LAPLACE TRANSFORMS

Definition of Laplace transform

Def: Let $f:(0,+\infty)\longrightarrow \mathbb{R}$ be a function. We define the Laplace transform $F(s)=\mathcal{L}(f(t))$ of $f(t)$ in terms of the following improper integral, if it converges,

$$F(s)=\int_0^{+\infty} e^{-st} f(t)\,dt \equiv \mathcal{L}(f(t))$$

Remarks

(a) The domain of $F(s)$ depends on the convergence of the Laplace integral, which in term depends on the function $f(t)$.

(b) It is possible to use a distribution for $f(t)$. Then $F(s)$ will sstill be a regular function. The theory of Laplace transforms of distributions requires some additional care.

(c) By convention, the original function is denoted as a function of t and represented with a lower-case letter. The transform is denoted as $F(s)=\mathcal{L}(f(t))$ as a function of s and represented with upper-case letter.

Convergence of the Laplace transform

We establish a sufficient (but not necessary) condition for the convergence of the Laplace integral as follows:

Def: Let $f:[0,+\infty)\to\mathbb{R}$ be a function. We say that

(a) f piecewise continuous on $(0,+\infty) \Leftrightarrow$

$$\Leftrightarrow \exists t_1,t_2,\dots,t_n\in(0,+\infty): \begin{cases} 0<t_1<t_2<\dots<t_n \\ f \text{ continuous on } [0,t_1)\cup(t_n,+\infty) \\ \forall k\in\mathbb{N}\cap[1,n-1]: f \text{ continuous on } (t_k,t_{k+1}) \end{cases}$$

(b) f has exponential order $\gamma \Leftrightarrow$

$$\Leftrightarrow \exists M>0:\exists\delta>0:\forall t\in(\delta,+\infty): |\exp(-\gamma t)f(t)|\leq M$$

notation: For convenience we introduce the following non-standard notation:

$f\in E_\gamma(\mathbb{R}_+) \Leftrightarrow f$ has exponential order γ

$f\in PC^0(\mathbb{R}_+) \Leftrightarrow f$ is piecewise continuous on $[0,+\infty)$

and compare with

$f\in C^0(\mathbb{R}_+) \Leftrightarrow f$ continuous on $[0,+\infty)$

We note that $\mathbb{R}_+=[0,+\infty)$.

Thm: Let $f:[0,+\infty)\to\mathbb{R}$ be a function and let $F(s)=\mathcal{L}(f(t))$. Then:

$$f\in E_\gamma(\mathbb{R}_+)\cap PC^0(\mathbb{R}_+) \Rightarrow \forall s\in(\gamma,+\infty): F(s)=\int_0^{+\infty} e^{-st}f(t)\,dt \text{ converges.}$$

Proof

Since $f\in E_\gamma(\mathbb{R}_+)\cap PC^0(\mathbb{R}_+) \Rightarrow f\in E_\gamma(\mathbb{R}_+) \Rightarrow$

$\Rightarrow \exists M>0:\exists\delta>0:\forall t\in(\delta,+\infty): |\exp(-\gamma t)f(t)|<M$

Choose some $M>0$ and $\delta>0$ such that

$\forall t\in(\delta,+\infty): |\exp(-\gamma t)f(t)|\leqslant M$

It follows that

$$\forall x\in(\delta,+\infty): \int_\delta^x |e^{-st}f(t)|\,dt = \int_\delta^x |e^{(\gamma-s)t}e^{-\gamma t}f(t)|\,dt =$$

$$= \int_\delta^x e^{(\gamma-s)t}|e^{-\gamma t}f(t)|\,dt \leqslant \int_\delta^x Me^{(\gamma-s)t}\,dt$$

$$= M\,\frac{\exp((\gamma-s)x)-\exp((\gamma-s)\delta)}{\gamma-s} \qquad (1)$$

Let $s\in(\gamma,+\infty)$ be given. Then we note that

$$s\in(\gamma,+\infty) \Rightarrow s>\gamma \Rightarrow \gamma-s<0 \Rightarrow \lim_{x\to+\infty}\exp((\gamma-s)x)=0 \Rightarrow$$

$$\Rightarrow \int_\delta^{+\infty} Me^{(\gamma-s)t}\,dt = \lim_{x\to+\infty}\left[M\,\frac{\exp((\gamma-s)x)-\exp((\gamma-s)\delta)}{\gamma-s}\right]$$

$$= \frac{-M\exp((\gamma-s)\delta)}{\gamma-s} \Rightarrow$$

$$\Rightarrow \int_\delta^{+\infty} Me^{(\gamma-s)t}\,dt \text{ converges.} \quad (2)$$

From Eq.(1) and Eq.(2), via the comparison test it follows that

$$\int_\delta^{+\infty} |e^{-st}f(t)|\,dt \text{ converges.} \quad (3)$$

From Eq.(3), via the absolute convergence test, it follows that

$$\int_\delta^{+\infty} e^{-st}f(t)\,dt \text{ converges} \Rightarrow \int_0^{+\infty} e^{-st}f(t)\,dt \text{ converges.} \quad \square$$

$\vdash$ Immediate consequences

a) Linearity of Laplace transform

It is easy, although tedious to show that

$f, g \in PC^0(\mathbb{R}_+) \cap E_\gamma(\mathbb{R}_+) \Rightarrow \forall \lambda_1, \lambda_2 \in \mathbb{R} : (\lambda_1 f + \lambda_2 g) \in PC^0(\mathbb{R}_+) \cap E_\gamma(\mathbb{R}_+)$

It follows that if the Laplace integral converges for $f(t)$ and $g(t)$ it also converges for $\lambda_1 f(t) + \lambda_2 g(t)$ and therefore

$\forall \lambda_1, \lambda_2 \in \mathbb{R} : \mathcal{L}[\lambda_1 f(t) + \lambda_2 g(t)] = \lambda_1 \mathcal{L}(f(t)) + \lambda_2 \mathcal{L}(g(t))$

Using proof by induction this generalizes as follows:

$$\boxed{\begin{array}{l} f_1, f_2, \ldots, f_n \in PC^0(\mathbb{R}_+) \cap E_\gamma(\mathbb{R}_+) \Rightarrow \\ \Rightarrow \forall \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R} : \mathcal{L}\left[\sum_{k=1}^{n} \lambda_k f_k(t)\right] = \sum_{k=1}^{n} \lambda_k \mathcal{L}(f_k(t)) \end{array}}$$

b) Uniform convergence of the Laplace integral

From the above convergence proof, it is also established that the Laplace integral, under the above conditions, converges both absolutely and uniformly with respect to $s \in (\gamma, +\infty)$. From the theory of uniform convergent integrals it follows that the Laplace integral can be exchanged with:

(1) A limit on s:

$$\boxed{\lim_{s \to \sigma} \int_0^{+\infty} e^{-st} f(t)\, dt = \int_0^{+\infty} \lim_{s \to \sigma} [e^{-st} f(t)]\, dt}$$

with $\sigma \in (\gamma, +\infty)$ or $\sigma = +\infty$.

(2) A derivative with respect to s for $s \in (\gamma, +\infty)$

$$\boxed{\forall s \in (\gamma, +\infty): \frac{d}{ds} \int_0^{+\infty} e^{-st} f(t)\, dt = \int_0^{+\infty} \frac{d}{ds} \left[e^{-st} f(t) \right] dt}$$

(3) An integral with respect to s:

$$\boxed{\forall s_1, s_2 \in (\gamma, +\infty): \int_{s_1}^{s_2} ds \int_0^{+\infty} dt\, e^{-st} f(t) = \int_0^{+\infty} dt \int_{s_1}^{s_2} ds\, e^{-st} f(t)}$$

In general, these operations are not allowed with respect to an arbitrary improper integral over $(0, +\infty)$. However, they ARE ALWAYS allowed with the Laplace integral as long as s satisfies the sufficient convergence condition $s > \gamma$.

• Laplace transforms of elementary functions

From the definition we can show that

$$\mathcal{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}, \quad \forall a \in \mathbb{R}-(-1)\mathbb{N}^*$$

↳ special cases: $\mathcal{L}(1) = \frac{1}{s}$; $\mathcal{L}(t) = \frac{1}{s^2}$

$$\forall n \in \mathbb{N}^* : \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}, \quad \forall a \in \mathbb{R} \qquad \mathcal{L}(\sinh(at)) = \frac{a}{s^2-a^2}, \quad \forall a \in \mathbb{R}$$

$$\mathcal{L}(\cos(at)) = \frac{s}{s^2+a^2}, \quad \forall a \in \mathbb{R} \qquad \mathcal{L}(\cosh(at)) = \frac{s}{s^2-a^2}, \quad \forall a \in \mathbb{R}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}, \quad \forall a \in \mathbb{R}$$

Using fundamental properties of Laplace transforms we can calculate the transforms of more complicated functions.

EXAMPLE

Show that $\mathcal{L}(\sin(at)) = \dfrac{a}{s^2+a^2}$

Proof

We note that

$$I(T) = \int_0^T \sin(at)e^{-st}\,dt = \int_0^T \sin(at)\left(\frac{e^{-st}}{-s}\right)' dt =$$

$$= \left[\frac{-\sin(at)e^{-st}}{s}\right]_0^T - \int_0^T (\sin(at))' \frac{e^{-st}}{-s}\,dt =$$

$$= \frac{-\sin(aT)e^{-sT} + \sin 0 \cdot e^0}{s} - \int_0^T a\cos(at)\frac{e^{-st}}{-s}\,dt$$

$$= \frac{-\sin(aT)e^{-sT}}{s} + \frac{a}{s}\int_0^T \cos(at)e^{-st}\,dt$$

$$= \frac{-\sin(aT)e^{-sT}}{s} + \frac{a}{s}\int_0^T \cos(at)\left(\frac{e^{-st}}{-s}\right)' dt =$$

$$= \frac{-\sin(aT)e^{-sT}}{s} + \frac{a}{s}\left[\left[\frac{\cos(at)e^{-st}}{-s}\right]_0^T - \int_0^T (\cos(at))'\frac{e^{-st}}{-s}\,dt\right]$$

$$= \frac{-\sin(aT)e^{-sT}}{s} + \frac{a}{s}\,\frac{\cos(aT)e^{-sT} - \cos 0\, e^0}{-s} + \frac{a}{s}\int_0^T \frac{(-a\sin(at))e^{-st}}{s}\,dt$$

$$= \frac{-\sin(aT)e^{-sT}}{s} + \frac{a}{s^2}\left[1 - \cos(aT)e^{-sT}\right] - \frac{a^2}{s^2}\int_0^T \sin(at)e^{-st}\,dt$$

$$= \frac{a}{s^2} - \frac{[a\cos(aT) + s\sin(aT)]\,e^{-sT}}{s^2} - \frac{a^2}{s^2}I(T) \quad (1)$$

