LINEAR DIFFERENTIAL EQUATIONS

▼ Basic Definitions - Terminology

- A linear differential equation is any equation of the form
$$p_n(x)y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = f(x). \quad (1)$$
- The functions $p_0, p_1, \ldots, p_n$ are called the coefficients of the linear differential equation and it is usually assumed that they are continuous functions.
- $n \in \mathbb{N}^*$ is the order of the linear differential equation.
- Given the linear differential equation of Eq.(1), we say that for a point $x_0 \in \mathbb{R}$:
$$x_0 \text{ is regular} \Leftrightarrow p_n(x_0) \neq 0$$
$$x_0 \text{ is singular} \Leftrightarrow p_n(x_0) = 0$$
- A linear differential equation of the form of Eq.(1) is homogeneous on a set $A \subseteq \mathbb{R}$ if and only if
$$\forall x \in A : f(x) = 0.$$
otherwise, we say that it is inhomogeneous.
- If an linear differential equation is regular for every point in some interval $A \subseteq \mathbb{R}$ (i.e. if $\forall x \in A : p_n(x) \neq 0$) then we can rewrite it as:
$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x) \quad (2)$$
with
$$\forall k \in [n-1] \cup \{0\} : a_k(x) = \frac{p_k(x)}{p_n(x)} \quad \text{and} \quad g(x) = \frac{f(x)}{p_n(x)}.$$

I Function operators and linear operators

- Let $A \subseteq \mathbb{R}$ be an interval. We define the following function spaces via belonging conditions as follows:

  a) Space of continuous functions $C^0(A)$:
  $$y \in C^0(A) \Leftrightarrow \begin{cases} y : A \to \mathbb{R} \\ y \text{ continuous on } A. \end{cases}$$
  b) Space of n-times continuously differentiable functions $C^n(A)$.
  $$y \in C^n(A) \Leftrightarrow \begin{cases} y : A \to \mathbb{R} \\ y \text{ n-times differentiable on } A \\ y^{(n)} \text{ continuous on } A \end{cases}$$
  c) Space of infinitely differentiable functions $C^\infty(A)$.
  $$y \in C^\infty(A) \Leftrightarrow \forall n \in \mathbb{N} : y \in C^n(A)$$

- Given the linear differential equation from Eq.(2) we define the mapping $L : C^n(A) \to C^0(A)$ such that
  $$\forall y \in C^n(A) : L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y \qquad (3)$$
  Then, the linear differential equation
  $$y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \dots + a_1(x) y'(x) + a_0(x) y(x) = g(x)$$
  can be rewritten as:
  $$L(y) = g. \quad \text{or also:} \quad Ly = g.$$
  Note that by analogy the <u>operator L</u> is to a function $y \in C^n(A)$ what a matrix A is to some vector $x \in \mathbb{R}^n$.

- The operator L defined by Eq. (3) satisfies the following definition of a <u>linear operator</u>

<u>Def</u>: Consider an operator $L : C^n(A) \to C^0(A)$. We say that L is a linear operator if and only if it satisfies the following conditions:

a) $\forall y_1, y_2 \in C^n(A) : L(y_1 + y_2) = Ly_1 + Ly_2$

b) $\forall \lambda \in \mathbb{R} : \forall y \in C^n(A) : L(\lambda y) = \lambda L(y)$.

<u>Prop</u>: Let $L : C^n(A) \to C^0(A)$ be a linear operator. Then:

$$\forall \lambda, \mu \in \mathbb{R} : \forall y_1, y_2 \in C^n(A) : L(\lambda y_1 + \mu y_2) = \lambda L(y_1) + \mu L(y_2).$$

<u>Proof</u>

Let $\lambda, \mu \in \mathbb{R}$ and $y_1, y_2 \in C^n(A)$ be given. Then:

$$\begin{aligned} L(\lambda y_1 + \mu y_2) &= L(\lambda y_1) + L(\mu y_2) \\ &= \lambda L(y_1) + \mu L(y_2). \end{aligned}$$

It follows that

$$\forall \lambda, \mu \in \mathbb{R} : \forall y_1, y_2 \in C^n(A) : L(\lambda y_1 + \mu y_2) = \lambda L(y_1) + \mu L(y_2).$$

→ Note that the definition

$$Ly = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$$

is given in terms of function algebra, i.e function addition and function multiplication. In terms of regular algebra, we write:

$$\forall x \in A : (Ly)(x) = y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x).$$

▼ Homogeneous linear differential equations

We begin by presenting the theory needed for solving homogeneous linear differential equations of the form $Ly = 0$ given a linear operator $L: C^n(A) \to C^0(A)$.

● Solution set of the homogeneous ODE

We begin by stating some needed definitions. Then we state the main result without proof.

Def: Let $y_1, y_2, \ldots, y_n \in C^0(A)$ be functions. We say that $y_1, y_2, \ldots, y_n$ linearly independent $\Leftrightarrow$

$$\Leftrightarrow \forall \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}: (\lambda_1 y_1 + \cdots + \lambda_n y_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0)$$

↳ We note that this definition is analogous to the linear independence of vectors on $\mathbb{R}^n$. However, the statement
$\lambda_1 y_1 + \cdots + \lambda_n y_n = 0$
is equivalent to the algebraic statement
$\forall x \in A: \lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x) = 0.$

Def: Let $y_1, y_2, \ldots, y_n \in C^0(A)$. We define the space spanned by the functions $y_1, \ldots, y_n$ as

$$\text{span}\{y_1, y_2, \ldots, y_n\} = \{\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n \mid \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}\}$$

The corresponding belonging condition reads:

$y \in \operatorname{span}\{y_1, y_2, \dots, y_n\} \Leftrightarrow$

$\Leftrightarrow \exists \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}: y = \lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_n y_n.$

Def: Let $L : C^n(A) \to C^0(A)$ be an operator. We define the null space of L as:

$$\operatorname{null}(L) = \{ y \in C^n(A) \mid Ly = 0 \}.$$

Thus, the problem of solving the homogeneous linear differential equation $Ly = 0$ is equivalent to the problem of finding the null space $\operatorname{null}(L)$ or the operator L.

Thm: Let $a_0, a_1, \dots, a_{n-1} \in C^0(A)$ for some interval $A \subseteq \mathbb{R}$ and define the operator $L : C^n(A) \to C^0(A)$ such that

$\forall y \in C^n(A): Ly = y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y$

Then there exist $y_1, y_2, \dots, y_n \in C^n(A)$ such that they satisfy the following conditions:

(a) $y_1, y_2, \dots, y_n$ are linearly independent

(b) $\operatorname{null}(L) = \operatorname{span}\{y_1, y_2, \dots, y_n\}$

It follows from this theorem that the general solution to the linear differential equation $Ly = 0$ takes the form

$\forall x \in A: y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \dots + \lambda_n y_n(x)$

where $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ are constant coefficients and $y_1, y_2, \dots, y_n$ are linearly independent functions.

EXAMPLES

a) Consider the functions

$\forall x \in \mathbb{R} : (f(x) = x \wedge g(x) = x^2 \wedge h(x) = x^3)$

Show that f, g, h are linearly independent.

Solution

- We give two different methods for solving this problem. Only one method is needed for a complete solution.

1st method : By definition.

It is sufficient to show that

$\forall \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} : (\lambda_1 f + \lambda_2 g + \lambda_3 h = \mathbf{0} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0)$

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be given and assume that $\lambda_1 f + \lambda_2 g + \lambda_3 h = \mathbf{0}$.

It follows that:

$\lambda_1 f + \lambda_2 g + \lambda_3 h = \mathbf{0} \Rightarrow \forall x \in \mathbb{R} : \lambda_1 f(x) + \lambda_2 g(x) + \lambda_3 h(x) = 0$

$\Rightarrow \forall x \in \mathbb{R} : \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 = 0 \qquad (1)$

For $x = 1$: $\lambda_1 + \lambda_2 + \lambda_3 = 0$

For $x = 2$: $2\lambda_1 + 4\lambda_2 + 8\lambda_3 = 0$

For $x = -1$: $-\lambda_1 + \lambda_2 - \lambda_3 = 0$

Consider the system of equations:

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ 2\lambda_1 + 4\lambda_2 + 8\lambda_3 = 0 \\ -\lambda_1 + \lambda_2 - \lambda_3 = 0 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1)$$

We note that:

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ -1 & 1 & -1 \end{vmatrix} \begin{matrix} (-2) & 1 \\ & \\ & \end{matrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 6 \\ 2 & 0 \end{vmatrix} = 2\cdot 0 - 6\cdot 2 = -12 \neq 0$$

$\Rightarrow$ Eq. (1) has a unique solution $(\lambda_1, \lambda_2, \lambda_3) = (0,0,0) \Rightarrow$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$

We have thus shown that

$\forall \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} : (\lambda_1 f + \lambda_2 g + \lambda_3 h = \mathbf{0} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0)$

$\Rightarrow f, g, h$ linearly independent.

b) Consider the functions $\forall x \in \mathbb{R} : (f(x) = e^{ax} \wedge g(x) = e^{bx})$

Show that: $a \neq b \Rightarrow f, g$ linearly independent.

