Linear Systems and Matrices

Linear Systems

- A linear system of equations, is a collection of equations of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

where the numbers a_{ij} and b_i are given and x_i are unknown.

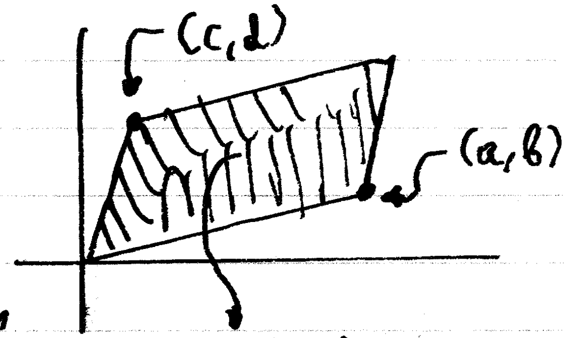
Solution Methods

1) Gaussian Elimination
2) Determinants Method (Cramer)
3) LU decomposition } numerical methods.
4) QR decomposition

2x2 system - Method of determinants.

The 2x2 determinant is defined as:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$



Geometric interpretation

area is $\underline{ad-bc}$!

To solve the system

$$\begin{cases} ax + by = A \\ cx + dy = B \end{cases}$$

•1 Calculate

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$D_x = \begin{vmatrix} A & b \\ B & d \end{vmatrix} = Ad - bB$$

$$D_y = \begin{vmatrix} a & A \\ c & B \end{vmatrix} = aB - cA$$

•2 If $D \neq 0$ then there is a unique solution:

$$\begin{cases} x = D_x / D \\ y = D_y / D \end{cases}$$

•3 If $D = 0$, solve for x or y and substitute to the other equation. The system either has no solutions or has multiple solutions.

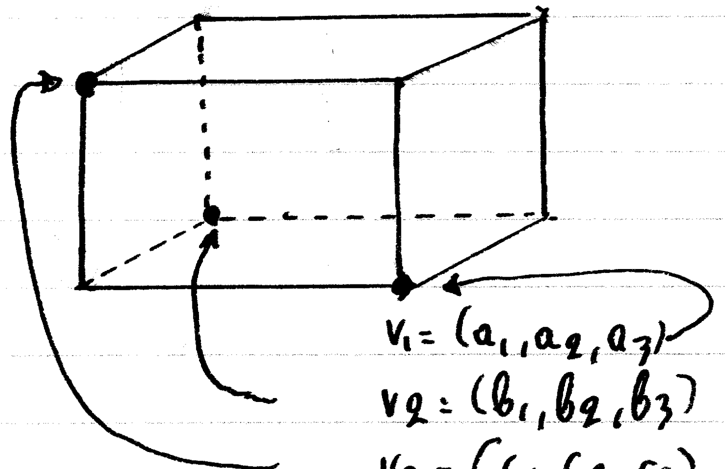
example

$$\begin{cases} 2x + y = 5 \\ 3x + 2y = 9 \end{cases} \longrightarrow (x, y) = (1, 3)$$

$$\begin{cases} x + 2y = 4 \\ 3x + 6y = 3 \end{cases} \longrightarrow \text{no solutions}$$

$$\begin{cases} 2x + 3y = 1 \\ 4x + 6y = 2 \end{cases} \longrightarrow \text{infinite solutions}$$

3x3 systems - Method of Determinants.



$v_1 = (a_1, a_2, a_3)$
$v_2 = (b_1, b_2, b_3)$
$v_3 = (c_1, c_2, c_3)$

- The determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is the volume of the <u>parallelepiped</u> defined by the vectors

$v_1 = (a_1, a_2, a_3)$
$v_2 = (b_1, b_2, b_3)$
$v_3 = (c_1, c_2, c_3)$.