From the zero bounded theorem we note that:

Define $b(T) = \dfrac{a\cos(aT) + s\sin(aT)}{s^2}$, $\forall T \in [0, +\infty)$

and note that

$$\forall T \in [0,+\infty): |b(T)| = \left| \frac{a\cos(aT) + s\sin(aT)}{s^2} \right| = \frac{|a\cos(aT) + s\sin(aT)|}{s^2} \leqslant$$

$$\leqslant \frac{|a\cos(aT)| + |s\sin(aT)|}{s^2} = \frac{|a|\cdot|\cos(aT)| + |s|\cdot|\sin(aT)|}{s^2}$$

$$\leqslant \frac{|a| + |s|}{s^2} \Rightarrow b \text{ bounded on } [0,+\infty) \qquad (2)$$

and $\lim\limits_{T\to+\infty} e^{-sT} = 0$ for $s > 0$. (3)

From Eq.(2) and Eq.(3):

$$\lim_{T\to+\infty} \frac{[a\cos(aT) + s\sin(aT)]e^{-sT}}{s^2} = 0$$

and from Eq.(1), taking the limit $T \to +\infty$:

$$\lim_{T\to+\infty} I(T) = \frac{a}{s^2} - \frac{a^2}{s^2} \lim_{T\to+\infty} I(T) \Leftrightarrow \left(1 + \frac{a^2}{s^2}\right) \lim_{T\to+\infty} I(T) = \frac{a}{s^2}$$

$$\Leftrightarrow \frac{s^2+a^2}{s^2} \lim_{T\to+\infty} I(T) = \frac{a}{s^2} \Leftrightarrow \lim_{T\to+\infty} I(T) = \frac{a}{s^2+a^2}$$

and therefore

$$\mathcal{L}(\sin(at)) = \int_0^{+\infty} \sin(at)e^{-st}\,dt = \lim_{T\to+\infty} I(T) = \frac{a}{s^2+a^2}$$

EXERCISES

① Use the definition of the Laplace transform to show that

a) $\mathcal{L}(t^a) = \dfrac{\Gamma(a+1)}{s^{a+1}}$ b) $\mathcal{L}(\cos(at)) = \dfrac{s}{s^2+a^2}$

c) $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$ d) $\mathcal{L}(\sinh(at)) = \dfrac{a}{s^2-a^2}$

e) $\mathcal{L}(\cosh(at)) = \dfrac{s}{s^2-a^2}$

② Find the Laplace transform of the following functions.

a) $f(t) = (t-2)^2$ b) $f(t) = (t-1)^2(t+1)^2$

c) $f(t) = \sin(2t)\cos(3t)$ d) $f(t) = \cos(t)\cos(3t)$

e) $f(t) = \sin(4t)\sin(3t)$ f) $f(t) = \sin^2(5t)$

g) $f(t) = \cos^2(3t)$ h) $f(t) = \sin(2t)[\sin(2t) - \cos(2t)]$

(Hint: For problems (b),..., (h) use trigonometric identities from Precalculus to eliminate the products).

i) $f(t) = 2\cosh(3t) - 3\sinh(3t)$

j) $f(t) = (\sin t + \cos t)^2$ k) $f(t) = \cosh^2(3t)$

Operational properties of Laplace Transforms

The following operational properties of Laplace transforms also follow from the definition and the uniform convergence of the Laplace transform.

Thm: Let $f \in PC^0(\mathbb{R}_+) \cap E_\gamma(\mathbb{R}_+)$ and assume that $\mathcal{L}(f(t)) = F(s)$. Then, it follows that

$$\mathcal{L}[e^{at} f(t)] = F(s-a), \quad \forall a \in \mathbb{R}$$

$$\mathcal{L}[f(t-a) H(t-a)] = e^{-as} F(s), \quad \forall a \in (0, +\infty)$$

$$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right), \quad \forall a \in (0, +\infty)$$

$$\lim_{t \to 0^+} \frac{f(t)}{t} \in \mathbb{R} \Rightarrow \mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{+\infty} F(\sigma)\, d\sigma$$

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s), \quad \forall n \in \mathbb{N}^*$$

▼ Evaluating Laplace Transforms

Method: We use a sequence of implications to build up the function $f(t)$ and its transform using the properties of the Laplace transform and the Laplace transform of fundamental functions.

EXAMPLES

a) Find $\mathcal{L}(f(t))$ for $f(t) = e^{2t}\sin^3 t$.

Solution

Since $\sin(3t) = -4\sin^3 t + 3\sin t \Rightarrow$

$$\Rightarrow \sin^3 t = (1/4)[3\sin t - \sin(3t)] \Rightarrow$$

$$\Rightarrow \mathcal{L}(\sin^3 t) = \mathcal{L}[(1/4)[3\sin t - \sin(3t)]]$$

$$= (1/4)[3\mathcal{L}(\sin t) - \mathcal{L}(\sin(3t))]$$

$$= (1/4)\left[3\frac{1}{s^2+1} - \frac{3}{s^2+3^2}\right] =$$

$$= \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right] = \frac{3}{4}\left[\frac{(s^2+9)-(s^2+1)}{(s^2+1)(s^2+9)}\right]$$

$$= \frac{3}{4}\frac{s^2+9-s^2-1}{(s^2+1)(s^2+9)} = \frac{3}{4}\frac{8}{(s^2+1)(s^2+9)} =$$

$$= \frac{6}{(s^2+1)(s^2+9)} \Rightarrow$$

$$\Rightarrow \mathcal{L}(e^{2t}\sin^3 t) = \frac{6}{[(s-2)^2+1][(s-2)^2+9]} =$$

$$= \frac{6}{(s^2-4s+4+1)(s^2-4s+4+9)} = \frac{6}{(s^2-4s+5)(s^2-4s+13)}$$

b) Find $\mathcal{L}(f(t))$ for $f(t) = t\cos(3t+\pi/6)$.

<u>Solution</u>

We note that

$$f(t) = t\cos(3t+\pi/6) = t\left[\cos(3t)\cos(\pi/6) - \sin(3t)\sin(\pi/6)\right] =$$
$$= t\left[(\sqrt{3}/2)\cos(3t) - (1/2)\sin(3t)\right] =$$
$$= (t/2)\left[\sqrt{3}\cos(3t) - \sin(3t)\right]$$

and

$$\mathcal{L}\left[\sqrt{3}\cos(3t) - \sin(3t)\right] = \sqrt{3}\,\mathcal{L}(\cos(3t)) - \mathcal{L}(\sin(3t)) =$$
$$= \sqrt{3}\,\frac{s}{s^2+3^2} - \frac{3}{s^2+3^2} = \frac{s\sqrt{3}-3}{s^2+9} \Rightarrow$$

$$\Rightarrow \mathcal{L}(f(t)) = \mathcal{L}\left\{(t/2)\left[\sqrt{3}\cos(3t) - \sin(3t)\right]\right\} =$$
$$= (1/2)(-1)\frac{d}{ds}\left(\frac{s\sqrt{3}-3}{s^2+9}\right) =$$
$$= \frac{-1}{2}\,\frac{(s\sqrt{3}-3)'(s^2+9) - (s\sqrt{3}-3)(s^2+9)'}{(s^2+9)^2} =$$
$$= \frac{-1}{2}\,\frac{\sqrt{3}(s^2+9) - (2s)(s\sqrt{3}-3)}{(s^2+9)^2} =$$
$$= \frac{-1}{2}\,\frac{s^2\sqrt{3}+9\sqrt{3} - s^2(2\sqrt{3}) + 6s}{(s^2+9)^2} =$$
$$= \frac{-1}{2}\,\frac{-s^2\sqrt{3}+6s+9\sqrt{3}}{(s^2+9)^2} = \frac{s^2\sqrt{3}-6s-9\sqrt{3}}{2(s^2+9)^2}$$

c) Find $\mathcal{L}(f(t))$ for $f(t) = \dfrac{e^t \sin(2t)}{t}$

Solution

Since,

$$\mathcal{L}(\sin(2t)) = \frac{2}{s^2+2^2} = \frac{2}{s^2+4} \qquad (1)$$

and

$$\lim_{t\to 0^+} \frac{\sin(2t)}{t} = 2\lim_{t\to 0^+} \frac{\sin(2t)}{2t} = 2\cdot 1 = 2 \in \mathbb{R} \qquad (2)$$

it follows, from Eq. (1) and Eq. (2), that

$$\mathcal{L}\left(\frac{\sin(2t)}{t}\right) = \int_s^{+\infty} \frac{2}{\sigma^2+4}\,d\sigma = 2\int_0^{+\infty} \frac{d\sigma}{\sigma^2+2^2} =$$

$$= 2\left[\frac{1}{2}\operatorname{Arctan}\left(\frac{\sigma}{2}\right)\right]_s^{+\infty} = \left[\operatorname{Arctan}\left(\frac{\sigma}{2}\right)\right]_s^{+\infty} =$$

$$= \lim_{\sigma\to+\infty}\left[\operatorname{Arctan}(\sigma/2)\right] - \operatorname{Arctan}(s/2) =$$

$$= \frac{\pi}{2} - \operatorname{Arctan}\left(\frac{s}{2}\right) = \operatorname{Arccot}\left(\frac{s}{2}\right) = \operatorname{Arctan}\left(\frac{2}{s}\right) \Longrightarrow$$

$$\Longrightarrow \mathcal{L}\left(\frac{e^t\sin(2t)}{t}\right) = \operatorname{Arctan}\left(\frac{2}{s-1}\right).$$

→ In this argument we have used the following trigonometric identities

$$\boxed{\begin{array}{l} \forall x\in\mathbb{R}: \operatorname{Arctan}(x) + \operatorname{Arccot}(x) = \pi/2 \\ \forall x\in\mathbb{R}^*: \operatorname{Arccot}(x) = \operatorname{Arctan}(1/x) \end{array}}$$

Laplace transform of piecewise defined functions

Method: To evaluate the Laplace transform of a function of the form

$$f(t) = \begin{cases} f_0(t), & \text{if } 0 \leqslant t < a_1 \\ f_1(t), & \text{if } a_1 < t < a_2 \\ f_2(t), & \text{if } a_2 < t < a_3 \\ \vdots & \\ f_n(t), & \text{if } a_n < t \end{cases}$$

