SERIES SOLUTION OF ODES

We begin by reviewing, and in some cases, extending, results from Calculus II needed for solving linear ODEs via convergent series methods.

▼ The Gamma function

We recall from my Calculus 2 lecture notes the definition of the factorial and the double factorial:

▸ Factorial:

$$\boxed{\begin{aligned} &0! = 1 \\ &\forall n \in \mathbb{N}^*: \; n! = \prod_{k=1}^{n} k = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \end{aligned}}$$

▸ Double Factorial:

$$\boxed{\begin{aligned} &0!! = 1 \text{ and } 1!! = 1 \\ &\forall n \in \mathbb{N}^*: \; (2n)!! = \prod_{k=1}^{n} (2k) = 2^n n! \\ &\forall n \in \mathbb{N}^*: \; (2n+1)!! = \prod_{k=1}^{n} (2k+1) = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1) \end{aligned}}$$

The Gamma function $\Gamma(n)$ generalizes the factorial and is defined, first on $(0,+\infty)$ and then on a wider set as follows.

Def: (Gamma function on $(0,+\infty)$)

$$\forall n \in (0,+\infty): \; \Gamma(n) = \int_{0^+}^{+\infty} x^{n-1} e^{-x} \, dx$$

Then, we show that:

$$\boxed{\begin{array}{l}\underline{\text{Prop}}: \\ \text{a) } \forall n \in (0,+\infty): \text{The } \Gamma(n) \text{ integral converges} \\ \text{b) } \Gamma(1) = 1 \\ \text{c) } \forall n \in (0,+\infty): \Gamma(n+1) = n\Gamma(n)\end{array}}$$

It immediately follows that

$$\boxed{\forall n \in \mathbb{N}^* : \Gamma(n) = (n-1)!}$$

But n is a continuous variable and $\Gamma(n)$ has been defined on $n \in (0,+\infty)$. So, $\Gamma(n)$ generalizes the factorial on a continuous set. We can now use the equation $\Gamma(n) = \Gamma(n+1)/(n+1)$ to extend the definition of the Gamma function for negative n as follows:

$$\forall n \in (-1,0): \Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\forall n \in (-2,-1): \Gamma(n) = \frac{\Gamma(n+1)}{n} = \frac{\Gamma(n+2)}{n(n+1)}$$

$$\forall n \in (-3,-2): \Gamma(n) = \frac{\Gamma(n+2)}{n(n+1)} = \frac{\Gamma(n+3)}{n(n+1)(n+2)}$$

and so on. The general definition of the Gamma function for negative numbers is:

Def : (Gamma function for negative numbers)

$$\forall k \in \mathbb{N}^* : \forall n \in (-k, -k+1) : \Gamma(n) = \frac{\Gamma(n+k)}{\prod_{a=0}^{k-1} (n-a)} = \frac{\Gamma(n+k)}{n(n+1)\cdots(n+k-1)}$$

Proof of proposition

The proof requires the following lemma.

Lemma: $\boxed{\forall a \in \mathbb{R} : \lim_{x \to +\infty} x^a e^{-x} = 0}$

Proof

Let $a \in \mathbb{R}$ be given. We distinguish between the following cases.

Case 1 : For $a \in (-\infty, 0)$, we have

$$\left(\lim_{x \to +\infty} x^a = 0 \wedge \lim_{x \to +\infty} e^{-x} = 0 \right) \Rightarrow \lim_{x \to +\infty} x^a e^{-x} = 0.$$

Case 2 : For $a = 0$, we have

$$\lim_{x \to +\infty} x^a e^{-x} = \lim_{x \to +\infty} x^0 e^{-x} = \lim_{x \to +\infty} e^{-x} = 0$$

Case 3 : For $a \in (0, +\infty)$, we define $n = \max\{k \in \mathbb{N} \mid a - k > 0\}$.

We evaluate the limit by applying De L'Hospital $n+1$ times:

$$\lim_{x \to +\infty} x^a e^{-x} = \lim_{x \to +\infty} \frac{x^a}{e^x} = \lim_{x \to +\infty} \frac{a(a-1)\cdots(a-n)\, x^{a-(n+1)}}{e^x} =$$

$$= a(a-1)\cdots(a-n) \lim_{x \to +\infty} x^{a-(n+1)} e^{-x} = 0$$

because, by definition of n, $a-(n+1) < 0$. $\square$

For the convergence proof we use the following theorems from Calculus II:

1) Comparison test

$$\left.\begin{array}{l}\forall x\in S: 0\leq f(x)\leq g(x)\\ \int_S g(x)\,dx \text{ converges}\end{array}\right\} \Rightarrow \int_S f(x)\,dx \text{ converges.}$$

2) Ratio test

$$\left.\begin{array}{l}\forall x\in S: (f(x)\geq 0 \wedge g(x)\geq 0)\\ \lim_{x\to\sigma} \dfrac{f(x)}{g(x)} = 0\end{array}\right\} \Rightarrow \left(\int_S g(x)\,dx \text{ converges} \Rightarrow \int_S f(x)\,dx \text{ converges}\right)$$

The proofs are as follows:

Proof of (a): Let $n\in(0,+\infty)$ be given.

We write

$$\Gamma(n) = \int_{0^+}^{+\infty} x^{n-1}e^{-x}\,dx = \int_{0^+}^{1} x^{n-1}e^{-x}\,dx + \int_{1}^{+\infty} x^{n-1}e^{-x}\,dx$$

For the $(1,+\infty)$ integral, we define

$$\begin{cases}\forall x\in(1,+\infty): f(x) = x^{n-1}e^{-x} > 0\\ \forall x\in(1,+\infty): g(x) = 1/x^2 > 0\end{cases}$$

and therefore:

$$\forall x\in(1,+\infty): \frac{f(x)}{g(x)} = \frac{x^{n-1}e^{-x}}{1/x^2} = x^2x^{n-1}e^{-x} = x^{n+1}e^{-x}$$

$$\Rightarrow \lim_{x\to+\infty}\frac{f(x)}{g(x)} = \lim_{x\to+\infty}(x^{n+1}e^{-x}) = 0 \qquad (1)$$

From Eq.(1) and the ratio test it follows that:

$$\int_1^{+\infty} \frac{dx}{x^2} \text{ converges} \Rightarrow \int_1^{+\infty} x^{n-1} e^{-x}\, dx \text{ converges.} \quad (2)$$

For the $(0,1)$ integral, let $x \in (0,1)$ be given. Then:

$x \in (0,1) \Rightarrow 0 < x < 1 \Rightarrow -1 < -x < 0 \Rightarrow 0 < e^{-x} < e^0 \Rightarrow$

$\Rightarrow 0 < e^{-x} < 1 \Rightarrow 0 < x^{n-1} e^{-x} < x^{n-1}$ (since $x^{n-1} > 0$).

It follows that

$\forall x \in (0,1) : 0 < x^{n-1} e^{-x} < x^{n-1} \quad (3)$

From Eq. (3) and via the comparison test, we argue that

$$n > 0 \Rightarrow n-1 > -1 \Rightarrow \int_{0^+}^1 x^{n-1}\, dx \text{ converges} \Rightarrow \int_{0^+}^1 x^{n-1} e^{-x}\, dx \text{ converges.} \quad (4)$$

From Eq. (2) and Eq. (4): $\Gamma(n) = \int_{0^+}^{+\infty} x^{n-1} e^{-x}\, dx$ converges. $\square$

<u>Proof of (b)</u> : Claim $\Gamma(1) = 1$

$$\Gamma(1) = \int_{0^+}^{+\infty} x^{1-1} e^{-x}\, dx = \int_{0^+}^{+\infty} x^0 e^{-x}\, dx = \int_{0^+}^{+\infty} e^{-x}\, dx = \Big[-e^{-x}\Big]_{0^+}^{+\infty} =$$

$$= \lim_{x \to +\infty} (-e^{-x}) - \lim_{x \to 0^+} (-e^{-x}) = (-0) - (-e^0) = 1$$

<u>Proof of (c)</u> : Claim $\forall n \in (0, +\infty) : \Gamma(n+1) = n\Gamma(n)$

Let $n \in (0, +\infty)$ be given. Then:

$$\Gamma(n+1) = \int_{0^+}^{+\infty} x^{(n+1)-1} e^{-x}\, dx = \int_{0^+}^{+\infty} x^n e^{-x}\, dx = \int_{0^+}^{+\infty} x^n (-e^{-x})'\, dx$$

$$= \Big[-x^n e^{-x}\Big]_{0^+}^{+\infty} - \int_{0^+}^{+\infty} (x^n)' (-e^{-x})\, dx =$$

$$= \lim_{x \to +\infty} (-x^n e^{-x}) - (-0^n e^0) - \int_{0^+} n x^{n-1} (-e^{-x})\, dx =$$

$$= 0 - 0 + n\int_{0^+}^{+\infty} x^{n-1} e^{-x}\,dx = n\Gamma(n)$$

and therefore $\forall n \in (0, +\infty): \Gamma(n+1) = n\Gamma(n)$. $\square$

● <u>Value of $\Gamma(1/2)$</u>: $\boxed{\Gamma(1/2) = \sqrt{\pi}}$

To show that $\Gamma(1/2) = \sqrt{\pi}$ we use the following result from Calculus 3:

$$\int_0^{+\infty} dx \int_0^{+\infty} dy\, f(x,y) = \int_0^{+\infty} r\,dr \int_0^{\pi/2} d\theta\, f(r\cos\theta, r\sin\theta)$$

<u>Proof</u>

We define $u = \sqrt{x} \Rightarrow du = \dfrac{dx}{2\sqrt{x}} \Rightarrow x^{-1/2}\,dx = 2\,du$

and note that $x = 0 \Leftrightarrow u = 0$ and $x \to +\infty \Leftrightarrow u \to +\infty$.

It follows that

$$\Gamma(1/2) = \int_{0^+}^{+\infty} x^{1/2-1} e^{-x}\,dx = \int_{0^+}^{+\infty} x^{-1/2} e^{-x}\,dx = \int_{0^+}^{+\infty} e^{-u^2}\,2\,du$$

$$= 2\int_0^{+\infty} \exp(-u^2)\,du \Rightarrow$$

$$\Rightarrow [\Gamma(1/2)]^2 = \left[2\int_0^{+\infty} \exp(-u^2)\,du\right]\left[2\int_0^{+\infty} \exp(-v^2)\,dv\right] =$$

$$= 4\int_0^{+\infty} du \int_0^{+\infty} dv\, \exp(-u^2 - v^2)$$

$$= 4\int_0^{+\infty} r\,dr \int_0^{\pi/2} d\theta\, \exp(-r^2\cos^2\theta - r^2\sin^2\theta)$$

$$= 4\int_0^{+\infty} r\,dr \int_0^{\pi/2} d\vartheta\, \exp(-r^2(\sin^2\vartheta + \cos^2\vartheta))$$

$$= 4\int_0^{+\infty} r\,dr \int_0^{\pi/2} d\vartheta\, \exp(-r^2) = 4\int_0^{+\infty} r\exp(-r^2)\left[\int_0^{\pi/2} d\vartheta\right] dr$$

$$= 4\int_0^{+\infty} r\exp(-r^2)(\pi/2)\,dr = \pi\int_0^{+\infty} 2r\exp(-r^2)\,dr =$$

$$= \pi\int_0^{+\infty} [-\exp(-r^2)]'\,dr = \pi\left[-\exp(-r^2)\right]_0^{+\infty} =$$

$$= \pi\left[\lim_{x\to+\infty}(-\exp(-r^2)) - (-\exp(-0))\right] = \pi[0-(-1)] = \pi$$

$$\Rightarrow \Gamma(1/2) = \sqrt{\pi} \lor \Gamma(1/2) = -\sqrt{\pi}. \qquad (1)$$

Since $(\forall u \in (0, +\infty) : \exp(-u^2) > 0) \Rightarrow$

$$\Rightarrow \Gamma(1/2) = 2\int_0^{+\infty} \exp(-u^2)\,du > 0 \qquad (2)$$

From Eq. (1) and Eq. (2) it follows that $\Gamma(1/2) = \sqrt{\pi}$.

