FIRST-ORDER ODEs

- A 1st-order ordinary differential equation (ODE) is an equation of the form $y' = f(x,y)$ satisfied by a function $y(x)$ of x. A corresponding 1st-order initial value problem is a problem of the form

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

with $x_0, y_0 \in \mathbb{R}$ given.

- An implicit solution to the initial value problem above is a solution of the form $F(x,y) = 0$ where we have shown that

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \Longleftrightarrow F(x,y) = 0$$

- An explicit solution to the initial value problem above is a solution of the form $y = g(x)$ such that

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \Longleftrightarrow y = g(x).$$

- There is no general solution method that can give an implicit or explicit solution to a 1st-order ODE. However, solution methods exist for some special cases, including the following:

(1) → Separable ODEs

These are problems of the form $\boxed{\begin{cases} y' = g(x)h(y) \\ y(x_0) = y_0 \end{cases}}$ (1)

Note that we say that

y_0 is a fixed point of Eq. (1) $\Leftrightarrow h(y_0) = 0$

If we initialize the system at a fixed point, then $y' = 0$, and we expect $y(x)$ to remain at the fixed point for all $x \in \mathbb{R}$. Furthermore, if we initialize at y_0 with $h(y_0) \neq 0$ then the solution cannot cross over any fixed point. We can therefore expect that $h(y(x)) \neq 0$ for all $x \in \mathbb{R}$ for which $y(x)$ can be obtained.

Methodology: Based on the above remarks we begin by assuming that $h(y) \neq 0$, and therefore:

$$y' = g(x)h(y) \Leftrightarrow \frac{y'}{h(y)} = g(x) \Leftrightarrow \int \frac{dy}{h(y)} = \int g(x)dx \Leftrightarrow$$

$$\Leftrightarrow H(y) = G(x) + c_1$$

To determine c we use the initial condition $y(x_0) = y_0$:

$H(y_0) = G(x_0) + c_1 \Leftrightarrow c_1 = H(y_0) - G(x_0)$.

Note that in the above argument we assume that the system has not been initialized at a fixed point. If the goal is to find a general solution, then it is necessary to explore whether the general solution continuous to hold when y_0 is a fixed point.

EXAMPLES

a) Solve the initial value problem

$$\begin{cases} y'(x) = (1+y^2(x))\cos x \\ y(0) = 1 \end{cases}$$

Solution

Since $1+y^2 > 0$, then the system has no fixed points.
We note that

$$y' = (1+y^2)\cos x \Leftrightarrow \frac{y'}{1+y^2} = \cos x \Leftrightarrow \int \frac{dy}{1+y^2} = \int \cos x\, dx \quad (1)$$

with $\int \cos x\, dx = \sin x + c_1$ and $\int \frac{dy}{1+y^2} = \operatorname{Arctan}(y) + c_2$

thus

$$(1) \Leftrightarrow \operatorname{Arctan}(y) = \sin x + c \Leftrightarrow y = \tan(\sin x + c)$$

From the initial condition:

$$y(0) = 1 \Leftrightarrow \operatorname{Arctan}(1) = \sin 0 + c \Leftrightarrow$$
$$\Leftrightarrow c = \operatorname{Arctan}(1) = \pi/4$$

and therefore: $y(x) = \tan(\sin x + \pi/4)$.
We note that with increasing x, this solution becomes singular

when:

$$\sin x + \pi/4 = \pi/2 \Leftrightarrow \sin x = \pi/4 - \pi/2 \Leftrightarrow \sin x = \pi/4 \in [-1,1]$$
$$\Leftrightarrow x = \operatorname{Arcsin}(\pi/4).$$

- We say that the solution has a <u>finite-time singularity</u> at $x = \operatorname{Arcsin}(\pi/4)$.

b) Solve the initial value problem

$$\begin{cases} y' = y^2 \\ y(0) = y_0 \end{cases}$$

Solution

We note that $y=0$ is a fixed point. We assume that initially $y_0 \neq 0$. Then $y \neq 0$, and it follows that

$$y' = y^2 \Leftrightarrow \frac{y'}{y^2} = 1 \Leftrightarrow \int \frac{dy}{y^2} = \int dx \Leftrightarrow \frac{y^{-1}}{-1} = x + c$$

$$\Leftrightarrow y^{-1} = -x - c \Leftrightarrow y = \frac{1}{-x-c} = \frac{-1}{x+c}$$