•1 We rewrite $f(t)$ in terms of the Heavyside function as:

$$\begin{aligned} f(t) &= f_0(t) + [f_1(t) - f_0(t)] H(t-a_1) + [f_2(t) - f_1(t)] H(t-a_2) \\ &\quad + \cdots + [f_n(t) - f_{n-1}(t)] H(t-a_n) \\ &= f_0(t) + \sum_{k=1}^{n} [f_k(t) - f_{k-1}(t)] H(t-a_k) \end{aligned}$$

•2 Define functions $g_k(t)$ such that

$\forall k \in [n]: \; g_k(t-a_k) = f_k(t) - f_{k-1}(t)$

The needed definition is:

$\forall k \in [n]: \; g_k(t) = f_k(t+a_k) - f_{k-1}(t+a_k)$

•3 Find the Laplace transforms $\mathcal{L}(g_k(t)) = G_k(s)$

•4 It follows that

$$\begin{aligned} \mathcal{L}(f(t)) &= \mathcal{L}\left[f_0(t) + \sum_{k=1}^{n} g_k(t-a_k) H(t-a_k) \right] = \\ &= \mathcal{L}[f_0(t)] + \sum_{k=1}^{n} \mathcal{L}[g_k(t-a_k) H(t-a_k)] \\ &= F_0(s) + \sum_{k=1}^{n} \exp(-a_k s) G_k(s). \end{aligned}$$

EXAMPLE

Find $\mathcal{L}(f(t))$ for $f(t) = \begin{cases} \sin t & , \text{ if } 0<t<\pi \\ e^t & , \text{ if } \pi<t \end{cases}$

Solution

We rewrite $f(t)$ as:

$$f(t) = \sin t + [e^t - \sin t]H(t-\pi) =$$
$$= \sin t + [\exp((t-\pi)+\pi) - \sin((t-\pi)+\pi)]H(t-\pi)$$
$$= \sin t + g(t-\pi)H(t-\pi)$$

with

$$g(t) = \exp(t+\pi) - \sin(t+\pi) = e^\pi e^t - (-1)\sin t =$$
$$= e^\pi e^t + \sin t.$$

It follows that

$$\mathcal{L}(g(t)) = \mathcal{L}(e^\pi e^t + \sin t) = e^\pi \mathcal{L}(e^t) + \mathcal{L}(\sin t) =$$
$$= e^\pi\left(\frac{1}{s-1}\right) + \frac{1}{s^2+1} = \frac{e^\pi(s^2+1)+(s-1)}{(s-1)(s^2+1)}$$
$$= \frac{e^\pi s^2 + e^\pi + s - 1}{(s-1)(s^2+1)} = \frac{e^\pi s^2 + s + (e^\pi - 1)}{(s-1)(s^2+1)} \Rightarrow$$
$$\Rightarrow \mathcal{L}(g(t-\pi)H(t-\pi)) = e^{-\pi s}\frac{e^\pi s^2 + s + (e^\pi-1)}{(s-1)(s^2+1)} \Rightarrow$$
$$\Rightarrow \mathcal{L}(f(t)) = \mathcal{L}(\sin t + g(t-\pi)H(t-\pi)) =$$
$$= \mathcal{L}(\sin t) + \mathcal{L}(g(t-\pi)H(t-\pi))$$
$$= \frac{1}{s^2+1} + \frac{e^{-\pi s}[e^\pi s^2 + s + (e^\pi - 1)]}{(s-1)(s^2+1)}$$

EXERCISES

③ Evaluate the Laplace transform for the following functions.

a) $f(t) = t^2 e^{3t}$ b) $f(t) = (t-2)^2 \sin t$

c) $f(t) = e^{-t} \cos(3t + \pi/6)$ d) $f(t) = e^{-4t} \sin(3t) \cos t$

e) $f(t) = e^{2t} \cos^2(t + \pi/3)$ f) $f(t) = e^{-3t} \cosh(2t)$

g) $f(t) = e^{-2t} [2\sinh(t) - 3\cosh(2t)]$

h) $f(t) = t^3 \sinh(2t)$ i) $f(t) = (\sin^2 t) H(t - \pi/4)$

j) $f(t) = te^{2t} H(t-1)$ k) $f(t) = \begin{cases} te^{2t}, & \text{if } t \in [0,3] \\ e^{2t}, & \text{if } t \in (3, +\infty) \end{cases}$

l) $f(t) = \begin{cases} \sin t, & \text{if } t \in (0, \pi/3) \\ t\cos t, & \text{if } t \in (\pi/3, +\infty) \end{cases}$

m) $f(t) = \dfrac{\sin(3t)}{t}$ n) $f(t) = \dfrac{1 - \cos(2t)}{3t}$

o) $f(t) = \dfrac{1 - \exp(3t)}{t}$ p) $f(t) = \dfrac{\sinh(2t)}{5t}$

q) $f(t) = \dfrac{1 - \cosh(4t)}{4t}$

Laplace transforms of functions defined as series

Given a function $f(t)$ defined as a power series
$$f(t) = \sum_{n=0}^{+\infty} a_n t^n$$

due to the uniform convergence of the power series, the Laplace integral can be done term by term and we get:

$$F(s) = \mathcal{L}(f(t)) = \int_0^{+\infty} e^{-st} \left[\sum_{n=0}^{+\infty} a_n t^n \right] dt = \sum_{n=0}^{+\infty} a_n \left[\int_0^{+\infty} e^{-st} t^n \, dt \right]$$
$$= \sum_{n=0}^{+\infty} a_n \mathcal{L}(t^n) = \sum_{n=0}^{+\infty} a_n \frac{\Gamma(n+1)}{s^{n+1}} = \sum_{n=0}^{+\infty} \frac{n! \, a_n}{s^{n+1}}$$

The same idea can be applied on Frobenius series, restricted at $t>0$ such that for
$$f(t) = t^\lambda \sum_{n=0}^{+\infty} a_n t^n = \sum_{n=0}^{+\infty} a_n t^{n+\lambda}$$

the corresponding Laplace transform of $f(t)$ is given by
$$F(s) = \mathcal{L}(f(t)) = \sum_{n=0}^{+\infty} \frac{a_n \Gamma(n+\lambda+1)}{s^{n+\lambda+1}}$$

EXAMPLE

Show that $\mathcal{L}(J_0(t)) = \dfrac{1}{\sqrt{s^2+1}}$

Solution

$$J_0(t) = \left(\frac{t}{2}\right)^0 \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+1+0)} \left(\frac{t}{2}\right)^{2n} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{t}{2}\right)^{2n} =$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{2n}(n!)^2} t^{2n} \Rightarrow$$

$$\Rightarrow \mathcal{L}(J_0(t)) = \mathcal{L}\left[\sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{2n}(n!)^2} t^{2n}\right] = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \mathcal{L}(t^{2n})$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \frac{(2n)!}{s^{2n+1}} = \frac{1}{s} \sum_{n=0}^{+\infty} \frac{(-1)^n (2n)!}{(2n)!!\,(2n)!!} \frac{1}{s^{2n}}$$

$$= \frac{1}{s}\left[1 + \sum_{n=1}^{+\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!} \left(\frac{1}{s^2}\right)^n\right] =$$

$$= \frac{1}{s}\left[1 + \sum_{n=1}^{+\infty} \binom{-1/2}{n} \left(\frac{1}{s^2}\right)^n\right] =$$

$$= \frac{1}{s} \sum_{n=0}^{+\infty} \binom{-1/2}{n} \left(\frac{1}{s^2}\right)^n =$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{s}\left(\frac{s^2+1}{s^2}\right)^{-1/2} =$$

$$= \frac{1}{s}\sqrt{\frac{s^2}{s^2+1}} = \sqrt{\frac{s^2}{s^2(s^2+1)}} = \sqrt{\frac{1}{s^2+1}} =$$

$$= \frac{1}{\sqrt{s^2+1}}$$

↳ Note that the key step is to use the binomial series expansion in reverse. We also recall from Calculus 2 that

$$\forall n \in \mathbb{N}^*: \binom{a}{n} = \prod_{k=1}^{n} \left(\frac{a+1-k}{k} \right)$$

and therefore

$$\forall n \in \mathbb{N}^*: \binom{-1/2}{n} = \prod_{k=1}^{n} \left(\frac{-1/2+1-k}{k} \right) = \prod_{k=1}^{n} \left(\frac{-1+2-2k}{2k} \right) =$$

$$= \prod_{k=1}^{n} \frac{1-2k}{2k} = (-1)^n \prod_{k=1}^{n} \frac{(2k-1)}{2k} =$$

$$= (-1)^n \frac{(2n-1)!!}{(2n)!!}$$

EXERCISES

(4) Use power series to show that

a) $\mathcal{L}(J_0(at)) = \dfrac{1}{\sqrt{s^2+a^2}}$

b) $\mathcal{L}(\sin(\sqrt{t})) = \dfrac{\sqrt{\pi}}{2s\sqrt{s}} \exp\left(\dfrac{-1}{4s}\right)$

c) $\mathcal{L}\left(\dfrac{\cos(\sqrt{t})}{\sqrt{t}}\right) = \exp\left(\dfrac{-1}{4s}\right)\sqrt{\dfrac{\pi}{s}}$

d) $\mathcal{L}\left(\dfrac{1}{\sqrt{\pi t}} \exp\left(\dfrac{-a^2}{4t}\right)\right) = \dfrac{\exp(-a\sqrt{s})}{\sqrt{s}}$

e) $\mathcal{L}(J_0(\sqrt{t})) = \dfrac{\exp(-s/4)}{s}$

f) $\mathcal{L}\left(\dfrac{\exp(-at)}{\sqrt{\pi t}}\right) = \dfrac{1}{\sqrt{s+a}}$

(5) Although these results can be established by evaluating the Laplace integral, provide an alternate proof using power series

a) $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$

b) $\mathcal{L}(\sin(at)) = \dfrac{a}{s^2+a^2}$

c) $\mathcal{L}(\cos(at)) = \dfrac{s}{s^2+a^2}$

(6) Consider the error function defined as

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-\tau^2)\, d\tau$$

Use power series to show that

$$\mathcal{L}(\operatorname{erf}(\sqrt{t})) = \frac{1}{s\sqrt{s+1}}$$

(7) Use the results from exercises 4, 5, 6 to find the Laplace transform of

a) $f(t) = t^2 J_0(3t)$

b) $f(t) = t \sin(\sqrt{t})$

c) $f(t) = \sqrt{t} \cos(\sqrt{t})$

d) $f(t) = t^3 \operatorname{erf}(\sqrt{t})$

e) $f(t) = t e^{2t} \operatorname{erf}(\sqrt{t})$

f) $f(t) = t e^{-t} J_0(t)$

Fundamental properties of Laplace transforms

Techniques based on Laplace transforms are founded on the following fundamental properties of Laplace transforms

① → Lerch's theorem

$$\boxed{\forall f, g \in PC^0(\mathbb{R}_+) \cap E_\gamma(\mathbb{R}_+) : \left(\mathcal{L}(f) = \mathcal{L}(g) \Rightarrow f = g \right)}$$

Remark: This theorem shows that two functions f, g that are piecewise continuous on $\mathbb{R}_+$ and of exponential order γ that have equal Laplace transforms have to be themselves equal. It follows that given $F(s) = \mathcal{L}(f(t))$ we can define the inverse Laplace transform operation $\mathcal{L}^{-1}$ such that given $F(s)$ we can find the original unique function $f(t)$ as $f(t) = \mathcal{L}^{-1}(F(s))$. Using complex analysis it is possible to represent the inverse Laplace transform in terms of an integral in the complex plane, however we will not need this representation in our work below.

