MAPPINGS AND FUNCTIONS

▼ Basic Definitions

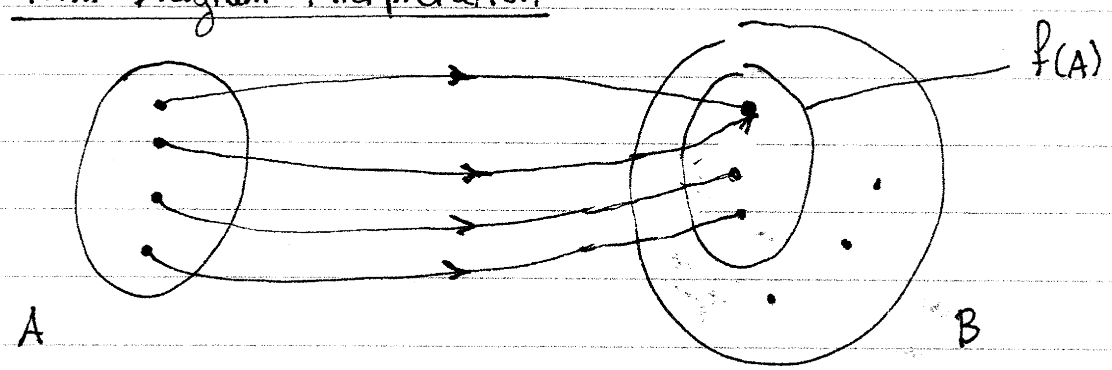
• Let A, B be two arbitrary sets. We say that f is a mapping that maps A to B (notation: $f: A \to B$) if and only if the following conditions are satisfied:

a) f is a relation $f \in Rel(A, B)$

b) $\forall x \in A : \exists y \in B : (x, y) \in f$

c) $\forall (x_1, y_1), (x_2, y_2) \in f : (x_1 = x_2 \Rightarrow y_1 = y_2)$

▶ Venn Diagram interpretation



Conditions (b) and (c) above have the following interpretations:

b) All elements of A have an outgoing arrow to some element of B

c) No element of A can have more than one outgoing arrow

Note that there are no restrictions on where the arrows go to as long as they go to some element of B.

▶ Special cases

- We denote the set of all mappings $f: A \to B$ as $Map(A,B) = \{f \in Rel(A,B) \mid f: A \to B\}$
- For $A \subseteq \mathbb{R}$ we define the set of all functions with domain A:

  $F(A) = Map(A, \mathbb{R})$.

- Also relevant are the following definitions

  $F(\mathbb{N})$ = the set of all real-valued sequences

  $Map(\mathbb{R}^n, \mathbb{R})$ = the set of all scalar fields

  $Map(\mathbb{R}^n, \mathbb{R}^n)$ = the set of all vector fields

▶ $f(x)$ notation

For every element $x \in A$, there is a unique $y \in B$ such that $(x,y) \in f$. We denote this unique y as $y = f(x)$.

EXAMPLE

For $f = \{(1,7), (2,5), (3,7)\}$, it follows that

$f(1) = 7$

$f(2) = 5$

$f(3) = 7$.

▶ $f(S)$ notation

Let $f: A \to B$ and let $S \subseteq A$. We define the image $f(S)$ of S as follows:

$f(S)=\{f(x)\mid x\in S\}$

The belonging condition corresponding to $f(S)$ is given by

$y\in f(S) \iff \exists x\in S:\ y=f(x)$

EXAMPLE

For $f=\{(1,7),(2,5),(3,7)\}$, it follows that

$f(\{1,2\})=\{5,7\}$

$f(\{1,3\})=\{7\}$

$f(\{1,2,3\})=\{5,7\}$

$f(\emptyset)=\emptyset$.

EXAMPLES

a) Let $f: A \to B$ be given and let $S \subseteq A$ and $T \subseteq A$.
Show that $f(S \cup T) = f(S) \cup f(T)$.

Solution

($\Rightarrow$): Let $y \in f(S \cup T)$ be given. Then

$y \in f(S \cup T) \Rightarrow \exists x \in S \cup T : f(x) = y$.

Choose $x_0 \in S \cup T$ such that $f(x_0) = y$.

Since $x_0 \in S \cup T \Rightarrow x_0 \in S \vee x_0 \in T$, we distinguish between the following cases:

Case 1 : Assume that $x_0 \in S$. Then

$$\begin{cases} x_0 \in S \\ f(x_0) = y \end{cases} \Rightarrow \exists x \in S : y = f(x) \Rightarrow y \in f(S)$$

$$\Rightarrow y \in f(S) \vee y \in f(T) \Rightarrow y \in f(S) \cup f(T).$$

Case 2 : Assume that $x_0 \in T$. Then

$$\begin{cases} x_0 \in T \\ f(x_0) = y \end{cases} \Rightarrow \exists x \in T : y = f(x) \Rightarrow y \in f(T)$$

$$\Rightarrow y \in f(S) \vee y \in f(T) \Rightarrow y \in f(S) \cup f(T).$$

In both cases we find $y \in f(S) \cup f(T)$ and therefore

$\forall y \in f(S \cup T) : y \in f(S) \cup f(T)$. (1)

($\Leftarrow$) : Let $y \in f(S) \cup f(T)$ be given. Then:

$y \in f(S) \cup f(T) \Rightarrow y \in f(S) \vee y \in f(T) \Rightarrow$

$\Rightarrow (\exists x \in S : y = f(x)) \vee (\exists x \in T : y = f(x))$

We distinguish between the following two cases:

Case 1 : Assume that $\exists x \in S : y = f(x)$.

Choose $x_0 \in S$ such that $y = f(x_0)$. Then:

$$\begin{cases} x_0 \in S \\ y = f(x_0) \end{cases} \Rightarrow \begin{cases} x_0 \in S \vee x_0 \in T \\ y = f(x_0) \end{cases} \Rightarrow \begin{cases} x_0 \in S \cup T \\ y = f(x_0) \end{cases} \Rightarrow$$

$$\Rightarrow \exists x \in S \cup T : y = f(x)$$

$$\Rightarrow y \in f(S \cup T).$$

Case 2 : Assume that $\exists x \in T : y = f(x)$.

Choose $x_0 \in T$ such that $y = f(x_0)$. Then:

$$\begin{cases} x_0 \in T \\ y = f(x_0) \end{cases} \Rightarrow \begin{cases} x_0 \in S \vee x_0 \in T \\ y = f(x_0) \end{cases} \Rightarrow \begin{cases} x_0 \in S \cup T \\ y = f(x_0) \end{cases} \Rightarrow$$

$$\Rightarrow \exists x \in S \cup T : y = f(x)$$

$$\Rightarrow y \in f(S \cup T).$$

In both cases we find $y \in f(S \cup T)$ and therefore

$$\forall y \in f(S) \cup f(T) : y \in f(S \cup T). \qquad (2)$$

From Eq. (1) and Eq. (2):

$$\begin{cases} \forall y \in f(S \cup T) : y \in f(S) \cup f(T) \\ \forall y \in f(S) \cup f(T) : y \in f(S \cup T) \end{cases} \Rightarrow \begin{cases} f(S \cup T) \subseteq f(S) \cup f(T) \\ f(S) \cup f(T) \subseteq f(S \cup T) \end{cases}$$

$$\Rightarrow f(S \cup T) = f(S) \cup f(T).$$

b) Let $f: A \to B$ be given. Use a counterexample to explain why we cannot prove that for $S \subseteq A$ and $T \subseteq A$ we have $f(S \cap T) = f(S) \cap f(T)$.

Solution

Consider the mapping

$f = \{(a,x),(b,x),(c,y),(d,y)\}$

and define $S = \{b,c\}$ and $T = \{a,d\}$.

Then:

$$f(S \cap T) = f(\{b,c\} \cap \{a,d\}) = f(\emptyset) = \emptyset \qquad (1)$$

but

$f(b) = x \wedge f(c) = y \Rightarrow f(S) = f(\{b,c\}) = \{x,y\}$

$f(a) = x \wedge f(d) = y \Rightarrow f(T) = f(\{a,d\}) = \{x,y\}$

and therefore

$$f(S) \cap f(T) = \{x,y\} \cap \{x,y\} = \{x,y\} \qquad (2)$$

From Eq.(1) and Eq.(2):

$f(S \cap T) \neq f(S) \cap f(T)$

Proof by counterexample can be very challenging. The statement $f(S \cap T) = f(S) \cap f(T)$ can be true for some choices of S, T and false for other choices of S, T. Can you find alternate choices for S, T for which the statement is true?

