Combinatorics

Fundamental counting principle

- If a task can be done in 2 stages and we have n_1 options for stage 1 and n_2 options for stage 2, then we have $n_1 n_2$ ways to do the task.

Explanation: using set theory.

Let A_1 = set of options for stage 1.
A_2 = set of options for stage 2.
The set of ways to do the task is
$$A = A_1 \times A_2 .$$
So if $a \in A_1$ and $b \in A_2$, then (a, b) represents doing the task by doing first a then b.

If $|A_1| = n_1$ and $|A_2| = n_2$, then

$$|A| = |A_1 \times A_2| = |A_1||A_2| = n_1 n_2$$

<u>example</u> : Roll a dice
then flip a coin

$A_1 = \{1,2,3,4,5,6\} = [6]$
$A_2 = \{h,t\}$

The set of possible events is

$A = A_1 \times A_2 =$
$= \{(1,h),(2,h),(3,h),(4,h),(5,h),(6,h)$
$(1,t),(2,t),(3,t),(4,t),(5,t),(6,t)\}$

- If a task can be done is s stages and stage $k \in [s]$ can be done in n_k different ways, then the number of ways N available for doing the task is

$$N = n_1 n_2 n_3 \cdots n_s$$

↳ We may use the "product" notation to write the above as:

$$N = \prod_{k=1}^{s} n_k .$$

example: How many 4-digit numbers can be formed with the digits 0, 1, 2, 3 ?

1st digit : $n_1 = 3$ (3 options: 1, 2, 3)
2nd digit : $n_2 = 4$ (4 options: 1, 2, 3, 0)
3rd digit $n_3 = 4$
4th digit $n_4 = 4$

Consequently: $N = n_1 n_2 n_3 n_4$
$= 3 \cdot 4 \cdot 4 \cdot 4.$

Method: Note that the first digit of a number cannot be 0!

Permutations

Let $[n] = \{1, 2, 3, \dots, n\}$.

- A <u>permutation σ</u> is a mapping
  $$\sigma : k \in [n] \longrightarrow \sigma(k) \in [n]$$
  such that
  $$\forall k_1, k_2 \in [n] : \sigma(k_1) = \sigma(k_2) \Longleftrightarrow k_1 = k_2$$

<u>example:</u> $\sigma = [3, 2, 1, 4]$
is the mapping

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 & 4 \end{array}$$

- The set of all permutations on $[n]$ is written S_n

▶ The cardinality of $|S_n| = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$

<u>Proof</u>

First number: n options
Second number: $n-1$ options
Third number: $n-2$ options, etc until

Final number has 1 option

Thus all permutations are

$$|S_n| = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$
$$= \prod_{k=1}^{n} k.$$

→ <u>Factorial</u> : $\boxed{\begin{array}{l} n! = \prod_{k=1}^{n} k \\ 0! = 1 \end{array}}$

Consequently $\underline{\underline{|S_n| = n!}}$

<u>examples with factorials</u>

Simplify: $\dfrac{7!\,5!}{6!\,4!} = \dots = 35$

$$\frac{(n-1)(n-1)!}{(n+1)!} = \dots = \frac{n-1}{n(n+1)}$$

Show that: $1! + 2! + 3! + \dots + 99! < 100!$

<u>example</u> : Number of sitting arrangements for 5 diplomats.
(Answer: $5!$)

<u>example</u> Number of anagrams of the word "popcorn"?
(Answer $7!$)
Number of anagrams of "popcorn" that start with "p" and end in "n".
(Answer: $1 \cdot 5! \cdot 1$)

Selections : Permutations and Combinations

- Select r objects out of n possible options.

  a) If the order of selection is important then it is called a <u>permutation</u>

  ↳ Let $P(n,r)$ = number of permutations when we select r out of n objects

  b) If the order of selection is NOT important then it is called a <u>combination</u>

  ↳ Let $C(n,r)$ = number of combinations when we select r out of n objects.

For the (n,r) problem a permutation has $r!$ rearrangements all of which represent the same combination. Therefore:

$$\boxed{C(n,r) = \frac{P(n,r)}{r!}}$$

▶ To calculate $P(n,r)$ note that once you choose r objects you have $n-r$ choices for object $r+1$.
Thus

$$P(n,r+1) = P(n,r)(n-r)$$

Also we have

$$P(n,1) = n$$

(obviously n options if we choose one object).

