Probability Theory.

Basic Concepts

- A random experiment is an experiment such that
  a) It can be repeated multiple times
  b) The outcome cannot be predicted.
- Unpredictability originates usually from insufficient knowledge of initial conditions.

- The set Ω of all possible outcomes is called the sample space of the experiment

→ We assume Ω is a finite set.
The more general probability theory on infinite sets was formalized by Kolmogorov.

- A subset $A \subseteq \Omega$ is called an event
- We say that the event $A \subseteq \Omega$ happened if the outcome of the experiment x satisfies $x \in A$.

Definition of probability

- We introduce a mapping
$$p : x \in \underline{\Omega} \longrightarrow p(x) \in [0,1]$$

such that $p(x)$ are the "chances" of the outcome being x.
We define p so that

$$\boxed{\sum_{x \in \underline{\Omega}} p(x) = 1}$$

- For each event $A \subseteq \underline{\Omega}$ we assign a probability $p(A)$ given by:

$$\boxed{p(A) = \sum_{x \in A} p(x)}$$

- The <u>odds in favor</u> $O_+(A)$ of an event $A \subseteq \underline{\Omega}$ are

$$\boxed{O_+(A) = \frac{p(A)}{p(\underline{\Omega} - A)}}$$

- The <u>odds against</u> the event $A \subseteq \Omega$ are defined as

$$\boxed{O_-(A) = \frac{p(\Omega - A)}{p(A)}}$$

Obviously : $\underline{\underline{O_+(A)\, O_-(A) = 1}}$

<u>example</u> : Rolling dice.

$\Omega = \{1,2,3,4,5,6\}$

A = odd outcome event

$= \{1,3,5\} \Rightarrow p(A) = p(1) + p(3) + p(5)$

B = even outcome event

$= \{2,4,6\} \Rightarrow p(A) = p(2) + p(4) + p(6)$

<u>Remarks</u>

1) By definition $p(\emptyset) = 0$

2) $p(\Omega) = \sum_{x \in \Omega} p(x) \underset{\uparrow}{=} 1$

by assumption

Events and set operations

Let $A, B \subseteq \Omega$ be two events.

Recall: The event A <u>happened</u> if and only if the outcome x of the random experiment satisfies $x \in A$.

1) The event $A \cap B$ happened $\Leftrightarrow$
$\Leftrightarrow$ A happened $\wedge$ B happened.

2) The event $A \cup B$ happened $\Leftrightarrow$
$\Leftrightarrow$ A happened $\vee$ B happened.

3) The event $\Omega - A$ happened $\Leftrightarrow$
$\Leftrightarrow$ A did NOT happen.

Probability and cardinality

- A sample space is <u>equiprobable</u> if all possible outcomes may occur with equal probability

$$\boxed{\Omega \text{ equiprobable} \Leftrightarrow \forall x \in \Omega : p(x) = 1/|\Omega|}$$

- If a sample space is equiprobable then the probability of an event $A \subseteq \Omega$ is given by

$$\boxed{p(A) = \frac{|A|}{|\Omega|}}$$

example : Probability of rolling an even number when 1 dice is rolled

$$\left.\begin{array}{ll} \Omega = \{1,2,3,4,5,6\} & \Rightarrow |\Omega| = 6 \\ A = \{2,4,6\} & \Rightarrow |A| = 3 \end{array}\right\} \Rightarrow p(A) = \frac{|A|}{|\Omega|} = \frac{3}{6} = \frac{1}{2}$$

Properties of probability.

1) $p(\emptyset) = 0$ 2) $p(\Omega) = 1$
3) $\forall A \subseteq \Omega : 0 \leq p(A) \leq 1$
4) $\forall A \subseteq \Omega : p(\Omega - A) = 1 - p(A)$
5) $A \cap B = \emptyset \Rightarrow p(A \cup B) = p(A) + p(B)$
6) $p(A \cup B) = p(A) + p(B) - p(A \cap B)$
7) $p(A - B) = p(A) - p(A \cap B)$
8) $A \subseteq B \Rightarrow p(A) \leq p(B)$

examples

① From a deck of 52 cards we pull 2 cards.

a) Probability to pull 2 aces?

b) Probability to pull at least one aces?

Solution

a) $$|\Omega| = C(52,2) = \frac{52!}{50!\,2!} = 1326$$

ways to choose 2 cards out of 52.

$$|A| = C(4,2) = \frac{4!}{2!\,2!} = 6$$ ways to choose 2 aces out of 2 aces.