EXERCISES

① Let $f: A \to B$ be given, and let $S \subseteq A$ and $T \subseteq A$. Show that

a) $f(S \cap T) \subseteq f(S) \cap f(T)$

b) $f(S) - f(T) \subseteq f(S - T)$

② Find a counterexample of an $f: A \to B$ and $S \subseteq A$ and $T \subseteq A$ such that the following statements are false:

a) $f(S \cap T) = f(S) \cap f(T)$

b) $f(S) - f(T) = f(S - T)$

$\hookrightarrow$ We will later show that these statements can be proved if additional assumptions about f are introduced.

③ Let $f: A \to B$ be given and let S_a such that $\forall a \in I : S_a \subseteq A$ with I an index set. Show that

a) $f\left(\bigcup_{a \in I} S_a\right) = \bigcup_{a \in I} f(S_a)$

b) $f\left(\bigcap_{a \in I} S_a\right) \subseteq \bigcap_{a \in I} f(S_a)$

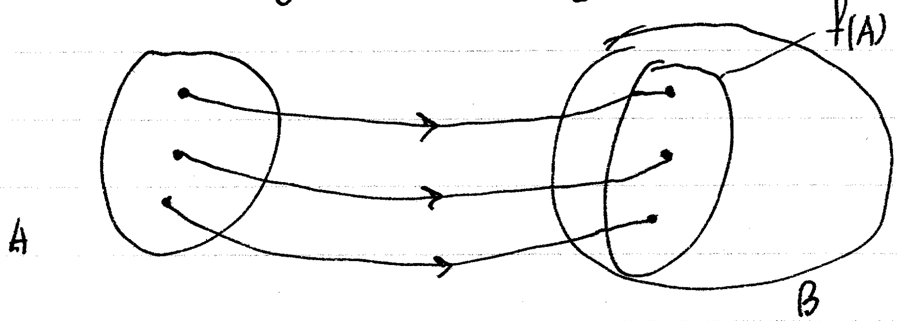
One-to-one and onto mappings

- Let $f: A \to B$ be given. We say that

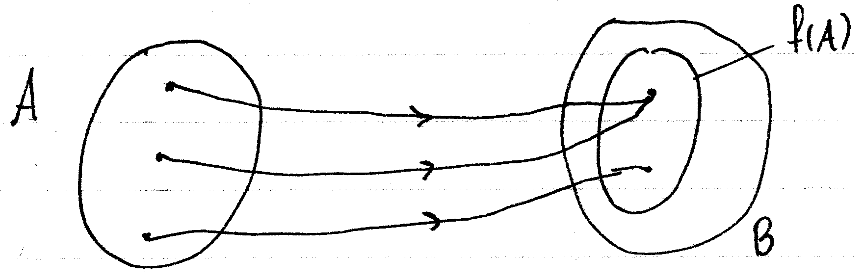
$$\boxed{\begin{array}{l} f \text{ one-to-one} \Leftrightarrow \forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2) \\ f \text{ onto} \Leftrightarrow f(A) = B \\ f \text{ Bijection} \Leftrightarrow f \text{ one-to-one} \wedge f \text{ onto} \end{array}}$$

Remarks

a) In a one-to-one mapping, every point in the range $f(A)$ receives only one incoming arrow.



f is one-to-one



f is NOT one-to-one

This interpetation becomes clear in terms of the negation of the one-to-one definition. Since $\overline{p \Rightarrow q} \equiv p \wedge \bar{q}$:

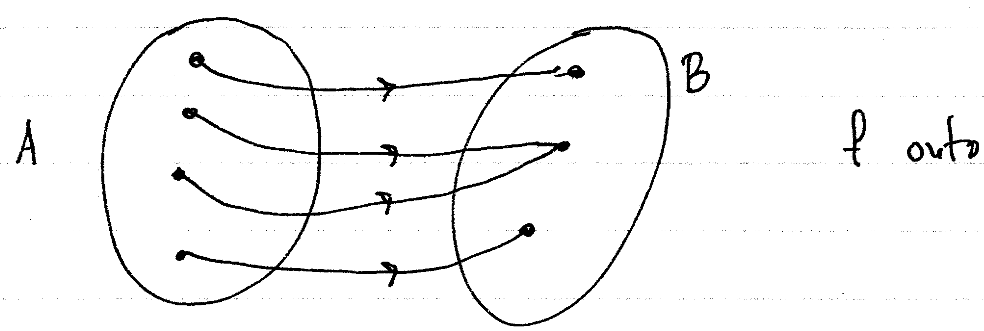
$$\boxed{f \text{ NOT one-to-one} \Leftrightarrow \exists x_1, x_2 \in A : (f(x_1) = f(x_2) \wedge x_1 \neq x_2)}$$

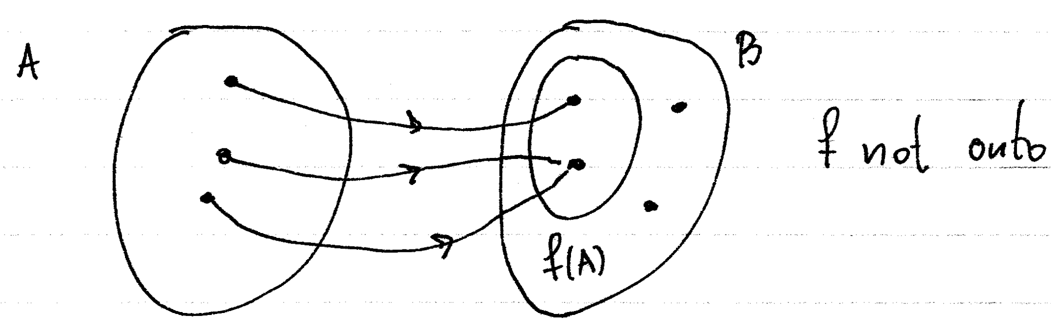
b) From the definition of $f(A)$, we always have $f(A) \subseteq B$.
It follows that the "onto" definition can be rewritten as:

$$f \text{ onto} \Leftrightarrow f(A) = B \Leftrightarrow f(A) \subseteq B \wedge B \subseteq f(A)$$
$$\Leftrightarrow B \subseteq f(A) \Leftrightarrow \forall y \in B : y \in f(A) \Leftrightarrow$$
$$\Leftrightarrow \forall y \in B : \exists x \in A : f(x) = y$$

This gives the following interpretation:
"f is onto if and only if for every element y of B, there is an element $x \in A$ such that $f(x) = y$"
or equivalently
"f is onto if and only if every element in B has at least one incoming arrow".
In summary:

$$\boxed{\begin{array}{l} f \text{ onto} \Leftrightarrow \forall y \in B : \exists x \in A : f(x) = y \\ f \text{ not onto} \Leftrightarrow \exists y \in B : \forall x \in A : f(x) \neq y \end{array}}$$



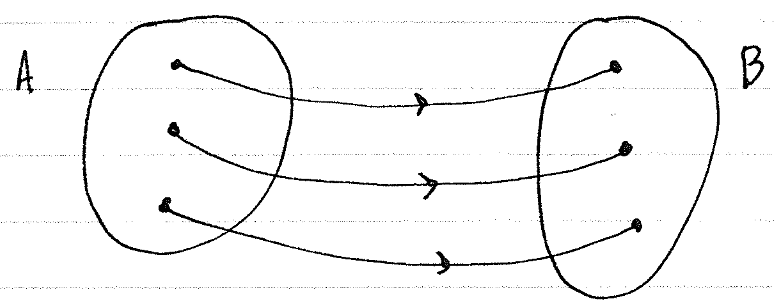


c) If f is a bijection then both conditions are satisfied:

(1) one-to-one: No element of B has more than 1 incoming arrow

(2) onto: Every element of B has at least one incoming arrow.

