LINEAR ALGEBRA REVIEW

General linear differential equations are analogous to linear systems of equations. It is therefore useful to briefly review basic concepts of linear algebra

▮ Vectors in $\mathbb{R}^n$

Consider two n-dimensional vectors $x, y \in \mathbb{R}^n$ with

$$x = (x_1, x_2, \dots, x_n)$$
$$y = (y_1, y_2, \dots, y_n)$$

We define the following vector operations:

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \leftarrow \text{vector addition}$$
$$\forall \lambda \in \mathbb{R}: \lambda x = (\lambda x_1, \lambda x_2, \dots, \lambda x_n) \leftarrow \text{scalar multiplication}$$

We also define the zero vector

$$\mathbf{0} = (0, 0, 0, \dots, 0)$$

● Linearly independent vectors

Def : Let $u_1, u_2, \dots, u_m \in \mathbb{R}^n$ be m n-dimensional vectors.

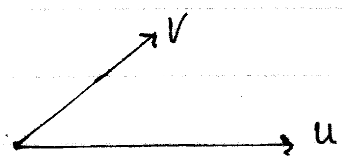
We say that

$$\boxed{\begin{aligned} & u_1, u_2, \dots, u_m \text{ linearly independent} \iff \\ & \iff \forall \lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}: (\lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_m u_m = \mathbf{0} \Rightarrow \\ & \qquad \Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_m = 0) \end{aligned}}$$

Interpretation

The equation $\lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_m u_m = \mathbf{0}$ implies that each of the m vectors can be written as a linear combination of the other vectors. If the vectors are linearly independent, it is impossible for the equation to be satisfied with non-zero coefficients, therefore none of the vectors can be written as a linear combination of the other vectors.

▸ In two dimensions:

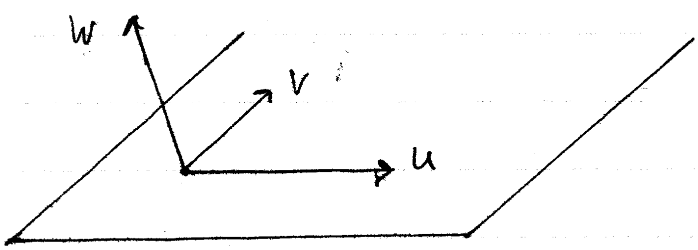


$u = (u_1, u_2)$

$v = (v_1, v_2)$

u, v are linearly independent if and only if they point in different directions.

▸ In three dimensions:



$u = (u_1, u_2, u_3)$

$v = (v_1, v_2, v_3)$

$w = (w_1, w_2, w_3)$

u, v, w are linearly independent if and only if u and v are not on the same line and w does not lie on the plane defined by u, v.

• Matrices

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be an arbitrary vector. A matrix $A \in M_n(\mathbb{R})$ represents a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined as:

$$\begin{cases} y_1 = (Ax)_1 = A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n \\ y_2 = (Ax)_2 = A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n \\ \vdots \\ y_n = (Ax)_n = A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nn}x_n \end{cases}$$

For $y = (y_1, y_2, \dots, y_n)$ we write: $y = Ax$

- The numbers A_{ab} are the components of the matrix A and we write

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

Alternatively, if $A_1, A_2, \dots, A_n \in \mathbb{R}^n$ are vectors representing the rows of A such that

$$A_1 = (A_{11}, A_{12}, \dots, A_{1n})$$
$$A_2 = (A_{21}, A_{22}, \dots, A_{2n})$$
$$\vdots$$
$$A_n = (A_{n1}, A_{n2}, \dots, A_{nn})$$

we write $A = (A_1, A_2, \dots, A_n)$.