EXAMPLE

Use proof by induction to show that given an $a \in \mathbb{R} - (-1)\mathbb{N}^*$ with $(-1)\mathbb{N}^* = \{-x \mid x \in \mathbb{N}^*\} = \{-1, -2, -3, \dots\}$, we have:

$$\boxed{\forall n \in \mathbb{N}^* : \prod_{k=1}^{n} (k+a) = \frac{\Gamma(n+1+a)}{\Gamma(a+1)}}$$

↳ This result is VERY useful for rewritting products in terms of Gamma functions.

Solution

For $n = 1$, we have:

$$\prod_{k=1}^{n} (k+a) = (1+a) = \frac{(a+1)\Gamma(a+1)}{\Gamma(a+1)} = \frac{\Gamma(a+2)}{\Gamma(a+1)} = \frac{\Gamma(1+1+a)}{\Gamma(a+1)} =$$

$$= \frac{\Gamma(n+1+a)}{\Gamma(a+1)}$$

For $n = m$, we assume that $\prod_{k=1}^{m} (k+a) = \dfrac{\Gamma(m+1+a)}{\Gamma(a+1)}$

For $n = m+1$, we will show that $\prod_{k=1}^{m+1} (k+a) = \dfrac{\Gamma((m+1)+1+a)}{\Gamma(a+1)}$ as follows:

$$\prod_{k=1}^{m+1} (k+a) = (m+1+a)\prod_{k=1}^{m} (k+a) = (m+1+a) \cdot \frac{\Gamma(m+1+a)}{\Gamma(a+1)} =$$

$$= \frac{(m+1+a)\Gamma(m+1+a)}{\Gamma(a+1)} = \frac{\Gamma(m+1+a+1)}{\Gamma(a+1)} = \frac{\Gamma((m+1)+1+a)}{\Gamma(a+1)}$$

EXERCISES

① Learn the proofs of the following statements:

a) The integrals $\Gamma(n) = \int_0^{+\infty} x^{n-1} e^{-x}\,dx$ converges for $n > 0$.

b) $\Gamma(1) = 1$

c) $\forall n \in \mathbb{N}^* : \Gamma(n+1) = n\Gamma(n)$

d) $\Gamma(1/2) = \sqrt{\pi}$

② Show that $\forall n \in \mathbb{N}^* : (2n+1)!! = \dfrac{2^{n+1}}{\sqrt{\pi}} \Gamma\left(\dfrac{2n+3}{2}\right)$

③ Recall from Calculus 2 that the binomial series is given by

$$\forall x \in (-1,1) : (1+x)^a = \sum_{n=0}^{+\infty} \binom{a}{n} x^n$$

with $\binom{a}{0} = 1$ and $\binom{a}{n} = \prod_{k=1}^{n} \dfrac{a+1-k}{k}$, $\forall n \in \mathbb{N}^*$

Show that:

a) $\forall n \in \mathbb{N}^* : \binom{-1/2}{n} = \dfrac{(-1)^n (2n-1)!!}{(2n)!!} = \dfrac{(-1)^n}{\sqrt{\pi}\,\Gamma(n+1)} \Gamma\left(\dfrac{2n+1}{2}\right)$

b) $\forall a \in (1,+\infty) : \forall n \in \mathbb{N}^* : \binom{1/a}{n} = \dfrac{(-1)^n \Gamma(n-1/a)}{n\Gamma(n)\Gamma(-1/a)}$

④ Show that:

a) $\displaystyle\int_0^{+\infty} x(2x+3)^2 e^{-x}\,dx = 57$

b) $\displaystyle\int_0^{+\infty} \frac{(x+1)(x-1)e^{-x}}{\sqrt{x}}\,dx = \frac{-\sqrt{\pi}}{4}$

c) $\displaystyle\int_0^{+\infty} (\sqrt{x}+2)^2 e^{-x}\,dx = 2\sqrt{\pi}+5$

d) $\displaystyle\int_0^{+\infty} \sqrt{x}\,(\sqrt{x}-1)^3 e^{-x}\,dx = 5 - \frac{11\sqrt{\pi}}{4}$

⑤ Use the method of substitution and the Gamma function integral to show that

a) $\displaystyle\int_0^{+\infty} \sqrt{x}\,\exp(-x^3)\,dx = \frac{\sqrt{\pi}}{3}$

b) $\displaystyle\int_0^{+\infty} 2^{-x^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{\ln 2}}$

c) $\displaystyle\int_{0^+}^{1^-} \frac{dx}{\sqrt{|\ln x|}} = \sqrt{\pi}$

d) $\displaystyle\int_{0^+}^{1} (\ln x)^3\,dx = -6$

e) $\displaystyle\int_{0^+}^{1} \sqrt{\ln(1/x)}\,dx = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$

f) $\displaystyle\int_{0^+}^{1} (x\ln x)^2\,dx = \frac{2}{27}$

▼ Review of power series

We review basic results from Calculus II concerning power series expansion of functions.

● Definitions

- A power series is a series of the form

$$\forall x \in A : f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n$$

with $a \in \mathrm{Seq}(\mathbb{R})$ and $x_0 \in \mathbb{R}$.

- The domain A is chosen to be the widest possible subset of $\mathbb{R}$ for which the series converges. If $A = (x_0 - \mu, x_0 + \mu)$ then we say that $\mu > 0$ is the radius of convergence.

Def: Let $f : A \to \mathbb{R}$ be a function with $x_0 \in A$. We say that f analytic at $x = x_0 \Leftrightarrow$

$$\Leftrightarrow \exists a \in \mathrm{Seq}(\mathbb{R}) : \exists \mu \in (0, +\infty) : \forall x \in (x_0 - \mu, x_0 + \mu) : f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n$$

f analytic on $S \subseteq A \Leftrightarrow \forall x_0 \in S : f$ analytic on $x = x_0$

- The space of all functions analytic on S is denoted as $C^\omega(S)$. Note that $C^\omega(S) \subseteq C^\infty(S)$ which means that in general

$$f \in C^{\omega}(S) \Rightarrow f \in C^{\infty}(S).$$

However, the converse statement is not always true.

General properties of power series

Let f, g be two functions that are analytic at $x = x_0$ such that

$$\forall x \in (x_0 - \mu, x_0 + \mu) : \left(f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n \wedge g(x) = \sum_{n=0}^{+\infty} b_n (x - x_0)^n \right)$$

Then, we can show that:

a) $(\forall x \in (x_0 - \mu, x_0 + \mu) : f(x) = g(x)) \Leftrightarrow (\forall n \in \mathbb{N} : a_n = b_n)$

b) $\forall x \in (x_0 - \mu, x_0 + \mu) : \quad f(x) + g(x) = \sum_{n=0}^{+\infty} (a_n + b_n)(x - x_0)^n$

c) $\forall x \in (x_0 - \mu, x_0 + \mu) : \quad f(x) g(x) = \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} a_k b_{n-k} \right] (x - x_0)^n$

d) $\forall x \in (x_0 - \mu, x_0 + \mu) : \quad f'(x) = \sum_{n=1}^{+\infty} n a_n (x - x_0)^{n-1}$

e) $\forall k \in \mathbb{N}^* : \forall x \in (x_0 - \mu, x_0 + \mu) : f^{(k)}(x) = \sum_{n=k}^{+\infty} \left[\prod_{l=0}^{k-1} (n - l) \right] a_n (x - x_0)^{n-k}$

$$= \sum_{n=k}^{+\infty} \frac{n!}{(n-k)!} a_n (x - x_0)^{n-k}$$

e) $\forall x_1, x_2 \in (x_0 - \mu, x_0 + \mu) : \quad \int_{x_1}^{x_2} f(t)\, dt = \sum_{n=0}^{+\infty} \left[a_n \int_{x_1}^{x_2} (t - x_0)^n \, dt \right]$

$$= \sum_{n=0}^{+\infty} \left[\frac{a_n \left[(x_2 - x_0)^{n+1} - (x_1 - x_0)^{n+1} \right]}{n+1} \right]$$

Some important power series

$$\forall x\in(-1,1):\quad \frac{1}{1-x}=\sum_{k=0}^{+\infty}x^k=1+x+x^2+\dots$$

$$\forall x\in(-1,1):\quad (1+x)^p=\sum_{n=0}^{+\infty}\binom{p}{n}x^n=1+px+\frac{p(p-1)}{2\cdot 1}x^2+\dots$$

$$\forall x\in\mathbb{R}:\quad e^x=\sum_{n=0}^{+\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

$$\forall x\in\mathbb{R}:\quad \sin x=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots$$

$$\forall x\in\mathbb{R}:\quad \cos x=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots$$

$$\forall x\in(-1,1]:\quad \ln(1+x)=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{n+1}}{n+1}=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots$$

$$\forall x\in[-1,1]:\quad \operatorname{Arctan}x=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\dots$$

The detailed theory on the above series is given in my Calculus 2 notes.

Convergence tests

The proofs of the relevant theorems for series solution of linear ODEs depend on the comparison test and the absolute ratio test. Applied on power series these tests reduce to the following statements:

1 Comparison test

Given $a, b \in Seq(\mathbb{R})$ and $x_0 \in \mathbb{R}$, then

$$\forall x \in \mathbb{R}: \left(\begin{cases} \forall n \in \mathbb{N} : |a_n| \leq b_n \\ \sum_{n=0}^{+\infty} b_n (x-x_0)^n \text{ converges} \end{cases} \Rightarrow \sum_{n=0}^{+\infty} a_n (x-x_0)^n \text{ converges} \right)$$

2 Absolute Ratio test

Given $a \in Seq(\mathbb{R})$ and $x_0 \in \mathbb{R}$, then:

$$\left(\lim_{n \in \mathbb{N}} \left| \frac{a_{n+1}(x-x_0)}{a_n} \right| < 1 \Rightarrow \sum_{n=0}^{+\infty} a_n (x-x_0)^n \text{ converges} \right), \forall x \in \mathbb{R}$$

$$\left(\lim_{n \in \mathbb{N}} \left| \frac{a_{n+1}(x-x_0)}{a_n} \right| > 1 \Rightarrow \sum_{n=0}^{+\infty} a_n (x-x_0)^n \text{ diverges} \right), \forall x \in \mathbb{R}$$

In practice we get convergence for free via the relevant theorems as we solve the linear ODE. Therefore the above convergence tests are only required in the proofs of the necessary theorems.

