POLYNOMIAL FUNCTIONS

Basic concepts – Definitions

- A <u>polynomial</u> f is a function $f: \mathbb{R} \to \mathbb{R}$ such that

$$\boxed{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \,, \forall x \in \mathbb{R}}$$

with $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$.

- The numbers $a_0, a_1, \ldots, a_n$ are called the <u>coefficients</u> of f.
- The expressions $a_n x^n, a_{n-1} x^{n-1}, \ldots, a_1 x, a_0$ are called the <u>terms</u> of f.
- n = the <u>degree</u> of f and we write:

$$\deg(f) = n.$$

- The set of all polynomials is denoted as
$\mathbb{R}[x]$ = all polynomials with <u>real</u> coefficients
$\mathbb{Q}[x]$ = all polynomials with <u>rational</u> coefficients
$\mathbb{Z}[x]$ = all polynomials with <u>integer</u> coefficients.
Thus, it follows that if

$$[n] = \{1, 2, 3, \ldots, n\}$$

then

$$\boxed{\begin{aligned} f\in\mathbb{R}[x] &\Leftrightarrow \forall k\in[n]: a_k\in\mathbb{R}\\ f\in\mathbb{Q}[x] &\Leftrightarrow \forall k\in[n]: a_k\in\mathbb{Q}\\ f\in\mathbb{Z}[x] &\Leftrightarrow \forall k\in[n]: a_k\in\mathbb{Z}. \end{aligned}}$$

Properties of degrees

Let $f, g \in \mathbb{R}[x]$. Then

$$\boxed{\begin{aligned} \deg(fg) &= \deg(f)+\deg(g)\\ \deg(f+g) &= \max\{\deg(f), \deg(g)\} \end{aligned}}$$

Polynomial Equality.

- In general, two polynomials $f, g \in \mathbb{R}[x]$ are equal if and only if they have the same degree and same coefficients. In other words, if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
$$g(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

then

$$\boxed{f = g \Leftrightarrow \forall k \in [n] : a_k = b_k}$$

Polynomial Division

- Let $f, g \in \mathbb{R}[x]$ be two polynomials with $\deg(f) \geq \deg(g)$. Then, there exist two unique polynomials $q, r \in \mathbb{R}[x]$ such that

$$\boxed{\forall x \in \mathbb{R}: \ f(x) = g(x)q(x) + r(x)}$$

Terminology - Remarks

(1) $q(x)$ = quotient of $f(x) \div g(x)$
$r(x)$ = remainder of $f(x) \div g(x)$

(2) Note that

$$\boxed{\begin{array}{l} \deg(q) = \deg(f) - \deg(g) \\ \deg(r) < \deg(g) \end{array}}$$

(3) We say that <u>g divides f</u> if and only if the corresponding remainder is 0:

$$\boxed{g \mid f \iff \exists q \in \mathbb{R}[x] : \forall x \in \mathbb{R} : f(x) = g(x)q(x)}$$

EXAMPLE

Perform the division
$$(2x^4-3x^3+5x^2-7x+1)\div(x^2-2x+3).$$

For $f(x)=2x^4-3x^3+5x^2-7x+1$

$g(x)=x^2-2x+3$

we have

$\deg(q)=\deg(f)-\deg(g)=4-2=2 \Rightarrow$

$\Rightarrow$ let $q(x)=ax^2+bx+c$.

$\deg(r)<\deg(g)=2 \Rightarrow \deg(r)\leqslant 1 \Rightarrow$

$\Rightarrow$ let $r(x)=dx+e$.

