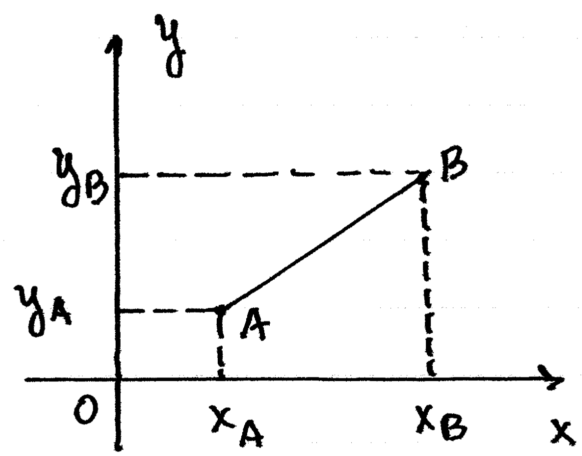
GRAPHING FUNCTIONS

Coordinate system



Let A, B be two points on the plane with coordinates

$A(x_A, y_A)$
$B(x_B, y_B)$

- The <u>slope</u> $m(AB)$ is defined as

$$\boxed{m(AB) = \frac{y_B - y_A}{x_B - x_A}}$$

when $x_A \neq x_B$.

- The <u>distance (AB)</u> between A and B is given by

$$\boxed{(AB) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}}$$

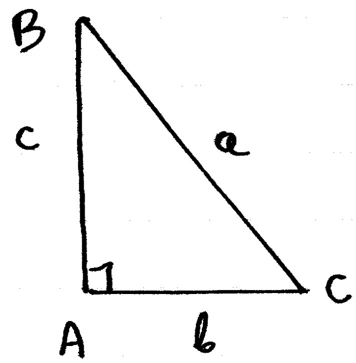
(distance formula)

Proof of distance formula

The distance formula is derived from the Pythagorean theorem:

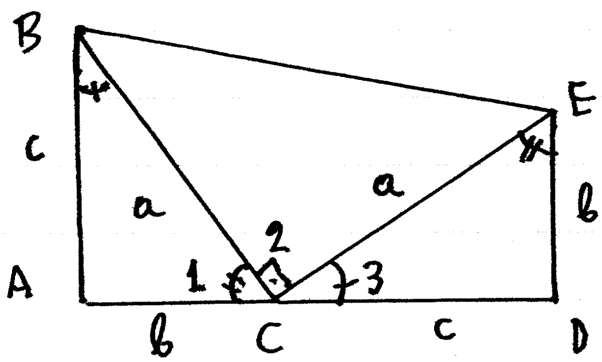
- Let $\overset{\triangle}{ABC}$ be a triangle. Then

$$\boxed{\hat{A} = 90^\circ \Rightarrow a^2 = b^2 + c^2}$$



with $a = (BC)$
$b = (CA)$
$c = (AB)$

Proof (by President Garfield)



Extend AC to D such that $CD = AB$
Let $DE \perp AD$ with $DE = AC$.
Draw CE, BE

Let $C_1 = A\hat{C}B$, $C_2 = B\hat{C}E$, $C_3 = E\hat{C}D$.
Note that
$AB = CD$ and $AC = ED$ and $\hat{A} = \hat{D} \Rightarrow \overset{\triangle}{ABC} = \overset{\triangle}{DCE} \Rightarrow$
$\Rightarrow CE = BC$ and $\hat{C}_3 = \hat{B}$

The area of the trapezoid

$$(ABED) = \frac{1}{2}(AB + DE)\,AD = \frac{1}{2}(c+b)(b+c) =$$

$$= \frac{(b+c)^2}{2}$$

The area of the three triangles:

$$(ABC) = (CDE) = \frac{1}{2}\,AB \cdot AC = \frac{bc}{2}.$$

Note that

$$\hat{C}_2 = 180 - \hat{C}_1 - \hat{C}_3 = 180 - \hat{C}_1 - \hat{B} = \hat{A} = 90^\circ \Rightarrow$$

$$\Rightarrow (BCE) = \frac{1}{2}\,BC \cdot CE = \frac{a^2}{2}.$$

Since

$$(ABED) = (ABC) + (CDE) + (BCE) \Rightarrow$$

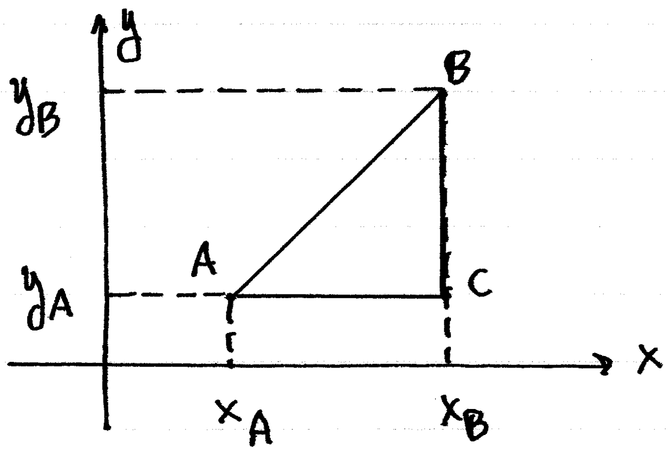
$$\Rightarrow \frac{(b+c)^2}{2} = \frac{bc}{2} + \frac{bc}{2} + \frac{a^2}{2} \Rightarrow$$

$$\Rightarrow a^2 = (b+c)^2 - bc - bc =$$

$$= b^2 + 2bc + c^2 - 2bc$$

$$= b^2 + c^2 \quad \square$$

- To show the distance formula note that:



$AC = x_B - x_A$ and $BC = y_B - y_A$.