Merten' theorem

Thm : Let (a_n) and (b_n) be two sequences with $n \in \mathbb{N}$

Then, we have:

$$\begin{cases} \sum_{n=0}^{+\infty} |a_n| \text{ converges} \\ \sum_{n=0}^{+\infty} b_n \text{ converges} \end{cases} \Rightarrow \left[\sum_{n=0}^{+\infty} a_n\right]\left[\sum_{n=0}^{+\infty} b_n\right] = \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n} a_k b_{n-k}\right]$$

Merten's theorem can be used safely to multiply power series because when they converge, they converge absolutely. A useful shortcut is to note that if

$$\forall x \in A : \left(f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n \wedge g(x) = \sum_{n=0}^{+\infty} b_n (x-x_0)^n \right)$$

then, it follows that

$$\forall x \in A : f(x) g(x) = \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} a_k b_{n-k} \right] (x-x_0)^n$$

For more details, see my Calculus 2 lecture notes.

EXAMPLES

a) Write the series expansion around $x_0 = 0$ of the function

$$f(x) = \frac{e^x}{2x+1}$$

and find the radius of convergence.

Solution

We have:

$$f(x) = \frac{e^x}{2x+1} = e^x \cdot \frac{1}{1-(-2x)} = \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-2x)^n\right]$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n 2^n x^n\right] =$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^k 2^k}{(n-k)!}\right] x^n$$

The series expansion of e^x converges on $\mathbb{R}$. The series expansion of $1/(1-(-2x))$ requires $|-2x| < 1$. Since:

$$|-2x| < 1 \Leftrightarrow |2x| < 1 \Leftrightarrow 2|x| < 1 \Leftrightarrow |x| < 1/2 \Leftrightarrow$$

$$\Leftrightarrow -1/2 < x < 1/2 \Leftrightarrow x \in (-1/2, 1/2).$$

b) Write the series expansion of the function $f(x) = e^x \cos x$ and find the radius of convergence

Solution

Since:

$$f(x) = e^x \cos x = \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right] =$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} \right] \left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} \right]$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \right] \left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} \right] + \left[\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} \right] \left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} \right]$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{2k}}{(2k)!} (-1)^{n-k} \frac{x^{2n-2k}}{(2n-2k)!} \right] +$$

$$+ \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{2k+1}}{(2k+1)!} (-1)^{n-k} \frac{x^{2n-2k}}{(2n-2k)!} \right] =$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(2k)!\,(2n-2k)!} \right] x^{2n} +$$

$$+ \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(2k+1)!\,(2n-2k)!} \right] x^{2n+1}$$

c) Write a series expansion of $f(x) = \sin(2x)$ around $x = \pi/8$ and find the radius of convergence.

<u>Solution</u>

$$f(x) = \sin(2x) = \sin(2x - \pi/4 + \pi/4) = \sin(2(x-\pi/8) + \pi/4) =$$

$$= \sin(2(x-\pi/8)) \cos(\pi/4) + \cos(2(x-\pi/8)) \sin(\pi/4) =$$

$$= (\sqrt{2}/2) [\cos(2(x-\pi/8)) + \sin(2(x-\pi/8))]$$

$$= \frac{\sqrt{2}}{2} \left[\sum_{n=0}^{+\infty} (-1)^n \frac{[2(x-\pi/8)]^{2n}}{(2n)!} + \sum_{n=0}^{+\infty} (-1)^n \frac{[2(x-\pi/8)]^{2n+1}}{(2n+1)!} \right]$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{\sqrt{2}\, 2^{2n}}{2(2n)!} (x-\pi/8)^{2n} + \sum_{n=0}^{+\infty} (-1)^n \frac{\sqrt{2}\, 2^{2n+1}}{2(2n+1)!} (x-\pi/8)^{2n+1}$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{2^{2n-1}\sqrt{2}}{(2n)!} (x-\pi/8)^{2n} + \sum_{n=0}^{+\infty} (-1)^n \frac{2^{2n}\sqrt{2}}{(2n+1)!} (x-\pi/8)^{2n+1}$$

The convergence set for all series expansions here is $\mathbb{R}$.

EXERCISES

(6) Show that

a) $\forall x \in \mathbb{R}: \ \sin^2 x = \sum_{n=1}^{+\infty} (-1)^{n+1} \dfrac{2^{2n-1}}{(2n)!} x^{2n}$

b) $\forall x \in (-2,2): \ \dfrac{x}{2-x} = \sum_{n=1}^{+\infty} \left(\dfrac{x}{2}\right)^n$

c) $\forall x \in \mathbb{R}: \ \sin^3 x = \dfrac{3}{4} \sum_{n=1}^{+\infty} \dfrac{(-1)^{n+1}(3^{2n}-1)}{(2n+1)!} x^{2n+1}$

d) $\forall x \in (-1,1): \ \ln\left(\sqrt{\dfrac{1+x}{1-x}}\right) = \sum_{n=0}^{+\infty} \dfrac{x^{2n+1}}{2n+1}$

e) $\forall x \in (-1,1): \ \dfrac{1}{x^2+x+1} = \dfrac{2}{\sqrt{3}} \sum_{n=0}^{+\infty} \sin\left(\dfrac{2\pi(n+1)}{3}\right) x^n$

(7) Derive the series expansions for the following functions around the indicated points, and find the convergence radius.

a) $f(x) = e^x \sin x$ (around $x_0 = 0$)

b) $f(x) = \sin(2x)$ (around $x_0 = \pi/6$)

c) $f(x) = e^x \ln(1+x)$ (around $x_0 = 0$)

d) $f(x) = \dfrac{\cos x}{1-x^2}$ (around $x_0 = 0$)

(8) Consider the function f defined by the power series

$$f(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{2^{2n}(n!)^2}$$

a) Show that the power-series converges on $\mathbb{R}$.

b) Show that $\forall x \in \mathbb{R}: \ x f''(x) + f'(x) = -x f(x)$

9 Consider the function f defined by the power series

$$f(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2^{2n+1}\, n!\,(n+1)!}$$

a) Show that the series converges on $\mathbb{R}$.

b) Show that $\forall x \in \mathbb{R}: \ x^2 f''(x) + x f'(x) = (1-x^2) f(x)$

Series solution of 2nd-order linear ODES

We consider a 2nd-order linear ordinary differential equation of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

and we seek the general solution approximated as a power series around the point $x = x_0$.

We distinguish between the following 3 cases:

1) $x = x_0$ is a <u>regular point</u> $\iff \begin{cases} p(x) \text{ analytic at } x = x_0 \\ q(x) \text{ analytic at } x = x_0 \end{cases}$

2) $x = x_0$ is a <u>regular singular point</u> $\iff \begin{cases} x = x_0 \text{ is NOT a regular point} \\ (x-x_0)p(x) \text{ analytic at } x = x_0 \\ (x-x_0)^2 q(x) \text{ analytic at } x = x_0 \end{cases}$

3) $x = x_0$ is an <u>irregular singular point</u> $\iff \big((x-x_0)p(x) \text{ NOT analytic at } x = x_0\big) \vee$
$\vee \big((x-x_0)^2 q(x) \text{ NOT analytic at } x = x_0\big).$

- The first two cases can be solved with convergent power series methods. The third case can be only investigated with asymptotic techniques or may be current research.

(1) → Regular linear ODEs

Thm: Consider an initial value problem of the form

$$\begin{cases} y''(x) + p(x)y'(x) + q(x)y(x) = 0 \\ y(x_0) = a_0 \wedge y'(x_0) = a_1 \end{cases}$$

with $p, q \in C^\omega((x_0-\mu, x_0+\mu))$ (i.e. p, q analytic at $x = x_0$) such that

$$\forall x \in (x_0-\mu, x_0+\mu): \left(p(x) = \sum_{n=0}^{+\infty} p_n (x-x_0)^n \wedge q(x) = \sum_{n=0}^{+\infty} q_n (x-x_0)^n \right)$$

The unique solution to this initial value problem is given by

$$\forall x \in (x_0-\mu, x_0+\mu): \ y(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n$$

with $a \in \mathrm{Seq}(\mathbb{R})$ a sequence defined recursively by

$$\forall n \in \mathbb{N}: \ a_{n+2} = \frac{-1}{(n+1)(n+2)} \sum_{k=0}^{n} \left[(k+1) a_{k+1} p_{n-k} + a_k q_{n-k} \right]$$

with $a_0, a_1 \in \mathbb{R}$ given via the above initial conditions.

Remarks

1) The unique sequence defined by the above recursion combined with initial values $a_0, a_1 \in \mathbb{R}$ will be denoted for convenience as: $a_n = A_n(a_0, a_1 | p, q)$.

2) The convergence of the power series for $y(x)$ is provided for by the theorem and has the same radius of convergence as the functions p, q. It is therefore not necessary to establish convergence when solving problems.

3) To find the two linearly independent solutions y_1, y_2 we solve, by convention, the following initial value problems:

$$\begin{cases} y(x_0) = 1 \\ y'(x_0) = 0 \end{cases} \longleftrightarrow y_1(x) = \sum_{n=0}^{+\infty} b_n (x-x_0)^n$$

$$\begin{cases} y(x_0) = 0 \\ y'(x_0) = 1 \end{cases} \longleftrightarrow y_2(x) = \sum_{n=0}^{+\infty} c_n (x-x_0)^n$$

with $\forall n \in \mathbb{N} : \begin{cases} b_n = A_n(1,0 \mid p,q) \\ c_n = A_n(0,1 \mid p,q) \end{cases}$

To show that y_1, y_2 are indeed linearly independent we note that

$$\begin{cases} y_1(x_0) = 1 \\ y_1'(x_0) = 0 \end{cases} \wedge \begin{cases} y_2(x_0) = 0 \\ y_2'(x_0) = 1 \end{cases}$$

and therefore:

$$W[y_1, y_2](x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 0 = 1$$

$\Rightarrow y_1, y_2$ linearly independent.

4) In practice it is customary to derive the recursion formula for the power series on a case by case basis. However, given the theorem, it is not necessary to prove convergence.

EXAMPLES

a) Find the general solution to the $\boxed{\text{Airy equation}}$ initial value problem

$$\begin{cases} y''(x) - xy(x) = 0 \\ y(0) = a_0 \wedge y'(0) = a_1 \end{cases}$$

Solution

Consider a solution of the form

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

and note that

$$y'(x) = \frac{d}{dx} \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n$$

and

$$y''(x) = \frac{d}{dx} \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{+\infty} n(n+1) a_{n+1} x^{n-1} =$$

$$= \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n .$$

Then, we have:

$$y''(x) - xy(x) = 0 \Leftrightarrow \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n - x \sum_{n=0}^{+\infty} a_n x^n = 0$$

$$\Leftrightarrow \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{+\infty} a_n x^{n+1} = 0$$

$$\Leftrightarrow \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{+\infty} a_{n-1} x^n = 0$$

$$\Leftrightarrow (0+1)(0+2)a_2 + \sum_{n=1}^{+\infty} [(n+1)(n+2)a_{n+2} - a_{n-1}] x^n = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a_2 = 0 \\ \forall n \in \mathbb{N}^* : (n+1)(n+2)a_{n+2} - a_{n-1} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a_2 = 0 \\ \forall n \in \mathbb{N}^* : a_{n+2} = \dfrac{a_{n-1}}{(n+1)(n+2)} \end{cases}$$

$$\Leftrightarrow \begin{cases} a_2 = 0 \\ \forall n \in \mathbb{N} - \{0,1,2\} : a_n = \dfrac{a_{n-3}}{n(n-1)} \end{cases}$$

▸ Now we can derive direct results for the sequence a_n. Starting from a_0, we get $a_3, a_6, \dots, a_{3k}, \dots$ we have:

$$\forall n \in \mathbb{N}^* : a_{3n} = a_0 \prod_{\lambda=1}^{n} \frac{1}{3\lambda(3\lambda-1)} = a_0 \prod_{\lambda=1}^{n} \frac{3\lambda-2}{3\lambda(3\lambda-1)(3\lambda-2)}$$

$$= a_0 \frac{1}{(3n)!} \prod_{\lambda=1}^{n} (3\lambda-2)$$

$$= a_0 \frac{3^n}{(3n)!} \prod_{\lambda=1}^{n} (\lambda - 2/3)$$

$$= a_0 \frac{3^n \Gamma(n-2/3+1)}{(3n)! \Gamma(-2/3+1)}$$

$$= a_0 \frac{3^n \Gamma(n+1/3)}{(3n)! \Gamma(1/3)}$$

and we note that this equation is also satisfied for $n=0$.