Then

$$\begin{aligned} f(x) &= g(x)q(x)+r(x) \\ &= (x^2-2x+3)(ax^2+bx+c)+(dx+e)=\cdots= \\ &= ax^4+(b-2a)x^3+(c-2b+3a)x^2+ \\ &\quad +(-2c+3b+d)x+(3c+e) \\ &= 2x^4-3x^3+5x^2-7x+1,\ \forall x\in\mathbb{R} \Leftrightarrow \end{aligned}$$

$$\begin{cases} a=2 \\ b-2a=-3 \\ c-2b+3a=5 \\ -2c+3b+d=-7 \\ 3c+e=1 \end{cases} \Leftrightarrow \cdots \Leftrightarrow \begin{cases} a=2 \\ b=1 \\ c=1 \\ d=-8 \\ e=-2 \end{cases} \Leftrightarrow$$

$\Leftrightarrow$ $q(x)=2x^2+x+1$ and $r(x)=-8x-2$.

2nd method

$$
\begin{array}{l|l}
x^2-2x+3 & 2x^4-3x^3+5x^2-7x+1 \\
\hline
2x^2+x+1 & \underline{-2x^4+4x^3-6x^2+0x+0} \\
 & \qquad\quad x^3 \quad -x^2-7x+1 \\
 & \qquad \underline{-x^3 \quad +2x^2-3x+0} \\
 & \qquad\qquad\qquad x^2-10x+1 \\
 & \qquad\qquad\quad \underline{-x^2+2x-3} \\
 & \qquad\qquad\qquad\quad -8x-2
\end{array}
$$

thus

quotient $q(x) = 2x^2+x+1$

remainder $r(x) = -8x-2$

EXERCISES

① Do the following divisions.

a) $(-x^2+81):(x-9)$
b) $(x^3+4x^2-11x-30):(-x^2+x+6)$
c) $(18x^3+9x^2-50x-25):(3x-5)$
d) $[(x^2-9)^2-(x+5)(x-3)^2]:(x^2+x-12)$
e) $(x^6-2x):(x^3+1)$
f) $(6x^4-19x^3+15x^2-x-6):(2x^2-3x+2)$
g) $(x^4-3x^2+5x-1):(2x-1)$
h) $(2x^5-11x^4+3x^3+31x^2+2x+5):(2x^3-5x^2-4x+1)$

② If $f(x)=x^2+x-2$, do the division
$[(f(x))^2-f(x+1)]:f(1-x)$

③ If $f(x)=x^2+5x-6$, do the division
$[f(x-2)f(x+2)-f(x)-10]:(x^2-x-2)$

Division with $x-c$

- Let $f \in \mathbb{R}[x]$ be a polynomial and let $c \in \mathbb{C}$ be an arbitrary complex number.

$$\boxed{c \text{ root of } f \iff f(c) = 0}$$

<u>Thm</u>: $x-c$ divides f if and only if c is a root of f.

$$\boxed{x-c \mid f(x) \iff c \text{ root of } f}$$

<u>Proof</u>

$(\Rightarrow)$: Assume $x-c \mid f(x) \Rightarrow$
$\Rightarrow \exists g \in \mathbb{R}[x] : \forall x \in \mathbb{R} : f(x) = (x-c)g(x)$
$\Rightarrow f(c) = (c-c)\,g(c) = 0 \cdot g(c) = 0$
$\Rightarrow c$ root of f.

$(\Leftarrow)$: Assume c root of $f \Rightarrow \underline{f(c) = 0}$. (1)
Let $q, r \in \mathbb{R}[x]$ such that
$\forall x \in \mathbb{R} : f(x) = (x-c)\,q(x) + r(x)$
Note that $\deg(r) < \deg(x-c) = 1 \Rightarrow$
$\Rightarrow \deg(r) = 0 \Rightarrow r(x) = \lambda, \forall x \in \mathbb{R}$.
Thus

$\forall x \in \mathbb{R} : f(x) = (x-c)\,q(x) + \lambda \Rightarrow$

$$\Rightarrow \left.\begin{array}{l} f(c) = (c-c)\,q(c) + \lambda = \lambda \\ f(c) = 0 \end{array}\right\} \Rightarrow \lambda = 0 \Rightarrow$$