Since $y(0) = y_0 \Leftrightarrow y_0^{-1} = -0 - c \Leftrightarrow c = -y_0^{-1} = \frac{-1}{y_0}$
it follows that

$$y = \frac{-1}{x+c} = \frac{-1}{x - y_0^{-1}} = \frac{-y_0}{y_0(x - y_0^{-1})} = \frac{-y_0}{y_0 x - 1}, \text{ with } y_0 \neq 0$$

For the fixed point initialization $y_0 = 0$, the above equation correctly gives $y = \frac{-0}{0x - 1} = 0$, therefore it is valid

for all $y_0 \in \mathbb{R}$.

The solution has a finite time singularity when
$y_0 x - 1 = 0 \Leftrightarrow y_0 x = 1 \Leftrightarrow x = 1/y_0$.

c) Solve the initial value problem

$$\begin{cases} y' = 2x(y-1) \\ y(1) = y_0 \end{cases}$$

Solution

We note that $y-1=0 \Leftrightarrow y=1$, so $y=1$ is the fixed point. We assume initialization $y_0 \neq 1$, thus $y \neq 1$. Then,

$$y' = 2x(y-1) \Leftrightarrow \frac{y'}{y-1} = 2x \Leftrightarrow \int \frac{dy}{y-1} = \int 2x dx$$

$$\Leftrightarrow \ln|y-1| = x^2 + c \qquad (1)$$

From the initial condition

$$y(1) = y_0 \Leftrightarrow \ln|y_0 - 1| = 1^2 + c \Leftrightarrow c = \ln|y_0-1| - 1$$

and therefore

$$\ln|y-1| = x^2 + \ln|y_0-1| - 1 \Leftrightarrow$$

$$\Leftrightarrow |y-1| = \exp(x^2 + \ln|y_0-1| - 1) = \exp(x^2-1)\exp(\ln|y_0-1|)$$

$$= |y_0-1|\exp(x^2-1) \Leftrightarrow$$

$$\Leftrightarrow y-1 = \pm|y_0-1|\exp(x^2-1) \qquad (2)$$

Since $y=1$ is a fixed point, for $y_0-1>0$ we will have $y-1>0$ and for $y_0-1<0$ we will have $y-1<0$. It follows that

$$(2) \Leftrightarrow y-1 = (y_0-1)\exp(x^2-1) \Leftrightarrow$$

$$\Leftrightarrow y = 1 + (y_0-1)\exp(x^2-1) \quad \text{for } y_0 \neq 1.$$

For $y_0=1$, the above solution gives $y=1$, so the general solution also works for $y_0=1$.

EXERCISES

① Solve the following initial value problems

a) $\begin{cases} y' = x^3/y \\ y(1) = y_0 \end{cases}$

b) $\begin{cases} (1+x^2)\, y' = y \\ y(0) = y_0 \end{cases}$

c) $\begin{cases} y' + y^2 \cos x = 0 \\ y(0) = y_0 \end{cases}$

d) $\begin{cases} (y+1)\, y' = x^2 - 4 \\ y(0) = y_0 \end{cases}$

e) $\begin{cases} xy' = \sqrt{1-y^2} \\ y(0) = y_0 \end{cases}$

f) $\begin{cases} e^{-x} y y' + x^2 = 0 \\ y(1) = y_0 \end{cases}$

g) $\begin{cases} y' = xy^3 (1+x^2)^{-1/2} \\ y(0) = y_0 \end{cases}$

h) $\begin{cases} y' = x^2 y \ln|x| \\ y(0) = y_0 \end{cases}$

i) $\begin{cases} y' = y^2 \operatorname{Arctan}(x) \\ y(0) = y_0 \end{cases}$

j) $\begin{cases} \cos(2x)\, y' + \sin y = 0 \\ y(\pi/2) = \pi/3 \end{cases}$

k) $\begin{cases} y' = \sqrt{y^2 + 3y + 2} \\ y(0) = y_0 \end{cases}$

l) $\begin{cases} e^{-x} y' = y^{-1} \cos(2x) \\ y(1) = y_0 \end{cases}$

m) $\begin{cases} dy/dt = y^2 - 4 \\ y(0) = y_0 \end{cases}$

↳ For the solution of the above ODEs it may be necessary to review techniques of integration from Calculus 2.