Since

$\hat{C} = 90^\circ \Rightarrow AB^2 = AC^2 + BC^2 =$

$= (x_B - x_A)^2 + (y_B - y_A)^2 \Rightarrow$

$$\Rightarrow AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2} \qquad \square$$

- The <u>midpoint M</u> between two points A and B is defineds as the unique point such that

  a) $AM = MB$

  b) $AM + MB = AB$

It can be shown that for $A(x_A, y_A)$ and $B(x_B, y_B)$, the midpoint M has coordinates

$$\boxed{x_M = \frac{x_A + x_B}{2} \quad \text{and} \quad y_M = \frac{y_A + y_B}{2}}$$

EXAMPLES

a) Find the slope, distance, and midpoint between $A(2,-3)$ and $B(-1,-5)$.

Solution

Slope: $m(AB) = \dfrac{y_B - y_A}{x_B - x_A} = \dfrac{(-5)-(-3)}{(-1)-2} = \dfrac{-5+3}{-1-2} =$

$$= \frac{-2}{-3} = \frac{2}{3}$$

Distance: $AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2} =$

$$= \sqrt{[2-(-1)]^2 + [(-3)-(-5)]^2} =$$

$$= \sqrt{(2+1)^2 + (-3+5)^2} =$$

$$= \sqrt{3^2 + 2^2} = \sqrt{9+4} = \sqrt{13}.$$

Midpoint: $$\left.\begin{aligned} x_M &= \frac{x_A + x_B}{2} = \frac{2+(-1)}{2} = \frac{1}{2} \\ y_M &= \frac{y_A + y_B}{2} = \frac{(-3)+(-5)}{2} = \frac{-8}{2} = -4 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow M(1/2, -4)$.

b) If $M(2a+1, a+2)$ is the midpoint between $A(a-1, 3a-1)$ and B, find all $a \in \mathbb{R}$ such that $AB = 8$.

Solution

First, we calculate AM:

$$AM^2 = (x_A - x_M)^2 + (y_A - y_M)^2 =$$
$$= [(a-1) - (2a+1)]^2 + [(3a-1) - (a+2)]^2$$
$$= (a-1-2a-1)^2 + (3a-1-a-2)^2 =$$
$$= (-a-2)^2 + (2a-3)^2 =$$
$$= (a^2 + 4a + 4 + 4a^2 - 12a + 9 =$$
$$= 5a^2 - 8a + 13 \qquad (1)$$

From (1):

$$AB = 8 \Leftrightarrow AM = 4 \Leftrightarrow AM^2 = 16 \Leftrightarrow 5a^2 - 8a + 13 = 16 \Leftrightarrow$$
$$\Leftrightarrow 5a^2 - 8a - 3 = 0$$

$$\Delta = b^2 - 4ac = (-8)^2 - 4 \cdot 5 \cdot (-3) = 64 + 60 = 124 = 2^2 \cdot 31$$

$$\Rightarrow a_{1,2} = \frac{-(-8) \pm 2\sqrt{31}}{2 \cdot 5} = \frac{8 \pm 2\sqrt{31}}{2 \cdot 5} =$$
$$= \frac{4 \pm \sqrt{31}}{5}.$$

EXERCISES

1. Find the slope, distance, and midpoint between the following points:

   a) $A(2,1)$, $B(4,3)$
   b) $A(0,0)$, $B(2,5)$
   c) $A(-1,-2)$, $B(3,-4)$
   d) $A(3,-4)$, $B(-1,-2)$
   e) $A(-2,-2)$, $B(-1,1)$

2. Let $A(a, a+1)$, $B(2a-1, a-1)$.
   Find all values of a such that
   a) $AB = 1$
   b) The slope $m(AB) = -2$

3. If $M(1,3)$ is the midpoint between $A(-1,1)$ and B, find the coordinates of B.

4. If $M(3a-1, a+1)$ is the midpoint between $A(a, a-1)$ and B, find all $a \in \mathbb{R}$ such that the distance $AB = 2$.

Curves represented by equations.

- The curve $(C): f(x,y) = g(x,y)$ consists of all the points of the plane that satisfy the equation
$$f(x,y) = g(x,y).$$
It follows that

$$\boxed{(x,y) \in (C) \iff f(x,y) = g(x,y)}$$

- We now consider 3 curves:
1) The line 2) Parabola 3) circle.

The line

- Every line (ℓ) can be represented as
$$(\ell): Ax + By + C = 0$$
with $|A| + |B| > 0$.
For $B \neq 0$, the <u>slope m</u> of the line (ℓ) is given by

$$\boxed{m = \frac{-A}{B}}$$

such that for any points $P_1, P_2 \in (\ell)$
$m(P_1P_2) = m = -A/B$.

