FUNCTIONS

▼ Preliminary Concepts

- An ordered pair (a,b) is a collection of two elements a and b in which a is the first element and b is the second element.
- By definition: (equality of ordered pairs).

$$(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2 \wedge b_1 = b_2 .$$

example

For $a \neq b$: $(a,b) \neq (b,a)$

$$\{a,b\} = \{b,a\}$$

i.e. in set equality the order with which the elements are listed is not important. In ordered-pair equality, the order with which the elements are listed is taken into account.

- Let A, B be two sets. The cartesian product $A \times B$ is defined as

$$A \times B = \{ (a,b) \mid a \in A \wedge b \in B \}$$

example

For $A = \{1, 3\}$ and $B = \{2, 4, 5\}$

$A \times B = \{(1,2), (1,4), (1,5), (3,2), (3,4), (3,5)\}$

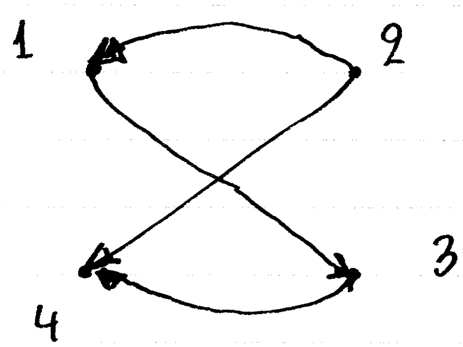
$B \times A = \{(2,1), (2,3), (4,1), (4,3), (5,1), (5,3)\}$

- Any subset $R \subseteq A \times B$ with $R \neq \emptyset$ is called a <u>relation</u> between elements of A and elements of B.

  A relation can be represented by a cartesian graph or Venn diagram as in the following example:

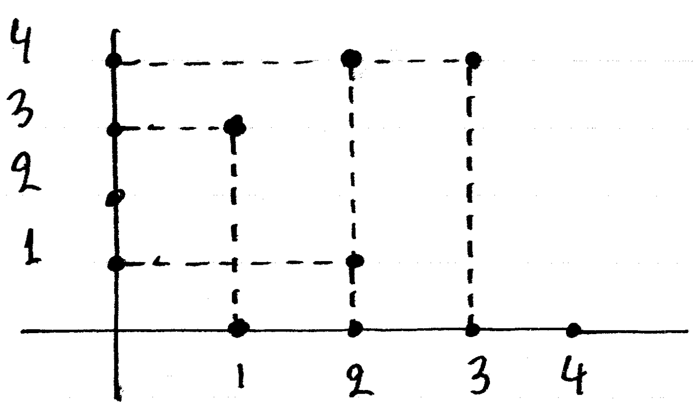
example

$R = \{(1,3), (2,4), (3,4), (2,1)\} \subseteq \mathbb{R} \times \mathbb{R}.$



<u>Venn Diagram</u>

Each ordered pair corresponds to an arrow.



<u>Cartesian graph</u>

Each ordered pair corresponds to an point on the axis system.

Quantified statements

- All definitions and theorems concerning relations, mapping, and functions require that we use the notation of quantified statements.
- Let $p(x)$ be some statement about x that is TRUE or FALSE depending on the value of the variable x. We now define the following statements:

  a) $\forall x \in A : p(x)$

  "For all $x \in A$, $p(x)$ is true"

  b) $\exists x \in A : p(x)$

  "There is at least one $x \in A$, such that $p(x)$ is true".
- We define $S = \{x \in A \mid p(x)\}$ as the set of all elements x of A for which $p(x)$ is true. It follows that:

  $$\forall x \in A : p(x) \Leftrightarrow S = \{x \in A \mid p(x)\} = A$$
  $$\exists x \in A : p(x) \Leftrightarrow S = \{x \in A \mid p(x)\} \neq \emptyset$$

EXAMPLE

$\forall a, b \in \mathbb{R} : \exists x \in \mathbb{R} : a + x = b$

"For all real numbers a, b there is another real number x such that $a + x = b$"

↳ This is an example of a well-known rule of regular algebra rewritten as a quantified statement.

Definitions of mappings

1) Algebraic Definition.
A mapping $f: A \to B$ is a relation $f \subseteq A \times B$ that satisfies the following properties:

- $\forall (a_1, b_1), (a_2, b_2) \in f : (a_1 = a_2 \Rightarrow b_1 = b_2)$
- $\forall a \in A : \exists b \in B : (a, b) \in f.$

2) Venn Diagram definition
A mapping $f: A \to B$ is a relation in whose Venn diagram every element of A has one and only one outgoing arrow.

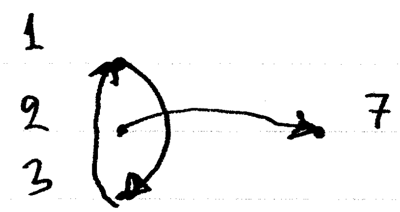
3) Cartesian graph definition
A mapping $f: A \to B$ is a relation whose graph has points such that no two points share the same x-coordinate.
(also see: vertical line test).

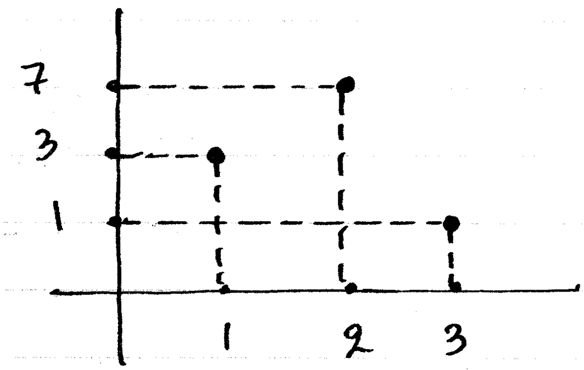
examples

The previous example was not a mapping.
The following is a mapping: $f: \{1, 2, 3\} \longrightarrow \mathbb{R}$

