RELATIONS AND FUNCTIONS

▼ Cartesian product

- An ordered pair (a,b) is defined as an ordered collection of two elements a and b such that it satisfies the axiom:
  $$(a_1,b_1)=(a_2,b_2) \Leftrightarrow a_1=a_2 \wedge b_1=b_2.$$
- Ordered pairs can be represented as sets:
  $$(a,b)=\{a,\{a,b\}\}$$
  Then ordered pair equality corresponds to set equality.
- Let A,B be two sets. We define the cartesian product $A\times B$ as:
  $$A\times B=\{(a,b) \mid a\in A \wedge b\in B\}$$
  The corresponding belonging condition is:
  $$x\in A\times B \Leftrightarrow \exists a\in A : \exists b\in B : x=(a,b).$$
  however, in practice we find it more useful to use the following statement
  $$(a,b)\in A\times B \Leftrightarrow a\in A \wedge b\in B.$$
- We also define $A^2=A\times A$.
- It is easy to see that
  $$\emptyset\times A=\emptyset$$
  $$A\times\emptyset=\emptyset.$$

EXAMPLES

a) For $A = \{1,2\}$ and $B = \{2,3\}$, evaluate $A \times B$, $B \times A$ and A^2.

<u>Solution</u>

$A \times B = \{1,2\} \times \{2,3\} =$
$= \{(1,2), (1,3), (2,2), (2,3)\}$

$B \times A = \{2,3\} \times \{1,2\} =$
$= \{(2,1), (2,2), (3,1), (3,2)\}$

$A^2 = A \times A = \{1,2\} \times \{1,2\} =$
$= \{(1,1), (1,2), (2,1), (2,2)\}$

b) Let A, B, C be sets. Show that
$A \times (B \cup C) = (A \times B) \cup (A \times C)$

<u>Solution</u>

Since,

$$
\begin{aligned}
(x,y) \in A \times (B \cup C) &\Leftrightarrow x \in A \wedge y \in B \cup C \\
&\Leftrightarrow x \in A \wedge (y \in B \vee y \in C) \\
&\Leftrightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \\
&\Leftrightarrow (x,y) \in A \times B \vee (x,y) \in A \times C \\
&\Leftrightarrow (x,y) \in (A \times B) \cup (A \times C),
\end{aligned}
$$

it follows that
$A \times (B \cup C) = (A \times B) \cup (A \times C)$.

c) Show that; for sets A, B, C:

$(C \neq \emptyset \wedge A \times C = B \times C) \Rightarrow A = B$.

Solution

Assume that $C \neq \emptyset$ and $A \times C = B \times C$.

Since $C \neq \emptyset$, choose a $y \in C$.

Let $x \in A$ be given. Then:

$$
\begin{aligned}
x \in A \wedge y \in C &\Rightarrow (x,y) \in A \times C && [\text{definition}] \\
&\Rightarrow (x,y) \in B \times C && [A \times C \subseteq B \times C] \\
&\Rightarrow x \in B \wedge y \in C && [\text{definition}] \\
&\Rightarrow x \in B
\end{aligned}
$$

and therefore:

$(\forall x \in A : x \in B) \Rightarrow A \subseteq B$. (1)

Let $x \in B$ be given. Then

$$
\begin{aligned}
x \in B \wedge y \in C &\Rightarrow (x,y) \in B \times C \\
&\Rightarrow (x,y) \in A \times C \\
&\Rightarrow x \in A \wedge y \in C \\
&\Rightarrow x \in A
\end{aligned}
$$

and therefore

$(\forall x \in B : x \in A) \Rightarrow B \subseteq A$. (2)

From (1) and (2): $A = B$.

d) Let A, B be sets with $A \neq \emptyset$ and $B \neq \emptyset$. Show that

$A \times B = B \times A \Rightarrow A = B$.

<u>Solution</u>

Assume that $A \neq \emptyset$ and $B \neq \emptyset$ and $A \times B = B \times A$.

Let $x \in A$ be given.

Since $B \neq \emptyset$, choose a $y \in B$. Then

$$\begin{aligned} x \in A \wedge y \in B &\Rightarrow (x,y) \in A \times B \\ &\Rightarrow (x,y) \in B \times A \quad [\text{via } A \times B \subseteq B \times A] \\ &\Rightarrow x \in B \wedge y \in A \\ &\Rightarrow x \in B. \end{aligned}$$

and therefore:

$(\forall x \in A : x \in B) \Rightarrow A \subseteq B.$ (1)

Let $x \in B$ be given.