- The equation of the line can be found as follows:

1) Given a point and slope:

$$\boxed{\left.\begin{array}{l} M(x_0, y_0) \in (\ell) \\ m = \text{slope of } (\ell) \end{array}\right\} \Rightarrow (\ell): y - y_0 = m(x - x_0)}$$

2) Given two points:

$$\boxed{\left.\begin{array}{l} A(x_1, y_1) \in (\ell) \\ B(x_2, y_2) \in (\ell) \end{array}\right\} \Rightarrow (\ell): \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}}$$

3) Vertical line:

$$\boxed{\left.\begin{array}{l} M(x_0, y_0) \in (\ell) \\ (\ell) \parallel y'y \end{array}\right\} \Rightarrow (\ell): x = x_0}$$

- <u>Distance of point from a line</u>

The distance of the point $A(x_0, y_0)$ from the line $(\ell): Ax + By + C = 0$ is given by

$$\boxed{d(A,(l)) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2+B^2}}}$$

▸ Relative position between two lines

- Consider two lines

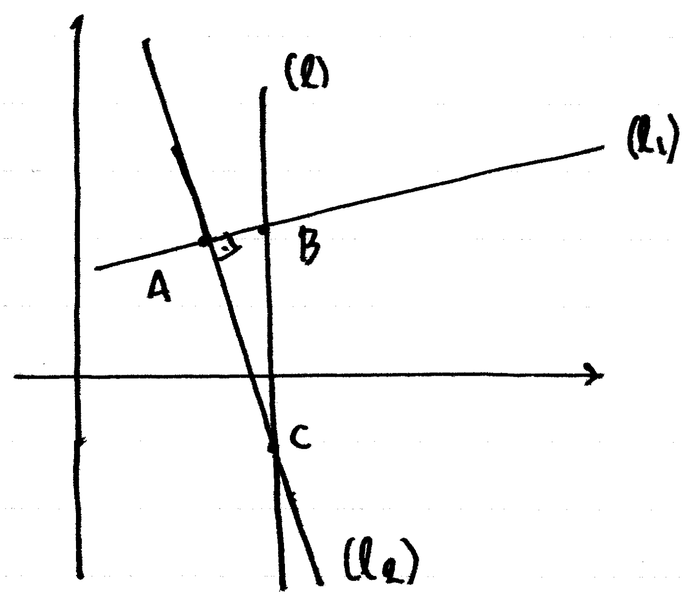
$$(l_1): \ y = m_1 x + b_1$$
$$(l_2): \ y = m_2 x + b_2$$

It can be shown that

$$\boxed{\begin{array}{l} (l_1) // l_2 \Leftrightarrow m_1 = m_2 \\ (l_1) \perp (l_2) \Leftrightarrow m_1 m_2 = -1 \end{array}}$$

(parallel)
(perpendicular)

- Proof of perpendicular condition:



Let $A(x_0, y_0) = (l_1) \cap (l_2)$.
Introduce the line
$(l): \ x = x_0 + 1$
and let
$B = (l) \cap (l_1)$
$C = (l) \cap (l_2)$
with $B(x_0+1, y_1)$
and $C(x_0+1, y_2)$

To find $y_1 - y_0$; and $y_2 - y_0$:

$$m(AB) = m_1 \Leftrightarrow \frac{y_1 - y_0}{(x_0+1) - x_0} = m_1 \Leftrightarrow y_1 - y_0 = m_1$$

$$m(AC) = m_2 \Leftrightarrow \frac{y_2 - y_0}{(x_0+1) - x_0} = m_2 \Leftrightarrow y_2 - y_0 = m_2$$

thus:

$$AB = \sqrt{((x_0+1) - x_0)^2 + (y_1 - y_0)^2} = \sqrt{1 + m_1^2}$$

$$AC = \sqrt{((x_0+1) - x_0)^2 + (y_2 - y_0)^2} = \sqrt{1 + m_2^2}$$

$$BC = \dots = y_2 - y_1 = m_2 - m_1$$

It follows that

$$\underline{(l_1) \perp (l_2)} \Leftrightarrow \hat{A} = 90^\circ \Leftrightarrow BC^2 = AB^2 + AC^2 \Leftrightarrow$$

$$\Leftrightarrow (m_2 - m_1)^2 = \left(\sqrt{1 + m_1^2}\right)^2 + \left(\sqrt{1 + m_2^2}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow m_2^2 - 2m_1 m_2 + m_1^2 = (1 + m_1^2) + (1 + m_2^2) \Leftrightarrow$$

$$\Leftrightarrow -2m_1 m_2 = 2 \Leftrightarrow \underline{m_1 m_2 = -1}$$

EXERCISES

⑤ Find the equation of the line
$(\ell): Ax + By + C = 0$
such that

a) (ℓ) goes through $A(1,3)$, $B(2,5)$
b) (ℓ) goes through $A(2,3)$, $B(2,4)$
c) (ℓ) goes through $A(1,4)$, $B(4,1)$
d) (ℓ) goes through $A(-2,3)$, $B(-3,-2)$
e) (ℓ) goes through $A(2,3)$ with slope -1
f) (ℓ) goes through $A(-1,2)$ with slope 4
g) $(\ell) \parallel (\ell_1): 3x+2y+4=0$ and goes through $A(-1,5)$
h) $(\ell) \parallel (\ell_2): x-3y+1=0$ and goes through $A(3,-2)$
i) $(\ell) \perp (\ell_1): 2x+5y+4=0$ and goes through $A(-2,3)$
j) $(\ell) \perp (\ell_1): 3x-y+2=0$ and goes through $A(1,-1)$

⑥ Find $a \in \mathbb{R}$ such that $(\ell_1): ax+5y+7=0$ and $(\ell_2): 2(a-1)x-3y+9=0$ are parallel.