<u>Solution</u>

Assume that $a \neq b$. It is sufficient to show that:

$\forall \lambda_1, \lambda_2 \in \mathbb{R} : (\lambda_1 f + \lambda_2 g = \mathbf{0} \Rightarrow \lambda_1 = \lambda_2 = 0)$

Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be given. Assume that $\lambda_1 f + \lambda_2 g = \mathbf{0}$. Then:

$\lambda_1 f + \lambda_2 g = \mathbf{0} \Rightarrow \forall x \in \mathbb{R} : \lambda_1 f(x) + \lambda_2 g(x) = 0$

$\qquad \Rightarrow \forall x \in \mathbb{R} : \lambda_1 e^{ax} + \lambda_2 e^{bx} = 0 \qquad (1)$

For $x = 0$: $\lambda_1 e^0 + \lambda_2 e^0 = 0 \Leftrightarrow \lambda_1 + \lambda_2 = 0$

For $x = 1$: $\lambda_1 e^a + \lambda_2 e^b = 0$

It follows that:

$$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ e^a \lambda_1 + e^b \lambda_2 = 0 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 \\ e^a & e^b \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (2)$$

and note that:

$$\begin{vmatrix} 1 & 1 \\ e^a & e^b \end{vmatrix} = e^b - e^a$$

Since: $a \neq b \Rightarrow e^a \neq e^b \Rightarrow e^b - e^a \neq 0 \Rightarrow \begin{vmatrix} 1 & 1 \\ e^a & e^b \end{vmatrix} \neq 0$

$\Rightarrow$ Eq. (2) has a unique solution $(\lambda_1, \lambda_2) = (0, 0)$

$\Rightarrow \lambda_1 = \lambda_2 = 0.$

We have thus shown that

$\forall \lambda_1, \lambda_2 \in \mathbb{R}: (\lambda_1 f + \lambda_2 g = 0 \Rightarrow \lambda_1 = \lambda_2 = 0)$

$\Rightarrow f, g$ linearly independent.

EXERCISES

(1) Show that the functions f, g, h, defined below, are linearly independent, using the definition.

a) $\begin{cases} \forall x \in \mathbb{R}: f(x) = 3x \\ \forall x \in \mathbb{R}: g(x) = x+2 \\ \forall x \in \mathbb{R}: h(x) = (x-1)^2 \end{cases}$

b) $\begin{cases} \forall x \in \mathbb{R}: f(x) = \sin x \\ \forall x \in \mathbb{R}: g(x) = \cos x \\ \forall x \in \mathbb{R}: h(x) = x \end{cases}$

c) $\begin{cases} \forall x \in \mathbb{R}: f(x) = 1-x \\ \forall x \in \mathbb{R}: g(x) = 1+x \\ \forall x \in \mathbb{R}: h(x) = 1-x^2 \end{cases}$

d) $\begin{cases} \forall x \in \mathbb{R}: f(x) = 1 \\ \forall x \in \mathbb{R}: g(x) = e^x \\ \forall x \in \mathbb{R}: h(x) = e^{2x} \end{cases}$

e) $\begin{cases} \forall x \in \mathbb{R}: f(x) = e^{3x} \\ \forall x \in \mathbb{R}: g(x) = xe^{3x} \\ \forall x \in \mathbb{R}: h(x) = x^2 e^{3x} \end{cases}$

The initial value problem

In an initial value problem we consider the homogeneous linear differential equation $Ly = 0$ where we introduce the restrictions

$$y(x_0) = a_0 \wedge y'(x_0) = a_1 \wedge y''(x_0) = a_2 \wedge \cdots \wedge y^{(n-1)}(x_0) = a_{n-1}$$

Given the general solution

$$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x)$$

the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ can be uniquely solved by the following system of equations:

$$\begin{cases} \lambda_1 y_1(x_0) + \lambda_2 y_2(x_0) + \cdots + \lambda_n y_n(x_0) = a_0 \\ \lambda_1 y_1'(x_0) + \lambda_2 y_2'(x_0) + \cdots + \lambda_n y_n'(x_0) = a_1 \\ \vdots \qquad \qquad \qquad \qquad \qquad \qquad \qquad \cdots \\ \lambda_1 y_1^{(n-1)}(x_0) + \lambda_2 y_2^{(n-1)}(x_0) + \cdots + \lambda_n y_n^{(n-1)}(x_0) = a_{n-1} \end{cases}$$

which can be rewritten in terms of matrices as follows:

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

The determinant of the matrix is called the <u>Wronskian</u> and we will prove later that it is non-zero. It follows that solving with respect to the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ will give a unique solution.

The Wronskian and its properties

Def: Let $y_1, y_2, \dots, y_n \in C^{n-1}(A)$, for some interval $A \subseteq \mathbb{R}$. We define:

a) The matrix $W[y_1, \dots, y_n](x)$ as:

$$\forall x \in A: W[y_1, \dots, y_n](x) = \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & & & \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix}$$

b) The Wronskian $w[y_1, \dots, y_n](x)$ as:

$$\forall x \in A: w[y_1, \dots, y_n](x) = \det W[y_1, \dots, y_n](x)$$

We now show that the Wronskian satisfies the following properties:

1 → Nonzero Wronskian implies linear independence

Thm: Let $y_1, y_2, \dots, y_n \in C^{n-1}(A)$ with $A \subseteq \mathbb{R}$ an interval. Then:

$$(\exists x \in A: w[y_1, \dots, y_n](x) \neq 0) \Longrightarrow$$
$$\Longrightarrow y_1, y_2, \dots, y_n \text{ linearly independent}$$

Proof

Assume that $\exists x \in A: w[y_1, \dots, y_n](x) \neq 0$. Choose an $x_0 \in A$ such that $w[y_1, \dots, y_n](x_0) \neq 0$. It is sufficient to show that $\forall \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}: (\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n = 0 \Longrightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0)$

Let $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ be given and assume that

$$\lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_n y_n = \mathbf{0} \Rightarrow$$

$$\Rightarrow \forall x \in A : \lambda_1 y_1(x) + \lambda_2 y_2(x) + \dots + \lambda_n y_n(x) = 0$$

Differentiating with respect to x gives the equations:

$$\forall x \in A : \lambda_1 y_1'(x) + \lambda_2 y_2'(x) + \dots + \lambda_n y_n'(x) = 0$$

$$\forall x \in A : \lambda_1 y_1''(x) + \lambda_2 y_2''(x) + \dots + \lambda_n y_n''(x) = 0$$

$$\vdots$$

$$\forall x \in A : \lambda_1 y_1^{(n-1)}(x) + \lambda_2 y_2^{(n-1)}(x) + \dots + \lambda_n y_n^{(n-1)}(x) = 0$$

These equations are equivalent to the matrix equation

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ y_1''(x) & y_2''(x) & \cdots & y_n''(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \forall x \in A$$

We define $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and the matrix equation is

$W[y_1, y_2, \dots, y_n](x)\, \Lambda = \mathbf{0}$, $\forall x \in A$.

For $x = x_0$, we have:

$$\begin{cases} W[y_1, \dots, y_n](x_0)\, \Lambda = \mathbf{0} \\ \det W[y_1, \dots, y_n](x_0) = w[y_1, \dots, y_n](x_0) \neq 0 \end{cases} \Rightarrow$$

$$\Rightarrow \Lambda = \mathbf{0} \Rightarrow (\lambda_1, \lambda_2, \dots, \lambda_n) = (0, 0, \dots, 0)$$

$$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$$

We have thus shown that

$$\forall \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R} : (\lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_n y_n = \mathbf{0} \Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0)$$

$\Rightarrow y_1, y_2, \dots, y_n$ linearly independent $\square$

(2) → Linearly independent solutions of a linear differential equation give a non-zero Wronskian

The previous property can be used to prove that a set of functions are linearly independent, if the corresponding Wronskian is nonzero for at least one point. The converse statement is not always true. However we will now show that if some functions $y_1, \dots, y_n$ solve the SAME linear differential equation and are linearly independent, then they will give a nonzero Wronskian for all points.

Thm: Define the operator $L: C^n(A) \to C^0(A)$, for some interval $A \subseteq \mathbb{R}$, such that:

$$\forall x \in A: Ly(x) = y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \dots + a_1(x) y_1'(x) + a_0(x) y(x)$$

We assume that

a) $y_1, y_2, \dots, y_n \in C^n(A)$ are linearly independent

b) $\forall k \in [n]: Ly_k = 0$

Then, it follows that

a) $\forall x \in A: w'[y_1, \dots, y_n](x) + a_{n-1}(x) w[y_1, \dots, y_n](x) = 0$

b) For some $c \in A$:

$$\forall c, x \in A: w[y_1, \dots, y_n](x) = w[y_1, \dots, y_n](c) \exp\left(-\int_c^x a_{n-1}(t)\,dt\right)$$

c) $\forall x \in A: w[y_1, \dots, y_n](x) \neq 0$

Proof

a) Define the vector-valued function $y : A \to \mathbb{R}^n$ with $y = (y_1, y_2, \dots, y_n)$. Since

$$(\forall k \in [n] : L y_k = 0) \Rightarrow \left(\forall k \in [n] : y_k^{(n)} = -\sum_{p=0}^{n-1} a_p y_k^{(p)}\right) \Rightarrow$$

$$\Rightarrow y^{(n)} = -\sum_{p=0}^{n-1} a_p y^{(p)} \qquad (1)$$

Note that $y^{(p)}$ is a vector-valued function whereas a_p is a scalar function. It follows that

$$(d/dx) w[y_1, \dots, y_n](x) = (d/dx) \det(y, y', y'', \dots, y^{(n-1)}) =$$

$$= \det(y, y', \dots, y^{(n-2)}, y^{(n)}) =$$

$$= \det\left(y, y', \dots, y^{(n-2)}, -\sum_{p=0}^{n-1} a_p y^{(p)}\right) =$$

$$= \sum_{p=0}^{n-1} \det(y, y', \dots, y^{(n-2)}, -a_p y^{(p)}) =$$

$$= -\sum_{p=0}^{n-1} a_p \det(y, y', \dots, y^{(n-2)}, y^{(p)}) =$$

$$= -\sum_{p=0}^{n-2} a_p \det(y, \dots, y^{(p)}, \dots, y^{(n-2)}, y^{(p)}) +$$

$$a_{n-1} \det(y, \dots, y^{(n-2)}, y^{(n-1)})$$

$$= 0 - a_{n-1} w[y] = -a_{n-1} w[y] \Rightarrow$$

$$\Rightarrow \forall x \in A : \; w'[y](x) + a_{n-1}(x) w[y](x) = 0$$

b) Define the integrating factor

$$\forall x \in A : h(x) = \exp\left(\int_c^x a_{n-1}(t)\,dt \right)$$

and note that

$$\forall x \in A : h'(x) = (d/dx) \exp\left(\int_c^x a_{n-1}(t)\,dt \right) =$$

$$= \exp\left(\int_c^x a_{n-1}(t)\,dt \right) \frac{d}{dx} \int_c^x a_{n-1}(t)\,dt =$$

$$= h(x)\, a_{n-1}(x).$$

We may now solve the differential equation satisfied by the Wronskian as follows:

$$w'[y](x) + a_{n-1}(x)\, w[y](x) = 0 \Leftrightarrow$$

$$\Leftrightarrow w'[y](x)\, h(x) + h(x)\, a_{n-1}(x)\, w[y](x) = 0 \Leftrightarrow$$

$$\Leftrightarrow w'[y](x)\, h(x) + w[y](x)\, h'(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow (d/dx)\, [w[y](x)\, h(x)] = 0 \Leftrightarrow w[y](x)\, h(x) = c_0$$

$$\Leftrightarrow w[y](x) = \frac{c_0}{h(x)} = c_0 \exp\left(- \int_c^x a_{n-1}(t)\,dt \right)$$