② → Laplace transform of derivatives

$$\boxed{\begin{cases} f \text{ differentiable on } \mathbb{R}_+ \\ f \in E_\gamma(\mathbb{R}_+) \wedge f' \in PC^0(\mathbb{R}_+) \\ F(s) = \mathcal{L}(f(t)) \end{cases} \Rightarrow \mathcal{L}(f'(t)) = sF(s) - f(0)}$$

Remarks: It is necessary to assume that $f' \in PC^0(\mathbb{R}_+)$ but we do not need to assume $f' \in E_\gamma(\mathbb{R}_+)$. The theorem can be proved using integration by parts.

The theorem generalizes for the second derivative $f''(t)$ and for the n^{th}-derivative $f^{(n)}(t)$ as follows:

Thm:

$$\left.\begin{array}{l} f \text{ twice differentiable on } \mathbb{R}_+ \\ f'' \in PC^0(\mathbb{R}_+) \wedge f, f' \in E_\gamma(\mathbb{R}_+) \\ F(s) = \mathcal{L}(f(t)) \end{array}\right\} \Rightarrow \mathcal{L}(f''(t)) = s^2 F(s) - s f(0) - f'(0)$$

Thm

$$\left.\begin{array}{l} f \text{ n-times differentiable on } \mathbb{R}_+ \\ f^{(n)} \in PC^0(\mathbb{R}_+) \\ \forall k \in [0, n-1] \cap \mathbb{N} : f^{(k)} \in E_\gamma(\mathbb{R}_+) \\ F(s) = \mathcal{L}(f(t)) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \mathcal{L}(f^{(n)}(t)) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0)$$

$$= s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

These theorems are immediate consequences and the second one can be proved via induction. Another immediate consequence is the following operational property:

Thm

$$\left.\begin{array}{l} f \in PC^0(\mathbb{R}_+) \cap E_\gamma(\mathbb{R}_+) \\ \mathcal{L}(f(t)) = F(s) \end{array}\right\} \Rightarrow \mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$$

(3) → Limit properties of Laplace transforms

An immediate consequence of the definition and the uniform convergence of the Laplace integral with respect to s is the following result

Prop: $\left.\begin{array}{l} f \in PC^0(\mathbb{R}_+) \cap E_\gamma(\mathbb{R}_+) \\ \mathcal{L}(f(t)) = F(s) \end{array}\right\} \Rightarrow \lim\limits_{s\to+\infty} F(s) = 0$

Using the previous theorem on the Laplace transform of the derivative of a function we can also prove the following theorem:

Thm: Assume that

$$\left\{\begin{array}{l} f \in E_\gamma(\mathbb{R}_+) \wedge f \text{ differentiable on } \mathbb{R}_+ \\ f' \in PC^0(\mathbb{R}_+) \\ \mathcal{L}(f(t)) = F(s) \end{array}\right.$$

Then it follows that:

a) $\lim\limits_{s\to+\infty} sF(s) = f(0)$

b) $\lim\limits_{s\to0} sF(s) \in \mathbb{R} \Rightarrow \lim\limits_{t\to+\infty} f(t) = \lim\limits_{s\to0^+} sF(s)$

EXERCISES

(8) Use the following steps to derive the Laplace transforms of the general Bessel function $J_n(t)$ with $n \in \mathbb{N}$.

a) Use the power series definition of $J_0(t)$ and $J_1(t)$ to show that $J_1'(t) = -J_0(t)$.

b) Show that $\mathcal{L}(J_1(t)) = \dfrac{\sqrt{s^2+1} - s}{\sqrt{s^2+1}}$

c) Use the identity $2J_n'(t) = J_{n-1}(t) - J_{n+1}(t)$ and proof by induction to show that
$$\mathcal{L}(J_n(t)) = \frac{(\sqrt{s^2+1} - s)^n}{\sqrt{s^2+1}}$$

d) Use the above results to show that $\lim\limits_{t \to +\infty} J_n(t) = 0$

(9) Alternate proof of Laplace transform of $f(t) = \sin(at)$

(a) Show that $f(t) = \sin(at)$ is the unique solution of the following initial value problem
$$\begin{cases} f''(t) + a^2 f(t) = 0 \\ f(0) = 0 \wedge f'(0) = a \end{cases}$$

(b) Use the previous solution to show that
$$\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}$$

c) Take the derivative of $\sin(at)$ to show that
$$\mathcal{L}(\cos(at)) = \frac{s}{s^2+a^2}$$

(10) Alternative proof of Laplace transform of $f(t)=e^{at}$.

a) Show that $f(t)=e^{at}$ is the unique solution of the initial value problem

$$\begin{cases} f'(t)-af(t)=0 \\ f(0)=1 \end{cases}$$

b) Use (a) to show that $\mathcal{L}(e^{at})=\dfrac{1}{s-a}$

(11) Use the theorem about the Laplace transform of the derivative of a function to show that

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)=\frac{F(s)}{s}$$

(12) The error function erf(t) and the complementary error function erfc(t) are defined as

$$\forall t\in\mathbb{R}:\ \mathrm{erf}(t)=\frac{2}{\sqrt{\pi}}\int_0^t \exp(-\tau^2)\,d\tau$$

$$\forall t\in\mathbb{R}:\ \mathrm{erfc}(t)=\frac{2}{\sqrt{\pi}}\int_t^{+\infty} \exp(-\tau^2)\,d\tau=1-\mathrm{erf}(t)$$

a) Use the Laplace integral to show that

$$\mathcal{L}(\exp(-t^2))=\exp\left(\frac{s^2}{4}\right)\mathrm{erfc}\left(\frac{s}{2}\right)$$

(Hint: it will be necessary to complete squares and use the method of substitution)

b) Show that $\mathcal{L}(\mathrm{erf}(t))=\dfrac{1}{s}\exp\left(\dfrac{s^2}{4}\right)\mathrm{erfc}\left(\dfrac{s}{2}\right)$

▼ Laplace transforms of functions defined as integrals

Such functions can be differentiated via the 1st Fundamental Theorem of Calculus, so we can find the Laplace transform of their derivative. If a differential equation is obtained, we can get initial conditions via the initial-value theorem or the final-value theorem. With this technique we can find the Laplace transform for the following special functions:

1) Sine integral

$$\boxed{Si(t) = \int_0^t \frac{\sin\tau}{\tau}\, d\tau \Rightarrow \mathcal{L}(Si(t)) = \frac{1}{s} \operatorname{Arctan}\left(\frac{1}{s}\right)}$$

2) Cosine integral

$$\boxed{Ci(t) = \int_t^{+\infty} \frac{\cos\tau}{\tau}\, d\tau \Rightarrow \mathcal{L}(Ci(t)) = \frac{\ln(s^2+1)}{2s}}$$

3) Exponential integral

$$\boxed{Ei(t) = \int_t^{+\infty} \frac{\exp(-\tau)}{\tau}\, d\tau \Rightarrow \mathcal{L}(Ei(t)) = \frac{\ln(s+1)}{s}}$$

EXAMPLES

Show that for the cosine integral function $Ci(t) = \int_t^{+\infty} \frac{\cos\tau}{\tau}\, d\tau$ its Laplace transform is

$$\mathcal{L}(Ci(t)) = \frac{\ln(s^2+1)}{2s}$$

Solution

Let $F(s) = \mathcal{L}(Ci(t))$. Since,

$$Ci'(t) = \frac{d}{dt}\int_t^{+\infty} \frac{\cos\tau}{\tau}\, d\tau = \frac{-\cos t}{t} \Rightarrow tCi'(t) = -\cos t$$

and

$$\mathcal{L}(-\cos t) = -\mathcal{L}(\cos t) = \frac{-s}{s^2+1^2} = \frac{-s}{s^2+1}$$

and

$$\mathcal{L}(Ci'(t)) = s\mathcal{L}(Ci(t)) - Ci(0) = sF(s) - Ci(0) \Rightarrow$$

$$\Rightarrow \mathcal{L}(tCi'(t)) = (-1)(d/ds)\mathcal{L}(Ci'(t)) = (-1)(d/ds)(sF(s) - Ci(0))$$

$$= -(d/ds)(sF(s))$$

it follows that

$$\mathcal{L}(tCi'(t)) = \mathcal{L}(-\cos t) \Leftrightarrow -(d/ds)(sF(s)) = \frac{-s}{s^2+1} \Leftrightarrow$$

$$\Leftrightarrow (d/ds)(sF(s)) = \frac{s}{s^2+1} \Leftrightarrow$$

$$\Leftrightarrow sF(s) = \int \frac{s\, ds}{s^2+1} = \frac{1}{2}\int \frac{2s\, ds}{s^2+1} = \frac{1}{2}\int \frac{(s^2+1)'}{s^2+1}\, ds =$$

$$= \frac{\ln|s^2+1|}{2} + c = \frac{\ln(s^2+1)}{2} + c$$

For $s \to 0^+$, we have

$$\lim_{s \to 0^+} sF(s) = \lim_{s \to 0^+} \left[\frac{\ln(s^2+1)}{2} + c \right] = \frac{\ln(0+1)}{2} + c = c \Rightarrow$$

$$\Rightarrow c = \lim_{s \to 0^+} sF(s) = \lim_{t \to +\infty} Ci(t) = \lim_{t \to +\infty} \int_t^{+\infty} \frac{\cos\tau}{\tau} d\tau = 0$$

and therefore

$$sF(s) = \frac{\ln(s^2+1)}{2} \Rightarrow F(s) = \frac{\ln(s^2+1)}{2s}$$

EXERCISES

(13) Use the properties of Laplace transforms to show that:

a) $\mathcal{L}(Si(t)) = \frac{1}{s} \operatorname{Arctan}\left(\frac{1}{s}\right)$

b) $\mathcal{L}(Ei(t)) = \frac{\ln(s+1)}{s}$

(14) Use the power-series method to show that
$\mathcal{L}(Si(t)) = \frac{1}{s} \operatorname{Arctan}\left(\frac{1}{s}\right)$

(15) Show that $\int_0^{+\infty} \frac{\sin \tau}{\tau}\, d\tau = \frac{\pi}{2}$

(16) Use the previous results and the properties of Laplace transforms to show the following generalizations

a) $\mathcal{L}(Ci(at)) = \frac{1}{2s} \ln\left(\frac{s^2+a^2}{a^2}\right)$

b) $\mathcal{L}(Si(at)) = \frac{1}{s} \operatorname{Arctan}\left(\frac{a}{s}\right)$

c) $\mathcal{L}(Ei(at)) = \frac{1}{s} \ln\left(\frac{s+a}{a}\right)$

Application to differential equations

The main idea for solving differential equations using the Laplace transform technique is that a wider range of ordinary differential equation initial value problems, to be solved on $t \in [0, +\infty)$ can be transformed to an algebraic problem. and the Laplace transform of the solution can be found via basic algebra. The challenge then is to apply an inverse Laplace transform and find the actual solution to the initial value problem.