Starting from a_1, we get $a_4, a_7, \ldots, a_{3n+1}, \ldots$ and therefore

$$\forall n \in \mathbb{N}^* : a_{3n+1} = a_1 \prod_{\lambda=1}^{n} \frac{1}{(3\lambda+1)((3\lambda+1)-1)} =$$

$$= a_1 \prod_{\lambda=1}^{n} \frac{1}{3\lambda(3\lambda+1)} = a_1 \prod_{\lambda=1}^{n} \frac{3\lambda-1}{(3\lambda-1)(3\lambda)(3\lambda+1)}$$

$$= a_1 \frac{1}{(3n+1)!} \prod_{\lambda=1}^{n} (3\lambda-1) = a_1 \frac{3^n}{(3n+1)!} \prod_{\lambda=1}^{n} (\lambda - 1/3)$$

$$= a_1 \frac{3^n \Gamma(n-1/3+1)}{(3n+1)!\,\Gamma(-1/3+1)} = a_1 \frac{3^n \Gamma(n+2/3)}{(3n+1)!\,\Gamma(2/3)}$$

and we note that this equation is also satisfied for $n=0$.

Since $a_2 = 0$, it follows that $\forall n \in \mathbb{N} : a_{3n+2} = 0$

It follows that the general solution is:

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} a_{3n} x^{3n} + \sum_{n=0}^{+\infty} a_{3n+1} x^{3n+1}$$

$$= \sum_{n=0}^{+\infty} a_0 \frac{3^n \Gamma(n+1/3)}{(3n)!\,\Gamma(1/3)} x^{3n} + \sum_{n=0}^{+\infty} a_1 \frac{3^n \Gamma(n+2/3)}{(3n+1)!\,\Gamma(2/3)} x^{3n+1}$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

with

$$y_1(x) = \sum_{n=0}^{+\infty} \frac{3^n \Gamma(n+1/3)}{(3n)!\,\Gamma(1/3)} x^{3n} \wedge y_2(x) = \sum_{n=0}^{+\infty} \frac{3^n \Gamma(n+2/3)}{(3n+1)!\,\Gamma(2/3)} x^{3n+1}$$

These series will converge uniformly on $\mathbb{R}$ and define the two linearly independent homogeneous solutions that span the null-space of the Airy equation.

$\hookrightarrow$ In the above argument, we have used the following identity

$$\boxed{\prod_{k=1}^{n} (k+a) = \frac{\Gamma(n+a+1)}{\Gamma(a+1)}}$$

to eliminate the products in the formula for $y_1(x)$ and $y_2(x)$ and extend their validity to from $n \in \mathbb{N}^*$ to $n \in \mathbb{N}$.

EXAMPLE

Solve the linear ODE: $y''(x) + \cos(x) y(x) = 0$.
with a series around $x=0$.

Solution

We consider a solution of the form

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

and note that

$$y'(x) = \frac{d}{dx} \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n \Rightarrow$$

$$\Rightarrow y''(x) = \frac{d}{dx} \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{+\infty} n(n+1) a_{n+1} x^{n-1} =$$

$$= \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n$$

and

$$(\cos x) y(x) = \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \right] \left[\sum_{n=0}^{+\infty} a_n x^n \right] =$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \right] \left[\sum_{n=0}^{+\infty} a_{2n} x^{2n} + \sum_{n=0}^{+\infty} a_{2n+1} x^{2n+1} \right] =$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \right] \left[\sum_{n=0}^{+\infty} a_{2n} x^{2n} \right] + \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \right] \left[\sum_{n=0}^{+\infty} a_{2n+1} x^{2n+1} \right]$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{(2n-2k)}}{(2n-2k)!} a_{2k} x^{2k} \right] + \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{2n-2k}}{(2n-2k)!} a_{2k+1} x^{2k+1} \right]$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{a_{2k}}{(2n-2k)!} \right] x^{2n} + \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{a_{2k+1}}{(2n-2k)!} \right] x^{2n+1}$$

It follows that

$y''(x) - \cos(x) y(x) = 0 \Leftrightarrow$

$$\Leftrightarrow \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{a_{2k}}{(2n-2k)!} \right] x^{2n} - $$

$$ - \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{a_{2k+1}}{(2n-2k)!} \right] x^{2n+1} = 0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{n=0}^{+\infty} \left[(2n+1)(2n+2) a_{2n+2} - \sum_{k=0}^{n} \frac{a_{2k}}{(2n-2k)!} \right] x^{2n} +$$

$$+ \sum_{n=0}^{+\infty} \left[((2n+1)+1)((2n+1)+2) a_{(2n+1)+2} - \sum_{k=0}^{n} \frac{a_{2k+1}}{(2n-2k)!} \right] x^{2n+1} = 0$$

$$\Leftrightarrow \sum_{n=0}^{+\infty} \left[(2n+1)(2n+2) a_{2n+2} - \sum_{k=0}^{n} \frac{a_{2k}}{(2n-2k)!} \right] x^{2n} +$$

$$+ \sum_{n=0}^{+\infty} \left[(2n+2)(2n+3) a_{2n+3} - \sum_{k=0}^{n} \frac{a_{2k+1}}{(2n-2k)!} \right] x^{2n+1} = 0$$

$$\Leftrightarrow \forall n \in \mathbb{N}: \begin{cases} (2n+1)(2n+2) a_{2n+2} - \sum_{k=0}^{n} \frac{a_{2k}}{(2n-2k)!} = 0 \\ (2n+2)(2n+3) a_{2n+3} - \sum_{k=0}^{n} \frac{a_{2k+1}}{(2n-2k)!} = 0 \end{cases}$$

$$\Leftrightarrow \boxed{\forall n \in \mathbb{N}: \begin{cases} a_{2n+2} = \dfrac{1}{(2n+1)(2n+2)} \sum_{k=0}^{n} \dfrac{a_{2k}}{(2n-2k)!} \\ a_{2n+3} = \dfrac{1}{(2n+2)(2n+3)} \sum_{k=0}^{n} \dfrac{a_{2k+1}}{(2n-2k)!} \end{cases} \qquad (1)}$$

Initializing the power series requires a_0 and a_1.

↳ Note that it is not possible to express the series in closed form. We can only use Eq. (1) to generate as many series terms as needed. To obtain two linearly independent solutions $y_1(x)$ and $y_2(x)$ we initialize Eq. (1) using $(a_0, a_1) = (1,0)$ and $(a_0, a_1) = (0,1)$ respectively, ~~to~~ This will yield the power series for $y_1(x)$ and $y_2(x)$.

↳ In order to multiply power series expansions to calculate $\cos(x)y(x)$ we used Merten's theorem from my Calculus 2 lecture notes:

$$\boxed{\left\{\begin{array}{l} \sum_{n=0}^{+\infty} |a_n| \text{ converges} \\ \sum_{n=0}^{+\infty} b_n \text{ converges} \end{array}\right. \Rightarrow \left[\sum_{n=0}^{+\infty} a_n\right]\left[\sum_{n=0}^{+\infty} b_n\right] = \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n} a_k b_{n-k}\right]}$$

The required assumptions are always satisfied by power series within their convergence interval.

EXERCISES

(10) Show that Hermite's equation $y''(x) - 2xy'(x) + 2ay(x) = 0$ with $a \in (0, +\infty)$ has the following linearly independent solutions:

$$y_1(x) = \Gamma(1 + a/2) \sum_{n=0}^{+\infty} \frac{(-1)^n (2x)^n}{(2n)!\, \Gamma(a/2 - n + 1)}$$

$$y_2(x) = \Gamma(1/2 + a/2) \sum_{n=0}^{+\infty} \frac{(-1)^n (2x)^{n+1}}{(2n+1)!\, \Gamma(a/2 - n + 1/2)}$$

(11) Show that Chebyshev's equation

$$(1 - x^2) y''(x) - x y'(x) + a^2 y(x) = 0$$

with $a \in (0, +\infty)$ has the following linearly independent solutions

$$y_1(x) = 1 + \sum_{n=1}^{+\infty} \frac{1}{(2n)!} \left[\prod_{k=0}^{n-1} (4k^2 - a^2) \right] x^{2n}$$

$$y_2(x) = x + \sum_{n=1}^{+\infty} \frac{1}{(2n+1)!} \left[\prod_{k=0}^{n-1} (4k^2 + 4k + 1 - a^2) \right] x^{2n+1}$$

(12) Show that the equation $y''(x) + xy'(x) + y(x) = 0$ has the following linearly independent solutions:

$$y_1(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n n!} x^{2n}$$

$$y_2(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n 2^n n!}{(2n+1)!} x^{2n+1}$$

(13) Show that the equation $y''(x)+x^2y'(x)+xy(x)=0$ has the following linearly independent solutions:

$$y_1(x) = 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n}{(3n)!}\left[\prod_{k=1}^{n}(3k-2)^2\right]x^{3n}$$

$$y_2(x) = x + \sum_{n=1}^{+\infty} \frac{(-1)^n}{(3n+1)!}\left[\prod_{k=1}^{n}(3k-1)^2\right]x^{3n+1}$$

(14) Show that the equation $y''(x)+x^2y(x)=0$ has the following linearly independent solutions:

$$y_1(x) = 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n}{4^n n!}\left[\prod_{k=1}^{n}\frac{1}{4k-1}\right]x^{4n}$$

$$y_2(x) = 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n}{4^n n!}\left[\prod_{k=0}^{n}\frac{1}{4k+1}\right]x^{4n+1}$$

(15) Consider the equation $y''(x)+a^2y(x)=0$ with $a\in\mathbb{R}$. Use the power-series method to "rediscover" the well-known general solution $y(x)=\lambda_1\cos(ax)+\lambda_2\sin(ax)$.