For $x = c$: $w[y](c) = c_0 \cdot 1 = c_0$, and therefore

$$\forall x \in A : w[y](x) = w[y](c) \exp\left(- \int_c^x a_{n-1}(t)\,dt \right)$$

c) From (b) we see that it is sufficient to show that $\exists c \in A : w[y](c) \neq 0$. To show a contradiction, we assume the opposite statement: $\forall c \in A : w[y](c) = 0$. Choose some $c \in A$ and consider the linear system of equations $W[y](c)\Lambda = \mathbf{0}$ with $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$. It follows that

$w[y](c)=0 \Rightarrow \det W[y](c)=0 \Rightarrow \exists \Lambda \in \mathbb{R}^n - \{0\} : W[y]\Lambda = \mathbf{0}$

Choose some $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n - \{\mathbf{0}\}$ such that $W[y](c)\Lambda = \mathbf{0}$ and define the function $f : A \to \mathbb{R}$ with

$\forall x \in A : f(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \dots + \lambda_n y_n(x)$

It follows that

$$Lf = L\left(\sum_{k=1}^{n} \lambda_k y_k\right) = \sum_{k=1}^{n} L(\lambda_k y_k) = \sum_{k=1}^{n} \lambda_k L y_k =$$

$$= \sum_{k=1}^{n} \lambda_k \cdot \mathbf{0} = \mathbf{0} \Rightarrow f \in \text{null}(L).$$

We also know that

$$\forall p \in \{0\} \cup [n-1] : f^{(p)}(c) = \sum_{k=1}^{n} \lambda_k y_k^{(p)}(c) = \sum_{k=1}^{n} [W[y](c)]_{pk} \Lambda_k =$$

$$= [W[y](c)\Lambda]_p = 0$$

We will now claim that given the initial condition

$f(c) = f'(c) = f''(c) = \dots = f^{(n-1)}(c) = 0$

the function f will satisfy $\forall x \in A : f(x) = 0$. To show this, we rewrite the equation as a system of first-order ODEs by defining

$\forall k \in [n] : \forall x \in A : g_k(x) = f^{(k-1)}(x)$.

The ODE $Lf = \mathbf{0}$ can be rewritten as

$$\begin{cases} g_1'(x) = g_2(x) \\ g_2'(x) = g_3(x) \\ \vdots \\ g_{n-1}'(x) = g_n(x) \\ g_n'(x) = -\sum_{k=1}^{n} a_{k-1}(x) g_k(x) \end{cases}$$

and the corresponding initial condition is

$$g_1(x) = g_2(x) = \cdots = g_n(x) = 0$$

It is easy to see that all derivatives $g_1'(x), g_2'(x), \ldots, g_n'(x)$ are then zero, and therefore all functions $g_1, \ldots, g_n$ will remain constant and be equal to zero for all $x \in A$. This proves the claim. From the claim we have:

$$(\forall x \in A : f(x) = 0) \Rightarrow f = \mathbf{0} \Rightarrow \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n = \mathbf{0} \quad (1)$$

By hypothesis, we also know that

$$y_1, y_2, \ldots, y_n \text{ linearly independent} \quad (2)$$

From Eq.(1) and Eq.(2):

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0 \Rightarrow \lambda = \mathbf{0}$$

This is a contradiction, since by construction λ satisfies $\lambda \in \mathbb{R}^n - \{\mathbf{0}\}$. It follows that

$$\exists c \in A : w[y](c) \neq 0$$

Fix a $c \in A$ such that $w[y](c) \neq 0$. Then, from (6), it follows that

$$\forall x \in A : w[y](x) = w[y](c) \exp\left(-\int_c^x a_{n-1}(t)\,dt\right) \neq 0$$

because $\forall x \in \mathbb{R} : \exp(x) > 0$. This concludes the proof. $\square$

EXAMPLES

a) Consider the functions

$\forall x \in \mathbb{R} : (f(x) = x \wedge g(x) = x^2 \wedge h(x) = x^3)$

Use the Wronskian to show that f, g, h are linearly independent

<u>Solution</u>

Since,

$$w[f,g,h](x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} (-x) =$$

$$= \begin{vmatrix} 0 & x^2 - 2x^2 & x^3 - 3x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = \begin{vmatrix} 0 & -x^2 & -2x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} =$$

$$= (-1) \begin{vmatrix} -x^2 & -2x^3 \\ 2 & 6x \end{vmatrix} = -[(-x^2)6x - 2(-2x^3)] =$$

$$= -(-6x^3 + 4x^3) = -(-2x^3) = 2x^3 , \ \forall x \in \mathbb{R} \Rightarrow$$

$\Rightarrow w[f,g,h](1) = 2 \neq 0 \Rightarrow \exists x \in \mathbb{R} : w[f,g,h](x) \neq 0$

$\Rightarrow f, g, h$ linearly independent.

b) Show that for $\forall x \in \mathbb{R}: (f(x) = e^{2x} \wedge g(x) = xe^{2x})$
f, g are linearly independent.

<u>Solution</u>

(We use the Wronskian)

Since,

$\forall x \in \mathbb{R}: f'(x) = (e^{2x})' = e^{2x}(2x)' = 2e^{2x}$

$\forall x \in \mathbb{R}: g'(x) = (xe^{2x})' = (x)'e^{2x} + x(e^{2x})' = e^{2x} + xe^{2x}(2x)' =$
$= e^{2x} + xe^{2x} \cdot 2 = (1+2x)e^{2x}$

it follows that

$$\forall x \in \mathbb{R}: w[f,g](x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix} =$$

$$= e^{2x} \begin{vmatrix} e^{2x} & xe^{2x} \\ 2 & 1+2x \end{vmatrix} = e^{2x}e^{2x} \begin{vmatrix} 1 & x \\ 2 & 1+2x \end{vmatrix} =$$

$$= e^{4x}[1(1+2x) - 2x] = e^{4x}(1+2x-2x) = e^{4x} > 0$$

$\Rightarrow \forall x \in \mathbb{R}: w[f,g](x) \neq 0$

$\Rightarrow \exists x \in \mathbb{R}: w[f,g](x) \neq 0$

$\Rightarrow f, g$ linearly independent

EXERCISES

(2) Use the Wronskian to show that the functions f, g, h, defined below, are linearly independent.

a) $\begin{cases} \forall x \in \mathbb{R}: f(x) = e^{ax} \\ \forall x \in \mathbb{R}: g(x) = x e^{ax} \\ \forall x \in \mathbb{R}: h(x) = x^2 e^{ax} \end{cases}$

with $a \in \mathbb{R}$

b) $\begin{cases} \forall x \in \mathbb{R}: f(x) = \sin(ax) \\ \forall x \in \mathbb{R}: g(x) = \cos(ax) \\ \forall x \in \mathbb{R}: h(x) = x^2 \end{cases}$

with $a \in \mathbb{R}$

c) $\begin{cases} \forall x \in \mathbb{R}: f(x) = e^{ax} \cos x \\ \forall x \in \mathbb{R}: g(x) = e^{ax} \sin x \\ \forall x \in \mathbb{R}: h(x) = e^{ax} \end{cases}$

with $a \in \mathbb{R}$

d) $\begin{cases} \forall x \in \mathbb{R}: f(x) = x^2 \\ \forall x \in \mathbb{R}: g(x) = x^2 \ln x \\ \forall x \in \mathbb{R}: h(x) = x^2 [\ln x]^2 \end{cases}$

e) $\begin{cases} \forall x \in \mathbb{R}: f(x) = x^3 \cos(2 \ln x) \\ \forall x \in \mathbb{R}: g(x) = x^3 \sin(2 \ln x) \\ \forall x \in \mathbb{R}: h(x) = x^2 \end{cases}$

(3) Consider a general linear differential equation of the form

$\forall x \in A: \ y''(x) + a_1(x) y'(x) + a_0(x) y(x) = 0$

for some interval $A \subseteq \mathbb{R}$ with $a_0, a_1 \in C^0(A)$. Assume that $y_1 \in C^2(A)$ is a solution, and define $y_2 \in C^2(A)$ as:

$$\forall x \in A: \quad y_2(x) = y_1(x) \int_c^x \frac{Q(t)}{[y_1(t)]^2} \, dt$$

with $c \in A$ and with $Q(t)$ given by

$$\forall t \in A: \ Q(t) = \exp\left(- \int a_1(t) \, dt \right)$$

a) Show that $y_2(x)$ is also a solution.
(Hint: start with $y_2(x) = y_1(x)\,u(x)$ and substitute to the ODE to derive a sufficient condition for $u(x)$)

b) Show that y_1, y_2 are linearly independent.
(Hint: Use the Wronskian)

$\hookrightarrow$ Note that an immediate consequence of (a) and (b) is that if we define an operator $L: C^2(A) \to C^0(A)$ with $Ly = y'' + a_1 y' + a_0 y$, then it follows that its null space is
$$\mathrm{null}(L) = \mathrm{span}\{y_1, y_2\}$$
The corresponding general solution of the equation $Ly = 0$ is given by
$$\forall x \in A: \ y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$$
This exercise shows that if we can guess one solution of the 2nd-order linear ODE $Ly = 0$, we have an equation that can be used to find a second linearly independent solution. Then given the aforementioned theorems, we have the null space and the general solution.