Since $A \neq \emptyset$, choose a $y \in A$. Then

$$\begin{aligned} x \in B \wedge y \in A &\Rightarrow (x,y) \in B \times A \\ &\Rightarrow (x,y) \in A \times B \quad [\text{via } B \times A \subseteq A \times B] \\ &\Rightarrow x \in A \wedge y \in B \\ &\Rightarrow x \in A. \end{aligned}$$

and therefore

$(\forall x \in B : x \in A) \Rightarrow B \subseteq A.$ (2)

From (1) and (2): $A = B$.

e) Let $\{A_a\}_{a\in I}$, $\{B_a\}_{a\in I}$ be indexed set collections and let G be a set. Show that

$$C \times \left[\bigcup_{a\in I}(A_a - B_a)\right] \subseteq \bigcup_{a\in I}\left[(C\times A_a) - (C\times B_a)\right]$$

Solution

Since

$$(x,y)\in G\times\left[\bigcup_{a\in I}(A_a - B_a)\right] \Rightarrow$$

$$\Rightarrow x\in G \wedge y\in \bigcup_{a\in I}(A_a - B_a) \Rightarrow$$

$$\Rightarrow x\in G \wedge \exists a\in I : y\in A_a - B_a$$

$$\Rightarrow x\in G \wedge \exists a\in I : (y\in A_a \wedge y\notin B_a)$$

$$\Rightarrow \exists a\in I : (x\in G \wedge y\in A_a \wedge y\notin B_a)$$

$$\Rightarrow \exists a\in I : [(x\in G \wedge y\in A_a) \wedge \underline{\underline{(x\notin G \vee y\notin B_a)}}] \quad (!!!)$$

$$\Rightarrow \exists a\in I : ((x,y)\in G\times A_a \wedge \overline{(x\in C \wedge y\in B_a)})$$

$$\Rightarrow \exists a\in I : ((x,y)\in G\times A_a \wedge (x,y)\notin G\times B_a)$$

$$\Rightarrow \exists a\in I : (x,y)\in (G\times A_a) - (G\times B_a)$$

$$\Rightarrow (x,y)\in \bigcup_{a\in I}\left[(G\times A_a) - (G\times B_a)\right]$$

it follows that:

$$C\times\left[\bigcup_{a\in I}(A_a - B_a)\right] \subseteq \bigcup_{a\in I}\left[(G\times A_a) - (G\times B_a)\right]$$

↳ Note that the (!!) step is valid but cannot be reversed.

EXERCISES

① Let $A = \{x \in \mathbb{Z} \mid 1 \leq x \leq 3\}$
$B = \{3x - 1 \mid x \in \mathbb{Z} \land 0 < x < 4\}$
List the elements of $A \times B$.

② Prove that for A, B, C sets
$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

③ Prove the following
a) $A \times B = \emptyset \Leftrightarrow A = \emptyset \lor B = \emptyset$
b) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
c) $(A \times B) \cap (C \times D) = \emptyset \Leftrightarrow A \cap C = \emptyset \lor B \cap D = \emptyset$.

④ Prove the following.
a) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$
b) $\{p, q\} \subseteq A \Rightarrow (A \times \{p\}) \cup (\{q\} \times A) \subseteq A \times A$

⑤ Prove the following:
a) $A \times B = B \times A \Leftrightarrow A = \emptyset \lor B = \emptyset \lor A = B$
b) $A \neq \emptyset \neq B \land (A \times B) \cup (B \times A) = C \times C \Rightarrow A = B = C$.

⑥ Let $\{A_a\}_{a\in I}$ and $\{B_a\}_{a\in I}$ be indexed set collections and let C be a set. Prove the following:

a) $$\left(\bigcup_{a\in I} A_a\right)\times C = \bigcup_{a\in I}(A_a\times C)$$

b) $$\left(\bigcap_{a\in I} A_a\right)\times C = \bigcap_{a\in I}(A_a\times C)$$

c) $$\bigcap_{a\in I}(A_a\times B_a) = \left(\bigcap_{a\in I} A_a\right)\times\left(\bigcap_{a\in I} B_a\right)$$

⑦ Show that for A, B sets

$$\bigcup_{S\in\mathcal{P}(A)}\left[\bigcup_{T\in\mathcal{P}(B)}\{S\times T\}\right]\subseteq\mathcal{P}(A\times B)$$

V Relations

- Let A, B be two sets with $A \neq \emptyset$ and $B \neq \emptyset$. We define the set of all relations from A to B via the following belonging condition:

$$R \in \mathrm{Rel}(A,B) \Leftrightarrow R \subseteq A \times B$$

- If $R \in \mathrm{Rel}(A,B)$, we say that R is a <u>relation</u> from A to B.
- Let $R \in \mathrm{Rel}(A,B)$ be a relation and let $x \in A$ and $y \in B$. Then we define the statements xRy and $x\not{R}y$ as follows:

$$\forall x \in A : \forall y \in B : (xRy \Leftrightarrow (x,y) \in R)$$
$$\forall x \in A : \forall y \in B : (x\not{R}y \Leftrightarrow (x,y) \notin R)$$

We say that:

xRy : x is related with y via relation R.