⑦ Find $a \in \mathbb{R}$ such that the lines
$(l_1): (a+3)x + y - 7 = 0$ and $(l_2): (1-a)x - 4y + 12 = 0$
are perpendicular.

⑧ Find $a \in \mathbb{R}$ such that $(l_1): (a-1)x - y + (a-2) = 0$
and $(l_2): (3a-7)x - y - 2a + 5 = 0$ are
parallel. Then find a line perpendicular
to (l_1) and (l_2) going through the point
$A(-1,1)$

⑨ Find $a \in \mathbb{R}$ such that the line
$$(l): (a-2)x - (a-1)y + (3a-2)(a-1) = 0$$
is:
a) Parallel to the x-axis
b) Parallel with $(q): 4x - y + 3 = 0$
c) Perpendicular to $(l_1): 2x + y - 5 = 0$

EXAMPLES

a) Find the equation of the line going through $A(1,2)$ and $B(-2,-3)$.

Solution

$$(AB): \frac{y-y_A}{x-x_A} = \frac{y_B-y_A}{x_B-x_A} \Leftrightarrow \frac{y-2}{x-1} = \frac{(-3)-2}{(-2)-1} \Leftrightarrow$$

$$\Leftrightarrow \frac{y-2}{x-1} = \frac{-5}{-3} \Leftrightarrow \frac{y-2}{x-1} = \frac{5}{3} \Leftrightarrow$$

$$\Leftrightarrow 3(y-2) = 5(x-1) \Leftrightarrow 3y-6 = 5x-5 \Leftrightarrow 5x-3y+6-5=0$$

$$\Leftrightarrow 5x-3y+1=0$$

thus $(AB): 5x-3y+1=0$.

b) Find the equation of the line through $A(3,-4)$ with slope -2.

Solution

$$(\ell): y-(-4) = (-2)(x-3) \Leftrightarrow y+4 = -2(x-3) \Leftrightarrow$$

$$\Leftrightarrow y+4 = -2x+6 \Leftrightarrow 2x+y+4-6=0 \Leftrightarrow 2x+y-2=0.$$

thus:

$(\ell): 2x+y-2=0$.

c) Find all values $a \in \mathbb{R}$ such that the lines

$(\ell_1): 3x + (a+1)y + 2a = 0$

$(\ell_2): ax - (2a-1)y + 1 = 0$

are perpendicular.

Solution

Let $m_1 =$ slope of (ℓ_1)

$m_2 =$ slope of (ℓ_2).

Then $m_1 = \dfrac{-3}{a+1}$ and $m_2 = \dfrac{a}{2a-1}$.

It follows that

$$(\ell_1) \perp (\ell_2) \Leftrightarrow m_1 m_2 = -1 \Leftrightarrow \frac{-3}{a+1} \frac{a}{2a-1} = -1$$

$$\Leftrightarrow \frac{-3a}{(a+1)(2a-1)} = -1 \Leftrightarrow \frac{3a}{(a+1)(2a-1)} = 1 \quad (1)$$

• Domain of equation: $A = \mathbb{R} - \{-1, 1/2\}$.

$$(1) \Leftrightarrow 3a = (a+1)(2a-1) \Leftrightarrow 3a = 2a^2 - a + 2a - 1 \Leftrightarrow$$

$$\Leftrightarrow 2a^2 + (-1+2-3)a - 1 = 0 \Leftrightarrow 2a^2 - 2a - 1 = 0.$$

$$\Delta = (-2)^2 - 4 \cdot 2 \cdot (-1) = 4 + 8 = 12 = 4 \cdot 3 \Rightarrow$$

$$\Rightarrow a_{1,2} = \frac{2 \pm 2\sqrt{3}}{2 \cdot 2} = \frac{1 \pm \sqrt{3}}{2} \leftarrow \text{Both accepted (they belong to A).}$$

Thus:

$$(\ell_1) \perp (\ell_2) \Leftrightarrow a = \frac{1+\sqrt{3}}{2} \vee a = \frac{1-\sqrt{3}}{2}.$$

The circle

- A circle (c) with center A and radius r is the set of all points M on the plane such that AM = r. Thus

  $$(c) = \{M \mid AM = r\}$$

  This circle can be represented as

  $$\boxed{(c): (x-x_A)^2 + (y-y_A)^2 = r^2}$$

  with $A(x_A, y_A)$.