(4) Find a solution of the form $\forall x \in \mathbb{R}: y_1(x) = e^{bx}$ for the linear ODE:
$$\forall x \in \mathbb{R}: \ y''(x) + 2a y'(x) + a^2 y(x) = 0$$
with $a \in \mathbb{R}$. Use exercise 2 to find the second solution and write the general solution.

(5) Find a solution of the form $\forall x \in (0,+\infty): y_1(x) = x^b$ for the linear ODE

$$\forall x \in (0,+\infty): x^2 y'' + (2\lambda+1) x y' + \lambda^2 y = 0$$

with $\lambda \in \mathbb{R}$. Use exercise 3 to find the second solution and write the general solution.

• Solving homogeneous linear differential equations

To solve a homogeneous linear differential equation
$y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \dots + a_1(x) y'(x) + a_0(x) y(x) = 0$
we need to find the linearly independent solutions
$y_1(x), y_2(x), \dots, y_n(x)$ that form the general solution

$$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \dots + \lambda_n y_n(x)$$

There is no general method for finding the functions $y_1(x), \dots, y_n(x)$. However, an exact solution is possible for the following cases.

(1) → Constant coefficient case

Consider the linear ODE
$y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \dots + a_1 y'(x) + a_0 y(x) = 0$
with $a_0, a_1, \dots, a_{n-1}, a_n \in \mathbb{R}$ given constants. Let L be the corresponding operator.

Solution method

- 1 Find the characteristic polynomial $P(b)$:

$$L(e^{bx}) = (b^n + a_{n-1} b^{n-1} + \dots + a_1 b + a_0) e^{bx}$$
$$= P(b) e^{bx}$$

- 2 Let $p_1, p_2, \dots, p_n \in \mathbb{C}$ be the zeroes of the characteristic polynomial P. Then:

  a) Each single zero p_k contributes a solution
  $$y_k(x) = \exp(p_k x)$$

b) Each zero ρ_k with multiplicity m (i.e. $P(b)$ has a factor $(x-\rho_k)^m$) contributes the following linearly independent solutions:

$$y_k(x) = \exp(\rho_k x)$$
$$y_{k+1}(x) = x\exp(\rho_k x)$$
$$y_{k+2}(x) = x^2\exp(\rho_k x)$$
$$\vdots$$
$$y_{k+m-1}(x) = x^{m-1}\exp(\rho_k x).$$

• 3 We write the general solution and apply the initial conditions if given.

↳ Remark : Complex zeroes appear as complex conjugate pairs $\rho_k = \gamma + i\omega$ and $\rho_{k+1} = \gamma - i\omega$, because the coefficients of the characteristic polynomial are real numbers. We use the De Moivre identity:

$$\forall \vartheta \in \mathbb{R}: e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$$

and note that the corresponding solutions satisfy:

$$y_k(x) = \exp(\rho_k x) = \exp((\gamma + i\omega)x) = \exp(\gamma x + i\omega x) =$$
$$= \exp(\gamma x)\exp(i\omega x) = e^{\gamma x}(\cos(\omega x) + i\sin(\omega x))$$
$$y_{k+1}(x) = \exp(\rho_{k+1} x) = \exp((\gamma - i\omega)x) = \exp(\gamma x - i\omega x) =$$
$$= \exp(\gamma x)\exp(-i\omega x) = e^{\gamma x}(\cos(-\omega x) + i\sin(-\omega x)) =$$
$$= e^{\gamma x}(\cos(\omega x) - i\sin(\omega x))$$

It follows that any linear combination of $y_k(x)$ and $y_{k+1}(x)$ can be rewritten as:

$$\lambda_k y_k(x) + \lambda_{k+1} y_{k+1}(x) =$$
$$= \lambda_k e^{\gamma x}(\cos(wx) + i\sin(wx)) + \lambda_{k+1} e^{\gamma x}(\cos(wx) - i\sin(wx)) =$$
$$= e^{\gamma x}[(\lambda_k + \lambda_{k+1})\cos(wx) + i(\lambda_k - \lambda_{k+1})\sin(wx)] =$$
$$= (\lambda_k + \lambda_{k+1})[e^{\gamma x}\cos(wx)] + i(\lambda_k - \lambda_{k+1})[e^{\gamma x}\sin(wx)]$$
$$= \mu_k e^{\gamma x}\cos(wx) + \mu_{k+1} e^{\gamma x}\sin(wx)$$

with

$$\begin{cases} \mu_k = \lambda_k + \lambda_{k+1} \\ \mu_{k+1} = i(\lambda_k - \lambda_{k+1}) \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} \lambda_k \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} \mu_k \\ \mu_{k+1} \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \lambda_k \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1}\begin{bmatrix} \mu_k \\ \mu_{k+1} \end{bmatrix} =$$

$$= \frac{1}{-i-i}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}\begin{bmatrix} \mu_k \\ \mu_{k+1} \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \lambda_k = \frac{-i\mu_k - \mu_{k+1}}{-2i} = \frac{\mu_{k+1} + i\mu_k}{2i} \wedge$$

$$\wedge\ \lambda_{k+1} = \frac{-i\mu_k + \mu_{k+1}}{-2i} = \frac{-\mu_{k+1} + i\mu_k}{2i}$$

It follows that an equivalent set of solutions are

$$z_k(x) = e^{\gamma x}\cos(wx)$$
$$z_{k+1}(x) = e^{\gamma x}\sin(wx)$$

In general: given complex conjugate zeroes $\gamma + iw$ and $\gamma - iw$ with multiplicity m, it is best practice to use the following set of linearly independent solutions:

$$y_k(x) = e^{\gamma x}\cos(wx), \quad y_{k+2} = x e^{\gamma x}\cos(wx), \quad \ldots,$$
$$y_{k+1}(x) = e^{\gamma x}\sin(wx) \quad y_{k+3} = x e^{\gamma x}\sin(wx)$$

$$y_{k+2m-2}(x) = x^{m-1} e^{\gamma x} \cos(\omega x)$$
$$y_{k+2m-1}(x) = x^{m-1} e^{\gamma x} \sin(\omega x).$$

EXAMPLE

a) Write the general solution to $y'''(x) - 2y'(x) = 0$.

Solution

Define $Ly(x) = y'''(x) - 2y'(x)$ and note that

$$L(e^{bx}) = (e^{bx})''' - 2(e^{bx})' = b^3 e^{bx} - 2b e^{bx} =$$
$$= (b^3 - 2b) e^{bx} = b(b^2 - 2) e^{bx} = b(b - \sqrt{2})(b + \sqrt{2}) e^{bx}$$

The characteristic polynomial $P(b) = b(b - \sqrt{2})(b + \sqrt{2})$ has zeroes: $0, \sqrt{2}, -\sqrt{2}$ and therefore

$$y(x) = \lambda_1 e^{0x} + \lambda_2 e^{\sqrt{2}x} + \lambda_3 e^{-\sqrt{2}x} =$$
$$= \lambda_1 + \lambda_2 e^{x\sqrt{2}} + \lambda_3 e^{-x\sqrt{2}}$$

b) Solve the initial value problem

$$\begin{cases} y''(x) - 8y'(x) + 16y(x) = 0 \\ y(0) = 1 \wedge y'(0) = 3 \end{cases}$$

Solution

Define $Ly(x) = y''(x) - 8y'(x) + 16y(x)$ and note that

$$L(e^{bx}) = (e^{bx})'' - 8(e^{bx})' + 16e^{bx} =$$
$$= b^2 e^{bx} - 8b e^{bx} + 16 e^{bx} = (b^2 - 8b + 16) e^{bx}$$
$$= (b-4)^2 e^{bx}$$

The characteristic polynomial $P(b) = (b-4)^2$ has zeroes: $4, 4$ and therefore:

$$y(x) = \lambda_1 e^{4x} + \lambda_2 x e^{4x}$$

To apply the initial condition, we note that

$$y'(x) = \lambda_1 (e^{4x})' + \lambda_2 (xe^{4x})' = 4\lambda_1 e^{4x} + \lambda_2 (e^{4x} + 4xe^{4x})$$
$$= (4\lambda_1 + \lambda_2) e^{4x} + 4\lambda_2 x e^{4x}$$

and therefore

$$\begin{cases} y(0)=1 \\ y'(0)=3 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 e^0 + \lambda_2 e^0 0 = 1 \\ (4\lambda_1+\lambda_2)e^0 + 4\lambda_2 \cdot 0 e^0 = 3 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 + 0\lambda_2 = 1 \\ 4\lambda_1 + \lambda_2 = 3 \end{cases}$$
$$\Leftrightarrow \begin{cases} \lambda_1 = 1 \\ 4\cdot 1 + \lambda_2 = 3 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 3-4 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}$$

It follows that the solution is

$$y(x) = e^{4x} - xe^{4x} = (1-x)e^{4x}.$$

c) Linear Oscillator problem:

Solve the initial value problem

$$\begin{cases} y''(x) + w^2 y(x) = 0 \\ y(0) = y_0 \wedge y'(0) = y_1 \end{cases}$$

Solution

Define $Ly(x) = y''(x) + w^2 y(x)$ and note that

$$L(e^{bx}) = (e^{bx})'' + w^2 e^{bx} = b^2 e^{bx} + w^2 e^{bx} = (b^2 + w^2) e^{bx}$$
$$= (b+iw)(b-iw)e^{bx}$$