(2) Logistic Population Model

The logistic population model is intended to model population growth under finite resources. If $y(t)$ is the population at time t, λ is the population growth rate, and N is the carrying capacity, then according to the logistic model, $y(t)$ is governed by

$$dy/dt = \lambda y(N-y)$$

Using initial condition $y(0) = y_0$, show that

$$y(t) = \frac{Ny_0}{y_0 + (N-y_0)\exp(-\lambda N t)}$$

2 → Homogeneous ODEs

Def: A homogeneous ODE is an equation of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Solution method: Let $y(x) = xu(x)$. It follows that:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \Leftrightarrow x\frac{du}{dx} + u = f(u) \Leftrightarrow x\frac{du}{dx} = f(u) - u \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{f(u)-u}\frac{du}{dx} = \frac{1}{x} \Leftrightarrow \int \frac{d\tilde{u}}{f(\tilde{u}) - \tilde{u}} = \int \frac{dx}{x} \Leftrightarrow \text{etc...}$$

EXAMPLE

Solve $\frac{dy}{dx} = \frac{2xy + y^2}{x^2}$ with $y_0 = -1/2$ for $x_0 = 1$.

Solution

We note that

$$\frac{dy}{dx} = \frac{2xy+y^2}{x^2} = \frac{2xy}{x^2} + \frac{y^2}{x^2} = 2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \quad (1)$$

Let $y = xu \Rightarrow u = y/x$. It follows that

$$(1) \Leftrightarrow x\frac{du}{dx} + u = 2u + u^2 \Leftrightarrow x\frac{du}{dx} = u^2 + 2u - u \Leftrightarrow$$

$$\Leftrightarrow x\frac{du}{dx} = u(u+1) \Leftrightarrow \frac{1}{u(u+1)}\frac{du}{dx} = \frac{1}{x} \Leftrightarrow$$

$$\Leftrightarrow \int \frac{du}{u(u+1)} = \int \frac{dx}{x} \quad (2)$$

Since $\dfrac{1}{u(u+1)} = \dfrac{A}{u} + \dfrac{B}{u+1}$ with

$$A = \left.\frac{1}{u+1}\right|_{u=0} = \frac{1}{0+1} = 1, \text{ and}$$

$$B = \left.\frac{1}{u}\right|_{u=-1} = \frac{1}{-1} = -1$$

it follows that

$$\int \frac{du}{u(u+1)} = \int \left(\frac{1}{u} - \frac{1}{u+1}\right) du = \ln|u| - \ln|u+1| + C_1$$

$$= \ln\left|\frac{u}{u+1}\right| + C_1$$

and

$$\int \frac{dx}{x} = \ln|x| + c_2$$

and therefore

$$(2) \Leftrightarrow \ln\left|\frac{u}{u+1}\right| = \ln|x| + C \qquad (3)$$

Apply the initial condition:

$$y(1) = -1/2 \Leftrightarrow u(1) = y(1)/1 = -1/2 \Leftrightarrow$$

$$\Leftrightarrow \ln\left|\frac{-1/2}{-1/2+1}\right| = \ln|1| + C \Leftrightarrow$$

$$\Leftrightarrow c = \ln\left|\frac{-1/2}{-1/2+1}\right| = \ln\left|\frac{-1}{-1+2}\right| = \ln|-1| = 0$$

and therefore:

$$(3) \Leftrightarrow \ln\left|\frac{u}{u+1}\right| = \ln|x| \Leftrightarrow \left|\frac{u}{u+1}\right| = |x| \Leftrightarrow$$

$$\Leftrightarrow \frac{u}{u+1} = x \quad \vee \quad \frac{u}{u+1} = -x \qquad (4)$$

From the initial condition $u(1) = -1/2$ we note that $\frac{u}{u+1} < 0$ and $x > 0$, and therefore we reject the first equation on (4) and have:

$$(4) \Leftrightarrow \frac{u}{u+1} = -x \Leftrightarrow u = -x(u+1) \Leftrightarrow u = -xu - x \Leftrightarrow$$

$$\Leftrightarrow (1+x)u = -x \Leftrightarrow u = \frac{-x}{1+x} \Leftrightarrow \frac{y}{x} = \frac{-x}{x+1}$$

$$\Leftrightarrow y = \frac{-x^2}{x+1}$$

EXERCISES

(3) Solve the following homogeneous ODEs using initial condition $y(1) = y_0$.