$$f = \{(1,3), (2,7), (3,1)\}$$

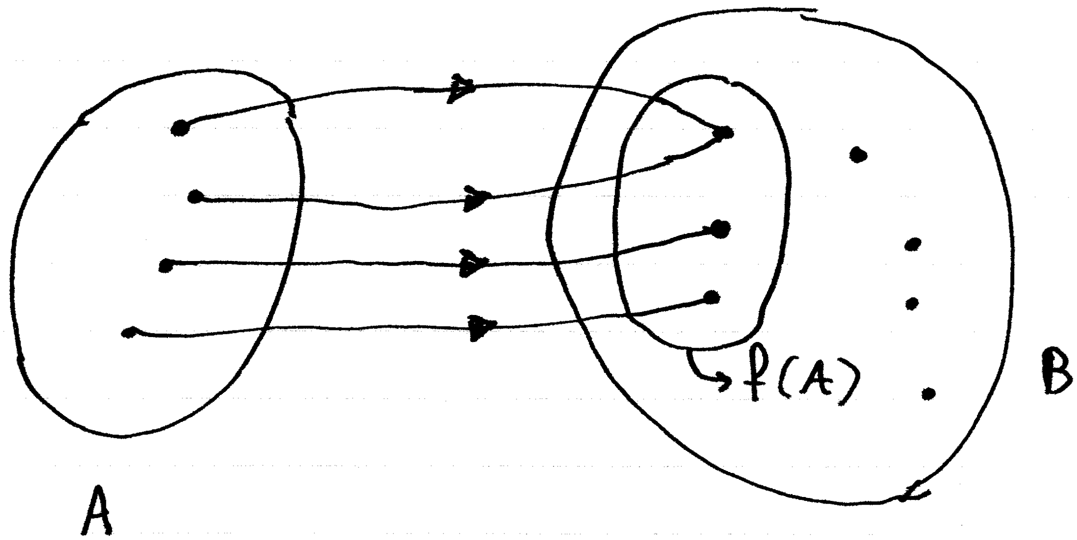
Venn Diagram





Cartesian Graph.

- Thus a mapping $f: A \to B$ maps every element $x \in A$ to a <u>unique</u> element of B, which we shall denote as $f(x)$. Obviously $f(x) \in B$. We call $f(x)$ the <u>image</u> of x under the mapping f.
  The set A is the <u>domain</u> of the mapping f and we write
  $$A = \mathrm{dom} f$$
- The <u>range $f(A)$</u> of the mapping f is the set of all elements of B which are an image of some element in A:
  $$f(A) = \{f(x) \mid x \in A\}$$
  Obviously $f(A) \subseteq B$. It is possible that some elements of B are NOT images of any element of A (see schematic)



example

For $f = \{(1,3), (2,7), (3,1)\}$
Domain $A = dom(f) = \{1,2,3\}$

$$\left.\begin{array}{l} f(1) = 3 \\ f(2) = 7 \\ f(3) = 1 \end{array}\right\} \Rightarrow f(A) = \{1,3,7\} \leftarrow \text{Range.}$$

Inverse Relations and Mappings

- Let $R \subseteq A \times B$ be a relation. We define the inverse relation as
  $$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$
  The Venn diagram of R^{-1} is obtained from the Venn diagram of R by reversing the direction of all arrows!!
- The inverse f^{-1} of a mapping $f: A \to B$ is a relation, of course, but there is no guarantee that it will also be a mapping.

- Let $f: A \to B$ be a mapping.
  We say that

$$\boxed{f \text{ one-to-one} \Leftrightarrow \forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)}$$

- $$\boxed{\left.\begin{array}{l} f \text{ mapping} \\ f \text{ one-to-one} \end{array}\right\} \Rightarrow \begin{array}{l} f^{-1} \text{ is also} \\ \text{a mapping} \end{array}}$$

<u>example</u>

For $f = \{(1,3), (2,7), (3,1)\}$
$f^{-1} = \{(3,1), (7,2), (1,3)\}$ is also a mapping.

For $f = \{(1,3), (2,4), (3,4)\}$
the inverse $f^{-1} = \{(3,1), (4,2), (4,3)\}$
is NOT a mapping.

- If $f: A \to B$ is a one-to-one mapping then f^{-1} is also the mapping $f^{-1}: f(A) \longrightarrow B$.
  Then, the range of f is the domain of f^{-1}.

  i.e: $\text{dom}(f^{-1}) = f(A) = f(\text{dom} f)$

EXERCISES

① Write out $A \times B$ and $B \times A$ for A, B defined as follows:

a) $A = \{1,3,7\}$ and $B = \{2,5\}$

b) $A = \{1,2\}$ and $B = \{1,2,3\}$

c) $A = \{2,9\}$ and $B = \{1,7\}$

d) $A = \{3\}$ and $B = \{2,6\}$

e) $A = \{5\}$ and $B = \{4\}$

f) $A = \emptyset$ and $B = \{2,3,4\}$

g) $A = \emptyset$ and $B = \emptyset$.

② Write out the following statements in complete English sentences.

a) $\forall x, y \in \mathbb{R}: x+y = y+x$

b) $\forall x, y, z \in \mathbb{R}: x(y+z) = xy + xz$

c) $\exists x \in \mathbb{R}: 3x+1 = 5$

d) $\forall x \in \mathbb{R} - \{0\}: \exists y \in \mathbb{R}: xy = 1$

e) $\exists a \in \mathbb{R}: \forall x \in \mathbb{R}: \dfrac{1}{1+x^2} < a$

f) $\forall x_1, x_2 \in A: (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$
(definition of "f is one-to-one")

g) $\exists x_1, x_2 \in A: (f(x_1) = f(x_2) \wedge x_1 \neq x_2)$
(definition of "f is not one-to-one")

(3) Make Venn diagrams for the following relations. Which of these relations are mappings? If yes, show the domain and range. Is the inverse relation a mapping?

a) $f = \{(1,2),(2,2),(3,1),(4,5)\}$

b) $f = \{(2,3),(1,5),(3,4),(2,5)\}$

c) $f = \{(2,1),(3,5),(5,3)\}$

d) $f = \{(3,7)\}$

e) $f = \{(1,5),(2,3),(3,7),(2,4)\}$

f) $f = \{(2,3),(3,2)\}$

g) $f = \{(1,3),(2,1),(3,2),(4,4),(5,6)\}$

h) $f = \{(2,4),(4,1),(1,1),(3,6),(5,5)\}$

i) $f = \{(3,7),(5,5),(6,2),(1,9),(2,7)\}$

Functions - Basic Concepts

- A real-valued function (or just function) f is a mapping $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and $A \neq \emptyset$
- $A = \mathrm{dom}(f)$ ⟵ Domain of f

  $f(A) = \{f(x) \mid x \in A\}$ ⟵ Range of f.

