f(C_n))\\ &\Rightarrow \forall n\in\mathbb{N}: g(y)\notin C_{n+1}\\ &\Rightarrow \forall n\in\mathbb{N}^*: g(y)\notin C_n \quad (5)\end{aligned}$$

For $n=0$, to show that $g(y)\notin C_0$, we will assume that $g(y)\in C_0$ and derive a contradiction. Then:

$$\begin{aligned}g(y)\in C_0 &\Rightarrow g(y)\in A-g(B)\\ &\Rightarrow g(y)\in A \wedge g(y)\notin g(B)\\ &\Rightarrow g(y)\notin g(B)\end{aligned}$$

which is a contradiction because

$y\in B \Rightarrow g(y)\in g(B)$

It follows that $g(y)\notin C_0$ (6)

From Eq.(5) and Eq.(6) we prove the claim. It follows that

$$\begin{aligned}h(g(y)) &= g^{-1}(g(y)) \quad [\text{because } \forall n\in\mathbb{N}: g(y)\notin C_n]\\ &= y \Rightarrow\end{aligned}$$

$\Rightarrow \exists x\in A: y=h(x)$ (for $x=g(y)$)

$\Rightarrow \underline{y\in h(A)}$

From the above argument we have:

$$\begin{cases} h(A)\subseteq B\\ \forall y\in B: y\in h(A)\end{cases} \Rightarrow \begin{cases} h(A)\subseteq B\\ B\subseteq h(A)\end{cases} \Rightarrow h(A)=B \Rightarrow \underline{h \text{ onto}} \quad (7)$$

From Eq.(4) and Eq.(7):

$$\begin{cases} h \text{ one-to-one}\\ h \text{ onto}\end{cases} \Rightarrow h: A\to B \text{ bijection}$$

$$\Rightarrow A\sim B \qquad \square$$

③ → Uncountability of $\mathbb{R}$

The Schroeder-Bernstein theorem can be used to derive the following characterization for the cardinality of $\mathbb{R}$:

$$\boxed{\mathbb{R} \sim \mathcal{P}(\mathbb{N})}$$

Once this result is established, we can use Cantor's theorem to argue that:

$$\left\{\begin{array}{l} \mathbb{R} \sim \mathcal{P}(\mathbb{N}) \\ \mathcal{P}(\mathbb{N}) > \mathbb{N} \end{array}\right. \Rightarrow \mathbb{R} > \mathbb{N} \Rightarrow \mathbb{R} \text{ uncountable.}$$

The argument below uses the previous result that $\mathbb{R} \sim (0,1)$.

► Proof of $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$

It is sufficient to show that $\mathcal{P}(\mathbb{N}) \leqslant \mathbb{R} \wedge \mathbb{R} \leqslant \mathcal{P}(\mathbb{N})$.

•$_1$ Proof of $\mathcal{P}(\mathbb{N}) \leqslant \mathbb{R}$.

We define a mapping $f : \mathcal{P}(\mathbb{N}) \to [0,1]$ as follows. Given $X \in \mathcal{P}(\mathbb{N})$ we define $f(X)$ via the expansion

$$f(X) = (0.a_0 a_1 a_2 \cdots)_{10} =$$

$$= \sum_{n=0}^{+\infty} a_n 10^{-n-1}$$

with

$$\forall n \in \mathbb{N} : a_n = \begin{cases} 1 & , \text{if } n \in X \\ 0 & , \text{if } n \notin X \end{cases}$$

To show that f is one-to-one, it is necessary to define it using a base representation that is greater than binary (i.e. base 2) while restricting the digits used to 0 and 1. This way, a number that terminates with an infinite sequence of 1s (e.g. 0.101111...) will not have an second alternate representation, as it would have in the binary system. We may therefore now argue as follows: Let $X_1, X_2 \in \mathcal{P}(\mathbb{N})$ be given and assume that $f(X_1) = f(X_2)$. Define the sequences (a_n) and (b_n) via the decimal representations:

$$f(X_1) = 0.a_0 a_1 a_2 \dots = \sum_{n=0}^{+\infty} a_n \cdot 10^{-n-1}$$

$$f(X_2) = 0.b_0 b_1 b_2 \dots = \sum_{n=0}^{+\infty} b_n \cdot 10^{-n-1}$$

We note that

$$f(X_1) = f(X_2) \Rightarrow 0.a_0 a_1 a_2 \dots = 0.b_0 b_1 b_2 \dots \Rightarrow$$
$$\Rightarrow \forall n \in \mathbb{N} : a_n = b_n .$$

We use this result to show that

$$\begin{aligned} n \in X_1 &\Leftrightarrow a_n = 1 && [\text{definition of } a_n] \\ &\Leftrightarrow b_n = 1 && [\text{via } a_n = b_n] \\ &\Leftrightarrow n \in X_2 && [\text{definition of } b_n] \end{aligned}$$