a) $3xy' + y = x$ b) $(x-2y)y' = x+y$

c) $(x+3y)y' = 3x+y$ d) $x^2y' = y(x+y)$

e) $xy^2y' = y^3 - x^3$ f) $(x^2+y^2)y' = xy$

g) $xy' + y\sqrt{x^2-y^2} = 0$ h) $y'\sqrt{x} = -\sqrt{x+y}$

(4) Consider an ordinary differential equation of the form $M(x,y) + N(x,y)\,y' = 0$ such that

$$\forall \lambda \in (0,+\infty): \begin{cases} M(\lambda x, \lambda y) = \lambda^a M(x,y) \\ N(\lambda x, \lambda y) = \lambda^a N(x,y) \end{cases}$$

with $a \in \mathbb{R}$.

a) Show that this ODE is homogeneous by reducing it to the form

$$\frac{dy}{dx} = \frac{-M(1, y/x)}{N(1, y/x)}$$

b) Show that the substitution $u = y/x$ reduces this ODE to the separable form:

$$\frac{1}{x} + \frac{N(1,u)}{M(1,u) + uN(1,u)}\frac{du}{dx} = 0$$

(3) Integrating Factors Method

This method can be applied to ODEs of the form:

$$\boxed{y' + f(x)y = g(x)}$$

with f, g continuous on $\mathbb{R}$.

Solution method

Define $h(x) = \exp\left(\int f(x)\,dx\right)$ and note that $h'(x) = f(x)h(x)$.
Then we multiply both sides of the ODE with $h(x)$:

$$y' + f(x)y = g(x) \Leftrightarrow y'h(x) + h(x)f(x)y = g(x)h(x) \Leftrightarrow$$

$$\Leftrightarrow y'h(x) + h'(x)\,y = g(x)h(x) \Leftrightarrow$$

$$\Leftrightarrow \frac{d}{dx}\left[y h(x)\right] = h(x)g(x) \Leftrightarrow$$

$$\Leftrightarrow h(x)y = \int h(x)g(x)\,dx + C_1$$

$$\Leftrightarrow y = \frac{1}{h(x)}\int h(x)g(x)\,dx + \frac{C_1}{h(x)} \qquad (1)$$

Note that for $g(x) = 0$, the above solution simplifies to

$$y = \frac{C_1}{h(x)} = C_1 \exp\left(-\int f(x)\,dx\right)$$

This is called the homogeneous term to Eq. (1).
The integral term is called the particular term.

EXAMPLE

a) Solve the ODE $y' + xy = x^2$ with $y(0) = y_0$.

Solution

Use the integrating factor

$$h(x) = \exp\left(\int x\,dx\right) = \exp(x^2/2) \Rightarrow h'(x) = x\,h(x)$$

and therefore:

$$y' + xy = x^2 \Leftrightarrow y'h(x) + xh(x)y = x^2 h(x) \Leftrightarrow y'h(x) + h'(x)y = x^2 h(x) \Leftrightarrow$$

$$\Leftrightarrow [y\,h(x)]' = x^2 h(x) \Leftrightarrow yh(x) = c + \int_0^x t^2 h(t)\,dt \qquad (1)$$

For $x = 0$: $y_0 h(0) = c + 0 \Leftrightarrow c = y_0 h(0) = y_0 \exp(0) = y_0$

and therefore,

$$(1) \Leftrightarrow yh(x) = y_0 + \int_0^x t^2 h(t)\,dt \Leftrightarrow$$

$$\Leftrightarrow y = \frac{y_0}{h(x)} + \frac{1}{h(x)} \int_0^x t^2 h(t)\,dt =$$

$$= \frac{y_0}{\exp(x^2/2)} + \frac{1}{\exp(x^2/2)} \int_0^x t^2 \exp(t^2/2)\,dt =$$

$$= y_0 \exp(-x^2/2) + \exp(-x^2/2) \int_0^x t^2 \exp(t^2/2)\,dt$$

$\looparrowright$ The integrating factor method can be applied to the more general problem of the form

$$\boxed{f(x)y' + g(x)y = h(x)}$$

However, if $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then x_0 is a singular point of the ODE and the ODE will only yield a unique solution if x is restricted to an interval between neighboring singular points.