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6}{1326} = \frac{1}{221}$$

b) Let A = at least one is an ace

B = both are aces

C = one is an ace and the other is one of the other 48 cards.

Then $B \cap C = \emptyset \Rightarrow p(A) = p(B \cup C) =$

$= p(B) + p(C)$.

Note that

$$\left.\begin{array}{l}|B| = C(4,2) \\ |C| = C(48,1)\,C(4,1)\end{array}\right\} \Rightarrow$$

$$\Rightarrow p(A) = p(B) + p(C) = \frac{|B|+|C|}{|\Omega|} =$$

$$= \frac{C(4,2) + C(48,1)\,C(4,1)}{C(52,2)}$$

$$= \dots = \frac{33}{221}$$

<u>example</u>: Roll a dice 3 times
Probability to get 3 consecutive numbers?

<u>Solution</u>

For $D = \{1,2,3,4,5,6\} \Rightarrow \Omega = D \times D \times D$

$\Rightarrow |\Omega| = |D|^3 = 6^3 = 216$

The possible consecutive numbers are
$(1,2,3)$ or $(2,3,4)$ or $(3,4,5)$ or $(4,5,6)$
Each can occur in $3!$ permutations. Thus

$$|A| = 4 \cdot 3! = 4 \cdot 6 = 24 \Rightarrow p(A) = \frac{|A|}{|\Omega|} = \frac{24}{216} = \frac{1}{9}$$

<u>example</u> : From a deck of 52 cards we pull 10 cards.
Probability of pulling at least one ace?

<u>Solution</u>

To choose 10 out of 52 cards we have
$|\underline{\Omega}| = C(52,10)$ combinations.

A: Pulling at least one ace

$\underline{\Omega} - A$ = Not pulling any ace
= Pull 10 out of 48 cards

$$\Rightarrow |\underline{\Omega} - A| = C(48,10)$$

$$\Rightarrow p(A) = 1 - p(\underline{\Omega} - A) = 1 - \frac{|\underline{\Omega} - A|}{|\underline{\Omega}|} =$$

$$= 1 - \frac{C(48,10)}{C(52,10)} =$$

$$= 1 - \frac{\frac{48!}{10!\,38!}}{\frac{52!}{10!\,42!}} = 1 - \frac{48!\,42!}{52!\,38!}$$

$$= 1 - \frac{48!\,38!(39\cdot 40\cdot 41\cdot 42)}{48!\,38!(49\cdot 50\cdot 51\cdot 52)} = 1 - \frac{246}{595} = \frac{349}{595}$$

__example__ : We draw ~~two~~ one card from standard deck.
Probability to draw a heart or 10?

__Solution__

$|\Omega| = 52 \longleftarrow$ number of cards we can choose.

A = choose a heart $\Rightarrow |A| = 13$

B = choose a 10 $\Rightarrow |B| = 4$

We want $\cancel{A \cap B}$ $A \cup B$. Note that $|A \cap B| = 1$.

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

$$= \frac{|A| + |B| - |A \cap B|}{|\Omega|} = \frac{13 + 4 - 1}{52}$$

$$= \frac{16}{52} = \frac{4}{13}.$$

Independent events : Two Events

Def : We say that two events $A, B \subseteq \Omega$ are **independent** iff

$$\boxed{P(A \cap B) = P(A)\,P(B)}$$

Interpretation : If A, B are independent it means that the realization of one event is not ~~realized by~~ influenced by the realization of the other event.

Properties

1) If $A, B \subseteq \Omega$ independent $\Rightarrow$ $A, \Omega - B$ independent.

Proof

Let $C_1 = A \cap B$ and $C_2 = A \cap (\Omega - B)$.

$$\begin{aligned} C_1 \cap C_2 &= (A \cap B) \cap (A \cap (\Omega - B)) = \\ &= (A \cap A) \cap (B \cap (\Omega - B)) \\ &= A \cap \emptyset = \emptyset \quad \text{and} \end{aligned}$$

$$\begin{aligned} C_1 \cup C_2 &= (A \cap B) \cup (A \cap (\Omega - B)) \\ &= A \cap [B \cup (\Omega - B)] = A \cap \Omega = A \end{aligned}$$

It follows that

$$p(A) = p(C_1) + p(C_2) = $$
$$= p(A \cap B) + p(C_2) =$$
$$= p(A)p(B) + p(C_2) \Rightarrow$$

$$\Rightarrow p(C_2) = p(A) - p(A)p(B)$$
$$= p(A)[1 - p(B)]$$
$$= p(A)\, p(\Omega - B) \Rightarrow$$

$$\Rightarrow p(A \cap (\Omega - B)) = p(A)\, p(\Omega - B) \Rightarrow$$

$\Rightarrow A, \Omega - B$ independent. $\square$

2) $\emptyset, A$ always independent

3) Ω, A always independent.