→ Methodology: Evaluating inverse Laplace transforms.

If $F(s) = \mathcal{L}(f(t))$, then the following properties are useful in the evaluation of inverse Laplace transforms

$$\boxed{\begin{aligned} &\mathcal{L}^{-1}[F(s-a)] = e^{at} f(t) \\ &\mathcal{L}^{-1}[e^{-as} F(s)] = f(t-a) H(t-a) \\ &\mathcal{L}^{-1}[F(as)] = \frac{1}{a} f\left(\frac{t}{a}\right) \end{aligned}}$$

To evaluate the inverse Laplace transform of functions of the form $P(s)/Q(s)$ where P, Q are polynomials we use partial fraction decomposition in conjunction with the following known inverse Laplace transforms:

$\mathcal{L}^{-1}\left(\dfrac{1}{s-a}\right) = e^{at}$	$\mathcal{L}^{-1}\left[\dfrac{1}{(s-a)^n}\right] = \dfrac{t^{n-1}e^{at}}{(n-1)!}$

If we encounter irreducible quadratic factors then the following additional inverse Laplace transforms can be very useful, when combined with s-shifting.

$$\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin(at) \longrightarrow \mathcal{L}^{-1}\left[\frac{a}{(s-b)^2+a^2}\right] = e^{bt}\sin(at)$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos(at) \longrightarrow \mathcal{L}^{-1}\left[\frac{s-b}{(s-b)^2+a^2}\right] = e^{bt}\cos(at)$$

More complicated cases can be handled by computer algebra systems or by inverse Laplace transform tables.

EXAMPLES

a) Use Laplace transforms to solve the initial value problem

$$\begin{cases} y''(t) - 3y'(t) + 2y(t) = 4e^{2t} \\ y(0) = -3 \wedge y'(0) = 5 \end{cases}$$

Solution

Define $Y(s) = \mathcal{L}(y(t))$. Then

$$\mathcal{L}[y''(t) - 3y'(t) + 2y(t)] = \mathcal{L}[y''(t)] - 3\mathcal{L}[y'(t)] + 2\mathcal{L}[y(t)] =$$

$$= [s^2 Y(s) - s y(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s)$$

$$= s^2 Y(s) - (-3)s - 5 - 3sY(s) + 3(-3) + 2Y(s)$$

$$= s^2 Y(s) + 3s - 5 - 3sY(s) - 9 + 2Y(s)$$

$$= (s^2 - 3s + 2)Y(s) + (3s - 5 - 9) =$$

$$= (s^2 - 3s + 2)Y(s) + (3s - 14)$$

and

$$\mathcal{L}(4e^{2t}) = 4\mathcal{L}(e^{2t}) = 4 \cdot \frac{1}{s-2} = \frac{4}{s-2}$$

so it follows that

$$\begin{cases} y''(t) - 3y'(t) + 2y(t) = 4e^{2t} \\ y(0) = -3 \wedge y'(0) = 5 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow (s^2 - 3s + 2)Y(s) + (3s - 14) = \frac{4}{s-2} \Leftrightarrow$$

$$\Leftrightarrow (s^2 - 3s + 2)Y(s) = \frac{4}{s-2} - (3s - 14) = \frac{4 - (3s-14)(s-2)}{s-2}$$

$$= \frac{4 - (3s^2 - 6s - 14s + 28)}{s-2} = \frac{4 - 3s^2 + 6s + 14s - 28}{s-2}$$

$$= \frac{-3s^2+(6+14)s+(4-28)}{s-2} = \frac{-3s^2+20s-24}{s-2} \Leftrightarrow$$

$$\Leftrightarrow Y(s) = \frac{-3s^2+20s-24}{(s^2-3s+2)(s-2)} = \frac{-3s^2+20s-24}{(s-2)(s-1)(s-2)} =$$

$$= \frac{-3s^2+20s-24}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{(s-2)^2} + \frac{C}{s-2} \quad (1)$$

with

$$A = \left.\frac{-3s^2+20s-24}{(s-2)^2}\right|_{s=1} = \frac{-3+20-24}{(1-2)^2} = -3+20-24 = -7$$

$$B = \left.\frac{-3s^2+20s-24}{s-1}\right|_{s=2} = \frac{-3\cdot 2^2+20\cdot 2-24}{2-1} =$$

$$= -3\cdot 4+40-24 = -12+40-24 = 40-36 = 4$$

- To find C, we multiply both sides with s and take the limit $s \to +\infty$:

$$\lim_{s\to+\infty}\left[\frac{As}{s-1} + \frac{Bs}{(s-2)^2} + \frac{Cs}{s-2}\right] = \lim_{s\to+\infty} \frac{s[-3s^2+20s-24]}{(s-1)(s-2)^2}$$

$$\Leftrightarrow A+0+C = -3 \Leftrightarrow -7+C = -3 \Leftrightarrow C = 7-3 = 4$$

It follows that

$$Y(s) = \frac{-7}{s-1} + \frac{4}{(s-2)^2} + \frac{4}{s-2}$$

Since:

$$\mathcal{L}^{-1}\left[\frac{-7}{s-1} + \frac{4}{s-2}\right] = -7\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + 4\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] =$$

$$= -7e^t + 4e^{2t}$$

and $\mathcal{L}^{-1}\left(\frac{4}{s^2}\right) = 4\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = 4t \Rightarrow \mathcal{L}^{-1}\left[\frac{4}{(s-2)^2}\right] = 4te^{2t}$

it follows that

$$y(t) = -7e^{t} + 4e^{2t} + 4t\,e^{2t} = -7e^{t} + 4(t+1)e^{2t}.$$

ODEs forced with generalized functions

A big advantage of the Laplace transform method is that it can be used to solve problems with discontinuous forcing or problems where the forcing is a generalized function. This requires extending the definition of the Laplace transform to generalized functions.

Def: Let $F \in \Delta^\infty(\mathbb{R})$ be a distribution with expansion

$$F(x) = f(x) + \sum_{n\in A} g_n(x) H(x-p_n) + \sum_{n\in B} a_n \delta^{(b_n)}(x-q_n)$$

with:

$$\begin{cases} A \subseteq \mathbb{N} \wedge B \subseteq \mathbb{N} \wedge f \in C^\infty(\mathbb{R}) \\ \forall n\in A : (g_n \in C^\infty(A) \wedge p_n \in \mathbb{R}) \\ \forall n \in B : (a_n, q_n \in \mathbb{R} \wedge b_n \in \mathbb{N}) \end{cases}$$

We say that F is of exponential order (notation: $F \in \mathcal{E}\Delta^\infty_+(\mathbb{R})$) if and only if all of the following conditions are satisfied

a) f is of exponential order (as a function)

b) $\forall n \in A$: g_n is of exponential order

c) b_n bounded sequence

d) B not finite $\Rightarrow \lim_{n\in\mathbb{N}} q_n = +\infty$

e) $\exists M > 0 : \forall n \in B : |a_n| \leq \exp(M q_n)$

Def: Given a distribution $f \in E\Delta^{\infty}_{+}(\mathbb{R})$ we define the $\mathcal{L}_+$ and $\mathcal{L}_-$ Laplace transforms as follows:

$$\mathcal{L}_+[f(t)] = \int_{0^+}^{+\infty} f(t)e^{-st}\,dt$$

$$\mathcal{L}_-[f(t)] = \int_{0^-}^{+\infty} f(t)e^{-st}\,dt$$

Remark: It can be shown that for any distribution $f \in E\Delta^{\infty}_{+}(\mathbb{R})$ we have:

- For $\mathcal{L}_+$ transforms:

Given $F(s) = \mathcal{L}_+(f(t))$ we have:

a) $\mathcal{L}_+[f'(t)] = sF(s) - f(0^+)$

$\mathcal{L}_+[f''(t)] = s^2F(s) - sf(0^+) - f'(0^+)$

$\forall n \in \mathbb{N}^*: \mathcal{L}_+[f^{(n)}(t)] = s^nF(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0^+)$

b) $\lim_{s\to+\infty} sF(s) = f(0^+)$

- For $\mathcal{L}_-$ transforms

Given $F(s) = \mathcal{L}_-(f(t))$ we have

a) $\mathcal{L}_-[f'(t)] = sF(s) - f(0^-)$

$\mathcal{L}_-[f''(t)] = s^2F(s) - sf(0^-) - f'(0^-)$

$\forall n \in \mathbb{N}^*: \mathcal{L}_-[f^{(n)}(t)] = s^nF(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0^-)$

b) $\lim_{s\to+\infty} sF(s) = f(0^+) \longleftarrow (!!!)$

- In practice, it is recommended to use the $\mathcal{L}_-$ transform because all other properties of Laplace transforms apply with no further modifications over the broader space $E\Delta^{\infty}_{+}(\mathbb{R})$ of generalized functions of exponential order.

- Laplace transforms of delta functions:

$\mathcal{L}_-[\delta(t)] = 1$	$\mathcal{L}_-[\delta(t-a)] = e^{-as}$
$\mathcal{L}_-[\delta^{(n)}(t)] = s^n$	$\mathcal{L}_-[\delta^{(n)}(t-a)] = e^{-as}\, s^n$

- Another advantage of the $\mathcal{L}_-$ transform is that for problems where the forcing function includes $\delta(t)$ or terms like $\delta^{(n)}(t)$, it makes physical sense to give the initial condition at 0^- instead of at 0^+.

