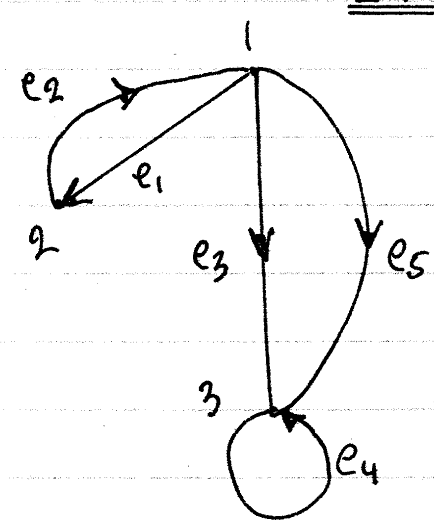
BASIC GRAPH THEORY

▼ Directed Graphs

Def: A directed graph G is an object that consists of
a) A set of vertices $V(G)$
b) A set of edges $E(G)$
c) An incidence mapping $\psi_G : E(G) \to V(G) \times V(G)$ that maps every edge $e \in E(G)$ to a unique pair of vertices $(u_1, u_2) \in V(G) \times V(G)$.

▸ Graphical representation: Each vertex $u \in V(G)$ is represented as a point on a plane. Each edge $e \in E(G)$ with $\psi_G(e) = (u_1, u_2)$ is represented as an arrow from u_1 to u_2. If $\psi_G(e) = (u, u)$ then the edge is a loop and is represented by an arrow that begins at u and loops back to terminate at u.

EXAMPLE



$V(G) = \{1, 2, 3\}$
$E(G) = \{e_1, e_2, e_3, e_4, e_5\}$
$\psi_G(e_1) = (1, 2)$ $\quad \psi_G(e_4) = (3, 3)$
$\psi_G(e_2) = (2, 1)$ $\quad \psi_G(e_5) = (1, 3)$
$\psi_G(e_3) = (1, 3)$

↳ In this example note that $e_3 \neq e_5$ but nonetheless $\psi_G(e_3) = \psi_G(e_5)$

↳ ψ_G can be also represented as a set $\psi_G \subseteq E(G) \times (V(G) \times V(G))$ with

$$\psi_G = \{(e_1,(1,2)),(e_2,(2,1)),(e_3,(1,3)),(e_4,(3,3)),(e_5,(1,3))\}.$$

▸ Successor vertices

Def: Let G be a graph and let $u_1, u_2 \in V(G)$ be two vertices. We say that u_2 is successor of $u_1 \Leftrightarrow \exists e \in E(G) : \psi_G(e) = (u_1, u_2)$

• notation: The set of all successors of a vertex $u \in V(G)$ is denoted as:

$$succ(u) = \{ w \in V(G) \mid \exists e \in E(G) : \psi_G(e) = (u,w) \}$$

EXAMPLE

In the previous example:

$succ(1) = \{2,3\}$

$succ(2) = \{1\}$

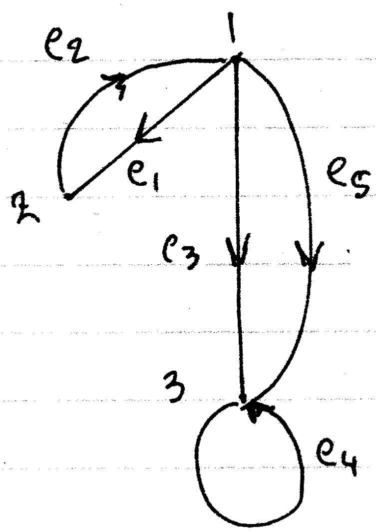
$succ(3) = \{3\}$

▸ Adjacency matrix

> Let G be a graph with $|V(G)| = n$ (i.e. with n vertices labeled as $V(G) = \{u_1, u_2, u_3, \dots, u_n\}$). The adjacency matrix $A(G) \in M_n(\mathbb{R})$ is an $n \times n$ square matrix such that
>
> $$\forall a, b \in [n]: \; [A(G)]_{ab} = |\{e \in E(G) \mid \psi_G(e) = (u_a, u_b)\}|$$

EXAMPLE

For the graph in the previous example:



$$A(G) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Adjacency matrices make it easy to define the concept of a simple graph. We say that a graph G is <u>simple</u> if and only if it contains no loops and no double or multiple edges. A rigorous definition is:

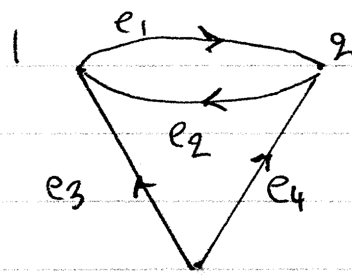
$$\boxed{G \text{ simple} \Leftrightarrow \begin{cases} \forall a,b \in [|V(G)|] : A_{ab}(G) \in \{0,1\} \\ \forall a \in [|V(G)|] : A_{aa}(G) = 0 \end{cases}}$$

The first condition rules out multiple edges and the second condition rules out loops.

EXERCISES

① Define the sets $V(G)$, $E(G)$, the mapping ψ_G, and the adjacency matrix $A(G)$ for the directed graphs shown below:

a)



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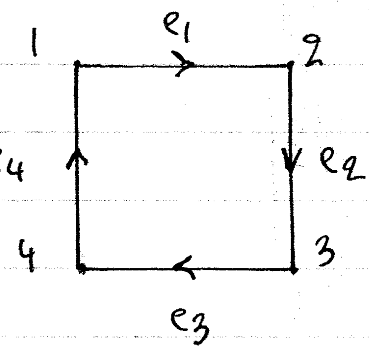
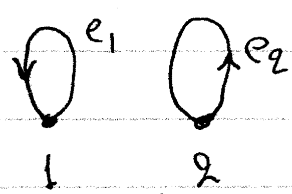
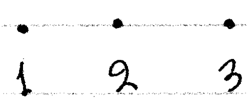


c)

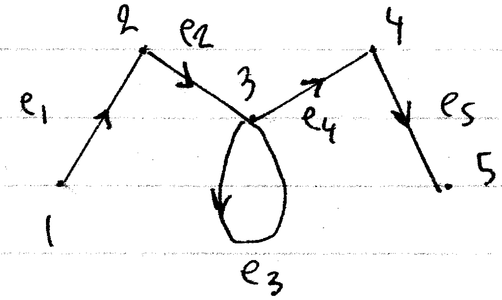


d)

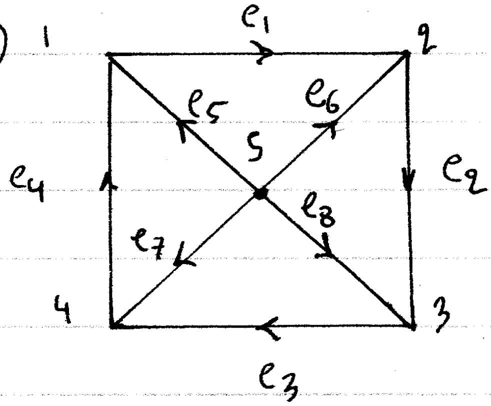
e)



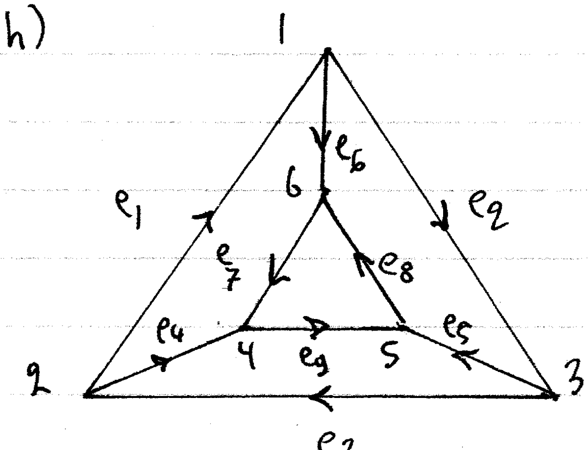
f)



g)



h)



(2) Identify which of the above directed graphs ~~is~~ are simple.