- The curve $(c): x^2+y^2+Ax+By = C$ is a circle if and only if

  $$\boxed{C+(A/2)^2+(B/2)^2 > 0}$$

Proof

$(c): x^2+y^2+Ax+By = C \Leftrightarrow$

$$\Leftrightarrow x^2+Ax+\left(\frac{A}{2}\right)^2 + y^2+By+\left(\frac{B}{2}\right)^2 = C+(A/2)^2+(B/2)^2$$

$$\Leftrightarrow \left(x+\frac{A}{2}\right)^2+\left(y+\frac{B}{2}\right)^2 = C+(A/2)^2+(B/2)^2$$

We see that for (c) to be a circle, the RHS has to be positive.

↳ We conclude from the argument above that if (c) is a circle then it has center $O(-A/2, -B/2)$ and radius r given by

$$\boxed{r = \sqrt{C + (A/2)^2 + (B/2)^2}}$$

EXERCISES

(10) Write the equation $(c): x^2 + y^2 + Ax + By = C$ for the circle with center O and radius r such that

a) $O(1,-1)$, $r = 2$
b) $O(3,-2)$, $r = 3$
c) $O(-1,-4)$, $r = 1/2$
d) $O(1/2, 1/3)$, $r = 1/3$
e) $O(a, a-1)$, $r = a+2$ with $a+2 > 0$.

(11) Find the center and radius for the following circles, if they are indeed circles:

a) $(c): x^2 + y^2 + 3x + 2y = 5$
b) $(c): x^2 + y^2 - 2x + 4y = 0$

c) $(c): x^2+y^2+3x+y=1$
d) $(c): x^2+y^2+6x+4y=-2$
e) $(c): x^2+y^2+x+y+10=0$
f) $(c): x^2+y^2-x+2y+9=0$

(12) For what values $a \in \mathbb{R}$ are the following curves circles?

a) $x^2+y^2+(a+1)x+(a-1)y=a$
b) $x^2+y^2+(2a-1)x+(3a+2)y=a+1$
c) $x^2+y^2+(a+2)x+(a-1)y=1-a^2$

(13) Find $a \in \mathbb{R}$ such that the circle

$$(c): x^2+y^2+(2a+1)x+(2a-1)y=a+3$$

passes through the point $A(1,2)$.
What is the radius and center of the circle?

(14) Find $a \in \mathbb{R}$ such that the radius r of the circle

$$(c): x^2+y^2+(a-1)x+(a+1)y=2a-1$$

satisfies $1 \leq r < 2$.

Graphing linear and quadratic functions.

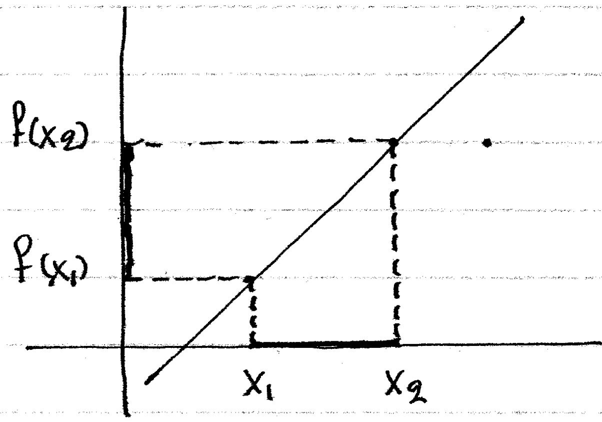
- Let $f: A \to \mathbb{R}$ be a function. The graph of the function f is a curve (c) given by

$$\boxed{(c): \begin{cases} y = f(x) \\ x \in A \end{cases}}$$

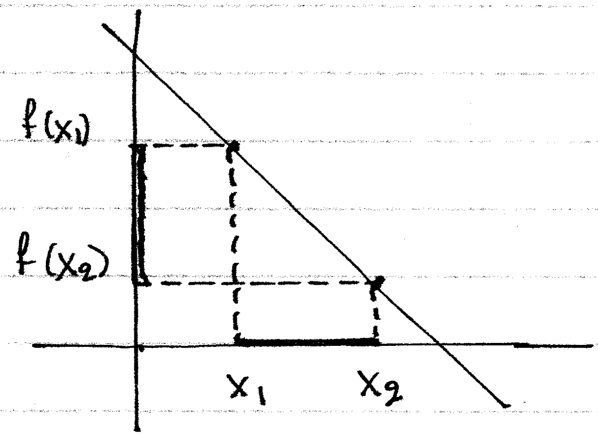
The linear function $\boxed{f(x) = ax + b}$

- Domain : $A = \mathbb{R}$
- Range : $f(A) = \mathbb{R}$
- Monotonicity : $f \nearrow \mathbb{R} \Leftrightarrow a > 0$
  $f \searrow \mathbb{R} \Leftrightarrow a < 0$
- Graph : The graph of $f(x) = ax + b$ is a line with slope a.
- To draw the graph of f it is sufficient to find two points on the graph. The line is then uniquely defined from these two points.
- Range on restricted intervals:

$$\boxed{\begin{array}{l} a > 0 \Rightarrow f([x_1, x_2]) = [f(x_1), f(x_2)] \\ a < 0 \Rightarrow f([x_1, x_2]) = [f(x_2), f(x_1)] \end{array}}$$



$a>0$



$a<0$

- Range on unrestricted intervals

$$a>0 \Rightarrow \begin{cases} f([x_1,+\infty)) = [f(x_1),+\infty) \\ f((-\infty,x_2]) = (-\infty, f(x_2)] \end{cases}$$

$$a<0 \Rightarrow \begin{cases} f([x_1,+\infty)) = (-\infty, f(x_1)] \\ f((-\infty,x_2]) = [f(x_2),+\infty) \end{cases}$$

Linear functions with absolute values.