- For problems that are forced with $H(t-a)$ or $\delta^{(n)}(t-a)$ terms, it is usually necessary to use the following property to find the corresponding inverse Laplace transform:

$$\boxed{F(s) = \mathcal{L}(f(t)) \Rightarrow \mathcal{L}^{-1}[e^{-as} F(s)] = f(t-a)\,H(t-a)}$$

EXAMPLES

a) Solve the initial value problem

$$\begin{cases} y''(t) + y'(t) - 2y(t) = \delta(t) + \delta(t-2) \\ y(0^-) = 0 \wedge y'(0^-) = 0 \end{cases}$$

Solution

Define $F(s) = \mathcal{L}_-(y(t))$ and note that

$\mathcal{L}_-(y'(t)) = sY(s) - y(0^-) = sY(s)$

$\mathcal{L}_-(y''(t)) = s^2Y(s) - sy(0^-) - y'(0^-) = s^2Y(s) - s\cdot 0 - 0 = s^2Y(s)$

$\mathcal{L}_-(\delta(t) + \delta(t-2)) = 1 + e^{-2s}$

It follows that

$$\begin{cases} y''(t) + y'(t) - 2y(t) = \delta(t) + \delta(t-2) \\ y(0^-) = 0 \wedge y'(0^-) = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow s^2Y(s) + sY(s) - 2Y(s) = 1 + e^{-2s} \Leftrightarrow$$

$$\Leftrightarrow (s^2 + s - 2)Y(s) = 1 + e^{-2s} \Leftrightarrow$$

$$\Leftrightarrow Y(s) = \frac{1 + e^{-2s}}{s^2 + s - 2} = \frac{1 + e^{-2s}}{(s+2)(s-1)} =$$

$$= (1 + e^{-2s})\left[\frac{A}{s+2} + \frac{B}{s-1}\right]$$

with $A = \left.\frac{1}{s-1}\right|_{s=-2} = \frac{1}{(-2)-1} = \frac{-1}{3}$

$B = \left.\frac{1}{s+2}\right|_{s=1} = \frac{1}{1+2} = \frac{1}{3}$

It follows that

$$Y(s) = (1+e^{-2s})\left[\frac{-1/3}{s+2} + \frac{1/3}{s-1}\right] =$$

$$= \frac{1+e^{-2s}}{3}\left[\frac{-1}{s+2} + \frac{1}{s-1}\right]$$

Since

$$\mathcal{L}^{-1}\left[\frac{-1}{s+2} + \frac{1}{s-1}\right] = -e^{-2t} + e^{t} = e^{t}(1-e^{-3t})$$

it follows that

$$y(t) = \mathcal{L}^{-1}\left[\frac{1+e^{-2s}}{3}\left(\frac{-1}{s+2} + \frac{1}{s-1}\right)\right] =$$

$$= (1/3)e^{t}(1-e^{-3t}) + (1/3)H(t-2)\,e^{t-2}(1-e^{-3(t-2)})$$

$$= (1/3)e^{t}(1-e^{-3t}) + (1/3)H(t-2)\,\frac{e^{t}}{e^{2}}\left[1-e^{6}e^{-3t}\right]$$

$$= \frac{1}{3}\left[e^{t}(1-e^{-3t}) + \frac{e^{t}(1-e^{6}e^{-3t})}{e^{2}}\,H(t-2)\right]$$

b) Solve the initial value problem

$$\begin{cases} y''(t) + 4y'(t) + 4y(t) = H(t-2) \\ y(0^-) = 1 \wedge y'(0^-) = 0 \end{cases}$$

<u>Solution</u>

Let $Y(s) = \mathcal{L}_-[y(t)]$ and note that

$$\mathcal{L}_-[y''(t) + 4y'(t) + 4y(t)] =$$
$$= [s^2 Y(s) - s y(0^-) - y'(0^-)] + 4[sY(s) - y(0^-)] + 4Y(s)$$
$$= [s^2 Y(s) - s] + 4[sY(s) - 1] + 4Y(s) =$$
$$= s^2 Y(s) - s + 4sY(s) - 4 + 4Y(s) =$$
$$= (s^2 + 4s + 4) Y(s) - (s+4) = (s+2)^2 Y(s) - (s+4)$$

and

$$\mathcal{L}_-(H(t-2)) = \frac{e^{-2s}}{s}$$

It follows that

$$\begin{cases} y''(t) + 4y'(t) + 4y(t) = H(t-2) \\ y(0^-) = 1 \wedge y'(0^-) = 0 \end{cases} \Leftrightarrow (s+2)^2 Y(s) - (s+4) = \frac{e^{-2s}}{s}$$

$$\Leftrightarrow Y(s) \cdot (s+2)^2 = (s+4) + \frac{e^{-2s}}{s} \Leftrightarrow$$

$$\Leftrightarrow Y(s) = \frac{s+4}{(s+2)^2} + \frac{e^{-2s}}{s(s+2)^2}$$

For the partial fraction decompositions, we note that

$$\frac{s+4}{(s+2)^2} = \frac{A}{(s+2)^2} + \frac{B}{s+2}$$

with $A = (s+4)|_{s=-2} = -2+4 = 2$. To find B, multiply both sides with s and take the limit $s \to +\infty$:

$$\lim_{s\to+\infty} \frac{s(s+4)}{(s+2)^2} = \lim_{s\to+\infty} \frac{As}{(s+2)^2} + \lim_{s\to+\infty} \frac{Bs}{s+2} \Leftrightarrow$$

$\Leftrightarrow 1 = 0 + B \Leftrightarrow B = 1$

We also note that

$$\frac{1}{s(s+2)^2} = \frac{C}{s} + \frac{D}{(s+2)^2} + \frac{E}{s+2}$$

with

$$C = \left.\frac{1}{(s+2)^2}\right|_{s=0} = \frac{1}{(0+2)^2} = \frac{1}{4}$$

$$D = \left.\frac{1}{s}\right|_{s=-2} = \frac{1}{-2} = \frac{-1}{2}$$

To find E, multiply both sides with s and take the limit $s\to+\infty$:

$$\lim_{s\to+\infty} \frac{Cs}{s} + \lim_{s\to+\infty} \frac{Ds}{(s+2)^2} + \lim_{s\to+\infty} \frac{Es}{s+2} = \lim_{s\to+\infty} \frac{s}{s(s+2)^2}$$

$\Leftrightarrow C + 0D + E = 0 \Leftrightarrow C + E = 0 \Leftrightarrow E = -C = -1/4$

From the above results:

$$Y(s) = \frac{2}{(s+2)^2} + \frac{1}{s+2} + e^{-2s}\left[\frac{1}{4s} - \frac{1}{2(s+2)^2} - \frac{1}{4(s+2)}\right]$$

To find $y(t)$ we note that

$$\mathcal{L}^{-1}\left[\frac{2}{(s+2)^2} + \frac{1}{s+2}\right] = 2\,\frac{t^{2-1}e^{-2t}}{(2-1)!} + e^{-2t} =$$

$$= 2t\,e^{-2t} + e^{-2t} = (2t+1)e^{-2t}$$

and

$$\mathcal{L}^{-1}\left[\frac{1}{4s} - \frac{1}{2(s+2)^2} - \frac{1}{4(s+2)}\right] = \frac{1}{4} - \frac{1}{2}\frac{t^{2-1}e^{-2t}}{(2-1)!} - \frac{1}{4}e^{-2t}$$

$$= (1/4) - (1/2)te^{-2t} - (1/4)e^{-2t} =$$

$$= (1/4)\left[1 - 2te^{-2t} - e^{-2t}\right] = (1/4)\left[1 - (2t+1)e^{-2t}\right]$$

and therefore

$$y(t) = \mathcal{L}^{-1}\left[\frac{2}{(s+2)^2} + \frac{1}{s+2} + e^{-2s}\left(\frac{1}{4s} - \frac{1}{2(s+2)^2} - \frac{1}{4(s+2)}\right)\right]$$

$$= (2t+1)e^{-2t} + (1/4)\left[1 - (2(t-2)+1)e^{-2(t-2)}\right]H(t-2)$$

$$= (2t+1)e^{-2t} + (1/4)\left(1 - (2t-4+1)e^{-2t}e^{4}\right)H(t-2)$$

$$= (2t+1)e^{-2t} + (1/4)\left(1 - (2t-3)e^{4}e^{-2t}\right)H(t-2)$$

EXERCISES

(17) Use Laplace transforms to follow the following initial value problem

a) $\begin{cases} y'(t) - a y(t) = 0 \\ y(0) = b \end{cases}$ with $a, b \in \mathbb{R}$

b) $\begin{cases} y''(t) - 5y'(t) + 6y(t) = e^{-t} \\ y(0) = 1 \wedge y'(0) = 3 \end{cases}$

c) $\begin{cases} y''(t) + y(t) = \sin^2 t \\ y(0) = 1 \wedge y'(0) = 0 \end{cases}$

d) $\begin{cases} y''(t) + 4y'(t) + 3y(t) = H(t-1) \\ y(0^-) = 0 \wedge y'(0^-) = 0 \end{cases}$

e) $\begin{cases} y''(t) + 7y'(t) + 12y(t) = H(t-1) + H(t-2) \\ y(0^-) = 0 \wedge y'(0^-) = 1 \end{cases}$

f) $\begin{cases} y''(t) + 6y'(t) + 9y(t) = t \\ y(0) = 0 \wedge y'(0) = 0 \end{cases}$

g) $\begin{cases} y''(t) + 10y'(t) + 25y(t) = \delta(t) \\ y(0^-) = 0 \wedge y'(0^-) = 0 \end{cases}$

h) $\begin{cases} y''(t) + 11y'(t) + 30y(t) = \delta(t) + \delta'(t-3) \\ y(0^-) = 1 \wedge y'(0^-) = 0 \end{cases}$

i) $\begin{cases} y'''(t) - 2y''(t) + y'(t) - 2y(t) = \delta''(t-a) \\ y(0^-) = 0 \wedge y'(0^-) = 0 \wedge y''(0^-) = 1 \end{cases}$ with $a > 0$

j) $\begin{cases} y'''(t) + 3y''(t) + 3y'(t) + y(t) = \delta''(t) + H(t-1) \\ y(0^-) = 0 \wedge y'(0^-) = 1 \wedge y''(0^-) = 1 \end{cases}$

k) $\begin{cases} y''''(t) - a^4 y(t) = \delta'(t) + \delta'(t-3) \\ y(0^-) = 1 \wedge y'(0^-) = y''(0^-) = y'''(0^-) = 0 \end{cases}$

k) $\begin{cases} y''''(t) - a^4 y(t) = \delta'(t) + \delta'(t-3) \\ y(0^-) = 1 \wedge y'(0^-) = y''(0^-) = y'''(0^-) = 0 \end{cases}$

l) $\begin{cases} y''''(t) - a^4 y(t) = \delta''(t) + H(t-2) \\ y(0^-) = y'(0^-) = y''(0^-) = 0 \wedge y'''(0^-) = 1 \end{cases}$

Systems of linear ODEs

c) Solve the linear system

$$\begin{cases} x'(t) = 2x(t) - 3y(t) \\ y'(t) = y(t) - 2x(t) \end{cases}$$

with $x(0) = 8 \wedge y(0) = 3$.