The characteristic polynomial $P(b) = (b+iw)(b-iw)$ has zeroes $+iw, -iw$. It follows that

$$y(x) = \lambda_1 e^{0x} \cos(wx) + \lambda_2 e^{0x} \sin(wx) =$$
$$= \lambda_1 \cos(wx) + \lambda_2 \sin(wx)$$

To apply the initial conditions, we calculate:

$$y'(x) = \lambda_1(-w\sin(wx)) + \lambda_2(w\cos(wx)) =$$
$$= -w\lambda_1 \sin(wx) + w\lambda_2 \cos(wx)$$

and therefore

$$\begin{cases} y(0) = y_0 \\ y'(0) = y_1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 \cos 0 + \lambda_2 \sin 0 = y_0 \\ -w\lambda_1 \sin 0 + w\lambda_2 \cos 0 = y_1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda_1 + 0\lambda_2 = y_0 \\ -0\lambda_1 + w\lambda_2 = y_1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = y_0 \\ w\lambda_2 = y_1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda_1 = y_0 \\ \lambda_2 = y_1/w \end{cases}$$

It follows that the solution is:

$$y(x) = y_0 \cos(wx) + (y_1/w)\sin(wx).$$

EXERCISES

(6) Find the general solution for the following linear differential equations

a) $y'''(x) - 5y''(x) + 6y'(x) = 0$

b) $y'''(x) - y'(x) = 0$

c) $y'''(x) - y(x) = 0$

d) $y''(x) + y'(x) + y(x) = 0$

e) $y^{(4)}(x) - 16y(x) = 0$

f) $y'''(x) - 4y'(x) + 3y(x) = 0$

g) $y^{(4)}(x) + 2y''(x) + y(x) = 0$

(7) Show that the initial value problem

$$\begin{cases} y''(x) - 2(p+a)y'(x) + p^2 y(x) = 0 \\ y(0) = 0 \land y'(0) = 1 \end{cases}$$

with $a, p \in (0, +\infty)$ has solution

$$y(x|a,p) = \frac{\exp(A(p,a)x) - \exp(B(p,a)x)}{2\sqrt{a(2p+a)}}$$

with
$$A(p,a) = p + a + \sqrt{a(2p+a)}$$
$$B(p,a) = p + a - \sqrt{a(2p+a)}$$

<u>without</u> substituting the solution to the ODE. Then show that $\lim_{a \to 0} y(x|a,p) = x e^{px}$

(8) Solve the following initial value problems; with $\mu \in (0, +\infty)$

a) $\begin{cases} y'''(x) - \mu y''(x) + \mu^2 y'(x) - \mu^3 y(x) = 0 \\ y(0) = 0 \land y'(0) = 0 \land y''(0) = 1 \end{cases}$

b) $\begin{cases} y'''(x) - \mu^2 y'(x) = 0 \\ y(0) = y'(0) = 0 \land y''(0) = 1 \end{cases}$

c) $\begin{cases} y^{(4)}(x) - \mu y(x) = 0 \\ y(0) = y'(0) = y''(0) = 0 \land y'''(0) = 1 \end{cases}$

2 → Equidimensional case (Euler-Cauchy equation)

Consider the linear ODE:

$$x^n y^{(n)}(x) + a_{n-1}x^{n-1}y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y(x) = 0$$

with $a_0, a_1, a_2, \ldots, a_{n-1} \in \mathbb{R}$ given constants. Let L be the corresponding operator.

Solution method

- 1 We evaluate the characteristic polynomial P from:
  $$L(x^b) = P(b)x^b$$
- 2 Let $p_1, p_2, \ldots, p_n \in \mathbb{C}$ be the zeroes of $P(b)$. Then

  (a) If p_k is a single zero, it contributes a solution
  $$y_k(x) = x^{p_k}.$$

  (b) If p_k is a zero with multiplicity m, it contributes the following linearly independent solutions.
  $$y_k(x) = x^{p_k}$$
  $$y_{k+1}(x) = x^{p_k}\ln x$$
  $$y_{k+2}(x) = x^{p_k}[\ln x]^2$$
  $$\vdots$$
  $$y_{k+m-1}(x) = x^{p_k}[\ln x]^{m-1}$$

  (c) Given a complex conjugate pair $p_k = \gamma + i\omega$ and $p_{k+1} = \gamma - i\omega$, from (a) we obtain (see remark below) the following linearly independent solutions:
  $$y_k(x) = x^\gamma \cos(\omega \ln x)$$
  $$y_{k+1}(x) = x^\gamma \sin(\omega \ln x)$$

(d) Given a complex conjugate pair $p_k = \gamma + i\omega$ and $p_k = \gamma - i\omega$ of multiplicity m, from (6), we obtain the following linearly independent solutions:

$$y_k(x) = x^\gamma \cos(\omega \ln x)$$
$$y_{k+1}(x) = x^\gamma \sin(\omega \ln x)$$
$$y_{k+2}(x) = x^\gamma \cos(\omega \ln x) \ln x$$
$$y_{k+3}(x) = x^\gamma \cos(\omega \ln x) \ln x$$
$$\vdots$$
$$y_{k+2m-2}(x) = x^\gamma \cos(\omega \ln x)[\ln x]^{m-1}$$
$$y_{k+2m-1}(x) = x^\gamma \sin(\omega \ln x)[\ln x]^{m-1}$$

•3 We write the general solution and apply the initial conditions, if given.

↳ <u>Remark</u>: For the case of a single pair of complex conjugate zeroes $p_k = \gamma + i\omega$ and $p_{k+1} = \gamma - i\omega$, we have the following contributed solutions:

$$y_k(x) = x^{p_k} = x^{\gamma + i\omega} = \exp((\gamma + i\omega)\ln x) = \exp(\gamma \ln x)\exp(i\omega \ln x) =$$
$$= x^\gamma [\cos(\omega \ln x) + i \sin(\omega \ln x)]$$

and similarly:

$$y_{k+1}(x) = x^{p_{k+1}} = x^{\gamma - i\omega} = x^\gamma [\cos(\omega \ln x) - i \sin(\omega \ln x)]$$

Via an argument similar to that of case 1, we obtain the following alternate linearly independent solutions:

$$z_k(x) = x^\gamma \cos(\omega \ln x)$$
$$z_{k+1}(x) = x^\gamma \sin(\omega \ln x)$$

EXAMPLES

a) Solve the initial value problem

$$\begin{cases} x^2 y''(x) + xy'(x) + 4y(x) = 0 \\ y(2) = p \wedge y'(2) = q \end{cases}$$

Solution

Define $Ly(x) = x^2 y''(x) + xy'(x) + 4y(x)$. It follows that

$$L(x^b) = x^2 (x^b)'' + x(x^b)' + 4x^b =$$
$$= x^2 b(b-1) x^{b-2} + x\, b x^{b-1} + 4x^b =$$
$$= b(b-1) x^b + b x^b + 4x^b = [b(b-1) + b + 4] x^b =$$
$$= (b^2 - b + b + 4) x^b = (b^2 + 4) x^b = (b+2i)(b-2i) x^b$$

which gives the characteristic polynomial

$$P(b) = (b+2i)(b-2i)$$

with zeroes $\rho_1 = 2i$ and $\rho_2 = -2i$. It follows that the general solution reads

$$y(x) = \lambda_1 \cos(2\ln x) + \lambda_2 \sin(2\ln x)$$

To apply the initial condition we note that

$$y(2) = \lambda_1 \cos(2\ln 2) + \lambda_2 \sin(2\ln 2)$$

and

$$y'(x) = \lambda_1 [\cos(2\ln x)]' + \lambda_2 [\sin(2\ln x)]' =$$
$$= \lambda_1 [-\sin(2\ln x)](2\ln x)' + \lambda_2 [\cos(2\ln x)](2\ln x)' =$$
$$= (2/x)[-\lambda_1 \sin(2\ln x) + \lambda_2 \cos(2\ln x)] \Longrightarrow$$
$$\Rightarrow y'(2) = (2/2)[-\lambda_1 \sin(2\ln 2) + \lambda_2 \cos(2\ln 2)] =$$
$$= -\lambda_1 \sin(2\ln 2) + \lambda_2 \cos(2\ln 2)$$

and it follows that:

$$\begin{cases} y(2) = p \\ y'(2) = q \end{cases} \Leftrightarrow \begin{cases} \lambda_1 \cos(2\ln 2) + \lambda_2 \sin(2\ln 2) = p \\ -\lambda_1 \sin(2\ln 2) + \lambda_2 \cos(2\ln 2) = q \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \cos(2\ln 2) & \sin(2\ln 2) \\ -\sin(2\ln 2) & \cos(2\ln 2) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \cos(2\ln 2) & \sin(2\ln 2) \\ -\sin(2\ln 2) & \cos(2\ln 2) \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \end{bmatrix} =$$

$$= \frac{1}{\cos^2(2\ln 2) + \sin^2(2\ln 2)} \begin{bmatrix} \cos(2\ln 2) & -\sin(2\ln 2) \\ \sin(2\ln 2) & \cos(2\ln 2) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

$$= \frac{1}{1} \begin{bmatrix} p\cos(2\ln 2) - q\sin(2\ln 2) \\ p\sin(2\ln 2) + q\cos(2\ln 2) \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda_1 = p\cos(2\ln 2) - q\sin(2\ln 2) \\ \lambda_2 = p\sin(2\ln 2) + q\cos(2\ln 2) \end{cases}$$