② → Regular singular linear ODEs (Frobenius method)

We consider a linear ODE of the form

$$y''(x) + \frac{p(x)}{x-x_0} y'(x) + \frac{q(x)}{(x-x_0)^2} y(x) = 0 \qquad (1)$$

or equivalently

$$(x-x_0)^2 y''(x) + (x-x_0)p(x)y'(x) + q(x)y(x) = 0$$

with p,q analytic at $x=x_0$ with power-series expansions

$$\forall x \in (x_0-\mu, x_0+\mu): \left(p(x) = \sum_{n=0}^{+\infty} p_n (x-x_0)^n \wedge q(x) = \sum_{n=0}^{+\infty} q_n (x-x_0)^n \right)$$

Since $x=x_0$ is not a regular point, the ODE does not admit linearly independent solutions $y_1(x), y_2(x)$ that can be expressed as a power series. Nonetheless, a general solution method for Eq.(1), where $x=x_0$ is a regular singular point, has been developed by Frobenius as follows.

Prop: Consider a function y defined as

$$y(x) = |x-x_0|^\lambda \sum_{n=0}^{+\infty} a_n (x-x_0)^n$$

If $y(x)$ solves Eq.(1), then:

(a) $F(\lambda | p_0, q_0) \equiv \lambda(\lambda-1) + p_0\lambda + q_0 = 0$

(b) $\forall n \in \mathbb{N}^*: F(\lambda+n | p_0, q_0) a_n = -\sum_{k=0}^{n-1} [(k+\lambda)p_{n-k} + q_{n-k}] a_k$

Remarks

(a) The polynomial $F(\lambda|p_0,q_0)$ is the indicial polynomial and the equation

$$\lambda(\lambda-1)+p_0\lambda+q_0=0$$

is the indicial equation associated with the linear ODE Eq.(1).

(b) Using the recurrence for the sequence a_n given by the above proposition with a given initial value a_0, we can show that $a_1, a_2, \ldots, a_n, \ldots$ are proportional to a_0 and the resulting sequence will be denoted as

$$\forall n \in \mathbb{N}^*: \quad a_n = a_0 \Phi_n(\lambda|p,q)$$

with $p,q \in \mathrm{Seq}(\mathbb{R})$ representing the sequences $p_1, p_2, \ldots, p_n, \ldots$ and $q_1, q_2, \ldots, q_n, \ldots$ Note that Φ_n is independent of x_0.

(c) We may now define the general function

$$y(x,\lambda|p,q) = |x-x_0|^\lambda \sum_{n=0}^{+\infty} \Phi_n(\lambda|p,q)(x-x_0)^n$$

For most values of λ this function does not solve Eq.(1). From the following propositions we see that $y(x,\lambda|p,q)$ solves Eq.(1) when λ is one of the zeroes of the indicial equation.

Prop: If p,q converge on $(x_0-\mu, x_0+\mu)$ then the series expansion for $y(x,\lambda|p,q)$ also converges both uniformly and absolutely on $(x_0-\mu, x_0+\mu)$.

Prop: Let $A = (x_0 - \mu, x_0 + \mu)$ and let $L : C^2(A) \to C^0(A)$ be the linear operator associated with the linear ODE Eq.(1) such that

$$\forall y \in C^2(A) : (Ly)(x) = y''(x) + \frac{p(x)}{x - x_0} y'(x) + \frac{q(x)}{(x-x_0)^2} y(x).$$

It follows that

$$\begin{aligned} Ly(x, \lambda | p, q) &= |x - x_0|^{\lambda - 2} F(\lambda | p_0, q_0) \\ &= |x - x_0|^{\lambda - 2} [\lambda(\lambda - 1) + p_0 \lambda + q_0] \end{aligned}$$

Using the above results and notations, and some additional considerations needed for the proofs, we establish the main result:

Thm: Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the zeroes of the indicial polynomial $F(\lambda | p_0, q_0)$ and with no loss of generality we assume that $\mathrm{Re}(\lambda_1) \geq \mathrm{Re}(\lambda_2)$. We then distinguish between the following cases:

Case 1: If $\lambda_1 \neq \lambda_2 \wedge \lambda_1 - \lambda_2 \notin \mathbb{N}^*$ then Eq.(1) has two linearly independent solutions given by:

$$\forall x \in (x_0 - \mu, x_0 + \mu) : \begin{cases} y_1(x) = y(x, \lambda_1 | p, q) = |x - x_0|^{\lambda_1} \sum_{n=0}^{+\infty} \phi_n(\lambda_1 | p, q)(x - x_0)^n \\ y_2(x) = y(x, \lambda_2 | p, q) = |x - x_0|^{\lambda_2} \sum_{n=0}^{+\infty} \phi_n(\lambda_2 | p, q)(x - x_0)^n \end{cases}$$

Case 2: If $\lambda_1 = \lambda_2$, then the two linearly independent solutions are

$$y_1(x) = y(x, \lambda_1 | p, q) = |x - x_0|^{\lambda_1} \sum_{n=0}^{+\infty} \phi_n(\lambda_1 | p, q)(x - x_0)^n$$

$$y_2(x) = \frac{\partial}{\partial \lambda} y(x, \lambda | p, q) \Big|_{\lambda = \lambda_1} =$$

$$= y_1(x) \ln|x - x_0| + |x - x_0|^{\lambda_1} \sum_{n=0}^{+\infty} b_n (x - x_0)^n$$

with $\forall n \in \mathbb{N} : b_n = \frac{\partial}{\partial \lambda} \phi_n(\lambda | p, q) \Big|_{\lambda = \lambda_1}$

Case 3: If $\lambda_1 - \lambda_2 = N \in \mathbb{N}^*$, then the two linearly independent solutions are

$$y_1(x) = y(x, \lambda_1 | p, q) = |x - x_0|^{\lambda_1} \sum_{n=0}^{+\infty} \phi_n(\lambda_1 | p, q)(x - x_0)^n$$

$$y_2(x) = \frac{\partial}{\partial \lambda} \left[(\lambda - \lambda_2) y(x, \lambda | p, q) \right] \Big|_{\lambda = \lambda_2} =$$

$$= C_1 y_1(x) \ln|x - x_0| + |x - x_0|^{\lambda_2} \sum_{n=0}^{+\infty} c_n (x - x_0)^n$$

with $C_1 = \lim_{\lambda \to \lambda_2} \left[(\lambda - \lambda_2) \phi_N(\lambda | p, q) \right]$

$$\forall n \in \mathbb{N} : \quad c_n = \frac{\partial}{\partial \lambda} \left[(\lambda - \lambda_2) \phi_n(\lambda | p, q) \right] \Big|_{\lambda = \lambda_2}$$

↳ Given the solutions $y_1(x)$ and $y_2(x)$, the general solution is:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

with $c_1, c_2 \in \mathbb{R}$.

Methodology / Remarks

(a) It is recommended that you use the above theorems and propositions to determine the indicial polynomial and the recurrence relationship defining the sequence $a_n = a_0 \Phi_n(\lambda | p, q)$. Although both can be obtained from substituting the solution forms to the original ODE, that tends to be cumbersome.

(b) An explicit expression for a_n as a function of λ is needed for cases 2,3 in order to differentiate them with respect to λ. For case 1 it is not needed, and it is sufficient to have explicit equations for a_n only for $\lambda = \lambda_1$ and $\lambda = \lambda_2$

(c) For the calculation of $y_2(x)$ in cases 2,3 it is often necessary to calculate the derivatives (with respect to λ) of a function defined as a product or ratio of a large number of factors. A technique known as logarithmic differentiation can be used to evaluate such products as follows:

$$\boxed{\frac{d}{dx} \prod_{a=1}^{n} [f_a(x)]^{c_a} = \prod_{a=1}^{n} [f_a(x)]^{c_a} \left[\sum_{a=1}^{n} c_a \frac{f_a'(x)}{f_a(x)} \right]}$$

as long as $\forall a \in [n]: f_a(x) \neq 0$.

(d) Gamma functions are used to simplify linear products:

$$\boxed{\prod_{k=1}^{n} (ak+b) = a^n \prod_{k=1}^{n} (k+b/a) = \frac{a^n \Gamma(n+1+b/a)}{\Gamma(1+b/a)}}$$

EXAMPLES

a) Solve the linear ODE

$$x^2 y''(x) + x(x - 1/2)y'(x) + (1/2)y(x) = 0$$

with a series around $x = 0$.

Solution

We rewrite the ODE as:

$$y''(x) + \frac{1}{x}(x - 1/2)y'(x) + \frac{1}{x^2}\frac{1}{2}y(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow y''(x) + \frac{p(x)}{x}y'(x) + \frac{q(x)}{x^2}y(x) = 0$$

with $p(x) = x - 1/2 \rightarrow p_0 = -1/2 \wedge p_1 = 1 \wedge p_2 = p_3 = \cdots = 0$

and $q(x) = 1/2 \Rightarrow q_0 = 1/2 \wedge q_1 = q_2 = \cdots = 0$

Consider a solution

$$y(x) = |x|^{\lambda} \sum_{n=0}^{+\infty} a_n x^n$$

Substituting to the ODE gives the indicial polynomial

$$F(\lambda) = \lambda(\lambda - 1) + p_0\lambda + q_0 = \lambda(\lambda-1) - (1/2)\lambda + 1/2 =$$

$$= \lambda(\lambda-1) - (1/2)(\lambda - 1) = (\lambda - 1/2)(\lambda - 1)$$

and the recurrence

$$\forall n \in \mathbb{N}^*: F(\lambda + n)a_n = -\sum_{k=0}^{n-1}[(k+\lambda)p_{n-k} + q_{n-k}]a_k =$$

$$= -[(n-1+\lambda)p_1 + q_1]a_{n-1} - \sum_{k=0}^{n-2}[(k+\lambda)p_{n-k} + q_{n-k}]a_k$$

$$= -[(n+\lambda-1)p_1 + 0]a_{n-1} = -(n+\lambda-1)a_{n-1} \Leftrightarrow$$

$$\Leftrightarrow (\lambda+n-1/2)(\lambda+n-1)a_n = -(\lambda+n-1)a_{n-1} \Leftrightarrow$$

$$\Leftrightarrow (\lambda+n-1/2)a_n = -a_{n-1} \Leftrightarrow a_n = \frac{-1}{\lambda+n-1/2} a_{n-1}.$$

It follows that

$$\forall n \in \mathbb{N}^*: a_n = a_0 \prod_{k=1}^{n} \left(\frac{-1}{\lambda+k-1/2} \right) = a_0 (-1)^n \prod_{k=1}^{n} \frac{1}{\lambda+k-1/2}.$$

Solving the indicial equation:

$$F(\lambda)=0 \Leftrightarrow (\lambda-1/2)(\lambda-1)=0 \Leftrightarrow \lambda-1/2=0 \vee \lambda-1=0 \Leftrightarrow$$

$$\Leftrightarrow \lambda=1/2 \vee \lambda=1.$$

For $\lambda=1/2$:

$$\forall n \in \mathbb{N}^*: a_n = a_0 \prod_{k=1}^{n} \frac{-1}{1/2+k-1/2} = a_0(-1)^n \prod_{k=1}^{n} \frac{1}{k} =$$

$$= a_0 \frac{(-1)^n}{n!}$$

and therefore the first homogeneous solution is:

$$y_1(x) = |x|^{1/2} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^n = |x|^{1/2} \sum_{n=0}^{+\infty} \frac{(-x)^n}{n!} = |x|^{1/2} e^{-x}$$

For $\lambda=1$:

$$\forall n \in \mathbb{N}^*: a_n = a_0 \prod_{k=1}^{n} \frac{-1}{1+k-1/2} = a_0(-1)^n \prod_{k=1}^{n} \frac{1}{k+1/2} =$$

$$= a_0(-1)^n \left[\prod_{k=1}^{n} (k+1/2) \right]^{-1} = a_0(-1)^n \left[\frac{\Gamma(n+1+1/2)}{\Gamma(1+1/2)} \right]^{-1}$$

$$= a_0 \frac{(-1)^n \Gamma(3/2)}{\Gamma(n+3/2)}$$

and therefore the second homogeneous solution is:

$$y_2(x) = |x| \sum_{n=0}^{+\infty} \frac{(-1)^n \Gamma(3/2)}{\Gamma(n+3/2)} x^n$$

The general solution is:

$$y(x) = \lambda_1 |x|^{1/2} e^{-x} + \lambda_2 |x| \sum_{n=0}^{+\infty} \frac{(-1)^n \Gamma(3/2)}{\Gamma(n+3/2)} x^n.$$

Since p & q converge on $\mathbb{R}$, the general solution $y(x)$ converges on $\mathbb{R}$.

b) Solve the linear ODE

$$x(1-x)y''(x)+(1-x)y'(x)-y(x)=0$$

around $x=0$.