$\Rightarrow \forall x \in \mathbb{R} : f(x) = (x-c)\,q(x) \Rightarrow$

$\Rightarrow x - c \mid f(x) \qquad \square$

$\mapsto$ Synthetic Division

Let $f \in \mathbb{R}[x]$ with

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

The division $f(x) : (x-c)$ has

a) Remainder :

$$r(x) = b_{-1}$$

b) Quotient

$$q(x) = b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_1 x + b_0$$

such that

$$\boxed{\begin{array}{l} b_{n-1} = a_n \\ b_{k-1} = a_k + c\, b_k \ , \ k = 0, 1, \dots, n-1 \end{array}}$$

Proof

We note that

$$(x-c)\,q(x)+r(x) = (x-c)\sum_{k=0}^{n-1} b_k x^k + b_{-1} =$$

$$= \sum_{k=0}^{n-1} \left(b_k x^{k+1} - c b_k x^k\right) + b_{-1}$$

$$= \sum_{k=1}^{n} b_{k-1} x^k + \sum_{k=1}^{n-1} c b_k x^k + (b_{-1} - c b_0)$$

$$= b_{n-1} x^n + \sum_{k=1}^{n-1} (b_{k-1} - c b_k) x^k + (b_{-1} - c b_0)$$

$$= a_n x^n + \cdots + a_1 x + a_0 \quad , \forall x \in \mathbb{R} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} b_{n-1} = a_n \\ b_{k-1} - c b_k = a_k , \quad k = 0, \ldots, n-1 \end{cases} \quad \square$$

- <u>Implementation - Example</u>

$$(2x^3 - 5x^2 + 6x - 1) : (x-2)$$

$$\begin{array}{c|cccc} 2 & 2 & -5 & 6 & -1 \\ & \downarrow & 4 & -2 & 8 \\ \hline & \underbrace{2 \quad\quad -1 \quad\quad 4}_{q(x)} & & & \underbrace{7}_{r(x)} \end{array} \quad \longleftarrow \underline{\text{Horner scheme}}$$

Thus

$$(2x^3 - 5x^2 + 6x - 1) = (x-2)(2x^2 - x + 4) + 7$$

Note that the last number in the Horner scheme is the remainder $r(x) = b_{-1}$.

Efficient polynomial evaluation

The following result suggests that we may use the Horner scheme to evaluate the polynomial $f(x)$ for specific values of x:

Thm : The remainder of the division
$f(x) : (x-c)$
is:
$$\boxed{r(x) = f(c)}$$

Proof

We know that

$$\deg(r) < \deg(x-a) = 1 \Rightarrow \deg(r) = 0 \Rightarrow$$
$$\Rightarrow r(x) = \lambda, \ \forall x \in \mathbb{R}.$$
$$\Rightarrow \exists q \in \mathbb{R}[x] : f(x) = (x-c)\, q(x) + \lambda$$
$$\Rightarrow f(c) = (c-c)\, q(c) + \lambda =$$
$$= 0\, q(c) + \lambda = \lambda \Rightarrow$$
$$\Rightarrow \forall x \in \mathbb{R} : \ r(x) = f(c). \qquad \square$$

Thus, to evaluate $f(c)$, it is sufficient to do the division $f(x) : (x-c)$ using the Horner scheme. The remainder will be equal to $f(c)$!

EXAMPLE

For $f(x) = 2x^3 - x^2 + 3x + 1$

To find $f(3)$:

$$
\begin{array}{c|cccc}
3 & 2 & -1 & 3 & 1 \\
\hline
 & & 6 & 15 & 54 \\
\hline
 & 2 & 5 & 18 & 55
\end{array}
\longrightarrow f(3) = 55 \;!!
$$

EXAMPLE

For what values $a \in \mathbb{R}$ does $x-1$ divide $f(x) = 2x^3 - (a+1)x^2 + (5a-2)x - 7$?