Thus, the solution reads

$$y(x) = [p\cos(2\ln 2) - q\sin(2\ln 2)]\cos(2\ln x) + [p\sin(2\ln 2) + q\cos(2\ln 2)]\sin(2\ln x)$$

$$= p[\cos(2\ln 2)\cos(2\ln x) + \sin(2\ln 2)\sin(2\ln x)] +$$

$$+ q[-\sin(2\ln 2)\cos(2\ln x) + \sin(2\ln x)\cos(2\ln 2)] =$$

$$= p\cos(2\ln x - 2\ln 2) + q\sin(2\ln x - 2\ln 2)$$

6) Solve the initial value problem

$$\begin{cases} 4x^2y''(x)+8xy'(x)+y(x)=0 \\ y(3)=p \wedge y'(3)=q \end{cases}$$

<u>Solution</u>

Define $Ly(x)=4x^2y''(x)+8xy'(x)+y(x)$. It follows that

$$\begin{aligned} L(x^b) &= 4x^2(x^b)''+8x(x^b)'+x^b= \\ &= 4x^2b(b-1)x^{b-2}+8x\,bx^{b-1}+x^b= \\ &= 4b(b-1)x^b+8bx^b+x^b=[4b(b-1)+8b+1]x^b= \\ &= (4b^2-4b+8b+1)x^b=(4b^2+4b+1)x^b=(2b+1)^2x^b \end{aligned}$$

which gives the characteristic polynomial $P(b)=(2b+1)^2$ with a double zero $p=-1/2$. Thus, the general solution reads:

$$y(x)=\lambda_1x^{-1/2}+\lambda_2x^{-1/2}\ln x=\frac{\lambda_1+\lambda_2\ln x}{\sqrt{x}}$$

To apply the initial condition, we note that

$$y(3)=\frac{\lambda_1+\lambda_2\ln 3}{\sqrt{3}}$$

and

$$\begin{aligned} y'(x) &= \frac{(\lambda_1+\lambda_2\ln x)'\sqrt{x}-(\lambda_1+\lambda_2\ln x)(\sqrt{x})'}{(\sqrt{x})^2}= \\ &= \frac{1}{x}\left[\lambda_2\frac{1}{x}\sqrt{x}-\frac{\lambda_1+\lambda_2\ln x}{2\sqrt{x}}\right]= \\ &= \frac{1}{x}\left[\lambda_2\frac{1}{\sqrt{x}}-\frac{\lambda_1+\lambda_2\ln x}{2\sqrt{x}}\right]= \\ &= \frac{1}{2x\sqrt{x}}\left[2\lambda_2-(\lambda_1+\lambda_2\ln x)\right]=\frac{(2\lambda_2-\lambda_1)-\lambda_2\ln x}{2x\sqrt{x}} \end{aligned}$$

$$\Rightarrow y'(3) = \frac{(2\lambda_2 - \lambda_1) - \lambda_2 \ln 3}{2 \cdot 3\sqrt{3}} = \frac{-\lambda_1 + (2-\ln 3)\lambda_2}{6\sqrt{3}}$$

and therefore

$$\begin{cases} y(3) = p \\ y'(3) = q \end{cases} \Leftrightarrow \begin{cases} \lambda_1 + \lambda_2 \ln 3 = p\sqrt{3} \\ -\lambda_1 + (2-\ln 3)\lambda_2 = 6q\sqrt{3} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} 1 & \ln 3 \\ -1 & 2-\ln 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} p\sqrt{3} \\ 6q\sqrt{3} \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & \ln 3 \\ -1 & 2-\ln 3 \end{bmatrix}^{-1} \begin{bmatrix} p\sqrt{3} \\ 6q\sqrt{3} \end{bmatrix} =$$

$$= \frac{1}{(2-\ln 3) + \ln 3} \begin{bmatrix} 2-\ln 3 & -\ln 3 \\ +1 & 1 \end{bmatrix} \begin{bmatrix} p\sqrt{3} \\ 6q\sqrt{3} \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} (2-\ln 3)p\sqrt{3} - 6q\sqrt{3}\ln 3 \\ p\sqrt{3} + 6q\sqrt{3} \end{bmatrix}$$

It follows that the solution to the initial value problem is:

$$y(x) = \frac{\lambda_1 + \lambda_2 \ln x}{\sqrt{x}} =$$

$$= \frac{1}{2\sqrt{x}} \left[(2-\ln 3)p\sqrt{3} - 6q\sqrt{3}\ln 3 + (p\sqrt{3} + 6q\sqrt{3})\ln x \right] =$$

$$= \frac{1}{2\sqrt{x}} \left[p\sqrt{3}(2 - \ln 3 + \ln x) + 6q\sqrt{3}(\ln x - \ln 3) \right]$$

c) Solve the initial value problem

$$\begin{cases} x^3 y'''(x) - xy'(x) - 3y(x) = 0 \\ y(1) = 0 \wedge y'(1) = 0 \wedge y''(1) = p \end{cases}$$

Solution

Define $Ly(x) = x^3 y'''(x) - xy'(x) - 3y(x)$. It follows that

$$L(x^b) = x^3 (x^b)''' - x(x^b)' - 3x^b =$$
$$= x^3 b(b-1)(b-2)x^{b-3} - x(bx^{b-1}) - 3x^b =$$
$$= b(b-1)(b-2)x^b - bx^b - 3x^b = [b(b-1)(b-2) - b - 3]x^b =$$
$$= [b(b^2-3b+2) - b - 3]x^b = (b^3 - 3b^2 + 2b - b - 3)x^b$$
$$= (b^3 - 3b^2 + b - 3)x^b$$

and therefore the characteristic polynomial is:

$$P(b) = b^3 - 3b^2 + b - 3 = b^2(b-3) + (b-3) = (b-3)(b^2+1)$$

with zeroes $\rho_1 = 3$; $\rho_2 = i$, and $\rho_3 = -i$. Thus, the general solution is given by:

$$y(x) = \lambda_1 x^3 + \lambda_2 \cos(\ln x) + \lambda_3 \sin(\ln x).$$

To apply the initial condition, we note that

$$y(1) = \lambda_1 \cdot 1^3 + \lambda_2 \cos(\ln 1) + \lambda_3 \sin(\ln 1) =$$
$$= \lambda_1 + \lambda_2 \cos 0 + \lambda_3 \sin 0 = \lambda_1 + \lambda_2$$

and

$$y'(x) = 3\lambda_1 x^2 + \lambda_2 [\cos(\ln x)]' + \lambda_3 [\sin(\ln x)]' =$$
$$= 3\lambda_1 x^2 + \lambda_2 [-\sin(\ln x)](\ln x)' + \lambda_3 [\cos(\ln x)](\ln x)'$$
$$= 3\lambda_1 x^2 + \frac{-\lambda_2 \sin(\ln x) + \lambda_3 \cos(\ln x)}{x} \Rightarrow$$

$$\Rightarrow y'(1) = 3\lambda_1 \cdot 1^2 + \frac{-\lambda_2 \sin(\ln 1) + \lambda_3 \cos(\ln 1)}{1} =$$

$$= 3\lambda_1 - \lambda_2 \sin 0 + \lambda_3 \cos 0 = 3\lambda_1 - 0\lambda_2 + \lambda_3.$$

and

$$y''(x) = 6\lambda_1 x + \frac{d}{dx}\left[\frac{-\sin(\ln x)}{x}\right]\lambda_2 + \frac{d}{dx}\left[\frac{\cos(\ln x)}{x}\right]\lambda_3 =$$

$$= (6x)\lambda_1 + \frac{-[(\sin(\ln x))'x - \sin(\ln x)(x)']}{x^2}\lambda_2$$

$$+ \frac{[\cos(\ln x)]'x - \cos(\ln x)(x)'}{x^2}\lambda_3 =$$

$$= (6x)\lambda_1 + \frac{-[\cos(\ln x)(\ln x)'x - \sin(\ln x)]}{x^2}\lambda_2$$

$$+ \frac{-\sin(\ln x)(\ln x)'x - \cos(\ln x)}{x^2}\lambda_3 =$$

$$= (6x)\lambda_1 + \frac{\sin(\ln x) - \cos(\ln x)}{x^2}\lambda_2 + \frac{-\sin(\ln x) - \cos(\ln x)}{x^2}\lambda_3$$

$$\Rightarrow y''(1) = 6\lambda_1 + [\sin(\ln 1) - \cos(\ln 1)]\lambda_2 - [\sin(\ln 1) + \cos(\ln 1)]\lambda_3 =$$

$$= 6\lambda_1 + (\sin 0 - \cos 0)\lambda_2 - (\sin 0 + \cos 0)\lambda_3 =$$

$$= 6\lambda_1 - \lambda_2 - \lambda_3$$

and therefore:

$$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ 3\lambda_1 + \lambda_3 = 0 \\ 6\lambda_1 - \lambda_2 - \lambda_3 = p \end{cases} \iff \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}$$

Apply Cramer rule:

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 6 & -1 & -1 \end{vmatrix} \begin{matrix} \\ \leftarrow \\ \cdot 1 \end{matrix} = \begin{vmatrix} 1 & 1 & 0 \\ 9 & -1 & 0 \\ 6 & -1 & -1 \end{vmatrix} = +(-1)\begin{vmatrix} 1 & 1 \\ 9 & -1 \end{vmatrix} = -(-1-9) = 10$$

$$D_1 = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & -1 & -1 \end{vmatrix} = p \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = p$$

$$D_2 = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 6 & p & -1 \end{vmatrix} = (-1)p \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = -p(1\cdot 1 - 0\cdot 3) = -p$$

$$D_3 = \begin{vmatrix} 1 & 1 & 0 \\ 3 & 0 & 0 \\ 6 & -1 & p \end{vmatrix} = (+1)p \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} = p(1\cdot 0 - 1\cdot 3) = -3p$$

thus

$$\lambda_1 = \frac{D_1}{D} = \frac{p}{10}$$

$$\lambda_2 = \frac{D_2}{D} = \frac{-p}{10}$$

$$\lambda_3 = \frac{D_3}{D} = \frac{-3p}{10}$$

and the solution reads

$$y(x) = \frac{px^3}{10} - \frac{p\cos(\ln x)}{10} - \frac{3p\sin(\ln x)}{10}$$

$$= (p/10)\left[x^3 - \cos(\ln x) - 3\sin(\ln x)\right]$$

EXERCISES

(9) Solve the following linear differential equations on the interval $(0,+\infty)$ using initial conditions $y(1)=y_0 \wedge y'(1)=y_1$.

a) $x^2y''(x) - 2xy'(x) + 2y(x) = 0$

b) $x^2y''(x) - 2y(x) = 0$

c) $x^2y''(x) - xy'(x) + y(x) = 0$

(10) Similarly, solve the following linear differential equations on the interval $(0,+\infty)$ using initial conditions $y(1)=y_0 \wedge y'(1)=y_1 \wedge y''(1)=y_2$:

a) $x^3y'''(x) - 6y(x) = 0$

b) $xy'''(x) + y''(x) = 0$

d) $x^3y'''(x) + 3x^2y''(x) + xy'(x) - 8y(x) = 0$

e) $x^3y'''(x) + 3x^2y''(x) - 6y(x) = 0$

(11) Consider the linear differential equation
$ax^2y''(x) + bxy'(x) + cy(x) = 0$
with $a,b,c \in \mathbb{R}$, and let Δ be the discriminant of the equation's characteristic polynomial $p(x) = Ax^2+Bx+C$.
Show that $\Delta = B^2-4AC = a^2+b^2-2a(b+2c)$.