<u>Solution</u>

We note that

$$x(1-x)y''(x)+(1-x)y'(x)-y(x)=0 \Leftrightarrow$$

$$\Leftrightarrow y''(x)+\frac{1-x}{x(1-x)}y'(x)-\frac{1}{x(x-1)}y(x)=0 \Leftrightarrow$$

$$\Leftrightarrow y''(x)+\frac{1}{x}y'(x)+\frac{1}{x^2}\frac{-x}{x-1}y(x)=0$$

$$\Leftrightarrow y''(x)+\frac{1}{x}p(x)y'(x)+\frac{1}{x^2}q(x)y(x)=0$$

with $p(x)=1=\sum_{n=0}^{+\infty}p_0x^n \Rightarrow p_0=1 \wedge p_1=p_2=\dots=0$

and $q(x)=\frac{-x}{1-x}=(-x)\frac{1}{1-x}=(-x)\sum_{n=0}^{+\infty}x^n=$

$$=\sum_{n=0}^{+\infty}(-x^{n+1})=\sum_{n=1}^{+\infty}(-1)x^n=\sum_{n=0}^{+\infty}q_nx^n \Rightarrow$$

$$\Rightarrow q_0=0 \wedge q_1=q_2=\dots=-1$$

Note that the convergence interval for $q(x)$ is $(-1,1)$.

Using a candidate solution

$$y(x)=|x|^\lambda\sum_{n=0}^{+\infty}a_nx^n$$

we find that the indicial polynomial is:

$$F(\lambda) = \lambda(\lambda-1) + p_0\lambda + q_0 = \lambda(\lambda-1) + \lambda + 0 = \lambda(\lambda-1+1) = \lambda^2$$

and the sequence a_n must satisfy

$$\forall n \in \mathbb{N}^*: F(\lambda+n)a_n = -\sum_{k=0}^{n-1} [(k+\lambda)p_{n-k} + q_{n-k}] a_k =$$

$$= -\sum_{k=0}^{n-1} (k+\lambda)p_{n-k} a_k - \sum_{k=0}^{n-1} q_{n-k} a_k =$$

$$= -0 - \sum_{k=0}^{n-1} (-1) a_k = a_0 + a_1 + \cdots + a_{n-1} \Leftrightarrow$$

$$\Leftrightarrow (\lambda+n)^2 a_n = a_0 + a_1 + \cdots + a_{n-1}, \ \forall n \in \mathbb{N}^* \qquad (1)$$

For $n=1$: $(\lambda+1)^2 a_1 = a_0 \Leftrightarrow a_1 = \dfrac{a_0}{(\lambda+1)^2}$ (2)

For $n \geq 2$ we note that $(\lambda+n-1)^2 a_{n-1} = a_0 + a_1 + \cdots + a_{n-2}$ and therefore from Eq. (1)

$$(1) \Leftrightarrow (\lambda+n)^2 a_n = (a_0 + a_1 + \cdots + a_{n-2}) + a_{n-1} =$$

$$= (\lambda+n-1)^2 a_{n-1} + a_{n-1} =$$

$$= [(\lambda+n-1)^2 + 1] a_{n-1} \Leftrightarrow$$

$$\Leftrightarrow a_n = \frac{(\lambda+n-1)^2+1}{(\lambda+n)^2} a_{n-1} \qquad (3)$$

Note that for $n=1$, $(\lambda+n-1)^2+1 = (\lambda+1-1)^2+1 = \lambda^2+1 \neq 0$ so equation (3) does not reduce to equation (2) for $n=1$. To mitigate that, we choose $a_0 = \lambda^2+1$. Then:

$$a_1 = \frac{\lambda^2+1}{(\lambda+1)^2} = \frac{(\lambda+1-1)^2+1}{(\lambda+1)^2}$$

and it follows that

$$\forall n \in \mathbb{N}^*: a_n = \prod_{k=1}^{n} \frac{(\lambda+k-1)^2+1}{(\lambda+k)^2}$$

Solving the indicial equation gives:
$F(\lambda)=0 \Leftrightarrow \lambda^2=0 \Leftrightarrow \lambda=0 \longleftarrow$ double zero.

For $\lambda=0$:

$$\forall n\in\mathbb{N}^*: \quad a_n=\prod_{k=1}^{n}\frac{(0+k-1)^2+1}{(0+k)^2}=\frac{1}{(n!)^2}\prod_{k=1}^{n}\left[(k-1)^2+1\right]$$
$$=\frac{1}{(n!)^2}\prod_{k=0}^{n-1}(k^2+1)$$

and $a_0=0^2+1=1$, therefore the first homogeneous solution is given by

$$y_1(x)=1+\sum_{n=1}^{+\infty}\left[\frac{1}{(n!)^2}\prod_{k=0}^{n-1}(k^2+1)\right]x^n.$$

and the second linearly independent solution is given by:

$$y_2(x)=y_1(x)\ln|x|+\sum_{n=0}^{+\infty}b_nx^n \quad \text{with } b_n=\left.\frac{\partial a_n}{\partial\lambda}\right|_{\lambda=0}$$

To calculate b_n, we note that

$$\frac{\partial a_0}{\partial\lambda}=\frac{\partial}{\partial\lambda}(\lambda^2+1)=2\lambda \Rightarrow b_0=\left.\frac{\partial a_0}{\partial\lambda}\right|_{\lambda=0}=\left.2\lambda\right|_{\lambda=0}=0$$

and

$$\forall n\in\mathbb{N}^*: \frac{\partial a_n}{\partial\lambda}=\frac{\partial}{\partial\lambda}\prod_{k=1}^{n}\frac{(\lambda+k-1)^2+1}{(\lambda+k)^2}=$$
$$=\prod_{k=1}^{n}\left(\frac{(\lambda+k-1)^2+1}{(\lambda+k)^2}\right)\left[\sum_{k=1}^{n}\frac{(\partial/\partial\lambda)[(\lambda+k-1)^2+1]}{(\lambda+k-1)^2+1}-\right.$$
$$\left.-2\sum_{k=1}^{n}\frac{(\partial/\partial\lambda)(\lambda+k)}{\lambda+k}\right]=$$

$$= \left[\prod_{k=1}^{n} \frac{(\lambda+k-1)^2+1}{(\lambda+k)^2} \right] \left[\sum_{k=1}^{n} \frac{2(\lambda+k-1)}{(\lambda+k-1)^2+1} - 2\sum_{k=1}^{n} \frac{1}{\lambda+k} \right] \Rightarrow$$

$$\Rightarrow b_n = \left.\frac{\partial a_n}{\partial \lambda}\right|_{\lambda=0} = a_n \sum_{k=1}^{n} \left[\frac{2(k-1)}{(k-1)^2+1} - \frac{2}{k} \right] =$$

$$= a_n \sum_{k=1}^{n} \left[\frac{2(k-1)k - 2[(k-1)^2+1]}{k[(k-1)^2+1]} \right] =$$

$$= a_n \sum_{k=1}^{n} \left[\frac{2k^2-2k-2(k^2-2k+1+1)}{k(k^2-2k+1+1)} \right] =$$

$$= a_n \sum_{k=1}^{n} \frac{2k^2-2k-2k^2+4k-4}{k(k^2-2k-2)} =$$

$$= a_n \sum_{k=1}^{n} \frac{2k-4}{k(k^2-2k-2)} =$$

$$= \frac{1}{(n!)^2} \prod_{k=0}^{n-1} (k^2+1) \left[\sum_{k=1}^{n} \frac{2(k-2)}{k(k^2-2k-2)} \right]$$

It follows that the second solution is given by

$$y_2(x) = y_1(x) \ln|x| + \sum_{n=1}^{+\infty} \left[\frac{1}{(n!)^2} \left(\prod_{k=0}^{n-1} (k^2+1) \right) \left(\sum_{k=1}^{n} \frac{2(k-2)}{k(k^2-2k-2)} \right) \right] x^n$$

and the general solution is $y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$.

The solution will converge on $(-1,1)$ since p converges on $\mathbb{R}$ and q converges on $(-1,1)$.

c) Solve the linear ODE $xy''(x)+2y'(x)-y(x)=0$ with a series around $x=0$

Solution

We note that

$$xy''(x)+2y'(x)-y(x)=0 \Leftrightarrow y''(x)+(2/x)y'(x)-(1/x)y(x)=0$$
$$\Leftrightarrow y''(x)+(1/x)2y'(x)+(1/x^2)(-x)y(x)=0$$
$$\Leftrightarrow y''(x)+(1/x)p(x)y'(x)+(1/x^2)q(x)y(x)=0$$

with

$$p(x)=2=\sum_{n=0}^{+\infty} p_n x^n \Rightarrow p_0=2 \wedge p_1=p_2=\dots=0$$

and

$$q(x)=-x=\sum_{n=0}^{+\infty} q_n x^n \Rightarrow q_0=0 \wedge q_1=-1 \wedge q_2=q_3=\dots=0.$$

Using a candidate solution $y(x)=|x|^\lambda \sum_{n=0}^{+\infty} a_n x^n$ the corresponding indicial polynomial is

$$F(\lambda)=\lambda(\lambda-1)+p_0\lambda+q_0=\lambda(\lambda-1)+2\lambda=\lambda(\lambda-1+2)=\lambda(\lambda+1)$$

and a_n satisfies:

$$\forall n\in\mathbb{N}^*: F(\lambda+n)a_n=-\sum_{k=0}^{n-1}[(k+\lambda)p_{n-k}+q_{n-k}]a_k=$$
$$=-\sum_{k=0}^{n-1}(k+\lambda)p_{n-k}a_k-\sum_{k=0}^{n-1}q_{n-k}a_k=$$
$$=-0-q_1 a_{n-1}=-(-1)a_{n-1}=a_{n-1}\Leftrightarrow$$
$$\Leftrightarrow (\lambda+n)(\lambda+n+1)a_n=a_{n-1}\Leftrightarrow a_n=\frac{1}{(\lambda+n)(\lambda+n+1)}a_{n-1}$$

and therefore $\forall n\in\mathbb{N}^*: a_n=a_0\prod_{k=1}^{n}\frac{1}{(\lambda+k+1)(\lambda+k)}$

Solving the indicial equation gives:

$F(\lambda)=0 \Leftrightarrow \lambda(\lambda+1)=0 \Leftrightarrow \lambda=0 \vee \lambda+1=0 \Leftrightarrow \lambda=0 \vee \lambda=-1.$

For $\lambda=0$, we have

$$a_n = a_0 \prod_{k=1}^{n} \frac{1}{(0+k+1)(0+k)} = a_0 \prod_{k=1}^{n} \frac{1}{k(k+1)} =$$

$$= a_0 \left[\prod_{k=1}^{n} \frac{1}{k}\right]\left[\prod_{k=1}^{n} \frac{1}{k+1}\right] =$$

$$= \frac{a_0}{n!} \prod_{k=2}^{n+1} \frac{1}{k} = \frac{a_0}{n!} \prod_{k=1}^{n+1} \frac{1}{k} = \frac{a_0}{n!\,(n+1)!} =$$

$$= \frac{a_0}{(n!)^2(n+1)}, \quad \forall n \in \mathbb{N}^*$$

and the corresponding solution is:

$$y_1(x) = \sum_{n=0}^{+\infty} \frac{x^n}{(n!)^2(n+1)}$$

Since $0-(-1)=1$, the second solution is

$$y_2(x) = C_1 y_1(x) \ln|x| + |x|^{-1} \sum_{n=0}^{+\infty} c_n x^n$$

Using $a_0(\lambda)=a_0$, we have:

$$C_1 = \lim_{\lambda\to-1} [(\lambda-(-1))\, a_1(\lambda)] = \lim_{\lambda\to-1} \left[(\lambda+1) \frac{a_0}{(\lambda+1)(\lambda+2)}\right] =$$

$$= \lim_{\lambda\to-1} \frac{a_0}{\lambda+2} = \frac{a_0}{-1+2} = a_0$$

and

$$c_n = \frac{\partial}{\partial\lambda}\left[(\lambda-(-1))a_n(\lambda)\right]\Big|_{\lambda=-1} = \frac{\partial}{\partial\lambda}\left[(\lambda+1)a_n(\lambda)\right]\Big|_{\lambda=-1}$$

$$= \frac{\partial}{\partial\lambda}\left[(\lambda+1)a_0\prod_{k=1}^{n}\frac{1}{(\lambda+k+1)(\lambda+k)}\right]\Big|_{\lambda=-1}, \quad \forall n\in\mathbb{N}^*$$

We distinguish between the following cases.

For $n=0$:

$$c_0 = \frac{\partial}{\partial\lambda}\left[(\lambda+1)a_0\right]\Big|_{\lambda=-1} = a_0\Big|_{\lambda=-1} = a_0$$

For $n=1$:

$$c_1 = \frac{\partial}{\partial\lambda}\left[(\lambda+1)a_0\frac{1}{(\lambda+1+1)(\lambda+1)}\right]\Big|_{\lambda=-1} = \frac{\partial}{\partial\lambda}\left[\frac{a_0}{\lambda+2}\right]\Big|_{\lambda=-1}$$

$$= \left[\frac{-a_0(\partial/\partial\lambda)(\lambda+2)}{(\lambda+2)^2}\right]\Big|_{\lambda=-1} = \left[\frac{-a_0}{(\lambda+2)^2}\right]_{\lambda=-1} =$$

$$= \frac{-a_0}{(-1+2)^2} = \frac{-a_0}{1^2} = -a_0.$$

For $n>1$:

$$c_n = \frac{\partial}{\partial\lambda}\left[(\lambda+1)a_n(\lambda)\right]\Big|_{\lambda=-1} =$$

$$= \frac{\partial}{\partial\lambda}\left[(\lambda+1)a_0\prod_{k=1}^{n}\left(\frac{1}{(\lambda+k+1)(\lambda+k)}\right)\right]\Big|_{\lambda=-1} =$$

$$= \frac{\partial}{\partial\lambda}\left[a_0\prod_{k=1}^{n}\left(\frac{1}{\lambda+k+1}\right)\prod_{k=2}^{n}\left(\frac{1}{\lambda+k}\right)\right]\Big|_{\lambda=-1} =$$

$$= a_0\frac{\partial}{\partial\lambda}\left[\frac{1}{\lambda+n+1}\left(\prod_{k=2}^{n}\frac{1}{\lambda+k}\right)^2\right]\Big|_{\lambda=-1} =$$

$$= a_0 \frac{1}{\lambda+n+1}\left(\prod_{k=2}^{n}\frac{1}{\lambda+k}\right)^2\left[\frac{-(\partial/\partial\lambda)(\lambda+n+1)}{\lambda+n+1}+\sum_{k=2}^{n}\frac{-2(\partial/\partial\lambda)(\lambda+k)}{\lambda+k}\right]\Bigg|_{\lambda=-1}$$

$$= -a_0 \frac{1}{\lambda+n+1}\left(\prod_{k=2}^{n}\frac{1}{\lambda+k}\right)^2\left[\frac{1}{\lambda+n+1}+\sum_{k=2}^{n}\frac{2}{\lambda+k}\right]\Bigg|_{\lambda=-1}$$

$$= -a_0 \frac{1}{-1+n+1}\left(\prod_{k=2}^{n}\frac{1}{-1+k}\right)^2\left[\frac{1}{-1+n+1}+\sum_{k=2}^{n}\frac{2}{-1+k}\right]$$

$$= -a_0 \frac{1}{n}\left(\prod_{k=1}^{n-1}\frac{1}{k}\right)^2\left[\frac{1}{n}+\sum_{k=1}^{n-1}\frac{2}{k}\right] =$$

$$= -a_0 \frac{1}{n\,[(n-1)!]^2}\left[\frac{-1}{n}+\sum_{k=1}^{n}\frac{2}{k}\right]$$

$$= \frac{-a_0}{n!\,(n-1)!}\left[\frac{-1}{n}+2\sum_{k=1}^{n}\frac{1}{k}\right]$$

Note that this result, for $n=1$, agrees with our previous result for $n=1$. It follows that the second solution is

$$y_2(x) = y_1(x)\ln|x| + |x|^{-1}\left[1-\sum_{n=1}^{+\infty}\frac{1}{n!\,(n-1)!}\left[\frac{-1}{n}+2\sum_{k=1}^{n}\frac{1}{k}\right]x^n\right]$$

The general solution is $y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$.

EXERCISES

(16) Show that the equation $4xy''(x)+2y'(x)+y(x)=0$ has the following linearly independent solutions

$$y_1(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^n$$

$$y_2(x) = |x|^{1/2} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^n$$

(17) Show that the equation $9x^2y''(x)+9xy'(x)+(9x^2-1)y(x)=0$ has the following linearly independent solutions

$$y_1(x) = |x|^{1/3}\left[1+\sum_{n=1}^{+\infty} \frac{(-1)^n 3^n}{(2n)!!}\left[\prod_{k=1}^{n} \frac{1}{6k+2}\right] x^{2n}\right]$$

$$y_2(x) = |x|^{-1/3}\left[1+\sum_{n=1}^{+\infty} \frac{(-1)^n 3^n}{(2n)!!}\left[\prod_{k=1}^{n} \frac{1}{6k-2}\right] x^{2n}\right]$$

(18) Show that the equation $x^2y''+(x^2-7/36)$

$$x^2y''(x)+(x^2-7/36)y(x)=0$$

has the following linearly independent solutions

$$y_1(x) = |x|^{7/6}\left[1+\sum_{n=1}^{+\infty} \frac{(-1)^n 3^n}{2^{2n}\, n!}\left[\prod_{k=1}^{n} \frac{1}{3k+2}\right] x^{2n}\right]$$

$$y_2(x) = |x|^{-1/6}\left[1+\sum_{n=1}^{+\infty} \frac{(-1)^n 3^n}{2^{2n}\, n!}\left[\prod_{k=1}^{n} \frac{1}{3k-2}\right] x^{2n}\right]$$

(19) Show that the equation $x^2y''(x)+(x^2-x)y'(x)+y(x)=0$ has the following linearly independent solution

$$y_1(x)=|x|\exp(-x)$$

$$y_2(x)=y_1(x)\ln|x|+|x|\left[\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}\varphi(n)}{n!}x^n\right]$$

with $\varphi(n)=\sum_{K=1}^{n}\frac{1}{K}$

(20) Show that the equation

$$x(1-x)y''(x)+(1-5x)y'(x)-4y(x)=0$$

has the following linearly independent solutions:

$$y_1(x)=\sum_{n=0}^{+\infty}(1+n)^2x^n$$

$$y_2(x)=y_1(x)\ln|x|-2\sum_{n=1}^{+\infty}n(n+1)x^n$$

(21) Show that the equation

$$(x^2+x^3)y''(x)-(x+x^2)y'(x)+y(x)=0$$

has the following linearly independent solutions:

$$y_1(x)=x(1+x)$$

$$y_2(x)=y_1(x)\ln|x|+|x|\left[-2x-\sum_{n=2}^{+\infty}\frac{(-1)^n}{n(n-1)}x^n\right]$$

(22) Show that the equation

$$x^2y''(x)+2xy'(x)+xy(x)=0$$

has the following linearly independent solutions:

$$y_1(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,(n+1)!} x^n$$

$$y_2(x) = -y_1(x)\ln|x| + |x|^{-1}\left[1 - \sum_{n=1}^{+\infty} \frac{(-1)^n\left(2\varphi(n-1)+1/n\right)}{n!\,(n-1)!} x^n\right]$$

with $\forall n \in \mathbb{N}^*: \varphi(n) = \sum_{k=1}^{n} (1/k)$.

23 Show that the equation $x(1-x)y''(x) - 3xy'(x) - y(x) = 0$ has the following linearly independent solutions

$y_1(x) = x(1-x)^{-2}$

$y_2(x) = y_1(x)\ln|x| + (1-x)^{-1}$

↳ Use the Frobenius method to solve the above differential equations.

▼ Theory of Bessel functions

● Summary of main results

The Bessel function $J_a(x)$ is defined via the following power series:

$$\boxed{\forall a\in\mathbb{R}: \forall x\in\mathbb{R}-\{0\}: J_a(x) = \left|\frac{x}{2}\right|^a \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+1+a)} \left(\frac{x}{2}\right)^{2n}}$$

For integer $a=m\in\mathbb{N}$, the above definition reduces to

$$\boxed{\forall x\in\mathbb{R}-\{0\}: \quad J_m(x) = \left|\frac{x}{2}\right|^m \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,(n+m)!} \left(\frac{x}{2}\right)^{2n}}$$

This function arises from using the Frobenius method to solve the Bessel equation, which is given by

$$\boxed{x^2 y''(x) + xy'(x) + (x^2-a^2)y(x) = 0, \quad \forall x\in\mathbb{R}-\{0\}}$$

With no loss of generality we will assume that $a\geq 0$ (since the transformation $a\to -a$ leaves the Bessel equation invariant). To write the general solution to the Bessel equation we distinguish between the following cases:

Case 1 : If $a \notin \mathbb{N}$ (with $a > 0$), then the general solution is
$\forall x \in \mathbb{R} - \{0\}: \; y(x) = \lambda_1 J_a(x) + \lambda_2 J_{-a}(x)$

Case 2 : If $a = 0$, then the general solution is
$\forall x \in \mathbb{R} - \{0\}: \; y(x) = \lambda_1 J_0(x) + \lambda_2 J^0(x)$
with $J^0(x)$ given by

$$\forall x \in \mathbb{R} - \{0\}: \; J^0(x) = J_0(x) \ln|x| - \sum_{n=1}^{+\infty} \frac{(-1)^n \varphi(n)}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$$

with $\forall n \in \mathbb{N}^*: \; \varphi(n) = \sum_{k=1}^{n} \frac{1}{k}$ and $\varphi(0) = 0$.