(3) Draw the directed graphs G given by the following set theory definitions: and define the corresponding $A(G)$

a) $V(G) = \{1,2,3,4\}$

$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

$\psi_G(e_1) = (1,3) \qquad \psi_G(e_4) = (2,4)$

$\psi_G(e_2) = (2,2) \qquad \psi_G(e_5) = (4,3)$

$\psi_G(e_3) = (3,1) \qquad \psi_G(e_6) = (1,1)$

b) $V(G) = \{2,3\}$, $E(G) = \emptyset$, $\psi_G = \emptyset$

c) $V(G) = \{1\}$, $E(G) = \{e_1\}$, $\psi_G(e_1) = (1,1)$

d) $V(G) = \{1,2,3\}$

$E(G) = \{e_1, e_2, e_3, e_4, e_5\}$

$\psi_G = \{(e_1,(1,1)), (e_2,(1,3)), (e_3,(2,3)), (e_4,(2,3)), (e_5,(3,3))\}$

e) $V(G) = \{1,2,3,4\}$

$E(G) = \{a,b,c,d,e,f,g,h\}$

$\psi_G(a) = (1,1) \qquad \psi_G(e) = (3,3)$

$\psi_G(b) = (1,2) \qquad \psi_G(f) = (3,4)$

$\psi_G(c) = (2,2) \qquad \psi_G(g) = (4,4)$

$\psi_G(d) = (2,3) \qquad \psi_G(h) = (4,1)$

f) $V(G) = \{1,2\}$

$E(G) = \{e_1, e_2, e_3\}$

$\psi_G = \{(e_1, (1,1)), (e_2, (1,2)), (e_3, (1,1))\}$

▼ Walks

Def: Let G be a directed graph. A walk w is an n-tuple of the form

$$w = (u_0, e_1, u_1, e_2, u_2, \dots, e_n, u_n)$$

of alternating edges/vertices such that

$$\begin{cases} \forall a \in [n] : e_a \in E(G) \\ \forall a \in \{0\} \cup [n] : u_a \in V(G) \\ \forall a \in [n] : \psi_G(e_a) = (u_{a-1}, u_a) \end{cases}$$

▶ Terminology

$|w| = n \longleftarrow$ length of the walk

$\$(w) = u_0 \longleftarrow$ initial vertex

$t(w) = u_n \longleftarrow$ terminal vertex

$u_a(w) = u_a \longleftarrow$ the a^{th} vertex, counting from 0

$e_a(w) = e_a \longleftarrow$ the a^{th} edge, counting from 1

$W(G) \longleftarrow$ the set of all walks on G.

Def: Let G be a graph and choose two vertices $u, v \in V(G)$. We define

a) The set of all walks that begin with u and terminate at v:

$$W(G|u,v) = \{w \in W(G) \mid \$(w) = u \wedge t(w) = v\}$$

b) The set of all walks with length n that begin with u and terminate at v:

$$W_n(G|u,v) = \{w \in W(G) \mid \$(w) = u \wedge t(w) = v \wedge |w| = n\}$$

▸ Enumeration of walks

The set $W(G|u,v)$ has an infinite number of elements. However, the set $W_n(G|u,v)$ can be enumerated using the adjacency matrix according to the following statement.

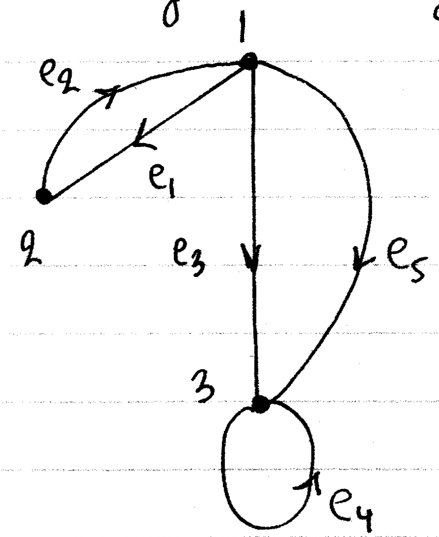
Thm: Let G be a graph with $V(G) = \{u_1, u_2, \ldots, u_m\}$ and corresponding adjacency matrix $A(G)$. Then
$$\forall a,b \in [m] : |W_n(G|u_a, u_b)| = [A^n(G)]_{ab}$$

The n^{th} power $A^n(G)$ of the adjacency matrix is defined recursively as follows:

$$\forall a,b \in [m] : [A^1(G)]_{ab} = [A(G)]_{ab}$$
$$\forall a,b \in [m] : \forall k \in \mathbb{N}^* : [A^{k+1}(G)]_{ab} = \sum_{c=1}^{m} [A^k(G)]_{ac} [A(G)]_{cb}$$

EXAMPLE

Use the adjacency matrix to enumerate the walks with length 3 from vertex 1 to 3 for the following directed graph.