$x\not{R}y$: x is NOT related with y via relation R.

EXAMPLE

Let $A = \{a,b,c\}$ and $B = \{d,e,f,g,h\}$. Then

$R = \{(a,e),(b,d),(c,g),(b,h),(c,d)\}$

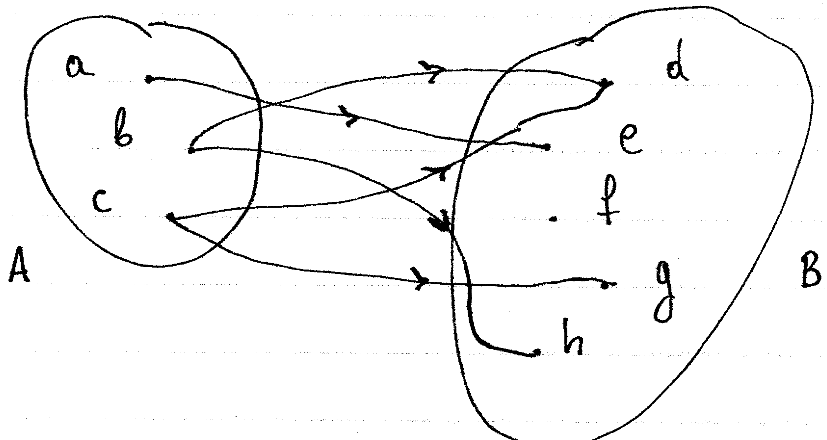
is a relation from A to B (i.e. $R \in \mathrm{Rel}(A,B)$). Then

$(a,e) \in R \Rightarrow aRe$ $\qquad$ $(b,h) \in R \Rightarrow bRh$

$(b,d) \in R \Rightarrow bRd$ $\qquad$ $(c,d) \in R \Rightarrow cRd$

$(c,g) \in R \Rightarrow cRg$

The relation R can be represented geometrically using a <u>Venn diagram</u>, as follows:



Each ordered pair (x,y) is represented by an arrow from x to y.

Domain and range of a relation

- Let $R \in Rel(A,B)$ be a relation from A to B. We define the domain $dom(R)$ and range $ran(R)$ of R as:

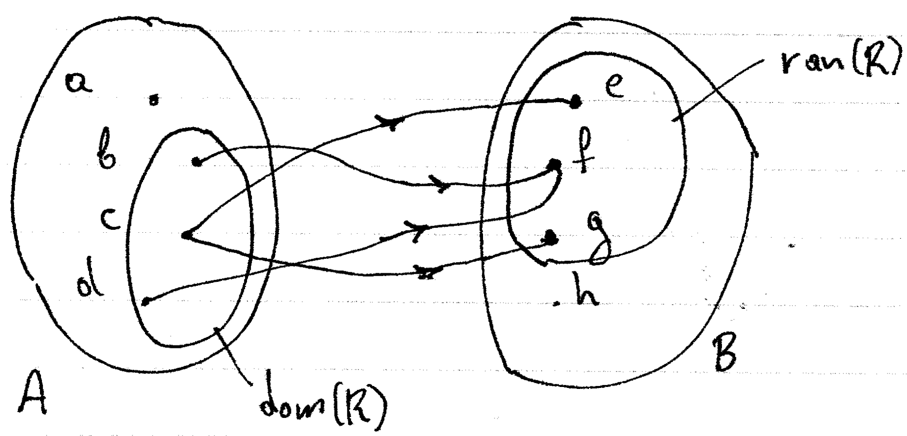
$$dom(R) = \{x \in A \mid \exists y \in B : xRy\} \subseteq A$$
$$ran(R) = \{y \in B \mid \exists x \in A : xRy\} \subseteq B$$

- $dom(R)$ contains all the elements of A that are related with some element of B. In terms of Venn diagrams, $dom(R)$ has all the elements of A that have an outgoing arrow.
- $ran(R)$ contains all the elements of B that are related with some element of A. In terms of Venn diagrams,

ran(R) has all the elements of B that have an incoming arrow.