<u>Def:</u> We say that the events $A, B, C \subseteq \Omega$ are independent iff all conditions bellow are satisfied:

a) $p(A \cap B) = p(A)\, p(B)$

b) $p(B \cap C) = p(B)\, p(C)$

c) $p(C \cap A) = p(C)\, p(A)$

d) $p(A \cap B \cap C) = p(A)\, p(B)\, p(C)$.

example : A box has 5 black balls and 3 white balls. Randomly pick one ball, put it back in the box, and pick a second ball.
Probability to pick 2 white balls?

Solution

Let A = 1st ball white
B = 2nd ball white. We want $p(A \cap B)$.
Since A, B are independent
(BECAUSE : the first ball is returned to box before choosing second ball)
then

$$p(A \cap B) = p(A)\, p(B) \quad (1)$$

Since

$$\left.\begin{array}{l} p(A) = 3/8 \\ p(B) = 3/8 \end{array}\right\} \Rightarrow p(A \cap B) = \frac{3}{8} \frac{3}{8} = \frac{9}{64}. \quad \square$$

example: We roll a dice 2 times. What is probability to have at least one 6.

Solution

Let $A =$ get 6 on first throw
$B =$ get 6 on second throw.

We want $A \cup B$.

Note that A, B independent (the two throws do not affect each other) thus

$$p(A \cap B) = p(A)p(B) \Rightarrow$$

$$\Rightarrow p(A \cup B) = p(A) + p(B) - p(A \cap B)$$
$$= p(A) + p(B) - p(A)p(B). \quad (1)$$

Since

$$\left.\begin{array}{l} p(A) = 1/6 \\ p(B) = 1/6 \end{array}\right\} \Rightarrow p(A \cup B) = \frac{1}{6} + \frac{1}{6} - \frac{1}{6}\,\frac{1}{6} = \dots =$$
$$= \frac{11}{36} \quad \square$$

Conditional probability.

Def : We define $p(A|B)$ the probability that A will happen if B has already happened and it is given by

$$\boxed{p(A|B) = \frac{p(A\cap B)}{p(B)}}$$

Remarks

1) This relation can be rewritten as
$$p(A\cap B) = p(B)\, p(A|B).$$

2) Since $A\cap B \subseteq B \Rightarrow p(A\cap B) \leq p(B)$, $p(B) \geq 0$ $\Rightarrow$

$$\Rightarrow p(A|B) = \frac{p(A\cap B)}{p(B)} \leq 1$$ ← Required for self-consistent definition!

3) If A, B independent then
$$p(A|B) = p(A) \text{ and } p(B|A) = p(B).$$

<u>example</u> : A box contains 5 black balls and 3 white balls. Choose at random 2 balls <u>without</u> putting the 1st ball back in the box.
Probability of choosing 2 white balls.

<u>Solution</u>

Let A = 1st ball is white
B = 2nd ball is white
We want $p(A \cap B)$.
Note that A, B are not independent and

$$\left.\begin{array}{l} p(A) = 3/8 \\ p(B|A) = 2/7 \end{array}\right\} \Rightarrow p(A \cap B) = p(A)\,p(B|A) = \frac{3}{8}\,\frac{2}{7} = \frac{3}{28}$$

<u>example</u> : If A, B, C independent show that
$$p(A | B \cap C) = p(A)$$

<u>Proof</u>

By definition $p(A | B \cap C) = \dfrac{p(A \cap B \cap C)}{p(B \cap C)}$ (1)

A, B, C independent $\Rightarrow$

$$\Rightarrow \begin{cases} p(A\cap B\cap C) = p(A)\,p(B)\,p(C) \\ p(B\cap C) = p(B)\,p(C) \end{cases} \Rightarrow$$

$$\Rightarrow p(A \mid B\cap C) = \frac{p(A\cap B\cap C)}{p(B\cap C)} = \frac{p(A)\,p(B)\,p(C)}{p(B)\,p(C)} = p(A) \quad \square.$$

Mathematical Expectation (Ensemble Average)

Consider a sample space Ω with probability function $p : \Omega \to \mathbb{R}$ and another mapping f such that

$$f : x \in \Omega \longrightarrow f(x) \in \mathbb{R}.$$

- The ensemble average of f is written $\langle f \rangle$ and it is given by

$$\boxed{\langle f \rangle = \sum_{x \in \Omega} f(x) p(x)}$$

<u>example</u>: If $\Omega = \{a, b, c, d\}$ then $\langle f \rangle = f(a)p(a) + f(b)p(b) + f(c)p(c) + f(d)p(d)$.