↳ To properly define a function f, we must define

a) The domain $A = \mathrm{dom}(f)$

b) The formula $y = f(x)$.

examples : How to write a function definition

a) Let $f: [0, +\infty) \to \mathbb{R}$ be a function with $f(x) = \sqrt{x}$

b) Let $f(x) = \sqrt{x}$, $\forall x \in [0, +\infty)$.

↳ When the domain A is not given, we assume by default the widest possible subset of $\mathbb{R}$ for which the formula $y = f(x)$ yields a real number.

- Method : To find the default domain of a function f we introduce the necessary constraints such that

a) There is NO DIVISION BY ZERO

b) There is NO ROOT OF A NEGATIVE NUMBER.

EXAMPLES

a) For $f(x) = x^2 + 3x + 1$, evaluate $f(1+\sqrt{2})$ and $f(2a-1)$.

Solution

$$f(1+\sqrt{2}) = (1+\sqrt{2})^2 + 3(1+\sqrt{2}) + 1 =$$
$$= (1 + 2\sqrt{2} + 2) + 3(1+\sqrt{2}) + 1 =$$
$$= 1 + 2\sqrt{2} + 2 + 3 + 3\sqrt{2} + 1 = 7 + 5\sqrt{2}$$
$$f(2a-1) = (2a-1)^2 + 3(2a-1) + 1 =$$
$$= 4a^2 - 4a + 1 + 6a - 3 + 1 =$$
$$= 4a^2 + 2a - 1.$$

b) Find the default domain for $f(x) = x^3(x^2+1)^4$.

Solution

No constraints, thus $A = \mathbb{R}$.

$\hookrightarrow$ For polynomial functions of the form
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
there are no constraints, therefore the default domain is always $A = \mathbb{R}$.

c) Find the default domain for $f(x) = \dfrac{x^2-9}{x^2+4x+3}$

Solution

We require $x^2+4x+3 \neq 0$

Solve: $x^2+4x+3=0 \Leftrightarrow (x+1)(x+3)=0 \Leftrightarrow$

$\Leftrightarrow x+1=0 \lor x+3=0 \Leftrightarrow x=-1 \lor x=-3.$

It follows that $A = \mathbb{R} - \{-1,-3\}.$

$= (-\infty,-3) \cup (-3,-1) \cup (-1,+\infty).$

$\hookrightarrow$ Note that this function can be simplified to:

$$f(x) = \frac{x^2-9}{x^2+4x+3} = \frac{(x-3)(x+3)}{(x+3)(x+1)} = \frac{x-3}{x+1}$$

However, the default domain of the simplified formula is wider: $A = \mathbb{R} - \{-1\}$. Thus, to find the correct default domain, you must NOT try to simplify or otherwise modify the formula for $f(x)$ before writing down the constraints.

d) Find the default domain of

$$f(x) = \sqrt{\frac{2x+1}{1+3x}} \quad \text{and} \quad g(x) = \frac{\sqrt{2x+1}}{\sqrt{1+3x}}$$

Solution

- **For $f(x)$**

Require $\dfrac{2x+1}{1+3x} \geqslant 0$. (1)

x		$-1/2$		$-1/3$	
$2x+1$	$-$	0	$+$		$+$
$1+3x$	$-$		$-$	0	$+$
ineq	$+$	0	$-$	$\Vert$	$+$

Thus (1) $\Leftrightarrow x \in (-\infty, -1/2] \cup (-1/3, +\infty)$
and therefore $A = (-\infty, -1/2] \cup (-1/3, +\infty)$.

- **For $g(x)$**

Require: $\begin{cases} 2x+1 \geqslant 0 \\ 1+3x > 0 \end{cases} \Leftrightarrow \begin{cases} 2x \geqslant -1 \\ 3x > -1 \end{cases} \Leftrightarrow \begin{cases} x \geqslant -1/2 \\ x > -1/3 \end{cases}$

$\Leftrightarrow x > -1/3$

therefore $A = (-1/3, +\infty)$.

EXERCISES

④ Find the default domain for the following functions

a) $f(x) = x^2(x+1)^3$

b) $f(x) = \dfrac{3}{x-1}$

c) $f(x) = \dfrac{x-2}{x^2-4}$

d) $f(x) = \dfrac{3x^2}{x^2-4x}$

e) $f(x) = \dfrac{2x^2-3}{x^2+5x+6}$

f) $f(x) = \dfrac{x^2-4}{x^2-5x+6}$

g) $f(x) = \dfrac{x+2}{x-3}$

$\left.\begin{array}{l} \text{f)} \\ \text{g)} \end{array}\right\}$ compare

h) $f(x) = \sqrt{3-x}$

i) $f(x) = \sqrt{x^2-x-6}$

j) $f(x) = \sqrt{x^3+5x^2-6x}$

k) $f(x) = \sqrt{x^4-x^3+x-1}$

l) $f(x) = \sqrt{\dfrac{x+2}{x-3}}$

m) $f(x) = \dfrac{\sqrt{x+2}}{\sqrt{x-3}}$

l) $f(x) = \sqrt{\dfrac{x^2-4}{x^2-9}}$

⑤ Find the default domain for the following functions

a) $f(x) = \dfrac{-3|x-1|}{|x+4}$

b) $f(x) = \dfrac{x-1}{|x-2|-2}$

c) $f(x) = \sqrt{|3x-1|-2}$

d) $f(x) = \dfrac{\sqrt{x}}{|x|}$

e) $f(x) = \sqrt{\dfrac{|x|-2}{|x|+1}}$

f) $f(x) = \dfrac{x-1}{\sqrt{2-|x+2|}}$

Algebra with functions

- Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ be two functions. We also define
$$C = \{x \in B \mid g(x) = 0\}$$
We now give the following definitions:

1) Equality:
$$\boxed{f = g \Leftrightarrow \begin{cases} dom(f) = dom(g) \\ \forall x \in dom(f): f(x) = g(x) \end{cases}}$$

2) Sum $f+g$

$$dom(f+g) = dom(f) \cap dom(g) = A \cap B$$
$$\forall x \in A \cap B: (f+g)(x) = f(x) + g(x)$$

3) Product fg

$$dom(fg) = dom(f) \cap dom(g) = A \cap B$$
$$\forall x \in A \cap B: (fg)(x) = f(x)g(x)$$

4) Scalar Product cf with $c \in \mathbb{R}$

$$dom(cf) = dom(f)$$
$$\forall x \in A: (cf)(x) = cf(x)$$

5) Division f/g

$$dom(f/g) = [dom(f) \cap dom(g)] - \{x \in B \mid g(x) = 0\}$$
$$= (A \cap B) - C$$
$$\forall x \in dom(f/g): (f/g)(x) = \frac{f(x)}{g(x)}$$

EXAMPLES

a) Given the functions:

$f = \{(1,3),(2,7),(3,6),(4,1)\}$

$g = \{(3,1),(5,2),(4,9),(6,3)\}$

define the function $h = f+g$.

Solution

$\mathrm{dom}(h) = \mathrm{dom}(f+g) = \mathrm{dom}(f) \cap \mathrm{dom}(g) =$

$= \{1,2,3,4\} \cap \{3,5,4,6\} = \{3,4\}$.

$h(3) = (f+g)(3) = f(3)+g(3) = 6+1 = 7$

$h(4) = (f+g)(4) = f(4)+g(4) = 1+9 = 10$

It follows that

$h = f+g = \{(3,7),(4,10)\}$.

b) Given the functions $f(x) = \sqrt{9-x^2}$ and $g(x) = \sqrt{x^2-1}$, define the functions $h = \dfrac{f}{g}$.

and $\varphi = f+g$

Solution

- **Domain of f**

Require $9-x^2 \geq 0 \Leftrightarrow (3-x)(3+x) \geq 0$
$\Leftrightarrow x \in [-3,3]$

x		-3		3	
$3-x$	$+$		$+$	0	$-$
$3+x$	$-$	0	$+$		$+$
	$-$	0	$+$	0	$-$

thus $\text{dom}(f) = [-3,3]$.

- **Domain of g**

Require $x^2-1 \geq 0 \Leftrightarrow (x-1)(x+1) \geq 0 \Leftrightarrow$
$\Leftrightarrow x \in (-\infty,-1] \cup [1,+\infty)$

x		-1		$+1$	
$x-1$	$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$
	$+$	0	$-$	0	$+$

- **Defining $h = f/g$**

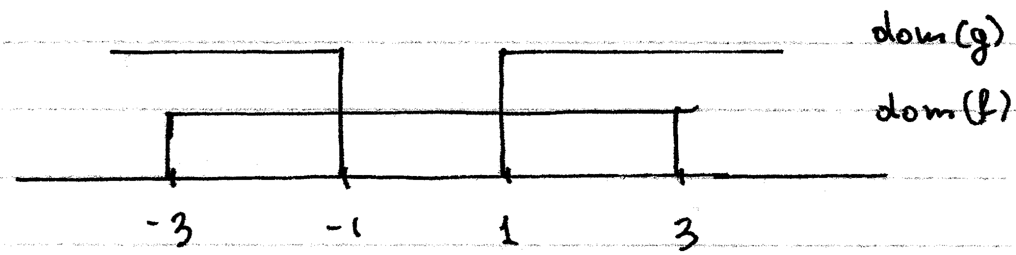
▸ First we find the domain of h.

Solve $g(x) = 0 \Leftrightarrow \sqrt{x^2-1} = 0 \Leftrightarrow x^2-1=0 \Leftrightarrow x^2 = 1$
$\Leftrightarrow x = 1 \vee x = -1 \Leftrightarrow x \in \{-1,1\}$.

It follows that

$$\begin{aligned}
\text{dom}(h) = \text{dom}(f/g) &= (\text{dom} f \cap \text{dom} g) - \{x \in \mathbb{R} \mid g(x) = 0\} \\
&= ([-3,3] \cap ((-\infty,-1] \cup [1,+\infty)) - \{-1,1\} = \\
&= ([-3,-1] \cup [1,3]) - \{-1,1\} = \\
&= [-3,-1) \cup (1,3]
\end{aligned}$$

On the 2nd line, to find the intersection we use:



$\mathrm{dom}\, f \cap \mathrm{dom}\, g = [-3,-1] \cup [1,3]$.

- The formula for h is:

$$h(x) = \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{9-x^2}}{\sqrt{x^2-1}}, \quad \forall x \in [-3,-1] \cup [1,3]$$

↳ Note again that to define a new function we must determine BOTH the domain and the formula of the new function

- <u>Defining $\varphi = f+g$</u>

$$\mathrm{dom}(\varphi) = \mathrm{dom}(f+g) = \mathrm{dom}(f) \cap \mathrm{dom}(g) =$$
$$= \dots = [-3,-1] \cup [1,3]$$

$$\varphi(x) = (f+g)(x) = f(x) + g(x) =$$
$$= \sqrt{9-x^2} + \sqrt{x^2-1}, \quad \forall x \in [-3,-1] \cup [1,3].$$

↳ In problems with multiple requests, it is sufficient to calculate $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ once, and then reuse the result, as we have done above.