Combining the two conditions together we see that f is a bijection if and only if every element of B has exactly 1 incoming arrow:



f Bijection

▸ Inverse mapping

If $f: A \to B$ is a bijection, then reversing all arrows gives the bijection $f^{-1}: B \to A$ such that for all $x \in A$ and for all $y \in B$,

$$\left(y = f^{-1}(x) \Leftrightarrow f(y) = x\right)$$

We say that f^{-1} is the inverse mapping of f. Note that

$\forall x \in A: f^{-1}(f(x)) = x$

$\forall x \in B: f(f^{-1}(x)) = x$

▸ Methodology

To derive statements of the form $A=B \Rightarrow C=D$ we use the following properties of real numbers

1) We can add/cancel any number to both sides of an equation:
$\forall a,x,y \in \mathbb{R} : (x=y \Leftrightarrow a+x=a+y)$
2) We can always add or multiply two equations
$\forall a,b,x,y \in \mathbb{R} : (a=b \wedge x=y \Rightarrow a+x=b+y)$
$\forall a,b,x,y \in \mathbb{R} : (a=b \wedge x=y \Rightarrow ax=by)$
3) We can multiply any number to both sides of an equation:
$\forall a,x,y \in \mathbb{R} : (x=y \Rightarrow ax=ay)$
However the converse does not work for $a=0$.
With the restriction $a \neq 0$ we have:
$\forall x,y \in \mathbb{R} : \forall a \in \mathbb{R}-\{0\} : (x=y \Leftrightarrow ax=ay)$
4) We can raise both sides of an equation to any integer power:
$\forall x,y \in \mathbb{R} : \forall n \in \mathbb{N} : (x=y \Rightarrow x^n=y^n)$
In general, the converse does not work. However, if we require $n \neq 0$ and distinguish between odd and even powers, we have:
$\forall x,y \in \mathbb{R} : \forall n \in \mathbb{Z} : (x^{2n+1}=y^{2n+1} \Leftrightarrow x=y)$
$\forall x,y \in \mathbb{R} : \forall n \in \mathbb{Z}-\{0\} : (x^{2n}=y^{2n} \Leftrightarrow x=y \vee x=-y)$
5) Factored equation:
$\forall a,b \in \mathbb{R} : (ab=0 \Leftrightarrow a=0 \vee b=0)$

EXAMPLES

a) Consider the function

$$\forall x \in \mathbb{R} - \{a\} : f(x) = \frac{x}{x-a}$$

Show that $a \neq 0 \Rightarrow f$ one-to-one.

Solution

Assume that $a \neq 0$. Let $x_1, x_2 \in \mathbb{R} - \{a\}$ be given such that $f(x_1) = f(x_2)$. Then

$$\underline{f(x_1) = f(x_2)} \Rightarrow \frac{x_1}{x_1 - a} = \frac{x_2}{x_2 - a} \Rightarrow$$

$$\Rightarrow (x_1 - a)(x_2 - a)\frac{x_1}{x_1 - a} = (x_1 - a)(x_2 - a)\frac{x_2}{x_2 - a} \Rightarrow$$

$$\Rightarrow x_1(x_2 - a) = x_2(x_1 - a) \Rightarrow x_1 x_2 - a x_1 = x_1 x_2 - a x_2$$

$$\left.\begin{array}{l} \Rightarrow -a x_1 = -a x_2 \\ \quad a \neq 0 \end{array}\right\} \Rightarrow \underline{x_1 = x_2}$$

It follows that

$$\forall x_1, x_2 \in \mathbb{R} - \{a\} : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$$

$\Rightarrow f$ one-to-one.

↳ Note that to cancel $-a$ in $-a x_1 = -a x_2$ we need the assumption $a \neq 0$, otherwise the cancellation cannot be justified.

b) Consider the function $f(x) = 2x^2 + 6x - 7, \ \forall x \in \mathbb{R}$

Show that f is not one-to-one.

Solution

Solve $f(x) = -7 \Leftrightarrow 2x^2 + 6x - 7 = -7 \Leftrightarrow 2x^2 + 6x = 0 \Leftrightarrow$

$\Leftrightarrow 2x(x+3) = 0 \Leftrightarrow 2x = 0 \vee x + 3 = 0$

$\Leftrightarrow x = 0 \vee x = -3$

It follows that

$f(0) = f(-3) = -7 \wedge 0 \neq -3 \Rightarrow$

$\Rightarrow \exists x_1, x_2 \in \mathbb{R} : f(x_1) = f(x_2) \wedge x_1 \neq x_2$

$\Rightarrow f$ not one-to-one.

c) Let $f: A \to B$ be given and let $S \subseteq A$ and $T \subseteq A$.

Show that

f one-to-one $\Rightarrow f(S \cap T) = f(S) \cap f(T)$.

Solution

Assume that f is one-to-one.

($\Rightarrow$): Let $y \in f(S \cap T)$ be given. Then,

$y \in f(S \cap T) \Rightarrow \exists x \in S \cap T : f(x) = y$

Choose $x_0 \in S \cap T$ such that $f(x_0) = y$. It follows that

$$\begin{cases} x_0 \in S \cap T \\ f(x_0) = y \end{cases} \Rightarrow \begin{cases} x_0 \in S \wedge x_0 \in T \\ f(x_0) = y \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_0 \in S \\ f(x_0) = y \end{cases} \wedge \begin{cases} x_0 \in T \\ f(x_0) = y \end{cases} \Rightarrow$$

$$\Rightarrow (\exists x \in S : f(x) = y) \wedge (\exists x \in T : f(x) = y)$$

$$\Rightarrow y \in f(S) \wedge y \in f(T) \Rightarrow$$

$$\Rightarrow y \in f(S) \cap f(T).$$

($\Leftarrow$): Let $y \in f(S) \cap f(T)$ be given. Then:

$$y \in f(S) \cap f(T) \Rightarrow y \in f(S) \wedge y \in f(T) \Rightarrow$$

$$\Rightarrow \begin{cases} \exists x \in S : f(x) = y \\ \exists x \in T : f(x) = y. \end{cases}$$

Choose $x_1 \in S$ and $x_2 \in T$ such that $f(x_1) = y$ and $f(x_2) = y$.

Then:

$$\begin{cases} f(x_1) = y = f(x_2) \\ f \text{ one-to-one} \end{cases} \Rightarrow x_1 = x_2 \in T \Rightarrow x_1 \in T.$$

and therefore:

$$\begin{cases} x_1 \in S \wedge x_1 \in T \\ f(x_1) = y \end{cases} \Rightarrow \begin{cases} x_1 \in S \cap T \\ f(x_1) = y \end{cases} \Rightarrow$$

$$\Rightarrow \exists x \in S \cap T : f(x) = y$$

$$\Rightarrow \underline{y \in f(S \cap T)}$$

From the above argument we have:

$$\begin{cases} \forall y \in f(S \cap T) : y \in f(S) \cap f(T) \\ \forall y \in f(S) \cap f(T) : y \in f(S \cap T) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f(S \cap T) \subseteq f(S) \cap f(T) \\ f(S) \cap f(T) \subseteq f(S \cap T) \end{cases} \Rightarrow$$

$$\Rightarrow f(S \cap T) = f(S) \cap f(T).$$

EXERCISES

(4) Show that the following functions are one-to-one

a) $\forall x \in \mathbb{R}: f(x) = 3x^5 + 2$

b) $\forall x \in (0, +\infty): f(x) = 2x^2 + 5$

c) $\forall x \in \mathbb{R}: f(x) = ax + b$ with $a, b \in \mathbb{R} \land a \neq 0$

d) $\forall x \in \mathbb{R}: f(x) = (2x^3 + 1)^5$

e) $\forall x \in \mathbb{R} - \{0\}: f(x) = a/x$ with $a \in \mathbb{R} \land a \neq 0$

f) $\forall x \in \mathbb{R} - \{-d/c\}: f(x) = \dfrac{ax+b}{cx+d}$ with $a, b, c, d \in \mathbb{R}$ $\land$ $ad - bc \neq 0$

(5) Show that for $\forall x \in \mathbb{R}: f(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$ and $a \neq 0$ is <u>not</u> one-to-one.

(6) Let $f: A \to B$ be given and let $S \subseteq A$ and $T \subseteq A$. Show that

$$f \text{ one-to-one} \Rightarrow f(S - T) = f(S) - f(T).$$

(7) Let $f: A \to B$ be given and let S_a be a set collection such that $\forall a \in I: S_a \subseteq A$, with I an index set. Show that

$$f \text{ one-to-one} \Rightarrow f\left(\bigcap_{a \in I} S_a\right) = \bigcap_{a \in I} f(S_a)$$

▼ Cardinality

- Given two finite sets A, B, if there is a bijection $f: A \to B$ then A and B have to have the same number of elements. Cantor proposed extending his observation to infinite sets according to the following definitions:

$$\boxed{\begin{array}{l}\underline{\text{Def}}: \text{ Let } A, B \text{ be two sets. We say that} \\ \qquad A \sim B \iff \exists f \in \mathrm{Map}(A,B) : f \text{ bijection}\end{array}}$$

- The statement $A \sim B$ reads "A, B are equipotent", or "A and B have the same cardinality".
- Recall the definition
  $[n] = \{x \in \mathbb{N}^* \mid x \leq n\} = \{1, 2, 3, \ldots, n\}$
  Based on that, we introduce the following characterizations:

$$\boxed{\begin{array}{l}A \text{ finite set} \iff A = \varnothing \lor (\exists n \in \mathbb{N}^* : A \sim [n]) \\ A \text{ infinite set} \iff A \text{ not finite set} \\ \qquad\qquad\qquad\quad \iff A \neq \varnothing \land (\forall n \in \mathbb{N}^* : \overline{A \sim [n]}) \\ A \text{ countable set} \iff \exists B \in \mathcal{P}(\mathbb{N}) : A \sim B \\ A \text{ countably infinite} \iff A \sim \mathbb{N} \\ A \text{ uncountable} \iff A \text{ not countable}\end{array}}$$

- A relative comparison of sets in terms of cardinality is:
  $\text{finite} \leq \text{countable} \leq \underbrace{\text{countably infinite} < \text{uncountable}}_{\text{infinite}}$

It should be stressed that since $\emptyset, \mathbb{N} \in \mathcal{P}(\mathbb{N})$ and
$\forall n \in \mathbb{N}^* : [n] \in \mathcal{P}(\mathbb{N})$
it follows that

A finite $\Rightarrow$ A countable

A countably infinite $\Rightarrow$ A countable

However, the converse statements do not hold.