- The 3x3 determinant can be calculated by the Sarrus rule:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}$$

$$= a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3$$
$$- a_3 b_2 c_1 - b_3 c_2 a_1 - c_3 a_2 b_1$$

- To solve the 3x3 system

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}$$

we calculate

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & d_1 & b_1 \\ a_2 & d_2 & b_2 \\ a_3 & d_3 & b_3 \end{vmatrix}, \quad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

If $D \neq 0$, then the system has the unique solution

$$\begin{cases} x = Dx/D \\ y = Dy/D \\ z = Dz/D. \end{cases}$$

example

$$\begin{cases} x + y + z = 2 \\ 2x + y + z = 3 \\ x + 2y + z = 3 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 1 \\ z = 0 \end{cases}$$

→ 2x3 systems

A system of the form

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \end{cases}$$

can be rewritten as a 2x2 system:

$$\begin{cases} a_1 x + b_1 y = d_1 - c_1 z \\ a_2 x + b_2 y = d_2 - c_2 z \end{cases}$$

which can be solved by 2x2 determinants. Thus we find x, y as functions of z which is a free variable

example : $\begin{cases} x - 7y + z = 3 \\ 2x - 14y + 3z = 4 \end{cases}$

Matrices

- An <u>$m \times n$ matrix</u> is a table of numbers of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- We represent a <u>vector</u> as an $m \times 1$ matrix:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- We use a <u>square matrix</u> ($n \times n$) to represent a <u>linear transformation</u> of a vector:

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{bmatrix}$$

Operations on square matrices

Let $A=[a_{ij}]$, $B=[b_{ij}]$, $C=[c_{ij}]$.
We define the following operations:

a) $C = A + B \Leftrightarrow c_{ij} = a_{ij} + b_{ij}$ (Addition)

- Motivation
  $Cx = (A+B)x = Ax + Bx$

b) $C = AB \Leftrightarrow c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ (Multiplication)

- Motivation
  $Cx = (AB)x = A(Bx)$
  AB is the linear transformation where we apply B on x to obtain Bx and then apply A on Bx to obtain A(Bx).

examples :

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

These definitions can be generalized for non-square matrices as follows:

1) If A, B are $m \times n$ matrices with $A = [a_{ij}]$ and $B = [b_{ij}]$ then $C \overset{= A+B}{}$ is an $m \times n$ matrix defined as

$$C = A + B \Leftrightarrow c_{ij} = a_{ij} + b_{ij}, \forall$$

2) If A is an $m \times k$ matrix and B is a $k \times n$ matrix with $A = [a_{ij}]$ and $B = [b_{ij}]$ then $C = AB$ is an $m \times n$ matrix defined as

$$C = AB \Leftrightarrow c_{ij} = \sum_{l=1}^{k} a_{ik} b_{kj}.$$

examples

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & 3 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 4 \\ 2 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Using matrix operations we may write a linear system of equations in terms of matrices in the form

$$Ax = b.$$

<u>example</u>

$$\begin{cases} 3x + y + 2z = 5 \\ 2x + 2y - z = 4 \\ x + 2y + 3z = -1 \end{cases} \Leftrightarrow \begin{bmatrix} 3 & 1 & 2 \\ 2 & 2 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -1 \end{bmatrix}$$

<u>Properties of matrix operations</u>

1) Properties of Addition
   a) $A+B = B+A$
   b) $A+(B+C) = (A+B)+C$
2) Properties of Multiplication
   a) $A(BC) = (AB)C$
   b) $A(B+C) = AB+BC$
   c) $(A+B)C = AC+BC$.

Note that matrix multiplication does not always satisfy:

$$AB = BA \quad \text{(NOT always true!)}$$

Identity matrix and matrix inverse

- A matrix $I = [\delta_{ij}]$ is an identity matrix if it is a square matrix and

$$\delta_{ij} = \begin{cases} 1 & , \text{ if } i=j \\ 0 & , \text{ if } i \neq j \end{cases}$$

example: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- If A is an $n \times n$ matrix and I is an $n \times n$ identity matrix then

$$\boxed{IA = AI = A}$$

- An $n \times n$ matrix A is invertable if and only if there is another $n \times n$ matrix B such that

$$AB = BA = I$$

If B exists, then it is unique and we write $B = A^{-1}$.

- If $\det A$ is the determinant of A then
A invertable $\Leftrightarrow$ $\det A \neq 0$.

Inverse of a 2x2 matrix

The 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertable if and only if
$\det A = ad - bc \neq 0$
with inverse:

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

example: $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Application to 2x2 systems

Consider a linear system $Ax = b$ with A a square invertable matrix. The solution x can be written as:

$$x = Ix = (A^{-1}A)x = A^{-1}(Ax)$$
$$= A^{-1}b.$$

Thus, we may use matrix inverses to solve 2×2 systems of equations.

example :

$$\begin{cases} 2x + y = 4 \\ x + 3y = 7 \end{cases}$$