c) Given the functions $f_1 : A \to \mathbb{R}$, $f_2 : A \to \mathbb{R}$, $g_1 : B \to \mathbb{R}$, and $g_2 : B \to \mathbb{R}$ with $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$, and $A \cap B \neq \emptyset$, show that

$$f_1 = f_2 \wedge g_1 = g_2 \Rightarrow f_1 + g_1 = f_2 + g_2$$

<u>Solution</u>

Assume that $f_1 = f_2$ and $g_1 = g_2$. Then

$f_1 = f_2 \Rightarrow \forall x \in A : f_1(x) = f_2(x)$ (1)

$g_1 = g_2 \Rightarrow \forall x \in B : g_1(x) = g_2(x)$ (2)

It follows that

$$\left.\begin{array}{l} \mathrm{dom}(f_1 + g_1) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, g_1 = A \cap B \\ \mathrm{dom}(f_2 + g_2) = \mathrm{dom}\, f_2 \cap \mathrm{dom}\, g_2 = A \cap B \end{array}\right\} \Rightarrow$$

$\Rightarrow \mathrm{dom}(f_1 + g_1) = \mathrm{dom}(f_2 + g_2)$ (3)

and

$$\begin{aligned} (f_1 + g_1)(x) &= f_1(x) + g_1(x) = && [\text{by def}] \\ &= f_2(x) + g_2(x) = && [\text{use (1) and (2)}] \\ &= (f_2 + g_2)(x), \ \forall x \in A \cap B. && (4) \end{aligned}$$

From (3) and (4): $f_1 + g_1 = f_2 + g_2$.

Note that to show that two functions f, g are equal (i.e. $f = g$), we have to show that:

1) Both functions have the same domain A

$\mathrm{dom}(f) = \mathrm{dom}(g) = A$

2) The formulas for f and g always agree:

$\forall x \in A : f(x) = g(x)$

d) Given the functions $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ with $A \cap B \neq \emptyset$ and the numbers $a, b \in \mathbb{R}$, show that $(af)(bg) = (ab)(fg)$.

Solution

$$\operatorname{dom}[(af)(bg)] = \operatorname{dom}(af) \cap \operatorname{dom}(bg) =$$
$$= \operatorname{dom}(f) \cap \operatorname{dom}(g) = A \cap B \quad (1)$$
$$\operatorname{dom}[(ab)(fg)] = \operatorname{dom}(fg) = \operatorname{dom}(f) \cap \operatorname{dom}(g) = A \cap B \quad (2)$$
From (1) and (2):
$$\operatorname{dom}[(af)(bg)] = \operatorname{dom}[(ab)(fg)]. \quad (3)$$
We also note that
$$[(af)(bg)](x) = (af)(x) \cdot (bg)(x) = [af(x)][bg(x)]$$
$$= ab\, f(x)\, g(x) = ab\,(fg)(x) =$$
$$= [(ab)(fg)](x), \ \forall x \in A \cap B \quad (4)$$
From (3) and (4): $(af)(bg) = (ab)(fg)$.

EXERCISES

⑥ Define the functions $h_1 = f+g$, $h_2 = fg$, $h_3 = f/g$, and $h_4 = g/f$, when f and g are defined as:

a) $f = \{(1,3),(2,4),(3,1),(4,7)\}$
 $g = \{(2,0),(3,2),(4,1),(5,3)\}$

b) $f = \{(1,1),(3,2),(2,5),(5,6)\}$
 $g = \{(2,0),(3,1),(4,3),(5,2),(0,2)\}$

c) $f = \{(1,3),(3,1)\}$
 $g = \{(1,0),(0,1),(2,2),(3,4)\}$

d) $f = \{(2,3),(3,6),(4,2),(5,1),(6,2)\}$
 $g = \{(1,5),(3,2),(6,3),(2,4)\}$

⑦ Let f, g, h be functions with $f: A \to \mathbb{R}$, $g: A \to \mathbb{R}$, and $h: B \to \mathbb{R}$. Show that

a) $f = g \Rightarrow f+h = g+h$

b) $f = g \Rightarrow fh = gh$

c) $(-f)(-g) = fg$.

⑧ Find the default domain for the functions f, g and define the functions $h_1 = f+g$, $h_2 = fg$, $h_3 = f/g$ with f and g given by:

a) $f(x) = \sqrt{1-x2}$ and $g(x) = 1/x$

b) $f(x) = x^2\sqrt{1-x}$ and $g(x) = \dfrac{2x}{\sqrt{1-x}}$

c) $f(x) = \sqrt{4-x^2}$ and $g(x) = \dfrac{3x+1}{\sqrt{x^2+x}}$

d) $f(x) = \sqrt{2x+3}$ and $g(x) = \sqrt{x^2+x-2}$

9 Consider the functions f, g, h with $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$, and $h: B \to \mathbb{R}$ where $A \cap B \neq \emptyset$. Show that

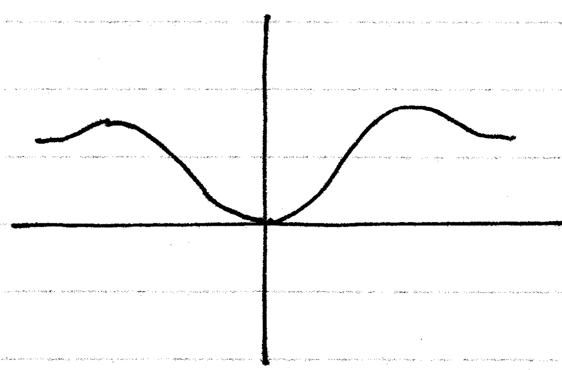
$$f(ag+bh) = a(fg) + b(fh)$$

▼ Odd and even functions

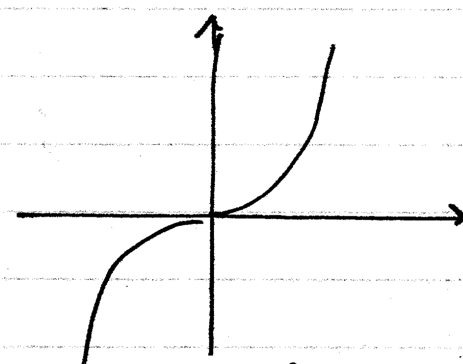
- Let $f: A \to \mathbb{R}$ be a function. We say that

$$\boxed{\begin{array}{l} f \text{ even} \Leftrightarrow \forall x \in A : (-x \in A \wedge f(-x) = f(x)) \\ f \text{ odd} \Leftrightarrow \forall x \in A : (-x \in A \wedge f(-x) = -f(x)) \end{array}}$$

- A prerequisite for the function f to be odd or even is that it has a domain A that is symmetric around the origin (i.e. $\forall x \in A : -x \in A$). If the domain is not symmetric, then the function can be neither even nor odd.
- An even function has graph that is symmetric accross the y-axis.
- An odd function has graph such that the $x<0$ part is obtained by first reflecting the $x>0$ part accross the y-axis and then the x-axis.



f even



f odd.