(12) Show that the linear differential equation
$ax^3y'''(x) + (b+3a)x^2y''(x) + (a+b+c)xy'(x) + dy(x) = 0$
with $a,b,c,d \in \mathbb{R}$ has characteristic polynomial
$p(x) = ax^3+bx^2+cx+d$.

Solving inhomogeneous linear differential equations

We will now consider the general problem of the linear inhomogeneous linear differential equation of the form

$$\forall x \in A : y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x) = f(x) \quad (1)$$

with $a_0, a_1, a_2, \ldots, a_{n-1}, f \in C^0(A)$. The general method is as follows:

1) Given the solutions $y_1, \ldots, y_n$ of the homogeneous equation and at least one solution y_p of the inhomogeneous equation we show that the general solution of Eq. (1) is:

$$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x) + y_p(x)$$

2) Given $y_1, y_2, \ldots, y_n$ there is a general result that gives the solution y_p.

<u>Terminology</u>: The terms $\lambda_1 y_1 + \cdots + \lambda_n y_n$ are the homogeneous solution and y_p are the particular solution to the problem.

We now give the details of the theory:

<u>Thm</u>: Consider the linear operator $L : C^n(A) \to C^0(A)$ for some interval $A \subseteq \mathbb{R}$ such that $Ly = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$ with $a_0, a_1, \ldots, a_n \in C^0(A)$. Let $f \in C^0(A)$, and assume that

(a) $\text{null}(L) = \text{span}\{y_1, y_2, \ldots, y_n\}$ with $y_1, y_2, \ldots, y_n \in C^n(A)$.

(b) $Ly_p = f$

Then: $Ly = f \Leftrightarrow \exists \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R} : y = y_p + \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n$

Proof

($\Rightarrow$): Assume that $Ly=f$. Then it follows that

$L(y-y_p) = Ly - Ly_p = f - f = 0 \Rightarrow (y-y_p) \in null(L) \Rightarrow$

$\Rightarrow y - y_p \in span\{y_1, y_2, \dots, y_n\} \Rightarrow$

$\Rightarrow \exists \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}: y - y_p = \lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_n y_n$

$\Rightarrow \exists \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}: y = y_p + \lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_n y_n$

($\Leftarrow$): Assume that: $\exists \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}: y = y_p + \lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_n y_n$

Then, it follows that:

$$Ly = L\left(y_p + \sum_{k=1}^{n} \lambda_k y_k\right) = Ly_p + L\left(\sum_{k=1}^{n} \lambda_k y_k\right) = f + \sum_{k=1}^{n} L(\lambda_k y_k) =$$

$$= f + \sum_{k=1}^{n} \lambda_k (Ly_k) = f + \sum_{k=1}^{n} \lambda_k 0 = f + 0 = f. \qquad \square$$

Thm: Let $L: C^n(A) \to C^0(A)$, with $A \subseteq \mathbb{R}$ an interval, be a linear operator defined as:

$\forall y \in C^n(A): Ly = y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y$

with $a_0, a_1, \dots, a_{n-1} \in C^0(A)$, and let $f \in C^0(A)$. Assume that $null(L) = span\{y_1, y_2, \dots, y_n\}$. Then, the inhomogeneous ODE $Ly = f$ has a particular solution $y_p \in C^n(A)$ such that $Ly_p = f$ given by:

$$\forall x \in A: \quad y_p(x) = \int_A G(x,t) f(t) \, dt$$

with $\forall x, t \in A: G(x,t) = \begin{cases} \sum_{k=1}^{n} B_k(t) y_k(x), & \text{if } x \geq t \\ 0, & \text{if } x < t \end{cases}$

where $B_1(t), B_2(t), \dots, B_n(t)$ is the unique solution of the system

$W[y_1, y_2, \dots, y_n](t) (B_1(t), B_2(t), \dots, B_n(t)) = (0, 0, \dots, 0, 1)$

Remarks :

a) The proof of this theorem is based on generalized functions and will be given later.

b) An alternative proof is to substitute the solution $y_p \in C^n(\mathbb{R})$ to the equation $Ly_p = f$ and confirm that the solution satisfies the equation. This method is known as "variation of parameters".

c) The function $G(x,t)$ is called the Green's function. It captures the effect of the value of the forcing function f at t to the solution y_p at x. The Green's function is not unique, but can be made unique if we introduce the assumption that $G(x,t) = 0$ for $x < t$. This is known as the causality assumption that "the future value $f(t)$ should not have an effect on the past solution $y_p(x)$".

→ Special case: 2nd-order linear ODE on $A = [c, d]$

Consider the 2nd-order linear ODE of the form

$$y''(x) + a_1(x) y'(x) + a_0(x) y(x) = f(x). \quad \text{with } a_0, a_1, f \in C^0(A)$$

Given two linearly independent solutions $y_1, y_2 \in C^2(A)$ such that

$$\begin{cases} y_1''(x) + a_1(x) y_1'(x) + a_0(x) y_1(x) = 0 \\ y_2''(x) + a_1(x) y_2'(x) + a_0(x) y_2(x) = 0 \end{cases}$$

a corresponding particular solution $y_p \in C^2(A)$ is given by

$$y_p(x) = -y_1(x) \int_c^x \frac{f(t) y_2(t)}{W(t)} dt + y_2(x) \int_c^x \frac{f(t) y_1(t)}{W(t)} dt$$

with $W(t) = y_1(t) y_2'(t) - y_1'(t) y_2(t)$

Proof

The Green's function is given by

$$G(x,t) = \begin{cases} B_1(t)\,y_1(x) + B_2(t)\,y_2(x) & , \text{ if } x \geq t \\ 0 & , \text{ if } x < t \end{cases}$$

with $B_1(t)$, $B_2(t)$ given by:

$$W[y_1, y_2](t)\,(B_1(t), B_2(t)) = (0,1) \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \frac{1}{y_1(t)y_2'(t) - y_1'(t)y_2(t)} \begin{bmatrix} y_2'(t) & -y_2(t) \\ -y_1'(t) & y_1(t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \frac{1}{w(t)} \begin{bmatrix} -y_2(t) \\ y_1(t) \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow B_1(t) = \frac{-y_2(t)}{w(t)} \wedge B_2(t) = \frac{y_1(t)}{w(t)}.$$

and therefore, a particular solution is:

$$y_p(x) = \int_c^d G(x,t) f(t)\,dt = \int_c^x [B_1(t)y_1(x) + B_2(t)y_2(x)]\, f(t)\,dt =$$

$$= y_1(x) \int_c^x B_1(t) f(t)\,dt + y_2(x) \int_c^x B_2(t) f(t)\,dt =$$

$$= -y_1(x) \int_c^x \frac{f(t)\,y_2(t)}{w(t)}\,dt + y_2(x) \int_c^x \frac{f(t)\,y_1(t)}{w(t)}$$

Note that the lower limit $-\infty$ can be replaced with any constant c. Then the $(-\infty, c)$ integrals gives a contribution that can be moved to the homogeneous solution.

EXAMPLES

a) Solve the initial-value problem:

$$\begin{cases} y''(x) - 2y'(x) + y(x) = (3x+2)e^x \\ y(0) = y_0 \wedge y'(0) = y_1 \end{cases}$$

Solution

Define $\forall y \in C^2(\mathbb{R}) : Ly = y'' - 2y' + y$, and note that

$$Le^{bx} = (e^{bx})'' - 2(e^{bx})' + e^{bx} = b^2 e^{bx} - 2be^{bx} + e^{bx} =$$
$$= (b^2 - 2b + 1)e^{bx} = (b-1)^2 e^{bx}.$$

The characteristic polynomial $P(b) = (b-1)^2$ has a double zero $b = 1$, therefore $null(L) = span\{y_1, y_2\}$ with $\forall x \in \mathbb{R} : (y_1(x) = e^x \wedge y_2(x) = xe^x)$.