▶ <u>Interpretation</u> : A countably infinite set contains an infinite number of elements, however the existence of some bijection $f : A \to \mathbb{N}$ allows us to enumerate each element of A by assigning it to a unique natural number from $\mathbb{N}$.

▶ <u>$\mathbb{Z}$ and $\mathbb{Q}$ are countable</u>

Recall that

$\mathbb{Z} = \mathbb{N} \cup \{-x \mid x \in \mathbb{N}^*\} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$

$\mathbb{Q} = \{(a/b) \mid a, b \in \mathbb{Z} \land b \neq 0\}$

with $\mathbb{Z}$ the set of integers and $\mathbb{Q}$ the set of rational numbers. The remarkable insight of Cantor is that even though $\mathbb{Z}$ and $\mathbb{Q}$ contain "more numbers" than $\mathbb{N}$, in the sense that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$, from the standpoint of cardinality, we can show that $\mathbb{Z} \sim \mathbb{N}$ and $\mathbb{Q} \sim \mathbb{N}$. Equivalently, we can show that

$$\begin{cases} \mathbb{Z} \text{ countably infinite} \\ \mathbb{Q} \text{ countably infinite} \end{cases}$$

▶ <u>$\mathbb{R}$ is uncountable</u>

With some additional theory we can show that the set $\mathbb{R}$ of all real numbers satisfies the following statements:

$$\begin{cases} \mathbb{R} \text{ is uncountable} \\ \mathbb{R} \sim \mathcal{P}(\mathbb{N}) \end{cases}$$

Proof of $\mathbb{Z} \sim \mathbb{N}$ ($\mathbb{Z}$ is countably infinite)

We define the mapping $f: \mathbb{Z} \to \mathbb{N}$ such that

$$\forall x \in \mathbb{Z} : f(x) = \begin{cases} 2x-1 & , \text{if } x > 0 \\ -2x & , \text{if } x \leq 0 \end{cases}$$

and note that

$$f = \{(0,0), (1,1), (-1,2), (2,3), (-2,4), (3,5), (-3,6), \ldots\}$$

which indicates that f is a bijection. To prove that, we show that f is one-to-one and that f is onto.

- <u>one-to-one</u>: Sufficient to show that

$$\forall x_1, x_2 \in \mathbb{Z} : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$$

Let <u>$x_1, x_2 \in \mathbb{Z}$</u> be given and assume that <u>$f(x_1) = f(x_2)$</u>.

We distinguish between the following cases.

<u>Case 1</u>: Assume that $f(x_1) = -2x_1$ and $f(x_2) = -2x_2$. Then,

$$f(x_1) = f(x_2) \Rightarrow -2x_1 = -2x_2 \Rightarrow x_1 = x_2.$$

<u>Case 2</u>: Assume that $f(x_1) = 2x_1 - 1$ and $f(x_2) = 2x_2 - 1$. Then

$$f(x_1) = f(x_2) \Rightarrow 2x_1 - 1 = 2x_2 - 1 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$$

<u>Case 3</u>: Assume that $f(x_1) = 2x_1 - 1$ and $f(x_2) = -2x_2$. Then

$$f(x_1) = f(x_2) \Rightarrow 2x_1 - 1 = -2x_2 \Rightarrow 2x_1 + 2x_2 = 1 \Rightarrow$$
$$\Rightarrow 2(x_1 + x_2) = 1 \Rightarrow x_1 + x_2 = 1/2$$

This is a contradiction, because

$$x_1, x_2 \in \mathbb{Z} \Rightarrow x_1 + x_2 \in \mathbb{Z} \Rightarrow x_1 + x_2 \neq 1/2$$

therefore case 3 does not materialize.

From the above cases we conclude that <u>$x_1 = x_2$</u> and therefore:

$$\forall x_1, x_2 \in \mathbb{Z} : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2) \quad (1)$$

• <u>Onto:</u> Sufficient to show that $\forall y \in \mathbb{N} : \exists x \in \mathbb{Z} : f(x) = y$.
Let $\underline{y \in \mathbb{N}}$ be given. From the division theorem we have:
$\exists k \in \mathbb{N} : (y = 2k \lor y = 2k+1)$
Choose a $k \in \mathbb{N}$ such that $y = 2k \lor y = 2k+1$ and distinguish between the following cases.

<u>Case 1:</u> Assume that $y = 2k$. Then:

$$k \in \mathbb{N} \Rightarrow k \geq 0 \Rightarrow -k \leq 0 \Rightarrow f(-k) = -2(-k) = 2k = y \Rightarrow$$
$$\Rightarrow \underline{\exists x \in \mathbb{Z} : f(x) = y} \quad (\text{for } x = -k)$$

<u>Case 2:</u> Assume that $y = 2k+1$. Then:

$$k \in \mathbb{N} \Rightarrow k \geq 0 \Rightarrow k+1 > 0 \Rightarrow$$
$$\Rightarrow f(k+1) = 2(k+1) - 1 = 2k+2-1 = 2k+1 = y \Rightarrow$$
$$\Rightarrow \underline{\exists x \in \mathbb{Z} : f(x) = y.} \quad (\text{for } x = k+1)$$

From the above argument, in all cases, we find that

$$(\forall y \in \mathbb{N} : \exists x \in \mathbb{Z} : f(x) = y) \Rightarrow \forall y \in \mathbb{N} : y \in f(\mathbb{Z}) \Rightarrow$$
$$\Rightarrow \mathbb{N} \subseteq f(\mathbb{Z}) \Rightarrow$$
$$\Rightarrow f(\mathbb{Z}) = \mathbb{N} \qquad (2)$$

From Eq.(1) and Eq.(2).

$$\begin{cases} \forall x_1, x_2 \in \mathbb{Z} : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2) \\ f(\mathbb{Z}) = \mathbb{N} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f \text{ one-to-one} \\ f \text{ onto} \end{cases} \Rightarrow f : \mathbb{Z} \to \mathbb{N} \text{ bijection}$$

$$\Rightarrow \mathbb{Z} \sim \mathbb{N} \Rightarrow \mathbb{Z} \text{ countably infinite.}$$

Sketch of proof that $\mathbb{Q} \sim \mathbb{N}$

A bijection $f: \mathbb{Q} \to \mathbb{N}$ can be constructed via the process of diagonalization, originally proposed by Cantor. We will explain this process and the overall argument informally, for the sake of clarity. We sequence the rational numbers using the diagonalizing pattern shown in the table below, making sure to skip any numbers previously encountered in an equivalent fractional representation:

	0	1	2	3	4	...
1	0/1	1/1	2/1	3/1	4/1	...
2	0/2	1/2	2/2	3/2	4/2	...
3	0/3	1/3	2/3	3/3	4/3	...
4	0/4	1/4	2/4	3/4	4/4	...
5	0/5	...	...	...	...	
⋮	⋮					

Consequently, we sequence the rational numbers of $\mathbb{Q}$ as follows:

$\underline{0/1}$, $\underline{1/1}$, $0/2$, $\underline{2/1}$, $\underline{1/2}$, $0/3$, $\underline{3/1}$, $2/2$, $\underline{1/3}$, $0/4$, $\underline{4/1}$, $\underline{3/2}$, $\underline{2/3}$, $\underline{1/4}$, $0/5$, etc.

where we have underlined all rational numbers that appear for the first time and thus are not being skipped. We can thus define a bijection $f: \mathbb{Z} \to \mathbb{Q}$

with the initial assignments:

$f(0) = 0/1 = 0$ $\quad f(4) = 3/1$ $\quad f(8) = 2/3$

$f(1) = 1/1 = 1$ $\quad f(5) = 1/3$ $\quad f(9) = 1/4$

$f(2) = 2/1 = 2$ $\quad f(6) = 4/1$

$f(3) = 1/2$ $\quad f(7) = 3/2$ $\quad$ etc.