EXAMPLES

Which of the following functions are odd or even?

a) $f(x) = |3x+2| + |3x-2|$

Solution

Domain $A = \mathbb{R}$ is symmetric.

$f(-x) = |3(-x)+2| + |3(-x)-2| =$

$= |-3x+2| + |-3x-2| =$

$= |3x-2| + |3x+2| =$

$= |3x+2| + |3x-2| = f(x)\ ,\ \forall x \in \mathbb{R} \Rightarrow$

$\Rightarrow f$ even.

b) $f(x) = \dfrac{5x^3}{|x|-1}$

Solution

Domain:

Solve $|x|-1=0 \Leftrightarrow |x| = 1 \Leftrightarrow x = 1 \lor x = -1$

thus $A = \mathbb{R} - \{-1, 1\}$ which is symmetric.

$f(-x) = \dfrac{5(-x)^3}{|-x|-1} = \dfrac{-5x^3}{|x|-1} = -f(x),\ \forall x \in A \Rightarrow$

$\Rightarrow f$ odd.

c) $f(x) = \dfrac{x^2}{x+1}$

Solution

- Domain.

Solve $x+1=0 \Leftrightarrow x=-1$,
thus $A = \mathbb{R} - \{-1\}$.

- Note that A is not symmetric because $-1 \notin A$ and $+1 \in A$. Therefore f is not even and f is not odd.

→ To establish that a function is even or odd, you have to first find the domain and show that it is symmetric before you calculate $f(-x)$. If the domain is not symmetric, then the function is neither odd nor even.

d) If f is an odd function and g an even function, show that fg is an odd function.

Solution

Define $A = \mathrm{dom}(f)$ and $B = \mathrm{dom}(g)$.
$\mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g) = A \cap B$

- Symmetry.

f odd $\Rightarrow A$ symmetric
g even $\Rightarrow B$ symmetric

$$-x \in A \cap B \Rightarrow -x \in A \wedge -x \in B \Rightarrow \quad \text{[set intersection]}$$
$$\Rightarrow x \in A \wedge x \in B \Rightarrow \quad \text{[A, B symmetric]}$$
$$\Rightarrow x \in A \cap B \quad \text{[set intersection]}$$

therefore $A \cap B$ is symmetric.

- Parity

$$(fg)(-x) = f(-x)g(-x) = [-f(x)]\,g(x) =$$
$$= -f(x)g(x) = -(fg)(x), \; \forall x \in A \cap B \Rightarrow$$
$\Rightarrow fg$ odd.

↳ In general to prove that a set A is symmetric we must prove

$$-x \in A \Rightarrow x \in A$$

or equivalently

$$x \in A \Rightarrow -x \in A$$

To do that we use the following properties:

$$x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$$
$$x \in A \cup B \Leftrightarrow x \in A \vee x \in B$$
$$x \in A - B \Leftrightarrow x \in A \wedge x \notin B.$$

as well as any pre-existing symmetry assumptions.

EXERCISES

(10) Which of the following functions are odd or even? Show that they are odd or even, when applicable, or show that they are neither odd nor even.

a) $f(x) = |x+4| + |x-4|$

b) $f(x) = 2x^3 - 3x$

c) $f(x) = 2x^4 - 3x^2 + 5$

d) $f(x) = \sqrt{1-x^2}$

e) $f(x) = \dfrac{x-1}{x+1}$

f) $f(x) = \dfrac{3x|x|}{2|x|+1}$

g) $f(x) = \dfrac{5x^5 - 4x^3}{x^4+3}$

h) $f(x) = \dfrac{x^2+4}{x^2+5x+6}$

i) $f(x) = \sqrt{1-x} + \sqrt{1+x}$

j) $f(x) = \dfrac{2x+1}{2x-1} + \dfrac{2x-1}{2x+1}$

(11) If f, g are even functions, show that $f+g$ and fg are also even.

(12) If f, g are odd functions, show that $f+g$ and fg are also odd.

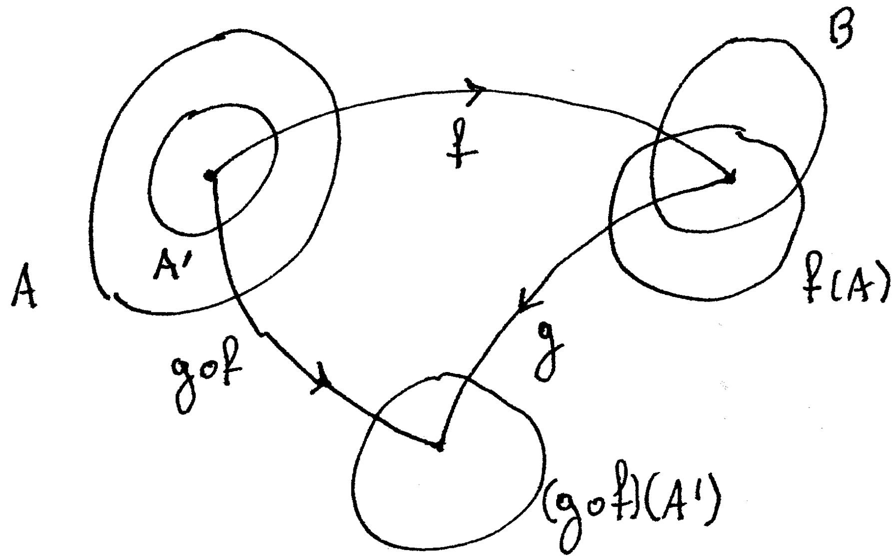
Function composition

- Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ be two functions. We assume that $f(A) \cap B \neq \emptyset$. Let A' be the subset of A whose elements are mapped by f into the intersection $f(A) \cap B$. Thus A' is given by

$$A' = \{x \in A \mid f(x) \in B\}.$$

We may therefore define the function $g \circ f : A' \to \mathbb{R}$ as follows:

$$\boxed{\begin{array}{l} \mathrm{dom}(g \circ f) = \{x \in \mathrm{dom}(f) \mid f(x) \in \mathrm{dom}(g)\} = A' \\ \forall x \in A' : \ (g \circ f)(x) = g(f(x)) \end{array}}$$