<u>example</u>: You draw 3 cards out of a deck of 52 cards. You gain \$100 for each ace. How much \$ do you get on average?

We have $C(52,3)$ different hands.

A_0 = pull no ace
A_1 = pull 1 ace
A_2 = pull 2 aces
A_3 = pull 3 aces.

$|A_0| = C(48,3)$ "choose 3 out of 48 cards"
$|A_1| = 4 \cdot C(48,2)$ "choose 1 out of 4 aces AND 2 out of 48 cards"
$|A_2| = C(4,2)\,C(48,1)$ "choose 2 out of 4 aces AND 1 out of 48 cards"
$|A_3| = C(4,3)$ "choose 3 out of 4 aces (and that's it!)"

Thus $\underline{0} = \{A_0, A_1, A_2, A_3\}$ and

$$\left.\begin{aligned} p(A_0) &= \frac{C(48,3)}{C(52,3)} \\ p(A_1) &= \frac{4\,C(48,2)}{C(52,3)} \\ p(A_2) &= \frac{C(4,2)\,C(48,1)}{C(52,3)} \\ p(A_3) &= \frac{C(4,3)}{C(52,3)} \end{aligned}\right\} \Rightarrow \langle f \rangle = 100\,p(A_1) + 200\,p(A_2) + 300\,p(A_3) = \ldots$$

Bernoulli trials

Def : Consider a random experiment with sample space Ω and probability function p. A trial series is an experiment in which the same random experiment is repeated n times.

Def : A trial series is a Bernoulli trial if it is satisfies the following conditions:

a) The outcome of each trial is SUCCESS or FAIL

b) The probability p of success is constant for each trial

c) Each trial is independent from previous and subsequent trials.

Let $B(n,p,k)$ be the probability that in doing n Bernoulli trials with success probability p we will have k successes. Then

$$\boxed{B(n,p,k) = \binom{n}{k} p^k (1-p)^{n-k}}$$

Let S = number of successes
Then

a) Probability of k successes

$$p(S=k) = B(n,p,k)$$

b) Probability of at least k successes

$$p(S \geq k) = \sum_{a=k}^{n} B(n,p,a)$$

c) Probability of at most k successes

$$p(S \leq k) = \sum_{a=0}^{k} B(n,p,a)$$

d) Probability of no success

$$p(S=0) = B(n,p,0) = \binom{n}{0} p^0 (1-p)^n$$
$$= 1 \cdot 1 \cdot (1-p)^n$$
$$= (1-p)^n.$$

e) Probability of ~~at least~~ one success

$$p(s=1) = \binom{n}{1} p(1-p)^{n-1}$$
$$= np(1-p)^{n-1}.$$

f) Probability of at least one success

$$p(s \geq 1) = 1 - p(s=0)$$
$$= 1 - (1-p)^{n}.$$

<u>example</u>

1) We toss a coin 5 times.
   Probability to get 3 heads?

Answer: $$p(s=3) = B(5, 1/2, 3)$$
$$= \binom{5}{3} \left(\frac{1}{2}\right)^{3} \left(1 - \frac{1}{2}\right)^{5-3}$$
$$= \dots$$

9) Probability of space shuttle to go BOOM is 1% per launch.

a) What is probability of NO BOOM after 60 launches?

b) What is probability of at least one BOOM after 60 launches?

Answer

a) $n = 60$ and $p = 1/100 = 10^{-2}$

$$p(s=0) = B(n, p, 0) = B(60, 1/100, 0)$$

$$= \binom{60}{0} (1/100)^0 (1 - 1/100)^{60}$$

$$= 1 \cdot 1 \cdot (99/100)^{60} = \left(\frac{99}{100}\right)^{60} \approx 0.54$$

thus $\approx 54\%$

b) $$p(s \geq 1) = 1 - p(s < 1) = 1 - p(s=0) =$$

$$= 1 - B(60, 1/100, 0)$$

$$= 1 - \binom{60}{0} (1/100)^0 (1 - 1/100)^{60}$$

$$= 1 - \left(\frac{99}{100}\right)^{60} \approx 1 - 0.54 = 0.46$$