It follows that $\underline{X_1 = X_2}$. We have thus shown that

$$\forall X_1, X_2 \in \mathcal{P}(\mathbb{N}) : (f(X_1) = f(X_2) \Rightarrow X_1 = X_2)$$
$$\Rightarrow f \text{ one-to-one} \Rightarrow \mathcal{P}(\mathbb{N}) \leq [0,1]$$

We also have: $[0,1] \subseteq \mathbb{R} \Rightarrow [0,1] \leq \mathbb{R}$

and therefore

$$\left\{\begin{array}{l} \mathcal{P}(\mathbb{N}) \leqslant [0,1] \\ [0,1] \leqslant \mathbb{R} \end{array}\right. \Rightarrow \underline{\mathcal{P}(\mathbb{N}) \leqslant \mathbb{R}.} \qquad (1)$$

• 2 Proof of $\mathbb{R} \leqslant \mathcal{P}(\mathbb{N})$.

We define a mapping $g : [0,1] \to \mathcal{P}(\mathbb{N})$ as follows.

Let $x \in [0,1]$ be given with binary representation

$$x = (0.a_0 a_1 a_2 \cdots)_2 = \sum_{n=0}^{+\infty} a_n 2^{-n-1}$$

To ensure uniqueness, we do not allow terminating the binary representation of x with an infinite sequence of 1s except for $x = 1$ (represented as $x = (0.1111\cdots)_2$)

Define $g(x) = \{n \in \mathbb{N} \mid a_n = 1\}$

Let $\underline{x_1, x_2 \in [0,1]}$ be given and assume that $\underline{g(x_1) = g(x_2)}$.

Define the sequences (a_n) and (b_n) via the unique binary representations (as explained above)

$$x_1 = (0.a_0 a_1 a_2 \cdots)_2$$
$$x_2 = (0.b_0 b_1 b_2 \cdots)_2$$

To show that $x_1 = x_2$, we assume that $x_1 \neq x_2$ and derive a contradiction. Then, we have

$$\begin{aligned} x_1 \neq x_2 &\Rightarrow (0.a_0 a_1 a_2 \cdots)_2 \neq (0.b_0 b_1 b_2 \cdots)_2 \\ &\Rightarrow \overline{\forall n \in \mathbb{N} : a_n = b_n} \\ &\Rightarrow \exists n \in \mathbb{N} : a_n \neq b_n \end{aligned}$$

Choose $n_0 \in \mathbb{N}$ such that $a_{n_0} \neq b_{n_0}$. It follows that

$$a_{n_0} \neq b_{n_0} \Rightarrow \left\{\begin{array}{l} a_{n_0} = 1 \\ b_{n_0} = 0 \end{array}\right. \veebar \left\{\begin{array}{l} a_{n_0} = 0 \\ b_{n_0} = 1 \end{array}\right. \Rightarrow$$

$$\Rightarrow \begin{cases} n_0 \in g(x_1) \\ n_0 \notin g(x_2) \end{cases} \veebar \begin{cases} n_0 \notin g(x_1) \\ n_0 \in g(x_2) \end{cases} \Rightarrow$$

$$\Rightarrow (\exists n \in g(x_1) : n \notin g(x_2)) \vee (\exists n \in g(x_2) : n \notin g(x_1))$$

$$\Rightarrow \overline{(\forall n \in g(x_1) : n \in g(x_2))} \veebar \overline{(\forall n \in g(x_2) : n \in g(x_1))}$$

$$\Rightarrow g(x_1) \notin g(x_2) \veebar g(x_2) \notin g(x_1)$$

which is a contradiction because

$$g(x_1) = g(x_2) \Rightarrow \begin{cases} g(x_1) \subseteq g(x_2) \\ g(x_2) \subseteq g(x_1) \end{cases}$$

We have thus shown that $\underline{x_1 = x_2}$

From the above argument we have show that

$$\forall x_1, x_2 \in [0,1] : (g(x_1) = g(x_2) \rightarrow x_1 = x_2)$$

$$\Rightarrow g \text{ one-to-one} \Rightarrow [0,1] \leqslant \mathcal{P}(\mathbb{N}).$$

and therefore:

$$\begin{aligned} \mathbb{R} &\sim (0,1) && [\text{previous result}] \\ &\leqslant [0,1] && [\text{via } (0,1) \subseteq [0,1]] \\ &\leqslant \mathcal{P}(\mathbb{N}) && [\text{above proof}] \end{aligned}$$

$$\Rightarrow \underline{\mathbb{R} \leqslant \mathcal{P}(\mathbb{N})} \qquad (2)$$

From Eq. (1) and Eq. (2) via the Schroeder-Bernstein theorem, it follows that

$$\begin{cases} \mathcal{P}(\mathbb{N}) \leqslant \mathbb{R} \\ \mathbb{R} \leqslant \mathcal{P}(\mathbb{N}) \end{cases} \Rightarrow \mathbb{R} \sim \mathcal{P}(\mathbb{N}). \qquad \square$$

EXERCISES

⑬ Study the proofs for

a) The Cantor theorem

b) The Schroder - Bernstein theorem

c) The statement $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$.