The corresponding wronskian is:

$$w(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t) =$$
$$= e^x(xe^x)' - (e^x)'(xe^x) = e^x(e^x + xe^x) - e^x xe^x =$$
$$= e^{2x} + xe^{2x} - xe^{2x} = e^{2x}$$

and a particular solution is:

$$y_p(x) = -y_1(x)\int_0^x \frac{f(t)y_2(t)}{w(t)}dt + y_2(x)\int_0^x \frac{f(t)y_1(t)}{w(t)}dt$$

with $f(t) = (3t+2)e^t$. It follows that

$$y_p(x) = -e^x\int_0^x \frac{(3t+2)e^t \cdot te^t}{e^{2t}}dt + xe^x\int_0^x \frac{(3t+2)e^t \cdot e^t}{e^{2t}}dt$$
$$= -e^x\int_0^x (3t^2+2t)dt + xe^x\int_0^x (3t+2)dt =$$

$$= -e^x \left[\frac{3t^3}{3} + \frac{2t^2}{2}\right]_0^x + xe^x\left[\frac{3t^2}{2} + 2t\right]_0^x =$$

$$= -e^x\left(x^3+x^2\right) + xe^x\left(\frac{3x^2}{2} + 2x\right) =$$

$$= -x^2e^x(x+1) + x^2e^x(3x/2+2) =$$

$$= \frac{+x^2e^x}{2}\left(-2(x+1)+3x+4\right) = (1/2)x^2e^x(-2x-2+3x+4)$$

$$= (1/2)x^2e^x(x+2).$$

and therefore the general solution is

$$y(x) = \lambda_1 e^x + \lambda_2 xe^x + (1/2)x^2e^x(x+2)$$

To apply the initial condition, we note that

$$y(0) = \lambda_1 e^0 + \lambda_2 \cdot 0e^0 + (1/2)0^2e^0(0+2) = \lambda_1$$

and

$$y'(x) = \lambda_1(e^x)' + \lambda_2(xe^x)' + (1/2)\left[x^2e^x(x+2)\right]' =$$

$$= \lambda_1 e^x + \lambda_2(e^x + xe^x) + (1/2)\left[e^x(x^3+2x^2)\right]' =$$

$$= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)\left[(e^x)'(x^3+2x^2) + e^x(x^3+2x^2)'\right] =$$

$$= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)\left[e^x(x^3+2x^2+3x^2+4x)\right]$$

$$= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)e^x(x^3+5x^2+4x)$$

$$= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)e^x x(x^2+5x+4)$$

$$= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)xe^x(x+4)(x+1) \Rightarrow$$

$$\Rightarrow y'(0) = (\lambda_1+\lambda_2)e^0 + \lambda_2\cdot 0\cdot e^0 + (1/2)\cdot 0e^0(0+4)(0+1) =$$

$$= \lambda_1+\lambda_2$$

and therefore:

$$\begin{cases} y(0) = y_0 \\ y'(0) = y_1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = y_0 \\ \lambda_1+\lambda_2 = y_1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = y_0 \\ y_0+\lambda_2 = y_1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = y_0 \\ \lambda_2 = y_1 - y_0 \end{cases}$$

thus $y(x) = y_0e^x + (y_1-y_0)xe^x + (1/2)x^2e^x(x+2)$.

6) Solve the ODE value problem

$$x^3y'''(x)+x^2y''(x)-2xy'(x)+2y(x)=f(x)\ ,\ \forall x\in[1,+\infty)$$

Solution

Define $Ly(x)=x^3y'''(x)+x^2y''(x)-2xy'(x)+2y(x)$. Then, since

$$Lx^b = x^3(x^b)'''+x^2(x^b)''-2x(x^b)'+2x^b=$$
$$=x^3b(b-1)(b-2)x^{b-3}+x^2b(b-1)x^{b-2}-2xbx^{b-1}+2x^b=$$
$$=[b(b-1)(b-2)+b(b-1)-2b+2]x^b$$

the characteristic polynomial is given by

$$P(b)=b(b-1)(b-2)+b(b-1)-2b+2=b(b^2-3b+2)+b^2-b-2b+2$$
$$=b^3-3b^2+2b+b^2-b-2b+2=$$
$$=b^3+(-3+1)b^2+(2-1-2)b+2$$
$$=b^3-2b^2-b+2=b^2(b-2)-(b-2)=(b^2-1)(b-2)$$
$$=(b-1)(b+1)(b-2)$$

and has single zeroes $b_1=-1 \wedge b_2=1 \wedge b_3=2$.

Thus the general solution is:

$$y(x)=\lambda_1x^{-1}+\lambda_2x+\lambda_3x^2+y_p(x)$$

Define: $y_1(x)=x^{-1} \wedge y_2(x)=x \wedge y_3(x)=x^2\ ,\ \forall x\in[1,+\infty)$

The particular solution is given by

$$y_p(x)=\int_1^{+\infty}G(x,t)f(t)\,dt\ ,\ \forall x\in[1,+\infty)$$

with
$$G(x,t)=\begin{cases} B_1(t)x^{-1}+B_2(t)x+B_3(t)x^2 &, \text{ if } x\geq t\\ 0 &, \text{ if } x<t\end{cases}$$

with $B_1(t), B_2(t), B_3(t)$ the solution of

$$W[y_1,y_2,y_3](B_1(t),B_2(t),B_3(t))=(0,0,1).$$

and therefore

$$y_p(x) = \int_1^{+\infty} G(x,t) f(t)\,dt = \int_1^x [B_1(t)x^{-1} + B_2(t)x + B_3(t)x^2] f(t)\,dt$$

$$= x^{-1}\int_1^x B_1(t) f(t)\,dt + x\int_1^x B_2(t) f(t)\,dt + x^2\int_1^x B_3(t) f(t)\,dt$$

Since

$$W[y_1,y_2,y_3](t) = \begin{bmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \\ y_1''(t) & y_2''(t) & y_3''(t) \end{bmatrix} = \begin{bmatrix} t^{-1} & t & t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{bmatrix}$$

it follows that

$$\begin{bmatrix} t^{-1} & t & t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{bmatrix}\begin{bmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We apply Cramer's rule:

$$D = \begin{vmatrix} t^{-1} & t & t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{vmatrix} (-t) = \begin{vmatrix} t^{-1}+t^{-1} & t-t & t^2-2t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{vmatrix} =$$

$$= \begin{vmatrix} 2t^{-1} & 0 & -t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2t^{-1} & -t^2 \\ 2t^{-3} & 2 \end{vmatrix} = (2t^{-1})2 - (-t^2)(2t^{-3})$$

$$= 4t^{-1} + 2t^{-1} = 6t^{-1}$$

and

$$D_1 = \begin{vmatrix} 0 & t & t^2 \\ 0 & 1 & 2t \\ 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t(2t) - t^2 = 2t^2 - t^2 = t^2.$$

$$D_2 = \begin{vmatrix} t^{-1} & 0 & t^2 \\ -t^{-2} & 0 & 2t \\ 2t^{-3} & 1 & 2 \end{vmatrix} = - \begin{vmatrix} t^{-1} & t^2 \\ -t^{-2} & 2t \end{vmatrix} = -\left[t^{-1}(2t) - t^2(-t^{-2})\right] =$$

$$= -(2+1) = -3$$

and

$$D_3 = \begin{vmatrix} t^{-1} & t & 0 \\ -t^{-2} & 1 & 0 \\ 2t^{-3} & 0 & 1 \end{vmatrix} = \begin{vmatrix} t^{-1} & t \\ -t^{-2} & 1 \end{vmatrix} = t^{-1} - t(-t^{-2}) = t^{-1} + t^{-1} = 2t^{-1}$$

and therefore:

$$B_1(t) = \frac{D_1(t)}{D(t)} = \frac{t^2}{6t^{-1}} = \frac{t^3}{6}$$

$$B_2(t) = \frac{D_2(t)}{D(t)} = \frac{-3}{6t^{-1}} = \frac{-t}{2}$$

$$B_3(t) = \frac{D_3(t)}{D(t)} = \frac{2t^{-1}}{6t^{-1}} = \frac{1}{3}$$

The particular solution is:

$$y_p(x) = x^{-1}\int_1^x \frac{t^3}{6} f(t)\,dt + x\int_1^x \frac{-t}{2} f(t)\,dt + x^2\int_1^x \frac{1}{3} f(t)\,dt =$$

$$= \frac{1}{6x}\int_1^x t^3 f(t)\,dt - \frac{x}{2}\int_1^x t f(t)\,dt + \frac{x^2}{3}\int_1^x f(t)\,dt.$$

It follows that the general solution is given by

$$y(x) = \left[\lambda_1 + \int_1^x \frac{t^3 f(t)}{6}\,dt\right] x^{-1} + \left[\lambda_2 - \int_1^x \frac{t f(t)}{2}\,dt\right] x$$
$$+ \left[\lambda_3 + \int_1^x \frac{f(t)}{3}\,dt\right] x^2$$

↳ Note that the integrals can start from numbers other than 1. This will result in a constant shift (i.e. independent of x) in the value of the integrals that can be absorbed by $\lambda_1, \lambda_2, \lambda_3$. In general, it is convenient for the integrals to begin at the location where the initial condition is given.

EXERCISES

(13) Derive the general solution for the following inhomogeneous linear differential equations

a) $y''(x) + y(x) = \sin(ax)$, with $a \in (0, +\infty)$

b) $y''(x) + y'(x) + y(x) = \sin(ax)$, with $a \in (0, +\infty)$

c) $y''(x) - 2y'(x) + y(x) = e^x / x$ on $x \in (0, +\infty)$

d) $x^2 y''(x) - 2xy'(x) + 2y(x) = x \ln x$ on $x \in [1, +\infty)$

e) $x^2 y''(x) - xy'(x) = x^3 e^x$ on $x \in [1, +\infty)$

f) $y'''(x) - y'(x) = x^2 - 3x$ on $x \in [1, +\infty)$

g) $x^3 y'''(x) + 3x^2 y''(x) = 1$ on $x \in [1, +\infty)$

(14) Solve the following initial value problem:

$$\begin{cases} x^2 y'' - 2xy' + 2y = f(x) \\ y(1) = y_0 \wedge y'(1) = y_1 \end{cases}$$

(15) Solve the general damped oscillator problem, which is defined as the following initial value problem.

$$\begin{cases} y''(x) + by'(x) + w^2 y(x) = f(x) \\ y(0) = y_0 \wedge y'(0) = y_1 \end{cases}$$

with $b, w \in (0, +\infty)$, $y_0, y_1 \in \mathbb{R}$. Distinguish 3 different cases:

Case 1: $b < 2w$ (underdamped oscillator)

Case 2: $b = 2w$ (critically damped oscillator)

Case 3: $b > 2w$ (overdamped oscillator)