EXAMPLE

For $A = \{a, b, c, d\}$ and $B = \{e, f, g, h\}$, let $R \in Rel(A, B)$ be a relation from A to B with
$R = \{(b, f), (c, e), (d, f), (c, g)\}$.
Then: $dom(R) = \{b, c, d\}$ and
$ran(R) = \{e, f, g\}$



Relations on A

We define $Rel(A) = Rel(A, A)$. Then:

$$R \in Rel(A) \Leftrightarrow R \subseteq A \times A$$

and we say that R is a <u>relation on A</u>.

Equivalence relations

- Let $R \in \mathrm{Rel}(A)$ be a relation on A with $A \neq \emptyset$. We say that

  R reflexive $\Leftrightarrow \forall x \in A : xRx$

  R symmetric $\Leftrightarrow \forall x, y \in A : (xRy \Rightarrow yRx)$

  R transitive $\Leftrightarrow \forall x, y, z \in A : ((xRy \wedge yRz) \Rightarrow xRz)$

  and

$$R \text{ equivalence} \Leftrightarrow \begin{cases} R \text{ reflexive} \\ R \text{ symmetric} \\ R \text{ transitive} \end{cases}$$

EXAMPLES

a) Let $R \in Rel(A)$ be a relation on A. Show that
R reflexive $\Rightarrow \text{dom}(R) = A$.

Solution

Assume that R is reflexive. Since
$\text{dom}(R) = \{x \in A \mid \exists y \in A : xRy\} \subseteq A \Rightarrow \underline{\text{dom}(R) \subseteq A}$ (1)
it is sufficient to show that $\forall x \in A : x \in \text{dom}(R)$.
Let $x \in A$ be given. Then:

$$\begin{aligned} R \text{ reflexive} &\Rightarrow xRx \\ &\Rightarrow \exists y \in A : xRy \\ &\Rightarrow x \in \text{dom}(R) \qquad [\text{via } x \in A] \end{aligned}$$

It follows that

$$\forall x \in A : x \in \text{dom}(R) \Rightarrow A \subseteq \text{dom}(R) \qquad (2)$$

From Eq.(1) and Eq.(2):

$$\left\{ \begin{array}{l} \text{dom}(R) \subseteq A \\ A \subseteq \text{dom}(R) \end{array} \right. \Rightarrow \text{dom}(R) = A.$$

b) Let $R \in Rel(A)$ be a relation on A. We define
R circular $\Leftrightarrow \forall x, y, z \in A : ((xRy \land yRz) \Rightarrow zRx)$
Show that:

$$\left\{ \begin{array}{l} R \text{ transitive} \\ R \text{ symmetric} \end{array} \right. \Rightarrow R \text{ circular}$$

Solution

Assume that R is transitive and symmetric.

Let $\underline{x, y, z \in A}$ be given and assume that $\underline{xRy \land yRz}$.

Then,

$$\begin{cases} xRy \\ yRz \end{cases} \Rightarrow xRz \qquad [R \text{ is transitive}]$$

$$\Rightarrow \underline{zRx} \qquad [R \text{ is symmetric}]$$

From the above argument, it follows that

$\forall x, y, z \in A : ((xRy \land yRz) \rightarrow zRx)$

$\Rightarrow R$ circular.

EXERCISES

(8) Show that the following relations are equivalences

a) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow a+2b \equiv 0 \pmod 3$

b) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow a^3 \equiv b^3 \pmod 4$

c) $R \in Rel(\mathbb{Z})$ with $aRb \Leftrightarrow 2a+3b \equiv 0 \pmod 5$

(9) Show that the following relations on $\mathbb{R}^* \times \mathbb{R}^*$ are equivalences

a) $(x_1, y_1) R (x_2, y_2) \Leftrightarrow x_1 y_2 - x_2 y_1 = 0$

b) $(x_1, y_1) R (x_2, y_2) \Leftrightarrow \exists \lambda \in \mathbb{R}^* : (x_1 = \lambda x_2 \wedge y_1 = \lambda y_2)$

(Recall that $\mathbb{R}^* = \mathbb{R} - \{0\}$).

(10) Let $R \in Rel(A)$ be a relation on A. Show that

a) R reflexive $\Rightarrow ran(R) = A$

b) R symmetric $\Rightarrow dom(R) = ran(R)$

c) $(R$ circular $\wedge R$ symmetric$) \Rightarrow R$ transitive

d) R equivalence $\Leftrightarrow (R$ reflexive $\wedge R$ circular$)$

$\hookrightarrow$ We use the definition

R circular $\Leftrightarrow \forall x, y, z \in A : ((xRy \wedge yRz) \Rightarrow zRx)$

(11) Let $R \in Rel(A)$. Write the definition, using quantifiers, for the following statements:

a) R is not reflexive

b) R is not symmetric

c) R is not transitive

d) R is not circular.