The algorithm for generating this bijection is as follows:

for $a = 0, 1, 2, 3, 4, \ldots$
    for $b = 0, 1, 2, \ldots, a$
        if it has not occured previously then add
        the number $(a-b)/(b+1)$ to the sequence.
    end for
end for.

To account for negative rational numbers, we extend the definition by the algorithm above as follows:

$\forall x \in \mathbb{N}^* : f(-x) = -f(x)$

and that completes the bijection $f : \mathbb{Z} \to \mathbb{Q}$. Skipping numbers that occured previously ensures that f is one-to-one. It is also clear that any rational number will be reached by this algorithm with a finite numbers of steps, which ensures that f is onto. Thus, it follows that

$f : \mathbb{Z} \to \mathbb{Q}$ bijection $\Rightarrow \mathbb{Q} \sim \mathbb{Z}$ $\quad$ [definition]
$\Rightarrow \mathbb{Q} \sim \mathbb{N}$ $\quad$ [via $\mathbb{Z} \sim \mathbb{N}$]
$\Rightarrow \mathbb{Q}$ countable $\quad \square$

EXAMPLE – APPLICATION

↳ The following problem is also a necessary first step towards proving that $\mathbb{R}$ is uncountable.

Show that $\boxed{\mathbb{R} \sim (0,1)}$

Solution

Define $\forall x \in \mathbb{R} : f(x) = (1/2) + (1/\pi)\,\mathrm{Arctan}(x)$.

We will show that $f : \mathbb{R} \to (0,1)$ is a bijection.

• Onto : Sufficient to show $\begin{cases} \forall y \in f(\mathbb{R}) : y \in (0,1) \\ \forall y \in (0,1) : y \in f(\mathbb{R}) \end{cases}$

$(\Rightarrow)$: Let $y \in f(\mathbb{R})$ be given. Then

$y \in f(\mathbb{R}) \Rightarrow \exists x \in \mathbb{R} : f(x) = y$.

Choose $x_0 \in \mathbb{R}$ such that $f(x_0) = y$. Then,

$$-\pi/2 < \mathrm{Arctan}(x_0) < \pi/2 \Rightarrow$$
$$\Rightarrow -1/2 < (1/\pi)\,\mathrm{Arctan}(x_0) < 1/2 \Rightarrow$$
$$\Rightarrow 0 < (1/2) + (1/\pi)\,\mathrm{Arctan}(x_0) < 1 \Rightarrow$$
$$\Rightarrow 0 < f(x_0) < 1 \Rightarrow 0 < y < 1 \Rightarrow \underline{y \in (0,1)}$$

It follows that $\forall y \in f(\mathbb{R}) : y \in (0,1)$. (1)

$(\Leftarrow)$: Let $y \in (0,1)$ be given. Then, we note that

$$f(x) = y \Leftrightarrow (1/2) + (1/\pi)\,\mathrm{Arctan}(x) = y \Leftrightarrow$$
$$\Leftrightarrow (1/\pi)\,\mathrm{Arctan}(x) = y - 1/2$$
$$\Leftrightarrow \mathrm{Arctan}(x) = \pi(y - 1/2) \qquad (2)$$

and also that

$$y \in (0,1) \Rightarrow 0 < y < 1 \Rightarrow -1/2 < y - 1/2 < 1/2 \Rightarrow$$
$$\Rightarrow -\pi/2 < \pi(y - 1/2) < \pi/2 \Rightarrow \tan \text{ is defined at } \pi(y-1/2).$$

Now we can define $x_0 = \tan(\pi(y - 1/2))$ and conclude that

$$\operatorname{Arctan}(x_0) = \operatorname{Arctan}(\tan(\pi(y-1/2))) = \pi(y-1/2) \overset{(2)}{\Longrightarrow}$$
$$\Rightarrow f(x_0) = y \Rightarrow \exists\, x \in \mathbb{R} : f(x) = y \Rightarrow$$
$$\Rightarrow \underline{y \in f(\mathbb{R})}$$

and therefore,

$$\forall y \in (0,1) : y \in f(\mathbb{R}) \qquad (3)$$

From Eq.(2) and Eq.(3):

$$\begin{cases} \forall y \in f(\mathbb{R}) : y \in (0,1) \\ \forall y \in (0,1) : y \in f(\mathbb{R}) \end{cases} \Rightarrow \begin{cases} f(\mathbb{R}) \subseteq (0,1) \\ (0,1) \subseteq f(\mathbb{R}) \end{cases} \Rightarrow f(\mathbb{R}) = (0,1)$$
$$\Rightarrow f \text{ onto.} \qquad (4)$$

- <u>One-to-one</u>

Let $\underline{x_1, x_2 \in \mathbb{R}}$ be given and assume that $\underline{f(x_1) = f(x_2)}$. Then,

$$f(x_1) = f(x_2) \Rightarrow (1/2) + (1/\pi)\operatorname{Arctan}(x_1) = (1/2) + (1/\pi)\operatorname{Arctan}(x_2) \Rightarrow$$
$$\Rightarrow (1/\pi)\operatorname{Arctan}(x_1) = (1/\pi)\operatorname{Arctan}(x_2) \Rightarrow$$
$$\Rightarrow \operatorname{Arctan}(x_1) = \operatorname{Arctan}(x_2) \Rightarrow$$
$$\Rightarrow \tan(\operatorname{Arctan}(x_1)) = \tan(\operatorname{Arctan}(x_2))$$
$$\Rightarrow \underline{x_1 = x_2}$$

and therefore, we have

$$\forall x_1, x_2 \in \mathbb{R} : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$$
$$\Rightarrow f \text{ one-to-one} \qquad (5)$$

From Eq.(4) and Eq.(5):

$$\begin{cases} f \text{ onto} \\ f \text{ one-to-one} \end{cases} \Rightarrow f : \mathbb{R} \to (0,1) \text{ bijection} \Rightarrow \mathbb{R} \sim (0,1).$$

EXERCISES

(8) Learn the proofs for the following statements

a) $\mathbb{Z}$ is countable

b) $\mathbb{Q}$ is countable

c) $\mathbb{R} \sim (0,1)$

(9) Let A, B be two sets. Show that
A countable $\wedge$ B countable $\Rightarrow$ $A \cup B$ countable.

(10) Let A_a with $a \in \mathbb{N}$ be a set collection. Show that:

a) $(\forall a \in \mathbb{N} : A_a \text{ finite}) \Rightarrow \bigcup_{a \in \mathbb{N}} A_a$ countable

b) Use part (a) to show that
$(\forall a \in \mathbb{N} : A_a \sim \mathbb{N}) \Rightarrow \bigcup_{a \in \mathbb{N}} A_a \sim \mathbb{N}$

(11) Given 3 sets A, B, C show that the set equivalence satisfies the reflexive, symmetric, and transitive properties.

a) $A \sim A$

b) $A \sim B \Rightarrow B \sim A$

c) $A \sim B \wedge B \sim C \Rightarrow A \sim C$

(12) Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$ and consider the intervals

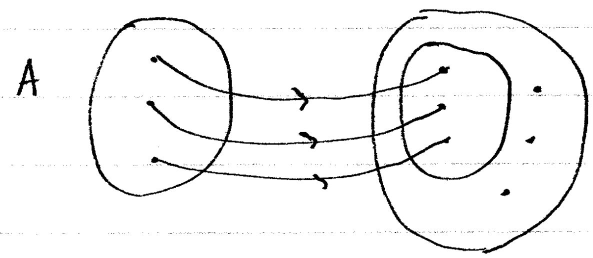
$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

$[c,d] = \{x \in \mathbb{R} \mid c \leq x \leq d\}$

Construct a bijection to show that $[a,b] \sim [c,d]$.

Cardinality inequalities

If we can define a mapping $f: A \to B$ which is one-to-one but not necessarily onto, then from an intuitive standpoint the only conclusion that can be drawn is that either A, B are of "equal cardinality" or "B has greater cardinality than A", as illustrated by the following figure:



$\begin{cases} f \text{ one-to-one} \\ f \text{ not onto} \end{cases}$

Consequently, we propose the following definitions:

$$\boxed{\begin{aligned} &A \leqslant B \iff \exists f \in \mathrm{Map}(A,B): f \text{ one-to-one} \\ &A < B \iff A \leqslant B \wedge A \not\sim B \end{aligned}}$$

Note that it is easy to show that:

$$A \sim B \wedge B \sim C \Rightarrow A \sim C$$
$$A \leqslant B \wedge B \leqslant C \Rightarrow A \leqslant C$$
$$A \subseteq B \Rightarrow A \leqslant B$$

which are left as homework problems. Starting from Cantor, the following two major theorems will be used to show that $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ uncountable.