Solution

We have

$$A(G) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow A^2(G) = A(G)A(G) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0\cdot 0 + 1\cdot 1 + 2\cdot 0 & 0\cdot 1 + 1\cdot 0 + 2\cdot 0 & 0\cdot 2 + 1\cdot 0 + 2\cdot 1 \\ 1\cdot 0 + 0\cdot 1 + 0\cdot 0 & 1\cdot 1 + 0\cdot 0 + 0\cdot 0 & 1\cdot 2 + 0\cdot 0 + 0\cdot 1 \\ 0\cdot 0 + 0\cdot 1 + 1\cdot 0 & 0\cdot 1 + 0\cdot 0 + 1\cdot 0 & 0\cdot 2 + 0\cdot 0 + 1\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow A^3(G) = A^2(G)\,A(G) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1\cdot 0 + 0\cdot 1 + 2\cdot 0 & 1\cdot 1 + 0\cdot 0 + 2\cdot 0 & 1\cdot 2 + 0\cdot 0 + 2\cdot 1 \\ 0\cdot 0 + 1\cdot 1 + 2\cdot 0 & 0\cdot 1 + 1\cdot 0 + 2\cdot 0 & 0\cdot 2 + 1\cdot 0 + 2\cdot 1 \\ 0\cdot 0 + 0\cdot 1 + 1\cdot 0 & 0\cdot 1 + 0\cdot 0 + 1\cdot 0 & 0\cdot 2 + 0\cdot 0 + 1\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow |W_3(G|1,3)| = [A^3(G)]_{13} = 4$$

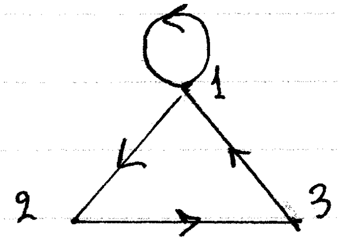
Remark: By inspection, the 4 walks from vertex 1 to vertex 3 with length 3 can be easily identified as follows:

$W_1 = (1, e_3, 3, e_4, 3, e_4, 3)$
$W_2 = (1, e_5, 3, e_4, 3, e_4, 3)$
$W_3 = (1, e_1, 2, e_2, 1, e_3, 3)$
$W_4 = (1, e_1, 2, e_2, 1, e_5, 3)$

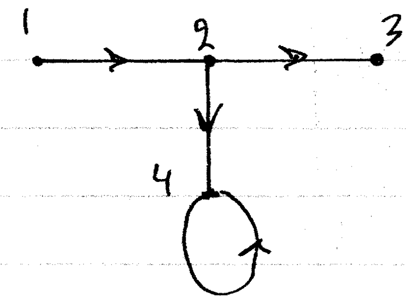
EXERCISES

(4) Enumerate the total number of open and closed walks of length 3 for the graphs shown below, using the adjacency matrix

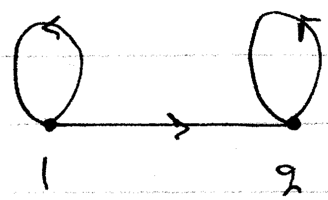
a)



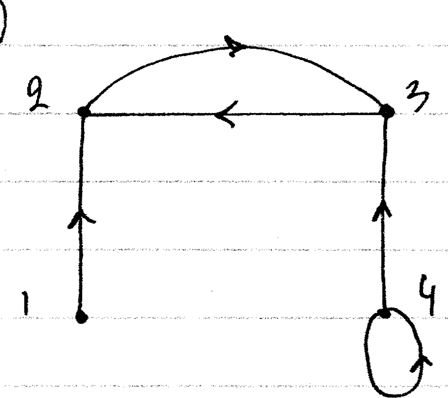
b)



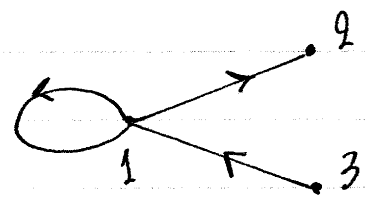
c)



d)



e)



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