Solution: $x-1 \mid f(x) \Leftrightarrow f(1) = 0 \Leftrightarrow$

$$\Leftrightarrow 2 \cdot 1^3 - (a+1) \cdot 1^2 + (5a-2) \cdot 1 - 7 = 0$$

$$\Leftrightarrow \cancel{2} - a - 1 + 5a - \cancel{2} - 7 = 0$$

$$\Leftrightarrow 4a - 8 = 0 \Leftrightarrow 4a = 8 \Leftrightarrow a = 2.$$

EXAMPLE

Show that $g(x) = 2x^3 + 3x^2 + x$ divides

$f(x) = (x+1)^{2n} - x^{2n} - 2x - 1.$

Solution:

$$g(x) = 2x^3 + 3x^2 + x = x(2x^2+3x+1) =$$
$$= x(x+1)(2x+1) \text{ , thus}$$

$$g(x) \mid f(x) \Leftrightarrow \begin{cases} x \mid f(x) \\ x+1 \mid f(x) \\ 2x+1 \mid f(x) \end{cases} \Leftrightarrow \begin{cases} f(0) = 0 \\ f(-1) = 0 \\ f(-1/2) = 0 \end{cases} \quad (1)$$

Note that

$$f(0) = (0+1)^{2n} - 0^{2n} - 2 \cdot 0 - 1 = 1 - 1 = 0$$
$$f(-1) = (-1+1)^{2n} - (-1)^{2n} - 2 \cdot (-1) - 1 =$$
$$= 0 - 1 + 2 - 1 = 0$$
$$f(-1/2) = (-1/2+1)^{2n} - (-1/2)^{2n} - 2(-1/2) - 1 =$$
$$= (1/2)^{2n} - (1/2)^{2n} + 1 - 1 = 0$$

Thus from (1): $g(x) \mid f(x)$.

EXAMPLE

Show that $11^{15} + 1$ is a multiple of 12.

Solution:

Define $f(x) = x^{15} + 1 \Rightarrow f(-1) = (-1)^{15} + 1 = 0$

$$\Rightarrow x+1 \mid f(x) \Rightarrow 11+1 \mid f(11) \Rightarrow$$
$$\Rightarrow 12 \mid 11^{15} + 1 .$$

EXERCISES

(4) Perform the following divisions

a) $(-x^2+2x^5+2x-3-2x^4):(x-1)$

b) $(3x^3-19x^2-11x+2):(3x+2)$

c) $(3x^3-x^2-x+1):(x+1)$

d) $(32x^5-243):(2x-3)$

(5) For what values of $\lambda \in \mathbb{R}$:

a) $x+2 \mid f(x)$ with $f(x)=\lambda x^2-(\lambda-1)x-4\lambda$

b) $x+1 \mid f(x)$ with $f(x)=x^3+\lambda x^2+2\lambda x-1$

c) $2x+4 \mid f(x)$ with $f(x)=2x^2-(\lambda-2)x+4$.

(6) Let $f(x)=x^3-\lambda x+1$ and let
$g(x)=f(x+1)f(x-2)-f(x)$
For what value of λ does $x-1$ divide $g(x)$?

(7) Let $f(x)=\lambda x^3+(\lambda-1)x^2+2\lambda x+3$ and let $g(x)=f(x-1)f(x+3)$ and let
$h(x)=g(x-1)+2f(x+1)$.
For what value of $\lambda \in \mathbb{R}$ does $x+1$ divide $h(x)$?

(8) Consider the polynomial
$$f(x) = ax^4 + bx^2 + c$$
Show that if $x+1$ divides $f(x)$ then $x-1$ also divides $f(x)$.

(9) Find the remainder of the division
$$[(3x-7)^{2n+1} - 5(x^2-3)^n + 6x-1] : (x-2)$$
with $n \in \mathbb{N}$, $n \geq 1$, without doing the division.

(10) Find the remainder of the division
$$[(2x-5)^{54} + (3x-8)^{23}] : (x-3)$$
without doing the division.

(11) Show that $(x-a)^2 + (x-a)$ divides
$$f(x) = (x-a)^{2n} + (x-a+1)^n - 1$$
with $n \in \mathbb{N}$, $n \geq 1$.