⑭ Use Exercise 9 and the previous results that $\mathbb{Q} \sim \mathbb{N}$ and $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$ to show that $\mathbb{R} - \mathbb{Q}$ (the set of irrational numbers) is uncountable.
(Hint: Use proof by contradiction)

⑮ Show that, given 3 sets A, B, C, we have:

a) $A \leq B \wedge B \leq C \Rightarrow A \leq C$

b) $(A \leq B \leq C \wedge A \sim C) \Rightarrow (B \sim C \wedge A \sim B)$

c) $A \sim B \wedge B \subseteq C \Rightarrow A \leq C$.

⑯ Consider the sets

$$\mathbb{R}_+^* = \{x \in \mathbb{R} \mid x > 0\}$$
$$\mathbb{R}_-^* = \{x \in \mathbb{R} \mid x < 0\}$$

Use the Schroder - Bernstein theorem to show that $\mathbb{R} \sim \mathbb{R}_+^*$ and $\mathbb{R} \sim \mathbb{R}_-^*$

(Hint: The needed one-to-one mappings can be constructed using the exponential function)

(Another hint: It is sufficient to show $\mathbb{R}_+^* \geq \mathbb{R}$ and $\mathbb{R}_-^* \geq \mathbb{R}$).

⑰ Use Exercise 16 to show that given two sets A, B we have:

$A \sim \mathbb{R} \wedge B \sim \mathbb{R} \Rightarrow A \cup B \sim \mathbb{R}$.

(Hint: Distinguish between the following cases. For case 1 assume that $A \cap B = \emptyset$. For case 2 assume that $A \cap B \neq \emptyset$. Define $B_1 = B - A$, show that $A \cup B = A \cup B_1$ and use Case 1 and the Schroeder–Bernstein theorem to show that $A \cup B_1 \sim \mathbb{R}$).

⑱ Use the Schoeder–Bernstein theorem to show that $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$.

(Hint: Use binary or decimal representations to show that $[0,1] \times [0,1] \sim [0,1]$ by defining one-to-one mappings $f : [0,1] \times [0,1] \to [0,1]$ and $g : [0,1] \to [0,1] \times [0,1]$. Then uplift this result to the statement $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$).

Cardinal numbers

- To introduce the concept of cardinality and cardinal numbers, we note first that

$$\forall n, m \in \mathbb{N}^* : \left(\begin{cases} A \sim [n] \\ A \sim [m] \end{cases} \Rightarrow n = m \right)$$

Thus, for finite sets A, we can define a unique integer $|A|$ such that $A \sim [|A|]$.

- $|A|$ is the number of elements in A and we call it the <u>cardinality</u> of A.
- Cantor proposed introducing "transfinite cardinal numbers" to denote the cardinality $|A|$ of infinite sets. A key requirement of this cardinal number arithmetic is that it should satisfy:

$$A \sim B \iff |A| = |B|$$
$$A < B \iff |A| \leq |B|$$
$$A \leq B \iff |A| < |B|$$

The Schroeder-Bernstein theorem ensures self-consistent behaviour of inequalities in cardinal arithmetic.

- Since $\mathbb{N} \sim \mathbb{Z} \sim \mathbb{Q}$, Cantor introduced the cardinal number $\aleph_0$ to represent the cardinality of countably infinite sets. Consequently, we may write

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \aleph_0$$

- <u>Aleph sequence</u>: Cantor proposed defining a sequence of cardinalities $\aleph_1, \aleph_2, \aleph_3, \ldots$ as follows.

Let V be the set of all sets that exist. We define:

$|A| = \aleph_1 \iff \forall B_1 \in V : \overline{\mathbb{N} < B_1 < A}$

$|A| = \aleph_2 \iff \forall B_1, B_2 \in V : \overline{\mathbb{N} < B_1 < B_2 < A}$

$|A| = \aleph_3 \iff \forall B_1, B_2, B_3 \in V : \overline{\mathbb{N} < B_1 < B_2 < B_3 < A}$

etc.

- Beth sequence : Another sequence of cardinal numbers is thee beth sequence. It is based on the Cantor theorem that tells us that $A < \mathcal{P}(A)$. The beth sequence is defined as follows:

  $\beth_0 = \aleph_0 = |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$

  $\beth_1 = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$

  $\beth_2 = |\mathcal{P}(\mathcal{P}(\mathbb{N}))|$

  $\beth_3 = |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))|$

  etc.

- Continuum hypothesis : With the above definitions, Cantor posed the question of whether the aleph and beth sequences coincide. This leads to two questions:

a) Continuum Hypothesis: The claim that $\beth_1 = \aleph_1$.

b) General Continuum Hypothesis: The claim that $\beth_a = \aleph_a$ for all a.

It was later found that these hypotheses are undecidable, i.e. it can neither be proved true or false. The underlying problem is that for the case of infinite sets, the mechanism for generating the powerset $\mathcal{P}(A)$ of an infinite set A is not precisely given. As a result, we have no way of deducing the correct "size" of $\mathcal{P}(\mathbb{N})$, $\mathcal{P}(\mathcal{P}(\mathbb{N}))$, etc.