- We note that

$$\boxed{\forall \lambda_1, \lambda_2 \in \mathbb{R} : \forall u, v \in \mathbb{R}^n : A(\lambda_1 u + \lambda_2 v) = \lambda_1 (Au) + \lambda_2 (Av).}$$

Matrix operations

Let $A, B \in M_n(\mathbb{R})$ be two matrices and let $\lambda \in \mathbb{R}$ be a number.
We define $A+B$, AB, and λA as follows:

$$\forall x \in \mathbb{R}^n : (A+B)x = Ax + By$$

$$\forall x \in \mathbb{R}^n : (AB)x = A(Bx)$$

$$\forall x \in \mathbb{R}^n : (\lambda A)x = \lambda (Ax)$$

It follows that the components of these new matrices are given by:

$$\forall a,b \in [n] : (A+B)_{ab} = A_{ab} + B_{ab}$$

$$\forall a,b \in [n] : (AB)_{ab} = \sum_{c \in [n]} A_{ac} B_{cb}$$

$$\forall a,b \in [n] : (\lambda A)_{ab} = \lambda A_{ab}$$

Identity Matrix

Given the unit vectors $e_1, e_2, \dots, e_n$ defined as:

$$e_1 = (1, 0, \dots, 0)$$

$$e_2 = (0, 1, \dots, 0)$$

$$\vdots$$

$$e_n = (0, 0, \dots, 1)$$

we define the $n \times n$ identity matrix as

$$I = (e_1, e_2, \dots, e_n)$$

or equivalently as

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

We note that

$$\forall A \in M_n(\mathbb{R}): \; IA = AI = A$$

Matrix Inverse

Let $A \in M_n(\mathbb{R})$ be a matrix. We say that

$$B = A^{-1} \Longleftrightarrow AB = BA = I.$$

- interpretation: The inverse matrix A^{-1} undoes the effect of the operation A on any vector x, since

$$A^{-1}(Ax) = (A^{-1}A)x = Ix = x$$

Not all matrices have an inverse. If a matrix A has an inverse, we say that A is non-singular

- inverse of a 2×2 matrix

Let A be a matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A non-singular if and only if $ad-bc \neq 0$ and A^{-1} is given by:

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinant of a matrix

The existence of an inverse can be queried via the determinant $\det(A)$ of the matrix A. We define the determinant as follows:

• 1 Permutations : A permutation σ is a mapping $\sigma : [n] \to [n]$ that rearranges the order of the elements of $[n]$.

e.g.: $\sigma = (3, 1, 2)$ is the permutation with $\sigma(1) = 3$, $\sigma(2) = 1$, and $\sigma(3) = 2$.

The set of all permutations $\sigma : [n] \to [n]$ is denoted as S_n.

• 2 Parity of a permutation :

Let $\sigma \in S_n$ be a permutation. We define the parity $s(\sigma)$ of σ as:

$$\boxed{\begin{gathered} s(\sigma) = \operatorname{sign}\left[\prod_{b=1}^{n-1} \prod_{a=b+1}^{n} (\sigma(a) - \sigma(b))\right] \\ \operatorname{sign}(x) = \begin{cases} +1 & , \text{ if } x > 0 \\ 0 & , \text{ if } x = 0 \\ -1 & , \text{ if } x < 0 \end{cases} \end{gathered}}$$

• 3 Determinant

Let $A \in M_n(\mathbb{R})$ be a matrix. We define the determinant $\det(A)$ of A as:

$$\boxed{\det A = \sum_{\sigma \in S_n} \left[s(\sigma) \prod_{a \in [n]} A_{a,\sigma(a)} \right]}$$

▶ Determinant of a 2x2 matrix

The determinant of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is given by: $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

▸ Properties of determinants

- 1 Determinant and matrix non-singularity.
  $\forall A \in M_n(\mathbb{R})$: A non-singular $\Leftrightarrow \det A \neq 0$
- 2 Determinant of a matrix product
  $\forall A \in M_n(\mathbb{R})$: $\det(AB) = \det(A)\det(B)$
- 3 Determinant of a matrix with two identical rows:
  Let $a_1, a_2, \dots, a_n \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ be vectors. Then
  $\det(a_1, a_2, \dots, b, \dots, b, \dots, a_n) = 0$
- 4 Determinant linearity:
  Let $a_1, a_2, \dots, a_n \in \mathbb{R}^n$ and $b, c \in \mathbb{R}^n$ be vectors. Then
  $$\forall \lambda, \mu \in \mathbb{R}: \det(a_1, \dots, \lambda b + \mu c, \dots, a_n) = \lambda \det(a_1, \dots, b, \dots, a_n) + \mu \det(a_1, \dots, c, \dots, a_n)$$

▸ Evaluation of determinants

The efficient evaluation of derivatives can be done using the following results:

(1) A 2×2 determinant can be evaluated as:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

(2) From properties 3 and 4 above it follows that we can a multiple of one row to another row without changing the value of the derivative. For example,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{matrix} \cdot\lambda \\ \leftarrow \\ \end{matrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + \lambda a_1 & b_2 + \lambda a_2 & b_3 + \lambda a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The same property also holds for columns. For example,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3+\lambda a_1 \\ b_1 & b_2 & b_3+\lambda b_1 \\ c_1 & c_2 & c_3+\lambda c_1 \end{vmatrix}$$

λ

We can use this to zero-out a row or column of the matrix.