(12) Let $f(x) = ax^n + bx^m + c$ with $n, m \in \mathbb{N}$ and $n \geq 1$, $m \geq 1$. Show that if $a+b+c=0$ then $x-1$ divides $f(x)$.

↳ Exercises 8-12 are short proof-type arguments.

(13) Show that

a) 8^9-1 is a multiple of 7

b) $15^{10}-1$ is a multiple of 14 and a multiple of 16

c) $5^{2n+1}-1$ with $n \in \mathbb{N}$, $n \geq 1$ is a multiple of 4.

d) $33^{20}-3\cdot 33^{10}+2$ is a multiple of 34.

(14) Let $f(x) = ax^3 + bx^2 + bx + a$.
Show that

$$x^2-1 \mid f(x) \iff a+b=0.$$

(15) Let $f \in \mathbb{R}[x]$ be a polynomial.
Let $g(x) = f(3x-5)$. Show that if $x+2$ divides $f(x)$ then $x-1$ divides $g(x)$.

(16) Let $f \in \mathbb{R}[x]$ be a polynomial and let

$$g(x) = f(x+1)\, f(x-2) + f(2x).$$

Show that:

$$x^2-x-2 \mid f(x) \Rightarrow x-1 \mid g(x).$$

Rational zero theorem

- Let $a \in \mathbb{Z}$ be an integer. We define the set of divisors of a, Δ_a, as

$$\boxed{\Delta_a = \{x \in \mathbb{Z} \mid x|a\}}$$

Here $x|a$: x divides a.

EXAMPLE

For $a=6$: $\Delta_6 = \{\pm 1, \pm 2, \pm 3, \pm 6\}$
$a=1$: $\Delta_1 = \{\pm 1\}$
$a=5$: $\Delta_5 = \{\pm 1, \pm 5\}$

- Let $a, b \in \mathbb{Z}$ be two integers. The greatest common divisor $GCD(a,b)$ is defined as

$$\boxed{GCD(a,b) = \max(\Delta_a \cap \Delta_b)}$$

- We say that a fraction a/b with $a \in \mathbb{Z}$ and $b \in \mathbb{Z} - \{0\}$ is
  a) irreducible $\Leftrightarrow$ $GCD(a,b) = 1$
  b) reducible $\Leftrightarrow$ $GCD(a,b) \geq 2$

EXAMPLE

For $4/3$:

$$\left.\begin{array}{l}\Delta_4 = \{\pm 1, \pm 2, \pm 4\} \\ \Delta_3 = \{\pm 1, \pm 3\}\end{array}\right\} \Rightarrow \Delta_3 \cap \Delta_4 = \{\pm 1\}$$

$\Rightarrow GCD(4,3) = \max\{\pm 1\} = 1 \Rightarrow$

$\Rightarrow 4/3$ irreducible.

For $8/4$

$$\left.\begin{array}{l}\Delta_8 = \{\pm 1, \pm 2, \pm 4, \pm 8\} \\ \Delta_4 = \{\pm 1, \pm 2, \pm 4\}\end{array}\right\} \Rightarrow$$

$\Rightarrow \Delta_4 \cap \Delta_8 = \{\pm 1, \pm 2, \pm 4\} \Rightarrow$

$\Rightarrow GCD(4,8) = \max\{\pm 1, \pm 2, \pm 4\}$

$= 4 \Rightarrow 8/4$ reducible.

- Let $f \in \mathbb{Z}[x]$ be a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ with integer coefficients, $a_k \in \mathbb{Z}$. We associate with f a set of rational numbers $\Delta(f)$ defined as:

$$\boxed{\Delta(f) = \{a/b \mid a \in \Delta_{a_0} \wedge b \in \Delta_{a_n}\}}$$

We now show our main result:

Thm : (Rational zero theorem)

$$\boxed{\left.\begin{array}{l} f \in \mathbb{Z}[x] \\ f(\rho) = 0 \\ \rho \in \mathbb{Q} \end{array}\right\} \Longrightarrow \rho \in \Delta(f)}$$

In words: If f is a polynomial with integer coefficients and $\rho \in \mathbb{Q}$ is a rational root of f (such that $f(\rho) = 0$), then ρ has to be an element of $\Delta(f)$.