(1) → Cantor's theorem

$\boxed{\underline{\text{Thm}} : \text{For any set } A, \quad A < \mathcal{P}(A)}$

Proof

Define $f_0 : A \to \mathcal{P}(A)$ such that $\forall x \in A : f_0(x) = \{x\}$. Then:

$\forall x_1, x_2 \in A : (\{x_1\} = \{x_2\} \Rightarrow x_1 = x_2) \Rightarrow$

$\Rightarrow \forall x_1, x_2 \in A : (f_0(x_1) = f_0(x_2) \Rightarrow x_1 = x_2)$

$\Rightarrow f_0$ one-to-one $\Rightarrow$

$\Rightarrow \exists f \in Map(A, \mathcal{P}(A)) : f$ one-to-one (for $f = f_0$)

$\Rightarrow A \leq \mathcal{P}(A)$. (1)

To show that $A \not\sim \mathcal{P}(A)$, assume that $A \sim \mathcal{P}(A)$. Then

$A \sim \mathcal{P}(A) \Rightarrow \exists f \in Map(A, \mathcal{P}(A)) : f$ bijection

Choose an $f \in Map(A, \mathcal{P}(A))$ such that ~~f is~~ $f : A \to \mathcal{P}(A)$ is a bijection. We define a set of "bad elements"

$B = \{x \in A \mid x \notin f(x)\} \subseteq A \Rightarrow B \in \mathcal{P}(A)$.

and note that

f bijection $\Rightarrow f$ onto $\Rightarrow f(A) = \mathcal{P}(A) \Rightarrow \mathcal{P}(A) \subseteq f(A)$

$\Rightarrow \forall y \in \mathcal{P}(A) : y \in f(A) \Rightarrow$

$\Rightarrow \forall y \in \mathcal{P}(A) : \exists x \in A : f(x) = y$

Let $y = B$ and choose a $b \in A$ such that $f(b) = B$.

We distinguish between the following cases.

Case 1 : Assume that $b \in B$. Then

$b \in B \Rightarrow b \in \{x \in A \mid x \notin f(x)\} \Rightarrow$

$\Rightarrow b \in A \wedge b \notin f(b) \Rightarrow b \notin f(b) \Rightarrow b \notin B$

which is a contradiction, therefore case 1 does not materialize.

Case 2 : Assume that $b \notin B$. We also now, by definition, that $b \in A$, and therefore:

$$\begin{cases} b \in A \\ b \notin B \end{cases} \Rightarrow \begin{cases} b \in A \\ b \notin f(b) \end{cases} \Rightarrow b \in \{x \in A \mid x \notin f(x)\} \Rightarrow$$

$$\Rightarrow b \in B$$

which is also a contradiction.

Since none of the possible cases materialize, it follows that

$A \not\sim \mathcal{P}(A)$. (2)

From Eq. (1) and Eq. (2):

$$\begin{cases} A \not\sim \mathcal{P}(A) \\ A \leqslant \mathcal{P}(A) \end{cases} \Rightarrow A < \mathcal{P}(A).$$

(2) → Schroeder – Bernstein theorem

Thm : Let A, B be two sets. Then
$A \leq B \wedge B \leq A \Rightarrow A \sim B$

Proof

Assume that $A \leq B$ and $B \leq A$. Then

$$\begin{cases} A \leq B \\ B \leq A \end{cases} \Rightarrow \begin{cases} \exists f \in Map(A,B) : f \text{ one-to-one} \\ \exists g \in Map(B,A) : g \text{ one-to-one} \end{cases} \quad (1)$$

Choose $f \in Map(A,B)$ and $g \in Map(B,A)$ such that f, g are one-to-one.

Define $C_0 = A - g(B)$ and distinguish between the following two cases.

Case 1 : Assume that $C_0 = \emptyset$. By construction, we have
$g \in Map(B,A) \Rightarrow g(B) \subseteq A$.

We will show that $A \subseteq g(B)$. (2)

Let $x \in A$ be given. To show that $x \in g(B)$, assume that $x \notin g(B)$ in order to derive a contradiction. It follows that

$$\begin{cases} x \in A \\ x \notin g(B) \end{cases} \Rightarrow x \in A - g(B) \Rightarrow x \in C_0 \Rightarrow x \in \emptyset$$

which is a contradiction. We conclude that $x \in g(B)$

We have thus shown that

$\forall x \in A : x \in g(B) \Rightarrow A \subseteq g(B)$ (3)

From Eq.(1), Eq.(2), Eq.(3) we conclude that:

$$\begin{cases} A \subseteq g(B) \wedge g(B) \subseteq A \\ g \text{ one-to-one} \end{cases} \Rightarrow \begin{cases} g(B) = A \\ g \text{ one-to-one} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} g \text{ onto} \\ g \text{ one-to-one} \end{cases} \Rightarrow g : B \to A \text{ bijection}$$

$$\Rightarrow B \sim A \Rightarrow \underline{A \sim B}.$$

Case 2 : Assume that $C_0 \neq \emptyset$. Then we define by recursion

$$\forall n \in \mathbb{N} : C_{n+1} = g(f(C_n)) = g(\{f(x) \mid x \in C_n\}) =$$
$$= \{g(f(x)) \mid x \in C_n\}$$

We construct the needed bijection $h : A \to B$ by the following definition:

$$\forall x \in A : h(x) = \begin{cases} f(x) & , \text{ if } \exists n \in \mathbb{N} : x \in C_n \\ g^{-1}(x) & , \text{ if } \forall n \in \mathbb{N} : x \notin C_n \end{cases}$$

Since we do not know if g is a bijection, we need to prove that $A - \bigcup_{n \in \mathbb{N}} C_n \subseteq g(B)$ to ensure that $g^{-1}(x)$ has a unique evaluation.

To show the claim, let $x \in A - \bigcup_{n \in \mathbb{N}} C_n$ be given. Then:

$$x \in A - \bigcup_{n \in \mathbb{N}} C_n \Rightarrow x \in A \wedge x \notin \bigcup_{n \in \mathbb{N}} C_n \Rightarrow x \notin \bigcup_{n \in \mathbb{N}} C_n \Rightarrow$$

$$\Rightarrow \overline{\exists n \in \mathbb{N} : x \in C_n} \Rightarrow$$
$$\Rightarrow \forall n \in \mathbb{N} : x \notin C_n \Rightarrow x \notin C_0.$$

To show that $x \in g(B)$, assume that $x \notin g(B)$. Then

$$\begin{cases} x \in A \\ x \notin g(B) \end{cases} \Rightarrow x \in A - g(B) \Rightarrow x \in C_0$$

which is a contradiction, since we previously showed that $x \notin C_0$.

We conclude that

$$\forall x \in A - \bigcup_{n \in \mathbb{N}} C_n : x \in g(B) \Rightarrow A - \bigcup_{n \in \mathbb{N}} C_n \subseteq g(B)$$

which proves the claim.

- We will show that <u>h is one-to-one.</u>

Let <u>$x_1, x_2 \in A$</u> be given and assume that <u>$h(x_1) = h(x_2)$.</u>
We distinguish between the following subcases.

<u>Case A:</u> Assume that $\begin{cases} \exists n \in \mathbb{N} : x_1 \in C_n \\ \exists n \in \mathbb{N} : x_2 \in C_n \end{cases}$

Then $h(x_1) = h(x_2) \Rightarrow f(x_1) = f(x_2)$ [definition of h]

$\Rightarrow \underline{x_1 = x_2}$ [f one-to-one]

<u>Case B:</u> Assume that $\begin{cases} \forall n \in \mathbb{N} : x_1 \notin C_n \\ \forall n \in \mathbb{N} : x_2 \notin C_n \end{cases}$. Then,

$$h(x_1) = h(x_2) \Rightarrow g^{-1}(x_1) = g^{-2}(x_2) \Rightarrow \text{[definition of } h\text{]}$$
$$\Rightarrow g(g^{-1}(x_1)) = g(g^{-1}(x_2)) \Rightarrow$$
$$\Rightarrow \underline{x_1 = x_2}.$$

<u>Case C:</u> Assume that $\begin{cases} \exists n \in \mathbb{N} : x_1 \in C_n \\ \forall n \in \mathbb{N} : x_2 \notin C_n \end{cases}$

Choose $n_0 \in \mathbb{N}$ such that $x_1 \in C_{n_0}$. We note that

$$\begin{cases} x_2 \in A \\ \forall n \in \mathbb{N} : x_2 \notin C_n \end{cases} \Rightarrow x_2 \in A - \bigcup_{n \in \mathbb{N}} C_n \Rightarrow g^{-1}(x_2) \text{ is defined}$$

and therefore:

$$\begin{aligned} x_2 &= g(g^{-1}(x_2)) \\ &= g(h(x_2)) && \text{[Definition of } h(x) \text{ - 2nd case]} \\ &= g(h(x_1)) && \text{[Hypothesis } h(x_1) = h(x_2)\text{]} \\ &= g(f(x_1)) && \text{[Definition of } h(x) \text{ - 1st case]} \end{aligned}$$

$\Rightarrow \exists x \in C_{n_0} : g(f(x)) = x_2 \Rightarrow$

$\Rightarrow x_2 \in \{g(f(x)) \mid x \in C_{n_0}\}$

$\Rightarrow x_2 \in g(f(C_{n_0}))$

$\Rightarrow x_2 \in C_{n_0+1}$

This is a contradiction because

$(\forall n \in \mathbb{N} : x_2 \notin C_n) \Rightarrow x_2 \notin C_{n_0+1}$

therefore Case C does not materialize. In all of the above cases we conclude that $x_1 = x_2$ and therefore:

$\forall x_1, x_2 \in A : (h(x_1) = h(x_2) \Rightarrow x_1 = x_2)$

$\Rightarrow h$ one-to-one. (4)

• 2 We will show that $h(A) = B$.