EXAMPLE

Solve the ODE $(x^2-1)y' + xy = 0$ with $y(x_0) = y_0$.

Solution

We have

$$(x^2-1)y' + xy = 0 \Leftrightarrow y' + \frac{x}{x^2-1}y = 0 \qquad (1)$$

We will use the integrating factor

$$h(x) = \exp\left(\int \frac{x}{x^2-1}dx\right) = \exp\left(\frac{1}{2}\int \frac{(x^2-1)'}{x^2-1}dx\right) =$$

$$= \exp\left(\frac{1}{2}\ln|x^2-1|\right) = \exp\left(\ln\sqrt{|x^2-1|}\right) =$$

$$= \sqrt{|x^2-1|}$$

$\Rightarrow h'(x) = h(x)\dfrac{x}{x^2-1}$. It follows that

$$(1) \Leftrightarrow y'h(x) + \frac{x}{x^2-1}h(x)y = 0 \Leftrightarrow y'h(x) + yh'(x) = 0$$

$$\Leftrightarrow (d/dx)[yh(x)] = 0 \Leftrightarrow (d/dx)\left[y\sqrt{|x^2-1|}\right] = 0$$

$$\Leftrightarrow y\sqrt{|x^2-1|} = c_1 \Leftrightarrow y = \frac{c_1}{\sqrt{|x^2-1|}}$$

We note that the ODE has singular points on $x=1$ and $x=-1$. From the initial condition:

$$y(x_0) = y_0 \Leftrightarrow \frac{c_1}{\sqrt{|x_0^2-1|}} = y_0 \Leftrightarrow c = y_0\sqrt{|x_0^2-1|}$$

and therefore:

$$y = \frac{y_0\sqrt{|x_0^2-1|}}{\sqrt{|x^2-1|}}$$

We distinguish between the following cases:

<u>Case 1</u> : If $x_0 \in (-\infty,-1)$, then $|x_0^2-1| = x_0^2-1$ and

$$y = \frac{y_0\sqrt{x_0^2-1}}{\sqrt{x^2-1}}, \quad \forall x \in (-\infty,-1)$$

<u>Case 2</u> : If $x_0 \in (-1,1)$, then $|x_0^2-1| = 1-x_0^2$ and

$$y = \frac{y_0\sqrt{1-x_0^2}}{\sqrt{1-x^2}}, \quad \forall x \in (-1,1)$$

<u>Case 3</u> : If $x_0 \in (1,+\infty)$, then $|x_0^2-1| = x_0^2-1$ and

$$y = \frac{y_0\sqrt{x_0^2-1}}{\sqrt{x^2-1}}, \quad \forall x \in (1,+\infty).$$

EXERCISES

(5) Solve the following initial value problems.

a) $\begin{cases} y' - 2y = x e^{-2x} \\ y(0) = y_0 \end{cases}$

b) $\begin{cases} xy' - 2y = x^4 \\ y(1) = y_0 \end{cases}$

c) $\begin{cases} y' + y \tan x = \sin(2x) \\ y(0) = y_0 \end{cases}$

d) $\begin{cases} y' - (\cot x)\, y = 3x \sin x \\ y(\pi/4) = y_0 \end{cases}$

e) $\begin{cases} xy' + y = 3x^3 - 1 \\ y(1) = y_0 \end{cases}$

f) $\begin{cases} y' + e^x y = 2e^x \\ y(0) = y_0 \end{cases}$

g) $\begin{cases} y' + 2xy = x \exp(-x^2) \\ y(0) = y_0 \end{cases}$

(6) Consider the initial value problem

$$\begin{cases} y' - 2xy = 1 \\ y(0) = y_0 \end{cases}$$

Show that its unique solution is:

$$y(x) = \exp(x^2) \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}(x) + y_0 \right]$$

with $\operatorname{erf}(t)$ the error function defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt$$

⑦ Bernoulli equations

A Bernoulli ordinary differential equation is an equation of the form

$$y' + p(x)y = q(x)y^n$$

with $n \in \mathbb{R}$.

a) Show that the substitution $u = y^{1-n}$ reduces the Bernoulli equation to a linear ordinary differential equation of the form

$$u' + (1-n)p(x)u = (1-n)q(x)$$

b) Use this substitution to solve the following Bernoulli initial value problem.

$$\begin{cases} y' + xy = xy^2 \\ y(0) = y_0 \end{cases}$$