An equivalent statement of the rational root theorem is that

$$\boxed{f \in \mathbb{Z}[x] \Rightarrow \{\rho \in \mathbb{Q} \mid f(\rho) = 0\} \subseteq \Delta(f)}$$

An immediate consequence is the integer zero theorem:

Thm: (Integer zero theorem)

$$\boxed{\left.\begin{array}{l} f \in \mathbb{Z}[x] \\ f(\rho) = 0 \\ \rho \in \mathbb{Z} \end{array}\right\} \Rightarrow \rho \in \Delta_{a_0}}$$

or:
$$\boxed{f \in \mathbb{Z}[x] \Rightarrow \{\rho \in \mathbb{Z} \mid f(\rho) = 0\} \subseteq \Delta_{a_0}}$$

Proof

Let $\rho = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{Z} - \{0\}$ such that $GCD(p,q) = 1$. (consequence of $\rho \in \mathbb{Q}$)
We assume that $f(\rho) = 0$. It follows that

$$f(p/q) = 0 \Leftrightarrow$$
$$\Leftrightarrow a_n (p/q)^n + a_{n-1}(p/q)^{n-1} + \cdots + a_1 (p/q) + a_0 = 0 \Leftrightarrow$$
$$\Leftrightarrow a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0 \qquad (1)$$

From (1):
$$p(a_n p^{n-1} + a_{n-1} p^{n-2} q + \cdots + a_1 q^{n-1}) = -a_0 q^n \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} p \mid a_0 q^n \\ GCD(p,q) = 1 \end{array}\right\} \Rightarrow p \mid a_0 \Rightarrow \underline{p \in \Delta a_0} \qquad (2)$$

From (1):
$$q(a_0 q^{n-1} + a_1 p q^{n-2} + \cdots + a_{n-1} p^{n-1}) = -a_n p^n \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} q \mid a_n p^n \\ GCD(p,q) = 1 \end{array}\right\} \Rightarrow q \mid a_n \Rightarrow \underline{q \in \Delta a_n} \qquad (3)$$

From (2) and (3):
$$\rho = p/q \in \{a/b \mid a \in \Delta a_0 \wedge b \in \Delta a_n\} = \Delta(f) \Rightarrow$$

$$\Rightarrow \underline{\rho \in \Delta(f)} \qquad \square$$

Method: Factorization of polynomial $f(x)$.

- 1 Find the set $\Delta(f)$ of candidate rational roots of f.
- 2 Test the candidates $\rho \in \Delta(f)$ by calculating $f(\rho)$ by Horner scheme (!!)
- 3 If you find a rational zero, then $x-\rho \mid f(x)$, so we may initiate the factorization
$$f(x) = (x-\rho)\,g(x)$$
from the division $f(x) : (x-\rho)$ which was already done in the previous step.
- 4 Repeat steps 2,3 until factorization is complete.

Then: We can solve $f(x)=0$ or inequalities like $f(x) \geq 0$, $f(x) < 0$, etc.

- If $\rho_1, \rho_2, \dots, \rho_n$ are roots of $f \in \mathbb{R}[x]$ with $\deg f = n$ then

$$\boxed{f(x) = a_n(x-\rho_1)(x-\rho_2)\cdots(x-\rho_n)}$$

EXAMPLE

a) $f(x) = 3x^3 - 22x^2 + 48x - 32 = 0$

$$\left.\begin{array}{l} \Delta_{32} = \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32\} \\ \Delta_3 = \{\pm 1, \pm 3\} \end{array}\right\} \Rightarrow$$

$\Rightarrow \Delta(f) = \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32, \pm 1/3, \pm 2/3, \pm 4/3, \pm 8/3, \pm 16/3, \pm 32/3\}$

Try:

$$\begin{array}{c|cccc} 1 & 3 & -22 & 48 & -32 \\ & & 3 & -19 & 29 \\ \hline & 3 & -19 & 29 & \boxed{-3} \end{array} \longrightarrow f(1) = \cancel{-3} \neq 0$$

etc.

$$\begin{array}{c|cccc} 2 & 3 & -22 & 48 & -32 \\ & & 6 & -32 & 32 \\ \hline & 3 & -16 & 16 & \boxed{0} \end{array}$$

Thus $x-2 \mid f(x) \Rightarrow f(x) = (x-2)(3x^2 - 16x + 16)$.