By definition, we have $h(A) \subseteq B$, so it is sufficient to show that $\forall y \in B : y \in h(A)$. Let $y \in B$ be given. We distinguish between the following cases.

Case 1 : Assume that $\exists n \in \mathbb{N} : y \in f(C_n)$.

Choose $n_0 \in \mathbb{N}$ such that $y \in f(C_{n_0})$. Since

$h(C_{n_0}) = \{h(x) \mid x \in C_{n_0}\}$

$= \{f(x) \mid x \in C_{n_0}\}$ [Definition of $h(x)$ – 1st case]

$= f(C_{n_0})$

it follows that

$y \in f(C_{n_0}) \Rightarrow y \in h(C_{n_0})$ [Because $h(C_{n_0}) = f(C_{n_0})$]

$\Rightarrow y \in h(A)$ [because $C_{n_0} \subseteq A$]

Case 2 : Assume that $\forall n \in \mathbb{N} : y \notin f(C_n)$.

We claim that $\forall n \in \mathbb{N} : g(y) \notin C_n$.

To show the claim, we note that:

$$\forall n \in \mathbb{N} : y \notin f(C_n) \Rightarrow \forall n \in \mathbb{N} : g(y) \notin g(f(C_n))$$
$$\Rightarrow \forall n \in \mathbb{N} : g(y) \notin C_{n+1}$$
$$\Rightarrow \forall n \in \mathbb{N}^* : g(y) \notin C_n \quad (5)$$

For $n=0$, to show that $g(y) \notin C_0$, we will assume that $g(y) \in C_0$ and derive a contradiction. Then:

$$g(y) \in C_0 \Rightarrow g(y) \in A - g(B)$$
$$\Rightarrow g(y) \in A \wedge g(y) \notin g(B)$$
$$\Rightarrow g(y) \notin g(B)$$

which is a contradiction because

$$y \in B \Rightarrow g(y) \in g(B)$$

It follows that $g(y) \notin C_0 \quad (6)$

From Eq.(5) and Eq.(6) we prove the claim. It follows that

$$h(g(y)) = g^{-1}(g(y)) \quad [\text{because } \forall n \in \mathbb{N} : g(y) \notin C_n]$$
$$= y \Rightarrow$$
$$\Rightarrow \exists x \in A : y = h(x) \quad (\text{for } x = g(y))$$
$$\Rightarrow \underline{y \in h(A)}$$

From the above argument we have:

$$\begin{cases} h(A) \subseteq B \\ \forall y \in B : y \in h(A) \end{cases} \Rightarrow \begin{cases} h(A) \subseteq B \\ B \subseteq h(A) \end{cases} \Rightarrow h(A) = B \Rightarrow \underline{h \text{ onto}} \quad (7)$$

From Eq.(4) and Eq.(7):

$$\begin{cases} h \text{ one-to-one} \\ h \text{ onto} \end{cases} \Rightarrow h : A \to B \text{ bijection}$$
$$\Rightarrow A \sim B \qquad \square$$

3 → Uncountability of $\mathbb{R}$

The Schroeder-Bernstein theorem can be used to derive the following characterization for the cardinality of $\mathbb{R}$:

$$\boxed{\mathbb{R} \sim \mathcal{P}(\mathbb{N})}$$

Once this result is established, we can use Cantor's theorem to argue that:

$$\begin{cases} \mathbb{R} \sim \mathcal{P}(\mathbb{N}) \\ \mathcal{P}(\mathbb{N}) > \mathbb{N} \end{cases} \Rightarrow \mathbb{R} > \mathbb{N} \Rightarrow \mathbb{R} \text{ uncountable.}$$

The argument below uses the previous result that $\mathbb{R} \sim (0,1)$.

▶ Proof of $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$

It is sufficient to show that $\mathcal{P}(\mathbb{N}) \leq \mathbb{R} \wedge \mathbb{R} \leq \mathcal{P}(\mathbb{N})$.

•$_1$ Proof of $\mathcal{P}(\mathbb{N}) \leq \mathbb{R}$.

We define a mapping $f : \mathcal{P}(\mathbb{N}) \to [0,1]$ as follows. Given $X \in \mathcal{P}(\mathbb{N})$ we define $f(X)$ via the expansion

$$f(X) = (0.a_0 a_1 a_2 \cdots)_{10} =$$

$$= \sum_{n=0}^{+\infty} a_n 10^{-n-1}$$

with

$$\forall n \in \mathbb{N} : a_n = \begin{cases} 1 & , \text{if } n \in X \\ 0 & , \text{if } n \notin X \end{cases}$$

To show that f is one-to-one, it is necessary to define it using a base representation that is greater than binary (i.e. base 2) while restricting the digits used to 0 and 1. This way, a number that terminates with an infinite sequence of 1s (e.g. 0.101111...) will not have an second alternate representation, as it would have in the binary system. We may therefore now argue as follows: Let $X_1, X_2 \in \mathcal{P}(\mathbb{N})$ be given and assume that $f(X_1) = f(X_2)$. Define the sequences (a_n) and (b_n) via the decimal representations:

$$f(X_1) = 0.a_0a_1a_2\ldots = \sum_{n=0}^{+\infty} a_n \cdot 10^{-n-1}$$

$$f(X_2) = 0.b_0b_1b_2\ldots = \sum_{n=0}^{+\infty} b_n \cdot 10^{-n-1}$$

We note that

$$f(X_1) = f(X_2) \Rightarrow 0.a_0a_1a_2\ldots = 0.b_0b_1b_2\ldots \Rightarrow$$
$$\Rightarrow \forall n \in \mathbb{N} : a_n = b_n.$$

We use this result to show that

$$\begin{aligned} n \in X_1 &\Leftrightarrow a_n = 1 && [\text{definition of } a_n] \\ &\Leftrightarrow b_n = 1 && [\text{via } a_n = b_n] \\ &\Leftrightarrow n \in X_2 && [\text{definition of } b_n] \end{aligned}$$

It follows that $\underline{X_1 = X_2}$. We have thus shown that

$$\forall X_1, X_2 \in \mathcal{P}(\mathbb{N}) : (f(X_1) = f(X_2) \Rightarrow X_1 = X_2)$$
$$\Rightarrow f \text{ one-to-one} \Rightarrow \mathcal{P}(\mathbb{N}) \preccurlyeq [0,1]$$

We also have: $[0,1] \subseteq \mathbb{R} \Rightarrow [0,1] \preccurlyeq \mathbb{R}$

and therefore

$$\left.\begin{cases} \mathcal{P}(\mathbb{N}) \leqslant [0,1] \\ [0,1] \leqslant \mathbb{R} \end{cases}\right. \Longrightarrow \underline{\mathcal{P}(\mathbb{N}) \leqslant \mathbb{R}.} \qquad (1)$$

•$_2$ Proof of $\mathbb{R} \leqslant \mathcal{P}(\mathbb{N})$.

We define a mapping $g : [0,1] \to \mathcal{P}(\mathbb{N})$ as follows.

Let $x \in [0,1]$ be given with binary representation

$$x = (0.a_0 a_1 a_2 \cdots)_2 = \sum_{n=0}^{+\infty} a_n 2^{-n-1}$$

To ensure uniqueness, we do not allow terminating the binary representation of x with an infinite sequence of 1s except for $x = 1$ (represented as $x = (0.1111\cdots)_2$)

Define $g(x) = \{n \in \mathbb{N} \mid a_n = 1\}$

Let $\underline{x_1, x_2 \in [0,1]}$ be given and assume that $\underline{g(x_1) = g(x_2)}$.

