INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

I Definitions

- An ordinary differential equation (ODE) is an equation that contains one or more derivatives of the unknown function. A function that satisfies the equation is called a solution of the ODE.
- The most general form of an ODE is:

$$\boxed{F(x, y(x), y'(x), y''(x), \dots, y^{(n)}(x)) = 0} \qquad (1)$$

with $F: \mathbb{R} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$.
If we define $Y(x) = (y(x), y'(x), y''(x), \dots, y^{(n)}(x))$, then the equation above can be rewritten as:

$$\boxed{F(x, Y(x)) = 0} \qquad (2)$$

- The natural number n is the order of the ODE.

II Linear vs. nonlinear ODEs

Let V be the set of all continuous functions $Y: \mathbb{R} \to \mathbb{R}^n$.
We say that the ODE $F(x, Y(x)) = 0$ is linear if and only

if F satisfies

$$\forall x, \lambda, \mu \in \mathbb{R} : \forall Y, Z \in V : F(x, \lambda Y + \mu Z) = \lambda F(x, Y) + \mu F(x, Z)$$

otherwise we say that the ODE is <u>nonlinear.</u>

- It can be shown that the most general form of a linear ODE is:

$$p_n(x) y^{(n)}(x) + \cdots + p_2(x) y''(x) + p_1(x) y'(x) + p_0(x) y(x) = q(x)$$

- <u>Types of ODE problems</u>

We distinguish between the following types of ODE problems:

(1) → <u>Initial Value Problem</u>

These are problems of the form:

$$\boxed{\begin{cases} F(x, y(x), y'(x), \ldots, y^{(n-1)}(x), y^{(n)}(x)) = 0 \\ y(x_0) = a_0 \wedge y'(x_0) = a_1 \wedge \cdots \wedge y^{(n-1)}(x_0) = a_{n-1} \end{cases}}$$

where $y, y', y'', \ldots, y^{(n-1)}$ are all fixed at the same point $x_0 \in \mathbb{R}$. These additional equations are called <u>initial conditions.</u>

2 → Boundary Value Problem

These are problems of the form

$$\boxed{\begin{array}{l} F(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0 \\ y^{(k_1)}(x_1) = a_1 \wedge y^{(k_2)}(x_2) = a_2 \wedge \cdots \wedge y^{(k_n)}(x_n) = a_n \end{array}}$$

where $y^{(k_1)}, y^{(k_2)}, \ldots, y^{(k_n)}$ are specified on more than just a unique point. These additional equations are called Boundary conditions.

- **Techniques for solving ODEs**

Solution techniques are classified under the following categories.

a) Exact analytic methods: We obtain an exact solution in closed form.

b) Approximate methods: We obtain an approximate solution in closed form.

   i) Local methods: We obtain an approximate solution which is good in a neighborhood of some special point.

   ii) Global methods: Obtain an approximate solution which is good on the entire domain of the ODE.

c) Numerical methods: We obtain an approximate discretized solution with the use of a computer.

d) Existence/Uniqueness: We prove rigorously that a given ODE problem has a unique solution, without actually being able to find the solution exactly or approximately.

Systems of ODEs

- A system of m ODEs is any problem of the form

$$\boxed{\begin{cases} F_1(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0 \\ F_2(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0 \\ \vdots \\ F_m(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0 \end{cases}}$$

where we require the logical conjunction of all equations.

- Every n^{th}-order ODE of the form

$$y^{(n+1)} = F(x, y(x), y'(x), \ldots, y^{(n)}(x))$$

can be rewritten as a system of 1st-order equations.

$$\begin{cases} y_0' = y_1 \\ y_1' = y_2 \\ \vdots \\ y_{n-1}' = y_n \\ y_n' = F(x, y_0, y_1, y_2, \ldots, y_n) \end{cases}$$