- We note that the belonging condition for $g \circ f$ is

$$\boxed{x \in \mathrm{dom}(g \circ f) \Leftrightarrow \begin{cases} x \in \mathrm{dom}(f) \\ f(x) \in \mathrm{dom}(g) \end{cases}}$$

<u>Method</u>: To define $g \circ f$:

•1 Find the domain $dom(g \circ f)$ by solving:

$$x \in dom(g \circ f) \Leftrightarrow \begin{cases} x \in dom(f) \\ f(x) \in dom(g) \end{cases} \Leftrightarrow \cdots$$

•2 Find the formula of $g \circ f$:

$$(g \circ f)(x) = g(f(x)) = \cdots$$

<u>example</u>

For $f(x) = \sqrt{1-x^2}$ and $g(x) = x^2 + 3x + 2$

$x \in dom(f) \Leftrightarrow 1 - x^2 \geq 0 \Leftrightarrow \cdots \Leftrightarrow x \in [-1, 1]$

thus $dom(f) = [-1, 1]$

Also $dom(g) = \mathbb{R}$.

(a) To find $g \circ f$:

$$x \in dom(g \circ f) \Leftrightarrow \begin{cases} x \in dom(f) \\ f(x) \in dom(g) \end{cases} \Leftrightarrow \begin{cases} x \in [-1, 1] \\ \sqrt{1-x^2} \in \mathbb{R} \end{cases}$$

$$\Leftrightarrow x \in [-1, 1]$$

thus $dom(g \circ f) = [-1, 1]$ and

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{1-x^2}) =$$

$$= (\sqrt{1-x^2})^2 + 3\sqrt{1-x^2} + 2$$

$$= 1 - x^2 + 3\sqrt{1-x^2} + 2 =$$

$$= 3 - x^2 + 3\sqrt{1-x^2}.$$

(b) To find $f \circ g$:

$$x \in dom(f \circ g) \Leftrightarrow \begin{cases} x \in dom(g) \\ g(x) \in dom(f) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x \in \mathbb{R} \\ x^2+3x+2 \in [-1,1] \end{cases} \Leftrightarrow -1 \leq x^2+3x+2 \leq 1$$

$$\Leftrightarrow \begin{cases} x^2+3x+3 \geq 0 & (1) \\ x^2+3x+1 \leq 0 & (2) \end{cases}$$

For (1): $\left.\begin{array}{l} \Delta = 9-4\cdot 1\cdot 3 = 9-12 < 0 \\ a = 1 > 0 \end{array}\right\} \Rightarrow$

$\Rightarrow x^2+3x+3 > 0 \ , \forall x \in \mathbb{R}$

$\Rightarrow$ (1) identity.

For (2): $\Delta = 9-4\cdot 1\cdot 1 = 9-4 = 5 \Rightarrow$

$\Rightarrow x_{1,2} = \dfrac{-3 \pm \sqrt{5}}{2}$

(2) $\Leftrightarrow x \in [x_1, x_2]$

Thus

$dom(f \circ g) = \left[\dfrac{-3-\sqrt{5}}{2}, \dfrac{-3+\sqrt{5}}{2}\right]$ and

$(f \circ g)(x) = f(x^2+3x+2) =$

$= \sqrt{1-(x^2+3x+2)^2}$

EXAMPLES

a) Given $f = \{(1,4),(2,2),(3,1),(4,1)\}$ and
$g = \{(2,3),(3,2),(4,5)\}$
define $f \circ g$ and $g \circ f$

Solution

We note that $dom(f) = \{1,2,3,4\}$ and $dom(g) = \{2,3,4\}$.

• $f \circ g$ definition:

$g(2) = 3 \in dom(f) \Rightarrow (f \circ g)(2) = f(g(2)) = f(3) = 1$
$g(3) = 2 \in dom(f) \Rightarrow (f \circ g)(3) = f(g(3)) = f(2) = 2$
$g(4) = 5 \notin dom(f) \Rightarrow 5 \notin dom(f \circ g)$.

It follows that $f \circ g = \{(2,1),(3,2)\}$.

• $g \circ f$ definition

$f(1) = 4 \in dom(g) \Rightarrow (g \circ f)(1) = g(f(1)) = g(4) = 5$
$f(2) = 2 \in dom(g) \Rightarrow (g \circ f)(2) = g(f(2)) = g(2) = 3$
$f(3) = 1 \notin dom(g) \Rightarrow 3 \notin dom(g \circ f)$
$f(4) = 1 \notin dom(g) \Rightarrow 4 \notin dom(g \circ f)$

It follows that $g \circ f = \{(1,5),(2,3)\}$

↳ To evaluate $f \circ g$ for a discrete problem, first we write down $dom(f)$ and $dom(g)$. Then, for each element $x \in dom(g)$ we do the following:

- 1 Calculate $g(x)$.
- 2 If $g(x) \in dom(f)$, then we can go ahead and calculate $(f \circ g)(x)$.
- 3 If $g(x) \notin dom(f)$, then $x \notin dom(f \circ g)$; in other words, $(f \circ g)(x)$ cannot be calculated and that x is not in the domain of $f \circ g$.

b) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Show that
f even and g odd $\Rightarrow g \circ f$ even $\wedge$ $f \circ g$ even.