Before working with a linear functions that has absolute values we rewrite it in the form :

$$f(x) = \begin{cases} a_1 x + b_1 , & x \in A_1 \\ a_2 x + b_2 , & x \in A_2 \\ \vdots & \vdots \\ a_n x + b_n , & x \in A_n \end{cases}$$

with $A_1, A_2, \dots, A_n$ intervals.

- Domain : $A = A_1 \cup A_2 \cup \cdots \cup A_n = \mathbb{R}$
- Range : $f(A) = f(A_1) \cup f(A_2) \cup \cdots \cup f(A_n)$
- Monotonicity : Depends on the intervals $A_1, A_2, \dots, A_n$
- Graph : A sequance of line segments with different slopes per segment.

EXAMPLES

a) For $f(x) = |x+1| + |x-2| - 1 \leftarrow \left\{ \begin{array}{l} \text{Domain} \\ \text{Range} \\ \text{Monotonicity} \\ \text{Graph.} \end{array} \right.$

x		-1		2	
$x+1$	$-$	0	$+$		$+$
$x-2$	$-$		$-$	0	$+$

$\forall x \in (-\infty, -1]$: $f(x) = -(x+1) - (x-2) - 1 =$
$= -x - 1 - x + 2 - 1 = -2x$

$\forall x \in [-1, 2]$: $f(x) = (x+1) - (x-2) - 1 =$
$= x + 1 - x + 2 - 1 = 2$

$\forall x \in [2, +\infty)$: $f(x) = (x+1) + (x-2) - 1 = 2x - 2$

therefore

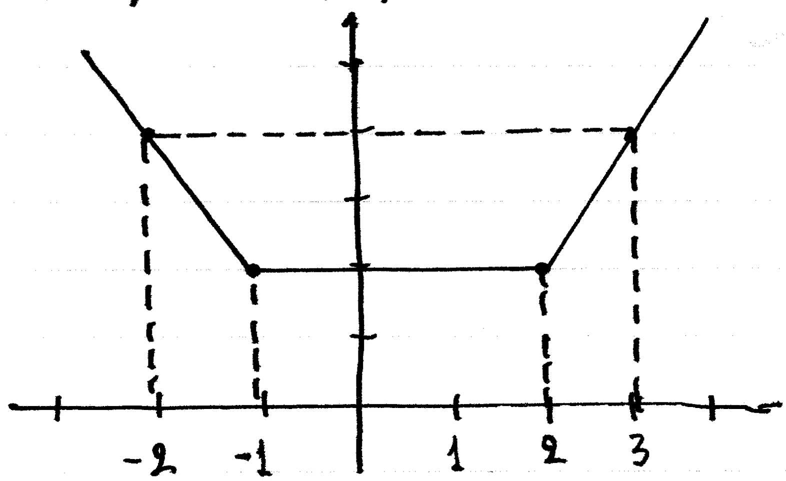
$$f(x) = \begin{cases} -2x & , \ x \in (-\infty, -1] \\ 2 & , \ x \in [-1, 2] \\ 2x - 2 & , \ x \in [2, +\infty) \end{cases}$$

Domain : $A = \mathbb{R}$

Range : $\left.\begin{array}{l} f((-\infty, -1]) = [2, +\infty) \\ f([-1, 2]) = \{2\} \\ f([2, +\infty)) = [2, +\infty) \end{array}\right\} \Rightarrow$

$\Rightarrow f(A) = [2, +\infty) \cup \{2\} \cup [2, +\infty)$
$= [2, +\infty)$

- Monotonicity: $f \searrow (-\infty, -1]$
  f const. $[-1, 2]$
  $f \nearrow [2, +\infty)$
- Graph.
  Use $f(-2) = -2 \cdot (-2) = 4$
  $f(-1) = 2$
  $f(2) = 2$
  $f(3) = 2 \cdot 3 - 2 = 4$



6) For what values of $a \in \mathbb{R}$ is the function $f(x) = 2x(a-1) + a^2(x-1)$ increasing in $\mathbb{R}$?