For
$$\left.\begin{array}{l} g(x) = 3x^2 - 16x + 16 \\ \Delta = 16^2 - 4 \cdot 3 \cdot 16 = 256 - 192 = 64 \end{array}\right\} \Rightarrow$$

$$\Rightarrow x_{2,3} = \frac{16 \pm 8}{6} = \begin{cases} 4 \\ 4/3 \end{cases}$$

Thus $\underline{f(x) = 3(x-2)(x-4)(x-4/3)}$.

B) $f(x) = x^4 - 4x^3 + 5x^2 - 4x + 4$

$$\left.\begin{array}{l} \Delta_4 = \{\pm 1, \pm 2, \pm 4\} \\ \Delta_1 = \{\pm 1\} \end{array}\right\} \Rightarrow A(f) = \Delta_4 = \{\pm 1, \pm 2, \pm 4\}$$

Note that $P(1) \neq 0$ and $P(-1) \neq 0$ (...)
but

$$\begin{array}{c|ccccc} 2 & 1 & -4 & 5 & -4 & 4 \\ & & 2 & -4 & 2 & -4 \\ \hline & 1 & -2 & 1 & -2 & \boxed{0} \end{array}$$

$$f(2) = 0 \Rightarrow x - 2 \mid f(x) \Rightarrow$$
$$\Rightarrow f(x) = (x-2)(x^3 - 2x^2 + x - 2)$$
$$= (x-2)(x^2(x-2) + (x-2))$$
$$= \underline{(x-2)(x-2)(x^2+1)}$$

Note that other factorization techniques can still be useful.

EXERCISES

(17) Solve the following equations or inequalities

a) $x^3 - x - 18 = 0$
b) $x^3 - 6x^2 + 11x - 6 \geq 0$
c) $x^4 + x^3 - 31x^2 - 25x + 150 = 0$
d) $x^4 - 6x^3 + 30x - 25 = 0$
e) $x^4 - 3x^3 + 12x - 16 \geq 0$
f) $2x^3 - 5x^2 + x + 2 = 0$
g) $6x^3 - 7x^2 + 1 > 0$
h) $2x^3 - 9x^2 + 7x + 6 \leq 0$
i) $3x^4 - 4x^3 + 1 = 0$
j) $6x^4 + 13x^3 - 2x^2 - 7x + 2 \leq 0$
k) $3x^4 - 8x^3 - 35x^2 - 4x + 20 = 0$

(18) Find $a \in \mathbb{R}$ such that

$$2x^3 + (a-4)x^2 - 5x + 1 - a = 0$$

has solution $x = 2$. Then find all other solutions.

(19) Find $a \in \mathbb{R}$ such that $ax - 1$ divides $f(x) = x^3 - 5x^2 - \frac{6}{a}$. Then solve $f(x) = 0$ for those values of a.

(20) Find $a \in \mathbb{R}$ such that
$f(x) = (a-1)x^5 + 3ax^4 - (a+1)x^3 - (a+1)x^2 + 3ax + (a-1)$
is divided by $x-2$.
Then solve the equation $f(x)=0$.

(21) Solve the equation
$(2x^2-x-2)^3 - 2(2x^2-x+3)^2 + 9(2x^2-x+2) + 26 = 0$

(22) Find all the integers $k \in \mathbb{Z}$ such that
the equation $x^3 - x^2 + kx + 4 = 0$
has at least one rational solution.

(23) Show that the equation
$x^4 + x^3 + x^2 + x + 1 = 0$
does not have rational solutions.