Solution

Let $X(s) = \mathcal{L}(x(t))$ and $Y(s) = \mathcal{L}(y(t))$. Then:

$$\begin{cases} x'(t) = 2x(t) - 3y(t) \\ y'(t) = y(t) - 2x(t) \end{cases} \Leftrightarrow \begin{cases} sX(s) - x(0) = 2X(s) - 3Y(s) \\ sY(s) - y(0) = Y(s) - 2X(s) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} sX(s) - 8 - 2X(s) + 3Y(s) = 0 \\ sY(s) - 3 - Y(s) + 2X(s) = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (s-2)X(s) + 3Y(s) = 8 \\ (s-1)Y(s) + 2X(s) = 3 \end{cases} \Leftrightarrow \begin{cases} (s-2)X(s) + 3Y(s) = 8 \\ 2X(s) + (s-1)Y(s) = 3 \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} s-2 & 3 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} s-2 & 3 \\ 2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 3 \end{bmatrix} =$$

$$= \frac{1}{(s-2)(s-1) - 3\cdot 2} \begin{bmatrix} s-1 & -3 \\ -2 & s-2 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$= \frac{1}{s^2 - 3s + 2 - 6} \begin{bmatrix} (s-1)8 - 3\cdot 3 \\ -2\cdot 8 + 3(s-2) \end{bmatrix}$$

$$= \frac{1}{s^2 - 3s - 4} \begin{bmatrix} 8s - 8 - 9 \\ 3s - 6 - 16 \end{bmatrix} = \frac{1}{s^2 - 3s - 4} \begin{bmatrix} 8s - 17 \\ 3s - 22 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s-4)} \begin{bmatrix} 8s - 17 \\ 3s - 22 \end{bmatrix}$$

$$\Leftrightarrow X(s) = \frac{8s-17}{(s+1)(s-4)} \wedge Y(s) = \frac{3s-22}{(s+1)(s-4)}.$$

With partial fraction decomposition, we have:

$$X(s) = \frac{8s-17}{(s+1)(s-4)} = \frac{A}{s+1} + \frac{B}{s-4}$$

$$Y(s) = \frac{3s-22}{(s+1)(s-4)} = \frac{C}{s+1} + \frac{D}{s-4}$$

with

$$A = \left.\frac{8s-17}{s-4}\right|_{s=-1} = \frac{8(-1)-17}{(-1)-4} = \frac{-8-17}{-1-4} = \frac{-25}{-5} = 5$$

$$B = \left.\frac{8s-17}{s+1}\right|_{s=4} = \frac{8\cdot 4-17}{4+1} = \frac{32-17}{5} = \frac{15}{5} = 3$$

$$C = \left.\frac{3s-22}{s-4}\right|_{s=-1} = \frac{3(-1)-22}{(-1)-4} = \frac{-3-22}{-5} = \frac{-25}{-5} = 5$$

$$D = \left.\frac{3s-22}{s+1}\right|_{s=4} = \frac{3\cdot 4-22}{4+1} = \frac{12-22}{5} = \frac{-10}{5} = -2$$

and therefore:

$$\begin{cases} X(s) = \dfrac{5}{s+1} + \dfrac{3}{s-4} \\ Y(s) = \dfrac{5}{s+1} + \dfrac{-2}{s-4} \end{cases} \Leftrightarrow \begin{cases} x(t) = 5e^{-t} + 3e^{4t} \\ y(t) = 5e^{-t} - 2e^{4t} \end{cases}$$

EXERCISES

(18) Use Laplace transforms to solve the following systems of ordinary differential equations

(a) $\begin{cases} x'(t) = x(t) - a y(t) \\ y'(t) = a x(t) + y(t) \\ x(0) = 1 \wedge y(0) = 0 \end{cases}$

b) $\begin{cases} x''(t) + y'(t) + 3x(t) = e^{-t} \\ y''(t) - x'(t) + 2y(t) = \cos(3t) \\ x(0) = 1 \wedge x'(0) = 0 \wedge y(0) = 0 \wedge y'(0) = 1 \end{cases}$

c) $\begin{cases} x'(t) - y(t) = \delta'(t) \\ y'(t) - x(t) = \delta(t-2) \\ x(0^-) = x'(0^-) = y(0^-) = y'(0^-) = 0 \end{cases}$

d) $\begin{cases} x''(t) + y(t) = \delta(t) \\ x''(t) - y'(t) = H(t-1) \\ x(0^-) = 1 \wedge x'(0^-) = 0 \wedge y(0^-) = 1 \wedge y'(0^-) = 0 \end{cases}$

(19) Linear damped oscillator.

Consider the linear damped oscillator governed by the following initial value problem

$\begin{cases} m x''(t) + b x'(t) + K x(t) = 0 \\ x(0) = x_0 \wedge x'(0) = u_0 \end{cases}$

a) Show that the Laplace transform $X(s) = \mathcal{L}(x(t))$ of the unique solution to the initial value problem is given by

$$X(s) = \frac{(s+a)x_0}{(s+a)^2 + (\omega^2 - a^2)} + \frac{u_0 + a x_0}{(s+a)^2 + (\omega^2 - a^2)}$$

with $a = b/(2m)$ and $\omega = k/m$.

b) Show that the solution $x(t)$ is given according to the following 3 cases:

Case I: If $\omega^2 - a^2 > 0$, (damped oscillatory case)

$$x(t) = x_0 e^{-at} \cos\left(t\sqrt{\omega^2 - a^2}\right) + \frac{u_0 + a x_0}{\sqrt{\omega^2 - a^2}} e^{-at} \sin\left(t\sqrt{\omega^2 - a^2}\right)$$

Case II: If $\omega^2 - a^2 = 0$, (critically damped case)

$$x(t) = x_0 e^{-at} + (u_0 + a x_0) t e^{-at}$$

Case III: If $\omega^2 - a^2 < 0$, (overdamped case)

$$x(t) = x_0 \cosh\left(t\sqrt{a^2 - \omega^2}\right) + \frac{u_0 + a x_0}{\sqrt{a^2 - \omega^2}} \sinh\left(t\sqrt{a^2 - \omega^2}\right)$$

(20) Consider the initial value problem

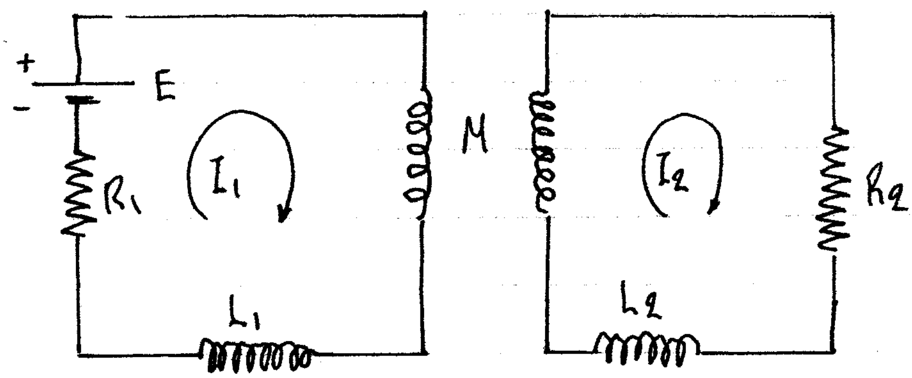
$$\begin{cases} x''(t) + k_1^2 y(t) = 0 \\ y''(t) + k_2^2 x(t) = 0 \\ x(0) = a \wedge y(0) = b \wedge x'(0) = y'(0) = 0 \end{cases}$$

Use Laplace transforms to show that $x(t)$ and $y(t)$ are given by

$$\begin{cases} x(t) = \left(\dfrac{a k_2 + b k_1}{2k_2}\right) \cos\left(t\sqrt{k_1 k_2}\right) + \left(\dfrac{a k_2 - b k_1}{2k_2}\right) \cosh\left(t\sqrt{k_1 k_2}\right) \\ y(t) = \left(\dfrac{a k_2 + b k_1}{2k_1}\right) \cos\left(t\sqrt{k_1 k_2}\right) - \left(\dfrac{a k_2 - b k_1}{2k_1}\right) \cosh\left(t\sqrt{k_1 k_2}\right) \end{cases}$$

(21) Inductively coupled circuits.

We consider two inductively coupled circuits of the form:



The currents satisfy the following system of differential equations:

$$\begin{cases} L_1 \dfrac{dI_1}{dt} + R_1 I_1 + M \dfrac{dI_2}{dt} = E \\ L_2 \dfrac{dI_2}{dt} + R_2 I_2 + M \dfrac{dI_1}{dt} = 0 \end{cases}$$

a) Using initial condition $I_1(0) = I_2(0) = 0$, show, using Laplace transforms, that $I_1(t)$ and $I_2(t)$ will satisfy

$$I_1(t) = \frac{EL_2}{L_1L_2 - M^2} \frac{e^{a_1 t} - e^{a_2 t}}{a_1 - a_2} + \frac{ER_2}{a_1 - a_2}\left(\frac{e^{a_1 t}}{a_1} - \frac{e^{a_2 t}}{a_2}\right) + \frac{E}{R_1}$$

$$I_2(t) = \frac{EM}{L_1L_2 - M^2} \frac{e^{a_1 t} - e^{a_2 t}}{a_2 - a_1}$$

where a_1, a_2 are the roots of the equation

$(L_1L_2 - M^2)a^2 + (L_1R_2 + L_2R_1)a + R_1R_2 = 0$

b) What happens when $L_1L_2 = M^2$?