▶ Consequently:

$$P(n,2) = P(n,1)(n-1) = n(n-1)$$
$$P(n,3) = P(n,2)(n-2) = n(n-1)(n-2)$$
$$P(n,4) = P(n,3)(n-3)$$
$$= n(n-1)(n-2)(n-3).$$

The overall pattern is:

$$P(n,r) = \prod_{k=0}^{r-1}(n-k) = \prod_{k=n-r+1}^{n} k = \frac{\prod_{k=1}^{n-r} k \prod_{k=n-r+1}^{n} k}{\prod_{k=1}^{n-r}}$$

$$= \frac{\prod_{k=1}^{n} k}{\prod_{k=1}^{n-r} k} = \frac{n!}{(n-r)!}$$

▶ To summarize:

$$\boxed{\begin{aligned} P(n,r) &= \frac{n!}{(n-r)!} \\ C(n,r) &= \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!} \end{aligned}}$$

▶ <u>Notation</u> : The "choose" symbol

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = C(n,r).$$

<u>example</u> : A club has 8 members
How many arrangements of
president – vice president ?

Answer:

$$P(8,2) = \frac{8!}{(8-2)!} = \frac{8!}{6!} = \frac{8\cdot 7\cdot 6!}{6!}$$

$$= 8\cdot 7 = 56$$

<u>example</u> : Out of 10 people how many
subcommittees of 8 people are
possible?

Answer:

$$C(10,8) = \frac{10!}{8!\,(10-8)!} = \frac{10!}{8!\,2!} =$$
$$= \frac{10 \cdot 9 \cdot 8!}{8!\,2!} = \frac{10 \cdot 9}{2} = \frac{90}{2} = 45.$$

example : You have 5 humans, 7 Klingons, 6 Romulans
How many away teams can you make with
1 human, 6 klingons, 4 Romulans?

Answer:
$C(5,1)\,C(7,6) \cdot C(6,4)$.

Mathematical properties of $C(n,r)$

1) $\boxed{C(n,1) = n} \ \boxed{C(n,0) = 1}$

Proof

$$C(n,1) = \frac{n!}{1!\,(n-1)!} = \frac{n \cdot (n-1)!}{(n-1)!}$$
$$= n. \quad \square$$

$$C(n,0) = \frac{n!}{0!\,n!} = 1$$

2) $\boxed{C(n,r) = C(n-1,r-1) + C(n-1,r)}$

<u>Proof</u>

$$C(n-1,r-1) = \frac{(n-1)!}{(r-1)![(n-1)-(r-1)]!} =$$

$$= \frac{(n-1)!}{(r-1)!(n-r)!} = \frac{r}{n}\frac{n!}{r!(n-r)!} = \frac{r}{n}C(n,r)$$

$$C(n-1,r) = \frac{(n-1)!}{r![(n-1)-r]!} = \frac{(n-1)!}{r!(n-r-1)!}$$

$$= \frac{n-r}{n}\frac{n!}{r!(n-r)!} = \frac{n-r}{n}C(n,r)$$

Therefore

$$C(n-1,r-1) + C(n-1,r) =$$

$$= \frac{r}{n}C(n,r) + \frac{n-r}{n}C(n,r)$$

$$= \left[\frac{r}{n} + \frac{n-r}{n}\right]C(n,r)$$

$$= \frac{n-r+r}{n}C(n,r) = C(n,r)$$

$\mapsto$ These properties motivate the construction of Pascal triangle:

0	1	2	3	4 $\rightarrow r$		$n \downarrow$
1	1					1
1	2	1				2
1	3	3	1			3
1	4	6	4	1		4
1	5	10	10	5	1.	5

1) We jumstart with
$$C(1,0) = C(1,1) = 1$$

2) Each element is the sum of the element above and the element to the left of that:

example: $\underbrace{3+3}_{6}$, $\underbrace{4+6}_{10}$, etc.

3) The symmetry property can be established as follows:

3) $\boxed{C(n, n-r) = C(n, r)}$

Proof

$$C(n, n-r) = \frac{n!}{(n-r)!\,[n-(n-r)]!}$$
$$= \frac{n!}{(n-r)!\,(r)!} = \frac{n!}{r!\,(n-r)!}$$
$$= C(n, r) \qquad \square$$

→ Another interesting property

4) $\boxed{C(n,0) + C(n,1) + C(n,2) + \cdots + C(n,n) = 2^n}$

Proof

Toss a coin n times.
Number of possible outcomes is 2^n.
Number of outcomes with k tails in $C(n,k)$.
If we add the outcomes with
$k = 0, 1, 2, \ldots, n$ tails
we must get $2^n \Rightarrow$ the equation to be shown.

example : Toss a coin 5 times.
How many outcomes
have at least 2 heads?

Answer: $C(5,2) + C(5,3) + C(5,4) + C(5,5)$

$= 2^5 - [C(5,0) + C(5,1)]$.

$= 2^5 - [\cancel{cC(5} 1 + 5]$

$= 32 - 6 = 26$