Define the sequences (a_n) and (b_n) via the unique binary representations (as explained above)

$$x_1 = (0.a_0 a_1 a_2 \cdots)_2$$
$$x_2 = (0.b_0 b_1 b_2 \cdots)_2$$

To show that $x_1 = x_2$, we assume that $x_1 \neq x_2$ and derive a contradiction. Then, we have

$$\begin{aligned} x_1 \neq x_2 &\Rightarrow (0.a_0 a_1 a_2 \cdots)_2 \neq (0.b_0 b_1 b_2 \cdots)_2 \\ &\Rightarrow \overline{\forall n \in \mathbb{N} : a_n = b_n} \\ &\Rightarrow \exists n \in \mathbb{N} : a_n \neq b_n \end{aligned}$$

Choose $n_0 \in \mathbb{N}$ such that $a_{n_0} \neq b_{n_0}$. It follows that

$$a_{n_0} \neq b_{n_0} \Rightarrow \begin{cases} a_{n_0} = 1 \\ b_{n_0} = 0 \end{cases} \vee \begin{cases} a_{n_0} = 0 \\ b_{n_0} = 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} n_0 \in g(x_1) \\ n_0 \notin g(x_2) \end{cases} \veebar \begin{cases} n_0 \notin g(x_1) \\ n_0 \in g(x_2) \end{cases} \Rightarrow$$

$$\Rightarrow (\exists n \in g(x_1) : n \notin g(x_2)) \vee (\exists n \in g(x_2) : n \notin g(x_1))$$

$$\Rightarrow \overline{(\forall n \in g(x_1) : n \in g(x_2))} \veebar \overline{(\forall n \in g(x_2) : n \in g(x_1))}$$

$$\Rightarrow g(x_1) \nsubseteq g(x_2) \veebar g(x_2) \nsubseteq g(x_1)$$

which is a contradiction because

$$g(x_1) = g(x_2) \Rightarrow \begin{cases} g(x_1) \subseteq g(x_2) \\ g(x_2) \subseteq g(x_1) \end{cases}$$

We have thus shown that $\underline{x_1 = x_2}$

From the above argument we have show that

$$\forall x_1, x_2 \in [0,1] : (g(x_1) = g(x_2) \Rightarrow x_1 = x_2)$$

$$\Rightarrow g \text{ one-to-one} \Rightarrow [0,1] \leqslant \mathcal{P}(\mathbb{N}).$$

and therefore:

$$\begin{aligned} \mathbb{R} &\sim (0,1) && [\text{previous result}] \\ &\leqslant [0,1] && [\text{via } (0,1) \subseteq [0,1]] \\ &\leqslant \mathcal{P}(\mathbb{N}) && [\text{above proof}] \end{aligned}$$

$$\Rightarrow \underline{\mathbb{R} \leqslant \mathcal{P}(\mathbb{N})} \qquad (2)$$

From Eq.(1) and Eq.(2) via the Schroeder-Bernstein theorem, it follows that

$$\begin{cases} \mathcal{P}(\mathbb{N}) \leqslant \mathbb{R} \\ \mathbb{R} \leqslant \mathcal{P}(\mathbb{N}) \end{cases} \Rightarrow \mathbb{R} \sim \mathcal{P}(\mathbb{N}). \qquad \square$$

EXERCISES

(13) Study the proofs for

a) The Cantor theorem

b) The Schroder – Bernstein theorem

c) The statement $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$.

(14) Use Exercise 9 and the previous results that $\mathbb{Q} \sim \mathbb{N}$ and $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$ to show that $\mathbb{R} - \mathbb{Q}$ (the set of irrational numbers) is uncountable.
(Hint: Use proof by contradiction)

(15) Show that, given 3 sets A, B, C, we have:

a) $A \leq B \wedge B \leq C \Rightarrow A \leq C$

b) $(A \leq B \leq C \wedge A \sim C) \Rightarrow (B \sim C \wedge A \sim B)$

c) $A \sim B \wedge B \subseteq C \Rightarrow A \leq C$.

(16) Consider the sets

$$\mathbb{R}_+^* = \{x \in \mathbb{R} \mid x > 0\}$$
$$\mathbb{R}_-^* = \{x \in \mathbb{R} \mid x < 0\}$$

Use the Schroder – Bernstein theorem to show that $\mathbb{R} \sim \mathbb{R}_+^*$ and $\mathbb{R} \sim \mathbb{R}_-^*$

(Hint: The needed one-to-one mappings can be constructed using the exponential function)

(Another hint: It is sufficient to show $\mathbb{R}_+^* \geq \mathbb{R}$ and $\mathbb{R}_-^* \geq \mathbb{R}$).

(17) Use Exercise 16 to show that given two sets A, B we have:

$A \sim \mathbb{R} \wedge B \sim \mathbb{R} \Rightarrow A \cup B \sim \mathbb{R}$.

(Hint: Distinguish between the following cases. For case 1 assume that $A \cap B = \emptyset$. For case 2 assume that $A \cap B \neq \emptyset$. Define $B_1 = B - A$, show that $A \cup B = A \cup B_1$ and use Case 1 and the Schroeder–Bernstein theorem to show that $A \cup B_1 \sim \mathbb{R}$).

(18) Use the Schoeder–Bernstein theorem to show that $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$.

(Hint: Use binary or decimal representations to show that $[0,1] \times [0,1] \sim [0,1]$ by defining one-to-one mappings $f: [0,1] \times [0,1] \to [0,1]$ and $g: [0,1] \to [0,1] \times [0,1]$. Then uplift this result to the statement $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$)

Cardinal numbers

- To introduce the concept of cardinality and cardinal numbers, we note first that

$$\forall n, m \in \mathbb{N}^* : \left(\begin{cases} A \sim [n] \\ A \sim [m] \end{cases} \Longrightarrow n = m \right)$$

Thus, for finite sets A, we can define a unique integer $|A|$ such that $A \sim [|A|]$.

- $|A|$ is the number of elements in A and we call it the <u>cardinality</u> of A.
- Cantor proposed introducing "transfinite cardinal numbers" to denote the cardinality $|A|$ of infinite sets. A key requirement of this cardinal number arithmetic is that it should satisfy:

$$A \sim B \iff |A| = |B|$$
$$A < B \iff |A| \leq |B|$$
$$A \lneqq B \iff |A| < |B|$$

The Schroeder-Bernstein theorem ensures self-consistent behaviour of inequalities in cardinal arithmetic.

- Since $\mathbb{N} \sim \mathbb{Z} \sim \mathbb{Q}$, Cantor introduced the cardinal number $\aleph_0$ to represent the cardinality of countably infinite sets. Consequently, we may write

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \aleph_0$$

- <u>Aleph sequence</u>: Cantor proposed defining a sequence of cardinalities $\aleph_1, \aleph_2, \aleph_3, \ldots$ as follows.

Let $\mathbb{V}$ be the set of all sets that exist. We define:

$|A| = \aleph_1 \Leftrightarrow \forall B_1 \in \mathbb{V} : \overline{\mathbb{N} < B_1 < A}$

$|A| = \aleph_2 \Leftrightarrow \forall B_1, B_2 \in \mathbb{V} : \overline{\mathbb{N} < B_1 < B_2 < A}$

$|A| = \aleph_3 \Leftrightarrow \forall B_1, B_2, B_3 \in \mathbb{V} : \overline{\mathbb{N} < B_1 < B_2 < B_3 < A}$

etc.

- Beth sequence : Another sequence of cardinal numbers is the beth sequence. It is based on the Cantor theorem that tells us that $A < \mathcal{P}(A)$. The beth sequence is defined as follows:

  $\beth_0 = \aleph_0 = |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$

  $\beth_1 = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$

  $\beth_2 = |\mathcal{P}(\mathcal{P}(\mathbb{N}))|$

  $\beth_3 = |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))|$

  etc.

- Continuum hypothesis : With the above definitions, Cantor posed the question of whether the aleph and beth sequences coincide. This leads to two questions:

a) Continuum Hypothesis: The claim that $\beth_1 = \aleph_1$.

b) General Continuum Hypothesis: The claim that $\beth_a = \aleph_a$ for all a.

It was later found that these hypotheses are undecidable, i.e. it can neither be proved true or false. The underlying problem is that for the case of infinite sets, the mechanism for generating the powerset $\mathcal{P}(A)$ of an infinite set A is not precisely given. As a result, we have no way of deducing the correct "size" of $\mathcal{P}(\mathbb{N})$, $\mathcal{P}(\mathcal{P}(\mathbb{N}))$, etc.