Solution

We note that

$$f(x) = 2x(a-1) + a^2(x-1) =$$
$$= 2ax - 2x + a^2x - a^2 =$$
$$= (a^2+2a-2)x - a^2 \;,\; \forall x \in \mathbb{R}.$$

It follows that

$$f \nearrow \mathbb{R} \Leftrightarrow a^2 + 2a - 2 > 0 \quad (1)$$

$$\Delta = 2^2 - 4 \cdot 1 \cdot (-2) = 4 + 8 = 12 = 4 \cdot 3 \Rightarrow$$

$$\Rightarrow a_{1,2} = \frac{-2 \pm 2\sqrt{3}}{2 \cdot 1} = -1 \pm \sqrt{3}$$

Thus

$$f \nearrow \mathbb{R} \Leftrightarrow a = -1+\sqrt{3} \vee a = -1-\sqrt{3}.$$

EXERCISES

(15) For the following functions, write the domain, range, monotonicity, and draw the corresponding graph:

a) $f(x) = 3x - 2$
b) $f(x) = -2x + 4$
c) $f(x) = |2x - 1|$
d) $f(x) = |2 - x| + x$
e) $f(x) = \frac{|x|}{x}$
f) $f(x) = |x-3| + |x+2| - 5$
g) $f(x) = |2x+1| + |x-2|$
h) $f(x) = |1-x| + |x-1| + 1$
i) $f(x) = |x| + |x+1| + |x+2|$
j) $f(x) = |x-1| + |2x| + |2x+1|$

(16) For what values $a \in \mathbb{R}$ are the following functions increasing in $\mathbb{R}$?

a) $f(x) = (a+1)x + (a-2)$
b) $f(x) = (a^2+3a+2)x + 2a$
c) $f(x) = a(x+a) + x(a-1)^2$
d) $f(x) = a(a+1)(a+x)$
e) $f(x) = 2x(a^2-1) + a(x-1)$
g) $f(x) = 2(x+a)(a-1) + (x-a)(a+1)^2$

Quadratic function $f(x) = ax^2 + bx + c$

- The graph of the quadratic function $f(x) = ax^2 + bx + c$ is a curve called <u>parabola</u> which has the following properties:

a) Its <u>vertex</u> is the point $A(-b/2a, -\Delta/4a)$ with $\Delta = b^2 - 4ac$ the discriminant of the quadratic $ax^2 + bx + c$.

   i) For $a > 0 \Rightarrow$ the vertex A is a minimum

   ii) For $a < 0 \Rightarrow$ the vertex A is the maximum.

b) It has <u>axis of symmetry</u> the line $(\ell): x = -b/2a$

c) Intersects the y-axis at $C(0, c)$

d) Intersects the x-axis at

   i) $A_1(x_1, 0)$ and $A_2(x_2, 0)$ with x_1, x_2 the zeroes of the quadratic, when $\Delta > 0$.

   ii) Tangent with the x-axis at the vertex, when $\Delta = 0$.

   iii) No intersection when $\Delta < 0$.

e) For $a > 0$: the parabola opens up
   $a < 0$: the parabola opens down.

- To justify the above claims, we rewrite the quadratic in the completed square form:

$$\boxed{f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}}$$

Proof

$$f(x) = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) =$$

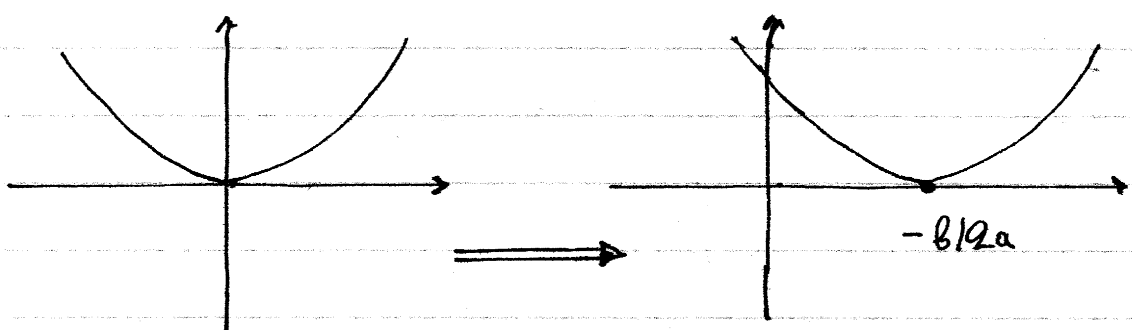
$$= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right] =$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right] =$$

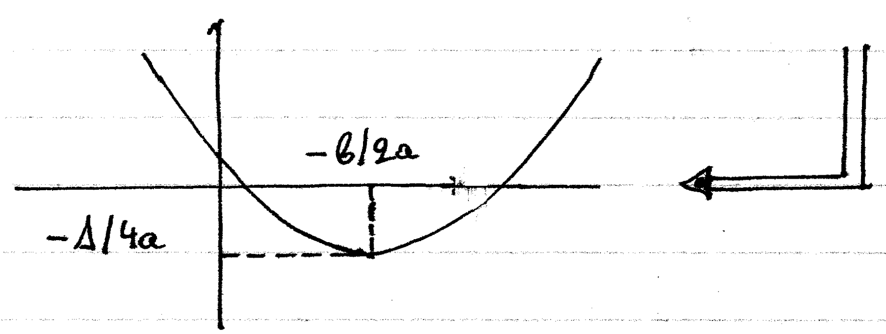
$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] =$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] =$$

$$= a(x + b/2a)^2 - \Delta/4a. \qquad \square$$



$f(x) = ax^2$ $\qquad\qquad$ $f(x) = a(x + b/2a)^2$



$f(x) = ax^2 + bx + c = a(x + b/2a)^2 - \Delta/4a$

- To plot the graph of a quadratic function, we first determine the coordinates of the vertex, and the points where the graph intersects the x-axis (if they exist) and the y-axis. If these are not sufficient, we find additional points by evaluating the function.