Laplace transform of a convolution

Def: Let $f, g \in PC(\mathbb{R}_+)$ be two piecewise-continuous functions. We define the convolution $f * g$ as:

$$\forall t \in [0, +\infty) : (f * g)(t) = \int_0^t f(\tau) g(t - \tau) \, d\tau$$

Remark: It can be shown that convolution satisfies the associative and commutative properties:

$$\forall f, g \in PC(\mathbb{R}_+) : f * g = g * f$$
$$\forall f, g, h \in PC(\mathbb{R}_+) : f * (g * h) = (f * g) * h$$

↳ The Laplace transform of a convolution is given by the following theorem:

Thm:

$$\left.\begin{array}{l} f, g \in PC(\mathbb{R}_+) \cap E_\gamma(\mathbb{R}_+) \\ F(s) = \mathcal{L}(f(t)) \\ G(s) = \mathcal{L}(g(t)) \end{array}\right\} \Rightarrow \mathcal{L}((f * g)(t)) = F(s) G(s)$$

Methodology: The convolution theorem can help with

(a) Inverse Laplace transforms of products.

(b) ODEs with general forcing term

(c) Integral and integrodifferential equations.

EXAMPLES

a) Solve the initial value problem

$$\begin{cases} y''(t) + w^2 y(t) = f(t) \\ y(0) = y_0 \wedge y'(0) = y_1 \end{cases}$$

using Laplace transforms, with $w \in (0, +\infty)$ and $y_0, y_1 \in \mathbb{R}$.

<u>Solution</u>

Define $Y(s) = \mathcal{L}(y(t))$, and note that

$$\mathcal{L}[y''(t) + w^2 y(t)] = s^2 Y(s) - s\,y(0) - y'(0) + w^2 Y(s)$$
$$= (s^2 + w^2) Y(s) - y_0 s - y_1 .$$

It follows that, with the definition $F(s) = \mathcal{L}(f(t))$,

$$y''(t) + w^2 y(t) = f(t) \Leftrightarrow (s^2+w^2) Y(s) - y_0 s - y_1 = F(s).$$
$$\Leftrightarrow (s^2+w^2) Y(s) = y_0 s + y_1 + F(s) \Leftrightarrow$$
$$\Leftrightarrow Y(s) = \frac{y_0 s + y_1 + F(s)}{s^2+w^2} = y_0 \frac{s}{s^2+w^2} + \frac{y_1}{w} \frac{w}{s^2+w^2} + \frac{1}{w} \underbrace{\frac{wF(s)}{s^2+w^2}} \quad (1)$$

Since $\mathcal{L}^{-1}\left(\frac{s}{s^2+w^2}\right) = \cos(wt)$ and

$$\mathcal{L}^{-1}\left(\frac{w}{s^2+w^2}\right) = \sin(wt)$$

$$\text{Eq.}(1) \Leftrightarrow y(t) = y_0 \cos(wt) + (y_1/w) \sin(wt) + \frac{1}{w} \underbrace{\int_0^t d\tau\, f(\tau) \sin(w(t-\tau))}$$

b) Evaluate the following inverse Laplace transform:

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] \quad \text{with } a>0.$$

Solution

We note that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] = \frac{1}{a}\sin(at) \equiv f(t) \Rightarrow$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = (f*f)(t) = \int_0^t f(\tau)f(t-\tau)\,d\tau =$$

$$= \int_0^t \left[\frac{\sin(a\tau)}{a}\right]\left[\frac{\sin(a(t-\tau))}{a}\right] d\tau$$

$$= \frac{1}{a^2}\int_0^t \sin(a\tau)\sin(at-a\tau)\,d\tau$$

$$= \frac{1}{a^2}\int_0^t (1/2)\left[\cos(a\tau-(at-a\tau)) - \cos(a\tau+(at-a\tau))\right] d\tau$$

$$= \frac{1}{a^2}\int_0^t (1/2)\left[\cos(a\tau-at+a\tau) - \cos(a\tau+at-a\tau)\right] d\tau$$

$$= \frac{1}{2a^2}\int_0^t \left[\cos(2a\tau-at) - \cos(at)\right] d\tau$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2a\tau-at)}{2a} - \tau\cos(at)\right]_{\tau=0}^{\tau=t} =$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2at-at)}{2a} - t\cos(at)\right] - \frac{1}{2a^2}\left[\frac{\sin(0-at)}{2a} - 0\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin(at)}{2a} - t\cos(at)\right] - \frac{1}{2a^2}\left[\frac{-\sin(at)}{2a}\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin(at)}{2a} - t\cos(at) + \frac{\sin(at)}{2a}\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin(at)}{a} - t\cos(at)\right] = \frac{\sin(at) - at\cos(at)}{2a^3}$$

c) Solve the integral equation

$$\int_0^t \frac{y(a)}{\sqrt{t-a}}\,da = t^n$$

with $n \in \mathbb{N}^*$.

Solution

Define $Y(s) = \mathcal{L}(y(t))$. We note that

$$\mathcal{L}\left(\int_0^t \frac{y(a)}{\sqrt{t-a}}\,da\right) = \mathcal{L}\left(y(t) * (1/\sqrt{t})\right) = \mathcal{L}(y(t))\,\mathcal{L}(t^{-1/2})$$

$$= Y(s)\,\frac{\Gamma(-1/2+1)}{s^{-1/2+1}} = \Gamma(1/2)\,\frac{Y(s)}{s^{1/2}} =$$

$$= \frac{Y(s)\sqrt{\pi}}{s^{1/2}}$$

and $\mathcal{L}(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}} = \dfrac{n!}{s^{n+1}}$ (because $n \in \mathbb{N}^*$)

and therefore,

$$\int_0^t \frac{y(a)}{\sqrt{t-a}}\,da = t^n \Leftrightarrow \frac{Y(s)\sqrt{\pi}}{s^{1/2}} = \frac{n!}{s^{n+1}} \Leftrightarrow$$

$$\Leftrightarrow Y(s) = \frac{(n!)\,s^{1/2}}{s^{n+1}\sqrt{\pi}} = \frac{(n!)}{s^{n+1/2}\sqrt{\pi}} = \frac{(n!)}{\Gamma(n+1/2)\sqrt{\pi}}\,\frac{\Gamma(n+1/2)}{s^{n+1/2}}$$

$$= \frac{(n!)}{\Gamma(n+1/2)\sqrt{\pi}}\,\frac{\Gamma(n-1/2+1)}{s^{n-1/2+1}} \Leftrightarrow$$

$$\Leftrightarrow y(t) = \mathcal{L}^{-1}\left[\frac{n!}{\Gamma(n+1/2)\sqrt{\pi}}\,\frac{\Gamma(n-1/2+1)}{s^{n-1/2+1}}\right] = \frac{n!}{\Gamma(n+1/2)\sqrt{\pi}}\,\mathcal{L}^{-1}\left[\frac{\Gamma(n-1/2+1)}{s^{n-1/2+1}}\right]$$

$$= \frac{n!}{\Gamma(n+1/2)\sqrt{\pi}}\,t^{n-1/2} = \frac{n!\,t^{n-1}\sqrt{t}}{\Gamma(n+1/2)\sqrt{\pi}}$$

We simplify further noting that

$$\Gamma(n+1/2) = \Gamma(n+1-1/2) = \Gamma(1/2)\prod_{k=1}^{n}(k-1/2) =$$

$$= \sqrt{\pi}\prod_{k=1}^{n}\frac{2k-1}{2} = \frac{\sqrt{\pi}}{2^n}\prod_{k=1}^{n}(2k-1) =$$

$$= \frac{\sqrt{\pi}\,(2n-1)!!}{2^n}$$

and therefore, we have:

$$y(t) = \frac{n!\,t^{n-1}\sqrt{t}}{\Gamma(n+1/2)\sqrt{\pi}} = \frac{n!\,t^{n-1}\sqrt{t}}{\left[\dfrac{\sqrt{\pi}\,(2n-1)!!}{2^n}\right]\sqrt{\pi}} =$$

$$= \frac{2^n\,n!}{\pi\,(2n-1)!!}\,t^{n-1}\sqrt{t}$$

EXERCISES

(22) Use the convolution theorem to evaluate the following inverse Laplace transforms:

a) $$\mathcal{L}^{-1}\left[\frac{s}{(s^2-a^2)(s-b)}\right] = \frac{1}{2}\left[\frac{e^{at}}{a-b} - \frac{e^{-at}}{a+b} - \frac{2b\,e^{bt}}{a^2-b^2}\right]$$

b) $$\mathcal{L}^{-1}\left[\frac{1}{(s-1)\sqrt{s}}\right] = e^{x}\,\mathrm{erf}(\sqrt{x})$$

c) $$\mathcal{L}^{-1}\left[\frac{s\exp(-\pi s/2)}{(s^2+1)(s^2+9)}\right] = \frac{H(t-\pi/2)}{8}\left[\sin(3t)+\sin t\right]$$

d) $$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{\sin(at)+at\cos(at)}{2a}$$

e) $$\mathcal{L}^{-1}\left[\frac{s}{(s^2-a^2)^2}\right] = \frac{t\sinh(at)}{2a}$$

(23) Use the Laplace transform in conjunction with the convolution theorem to solve the following initial value problems or integrodifferential equations under the forcing functions $f(t)$ and $g(t)$ (whenever it applies).

a) $$\begin{cases} x'(t) = x(t)+y(t)+f(t) \\ y'(t) = x(t)-y(t) \\ x(0)=y(0)=0 \end{cases}$$

b) $$\begin{cases} x'(t) = 2x(t)-y(t)+f(t) \\ y'(t) = x(t)-y(t)+g(t) \\ x(0)=y(0)=1. \end{cases}$$

c) $$x(t)+\int_0^t (t-a)\,x(a)\,da = f(t)$$

d) $x(t) = f(t) - \int_0^t \sin(a)\, x(t-a)\, da$

e) $x(t) + \int_0^t x(a)\, da = 1$

f) $x(t) = \cos t + \int_0^t e^{-a} x(t-a)\, da$

g) $\begin{cases} x'(t) = 1 - \sin t - \int_0^t x(a)\, da \\ x(0) = 0 \end{cases}$

h) $x(t) = f(t) + \int_0^t a\, x(t-a)\, da$

i) $\int_0^t x(a) x(t-a)\, da = 2x(t) + t - 2$

j) $\int_0^t x(a) \sin(t-a)\, da = x(t)$

24 Show that the integrodifferential equation

$$\int_0^t \frac{y(a)}{(t-a)^n}\, da = f(t)$$

with $0 < a < 1$ and $f(0) = 0$ has the solution

$$y(t) = \frac{\sin(n\pi)}{\pi} \int_0^t f'(a)(t-a)^{n-1}\, da$$