(3) Determinants with a row or column of the form $(0,0,\dots,0,a,0,\dots,0)$ can be reduced into an equal determinant of smaller size by deleting both the row and column that pass through a. We then multiply with a ± 1 factor, depending on the location of a, according to a "chessboard pattern" of the form

$$\begin{vmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{vmatrix}$$

in which the upper-left corner is always "+". For example,

$$\begin{vmatrix} a_1 & 0 & b_1 \\ a_2 & 0 & b_2 \\ a_3 & \lambda & b_3 \end{vmatrix} = (-1)\lambda \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\lambda(a_1 b_2 - a_2 b_1).$$

$$\begin{vmatrix} 0 & 0 & \lambda \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (+1)\lambda \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \lambda(a_1 b_2 - a_2 b_1).$$

Linear system of equations

Consider the linear system $Ax = b$ with $A \in M_n(\mathbb{R})$ and $x, b \in \mathbb{R}^n$ which can be expanded as:

$$\begin{cases} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = b_2 \\ \vdots \\ A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nn}x_n = b_n \end{cases}$$

Cramer rule

If $\det A \neq 0$, then the system $Ax = b$ has a unique solution $x = (x_1, x_2, \ldots, x_n)$ with

$\forall k \in [n]: \ x_k = D_k / D$

where $D = \det A$ and

$$D_1 = \begin{vmatrix} b_1 & A_{12} & \cdots & A_{1n} \\ b_2 & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & A_{n2} & \cdots & A_{nn} \end{vmatrix}, \quad D_2 = \begin{vmatrix} A_{11} & b_1 & \cdots & A_{1n} \\ A_{21} & b_2 & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & b_n & \cdots & A_{nn} \end{vmatrix}, \ldots,$$

$$D_n = \begin{vmatrix} A_{11} & A_{12} & \cdots & b_1 \\ A_{21} & A_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & b_n \end{vmatrix}$$

In other words D_k is the definition of the matrix obtained by replacing the column k of A with the components of b.

▸ <u>null space</u>

We now consider the case $\det A = 0$. We define the <u>null-space</u> of the matrix A as:

$$\text{null}(A) = \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\}$$

with corresponding belonging condition given by

$$x \in \text{null}(A) \iff Ax = \mathbf{0}.$$

- Given a <u>particular solution</u> $p \in \mathbb{R}^n$ of $Ax = b$, the entire solution set of the system is given by:

  $$S = \{x \in \mathbb{R}^n \mid Ax = b\} = \{p + x \mid x \in \text{null}(A)\}.$$

  We will see that an analogous result holds for linear differential equations with respect to homogeneous and particular solutions.

- We can also show that

  $$\text{null}(A) = \{\mathbf{0}\} \iff \det A \neq 0$$

  therefore $\text{null}(A)$ has non-trivial content only if $\det A = 0$.

  Specifically we can show that:

a) $\text{null}(A) \cap (\mathbb{R}^n - \{\mathbf{0}\}) \neq \varnothing \iff \det A = 0$

   or equivalently:

   $(\exists x \in \mathbb{R}^n - \{\mathbf{0}\} : Ax = \mathbf{0}) \iff \det A = 0$

b) If $\det A = 0$, then:

$$\exists u_1, \ldots, u_k \in \mathbb{R}^n : \begin{cases} u_1, u_2, \ldots, u_k \text{ linearly independent} \\ \text{null}(A) = \text{span}\{u_1, u_2, \ldots, u_k\} \end{cases}$$

   where we define:

$$\text{span}\{u_1, u_2, \ldots, u_n\} = \{\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k \mid \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}\}.$$