EXAMPLE

Graph the function $f(x) = x^2 + 5x + 6$.

Solution

Discriminant $\Delta = b^2 - 4ac = 5^2 - 4 \cdot 1 \cdot 6 = 25 - 24 = 1$

$$\left.\begin{aligned} \frac{-b}{2a} &= \frac{-5}{2 \cdot 1} = \frac{-5}{2} \\ \frac{-\Delta}{4a} &= \frac{-1}{2 \cdot 1} = \frac{-1}{2} \end{aligned}\right\} \Rightarrow \text{Vertex } A(-5/2, -1/2)$$

x-axis intercepts:

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-5 \pm \sqrt{1}}{2 \cdot 1} = \begin{cases} -6/2 = -3 \\ -4/2 = -2 \end{cases}$$

thus $(-3, 0)$, $(-2, 0)$.

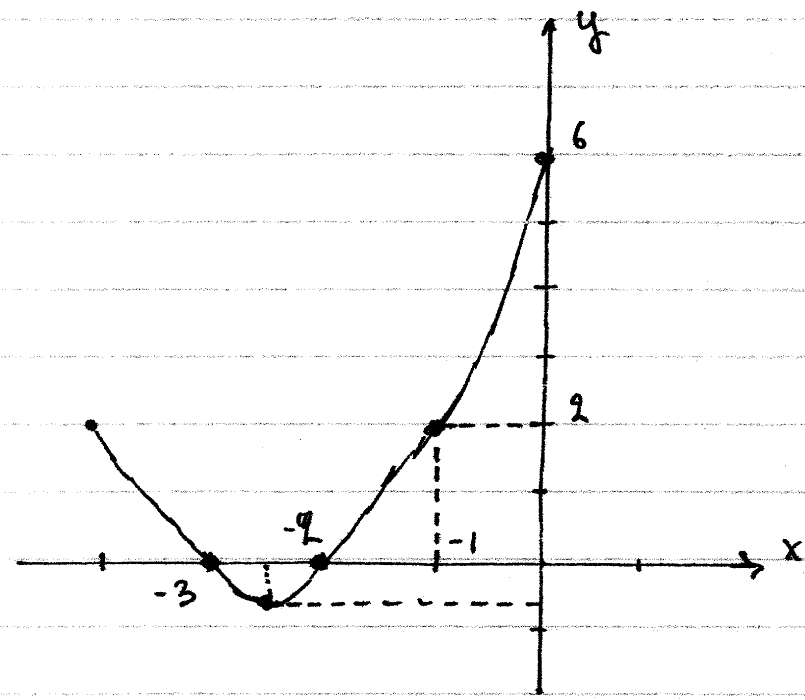
y-axis intercept: $(0, 6)$

Also need at $x = -1$: $f(-1) = (-1)^2 + 5 \cdot (-1) + 6 =$

$= 1 - 5 + 6 = 2$

Thus, to summarize:

x	-3	-5/2	-2	-1	0
y	0	-1/2	0	2	6


y
6
2
-2
-1
-3
x

▶ Range of a quadratic function

It can be shown, directly by definition, and also seen via the graph, that for the quadratic function $f(x) = ax^2 + bx + c$, the range $f(\mathbb{R})$ is given by:

$$\boxed{\begin{array}{l} a>0 \Rightarrow f(\mathbb{R}) = [-\Delta/4a, +\infty) \\ a<0 \Rightarrow f(\mathbb{R}) = (-\infty, -\Delta/4a] \end{array}}$$

EXAMPLE

Find the range of $f(x) = 3x^2 - 2x + 5$

Solution

$$\Delta = b^2 - 4ac = (-2)^2 - 4\cdot 3\cdot 5 = 4 - 60 = -56 \Rightarrow$$

$$\Rightarrow \frac{\Delta}{4a} = \frac{-56}{4\cdot 3} = \frac{-56}{12} = \frac{-2^3\cdot 7}{2^2\cdot 3} = \frac{-2\cdot 7}{3} = \frac{-14}{3}$$

$$\Rightarrow f(\mathbb{R}) = [-\Delta/4a, +\infty) = [14/3, +\infty).$$

EXERCISES

(17) For the following functions, write the domain, range, axis of symmetry, vertex, intercepts, and the corresponding completed square form. Then graph the function.

a) $f(x) = 2x^2$

b) $f(x) = -3x^2$

c) $f(x) = x^2/2$

d) $f(x) = -x^2/3$

e) $f(x) = 3x^2+1$

f) $f(x) = -2x^2+3$

g) $f(x) = x^2-2$

h) $f(x) = x^2-2x$

i) $f(x) = -2x^2+3x$

j) $f(x) = 2x^2+4x+2$

k) $f(x) = x^2-4x+3$

l) $f(x) = -x^2+x+2$

m) $f(x) = -2x^2-6x-4$

n) $f(x) = x^2+x+1$

o) $f(x) = x^2+3x+4$

p) $f(x) = -x^2+2x-3$