Case 3 : If $a \in \mathbb{N}^*$, then the general solution is
$\forall x \in \mathbb{R} - \{0\}: \; y(x) = \lambda_1 J_a(x) + \lambda_2 J^a(x)$
with $J^a(x)$ given by

$$\forall x \in \mathbb{R} - \{0\}: \; J^a(x) = J_a(x) \ln|x| - \frac{1}{2}\left(\frac{x}{2}\right)^{-a} \sum_{n=0}^{a-1} \frac{(a-n-1)!}{n!} \left(\frac{x}{2}\right)^{2n}$$
$$- \frac{1}{2}\left(\frac{x}{2}\right)^{a} \sum_{n=0}^{+\infty} \frac{(-1)^n \left[\varphi(n) + \varphi(n+a)\right]}{n!\,(n+a)!} \left(\frac{x}{2}\right)^{2n}$$

The above results can be obtained by application of the Frobenius method and a lot of tedious calculations.

Properties of Bessel functions

We prove some interesting properties of Bessel functions and leave the rest as exercises.

① → $$\boxed{\forall x \in \mathbb{R}^* : \forall t \in \mathbb{R}^* ;\ G(x,t) \equiv \exp\left(\frac{1}{2}x\left(t-\frac{1}{t}\right)\right) = \sum_{n=-\infty}^{+\infty} J_n(x) t^n}$$

Proof

Let $x \in \mathbb{R}^*$ and $t \in \mathbb{R}^*$ be given. It follows that

$$G(x,t) = \exp((1/2)x(t-1/t)) = \exp((1/2)xt)\exp(-(1/2)(x/t))$$

$$= \left[\sum_{p=0}^{+\infty} \frac{1}{p!}\left(\frac{xt}{2}\right)^p\right]\left[\sum_{q=0}^{+\infty}\frac{1}{q!}\left(\frac{-x}{2t}\right)^q\right] =$$

$$= \left[\sum_{p=0}^{+\infty}\frac{x^p t^p}{2^p p!}\right]\left[\sum_{q=0}^{+\infty}\frac{(-1)^q x^q}{2^q q! t^q}\right] =$$

$$= \sum_{p=0}^{+\infty}\sum_{q=0}^{+\infty}\left[\frac{(-1)^q x^{p+q} t^{p-q}}{2^{p+q} p! q!}\right]$$

Let $n=p-q$. Then n ranges from $-\infty$ to $+\infty$ and we replace the sum over p with a sum over n. The sum over q is retained. We note that $p=n+q$ and $p+q=n+2q$. and therefore

$$G(x,t) = \sum_{n=-\infty}^{+\infty}\sum_{q=0}^{+\infty}\left[\frac{(-1)^q x^{n+2q} t^n}{2^{n+2q}(n+q)!\, q!}\right] =$$

$$= \sum_{n=-\infty}^{+\infty}\left[t^n \frac{x^n}{2^n}\sum_{q=0}^{+\infty}\left(\frac{(-1)^q x^{2q}}{2^{2q} q!(n+q)!}\right)\right] =$$

$$= \sum_{n=-\infty}^{+\infty}\left[t^n\left(\frac{x}{2}\right)^n\sum_{q=0}^{+\infty}\left((-1)^q\frac{1}{q!\,\Gamma(n+q+1)}\left(\frac{x}{2}\right)^{2q}\right)\right]$$

$$= \sum_{n=-\infty}^{+\infty} t^n J_n(x).$$

(2) $\boxed{\forall n \in \mathbb{N} : \forall x \in \mathbb{R}^* : J_n(x) = (-1)^n J_{-n}(x)}$

Proof

Let $n \in \mathbb{N}$ and $x \in \mathbb{R}^*$ be given. Using the previous result, we note that

$$G(x, -1/t) = \sum_{n=-\infty}^{+\infty} J_n(x)(-1/t)^n = \sum_{n=-\infty}^{+\infty} (-1)^{-n} J_n(x) t^{-n} =$$

$$= \sum_{n=-\infty}^{+\infty} (-1)^n J_{-n}(x) t^n \quad , \forall t \in \mathbb{R}^*$$

and

$$G(x,-1/t) = \exp\left(\frac{1}{2} x \left((-1/t) - \frac{1}{-1/t}\right)\right) = \exp\left(\frac{x}{2}\left(-\frac{1}{t} - (-t)\right)\right)$$

$$= \exp\left(\frac{1}{2} x \left(t - \frac{1}{t}\right)\right) = G(x,t) = \sum_{n=-\infty}^{+\infty} J_n(x) t^n, \forall t \in \mathbb{R}^*$$

It follows that

$$\forall t \in \mathbb{R}^* : \sum_{n=-\infty}^{+\infty} (-1)^n J_{-n}(x) t^n = \sum_{n=-\infty}^{+\infty} J_n(x) t^n$$

$$\Rightarrow \forall n \in \mathbb{N} : J_n(x) = (-1)^n J_{-n}(x). \qquad \square$$

It follows from the above result that for integer order, $J_n(x)$ and $J_{-n}(x)$ are not linearly independent. This is the reason why it becomes necessary to introduce the function $I^a(x)$ when $a \in \mathbb{N}$ for the second solution.

(3) $\rightarrow$ $$\boxed{\forall a \in \mathbb{R} : \forall x \in (0,+\infty) : \ x J_a'(x) = a J_a(x) - x J_{a+1}(x)}$$

<u>Proof</u>

Let $a \in \mathbb{R}$ and $x \in (0,+\infty)$ be given. Then since

$$J_a(x) = \left|\frac{x}{2}\right|^a \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n} =$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a} \Rightarrow$$

$$\Rightarrow x J_a'(x) = x \frac{d}{dx} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a}$$

$$= x \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+a+1)} \frac{d}{dx} \left(\frac{x}{2}\right)^{2n+a}$$

$$= x \sum_{n=0}^{+\infty} \frac{(-1)^n (2n+a)}{n!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a-1} \cdot \left(\frac{1}{2}\right)$$

$$= \sum_{n=0}^{+} \frac{(-1)^n (2n+a)}{n!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a} =$$

$$= a \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a} + \sum_{n=1}^{+\infty} \frac{(-1)^n (2n)}{n!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a}$$

$$= a J_a(x) + x \sum_{n=1}^{+\infty} \frac{(-1)^n}{(n-1)!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a-1}$$

$$= a J_a(x) + x \sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{n!\,\Gamma((n+1)+a+1)} \left(\frac{x}{2}\right)^{2(n+1)+a-1}$$

$$= a J_a(x) - x \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+(a+1)+1)} \left(\frac{x}{2}\right)^{2n+(a+1)}$$

$$= a J_a(x) - x J_{a+1}(x).$$

$\square$

4 $\longrightarrow$ $\boxed{\forall x \in (0,+\infty): \ J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x}$

Proof

Let $x \in (0,+\infty)$ be given. Then

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \sum_{n=0}^{+\infty} (-1)^n \frac{1}{n!\,\Gamma(n+1/2+1)} \left(\frac{x}{2}\right)^{2n} =$$

$$= \left(\frac{x}{2}\right)^{1/2} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \left[\prod_{k=0}^{n} \frac{1}{k+1/2}\right] \frac{1}{\Gamma(1/2)} \left(\frac{x}{2}\right)^{2n} =$$

$$= \left(\frac{x}{2}\right)^{1/2} \frac{1}{\Gamma(1/2)} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \left[2^{n+1} \prod_{k=0}^{n} \frac{1}{2k+1}\right] \frac{x^{2n}}{2^{2n}} =$$

$$= \left(\frac{x}{2}\right)^{1/2} \frac{1}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{2(-1)^n}{2^n n!} \frac{1}{(2n+1)!!} x^{2n} =$$

$$= \left(\frac{x}{2\pi}\right)^{1/2} \sum_{n=0}^{+\infty} \frac{2(-1)^n}{(2n)!!\,(2n+1)!!} x^{2n} =$$

$$= 2\left(\frac{x}{2\pi}\right)^{1/2} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} =$$

$$= \left(\frac{2}{\pi x}\right)^{1/2} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} =$$

$$= \left(\frac{2}{\pi x}\right)^{1/2} \sin x = \sqrt{\frac{2}{\pi x}} \sin x. \qquad \square$$

EXERCISES

(24) Given $n \in \mathbb{N}$ and $x \in (0, +\infty)$, show the following identities.

a) $x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$

b) $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

c) $2n J_n(x) = x \left[J_{n-1}(x) + J_{n+1}(x) \right]$

d) $(d/dx)(x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x)$

e) $(d/dx)(x^n J_n(x)) = x^n J_{n-1}(x)$

f) $(d/dx)(x J_n(x) J_{n+1}(x)) = x (J_n^2(x) - J_{n+1}^2(x))$

↳ We have already showed that
$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$.
This result can be combined with (a) to prove the other results, without directly using power series expansions.

(25) Given $x \in (0, +\infty)$, show that

a) $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$

b) $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \cos x$

c) $J_{3/2}(x) = \sqrt{\dfrac{2}{\pi x}} \left(\dfrac{\sin x}{x} - \cos x \right)$

d) $J_{-3/2}(x) = \sqrt{\dfrac{2}{\pi x}} \left(-\dfrac{\cos x}{x} - \sin x \right)$

(26) Mini-project

The goal of this mini-project is to establish the following integral representation:

$$\boxed{\forall n\in\mathbb{N}: \forall x\in\mathbb{R}^*: J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\vartheta - x\sin\vartheta)\,d\vartheta}$$

We recall that $\forall \vartheta\in\mathbb{R}: e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$.

a) Show that $\exp(ix\sin\vartheta) = G(x, e^{i\vartheta})$

b) Use (a) to establish the following identities:

$$\cos(x\sin\vartheta) = \left[\sum_{n=0}^{+\infty} 2J_{2n}(x)\cos(2n\vartheta)\right] - J_0(x)$$

$$\sin(x\sin\vartheta) = \sum_{n=0}^{+\infty} 2J_{2n+1}(x)\sin((2n+1)\vartheta)$$

c) Show that

$$\forall a,b\in\mathbb{N}: \int_0^{\pi}\cos(a\vartheta)\cos(b\vartheta)\,d\vartheta = \begin{cases} 0, & \text{if } a\neq b \\ \pi/2, & \text{if } a=b\end{cases}$$

$$\forall a,b\in\mathbb{N}: \int_0^{\pi}\sin(a\vartheta)\sin(b\vartheta)\,d\vartheta = \begin{cases} 0, & \text{if } a\neq b \\ \pi/2, & \text{if } a=b\end{cases}$$

d) Combine the results (b) and (c) to show that

$$\int_0^{\pi}\cos(n\vartheta - x\sin\vartheta)\,d\vartheta = \pi J_n(x)$$

(Hint: it will be necessary to distinguish between two cases: n even vs. n odd).