<u>Solution</u>

- <u>For $g \circ f$.</u>

Since $dom(f) = \mathbb{R}$ and $dom(g) = \mathbb{R}$, it follows that:

$$dom(g \circ f) = \{x \in dom(f) \mid f(x) \in dom(g)\} =$$
$$= \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\} = \mathbb{R}.$$

which is symmetric. Furthermore:

$$(g \circ f)(-x) = g(f(-x)) \qquad [\text{def.}]$$
$$= g(f(x)) \qquad [f \text{ even}]$$
$$= (g \circ f)(x), \forall x \in \mathbb{R} \qquad [\text{def.}]$$

$\Rightarrow g \circ f$ even.

- <u>For $f \circ g$</u>

$$dom(f \circ g) = \{x \in dom(g) \mid g(x) \in dom(f)\}$$
$$= \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\} = \mathbb{R}$$

which is symmetric. Furthermore:

$$(f \circ g)(-x) = f(g(-x)) \qquad [\text{def}]$$
$$= f(-g(x)) \qquad [g \text{ odd}]$$

$$= f(g(x)) \qquad [f \text{ even}]$$

$$= (f \circ g)(x), \forall x \in \mathbb{R} \qquad [\text{def}]$$

$\Rightarrow f \circ g$ even.

EXERCISES

(13) Find $f \circ g$ and $g \circ f$ for the following functions

a) $f(x) = x^2+1$, $g(x) = \sqrt{3-x}$

b) $f(x) = 2x+1$, $g(x) = x^2+2$

c) $f(x) = \sqrt{4-x^2}$, $g(x) = \sqrt{1-x^2}$

d) $f(x) = \dfrac{x+2}{x+1}$, $g(x) = \dfrac{1}{x}$

• (14) Let f, g, h be three functions. Show that

$$f = g \Rightarrow f \circ h = g \circ h$$

• (15) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Show that

a) f even and g even $\Rightarrow f \circ g$ even

b) f odd and g odd $\Rightarrow f \circ g$ odd

c) f even and g odd $\Rightarrow f \circ g$ even

(16) Let $f = \{(1,3), (2,4), (3,1), (4,2)\}$

$g = \{(2,4), (3,1), (4,2)\}$

Define $f \circ g$ and $g \circ f$.

(17) Let $f = \{(1,2), (3,2), (2,4), (4,4)\}$

$g = \{(1,3), (2,1), (3,5), (4,2)\}$

Define $f \circ g$ and $g \circ f$.

Functions and Monotonicity

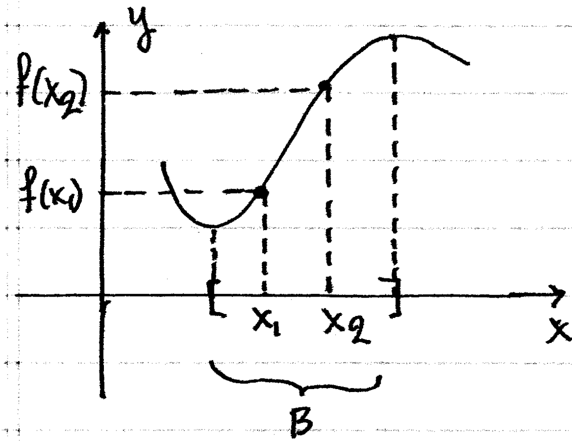
Let f be a function with $f: A \to \mathbb{R}$ and let $B \subseteq A$. We make the following definitions:

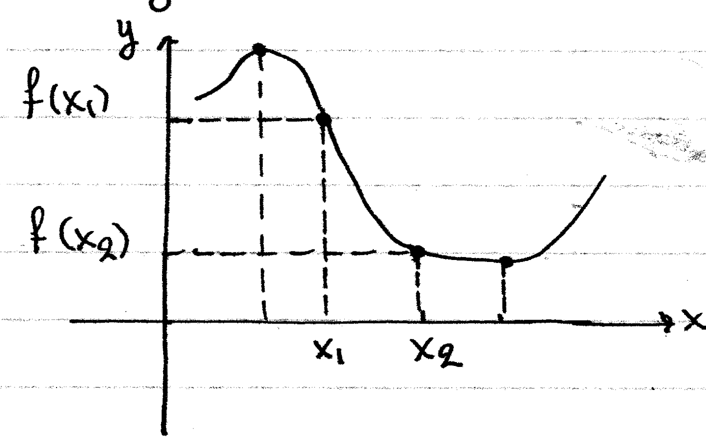
$$\boxed{\begin{array}{l} f \nearrow B \Leftrightarrow \forall x_1, x_2 \in B : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2)) \\ f \searrow B \Leftrightarrow \forall x_1, x_2 \in B : (x_1 < x_2 \Rightarrow f(x_1) > f(x_2)) \end{array}}$$

We read:

$f \nearrow B$: f is strictly increasing in B

$f \searrow B$: f is strictly decreasing in B.





Monotonicity can be determined directly from the definition with 2 methods:

1) Analytic Method
2) Synthetic Method.

In Calculus, monotonicity can also be determined using Differential Calculus.

↳ Analytic Method

To show $f \nearrow B$ or $f \searrow B$.

- 1 Let $x_1, x_2 \in B$ be given with $x_1 < x_2$.
- 2 Calculate and factor $\Delta f(x_1, x_2) = f(x_2) - f(x_1)$
- 3 Determine the sign of each factor of Δf and then conclude whether $\Delta f > 0$ or $\Delta f < 0$.
- 4 Finish the argument.

EXAMPLES

a) Show that $f(x) = 3x + 5$ is strictly increasing in $\mathbb{R}$.

Solution

$dom(f) = \mathbb{R}$.

Let $x_1, x_2 \in \mathbb{R}$ be given with $x_1 < x_2$.

$$\Delta f(x_1, x_2) = f(x_2) - f(x_1) = (3x_2 + 5) - (3x_1 + 5) =$$
$$= 3(x_2 - x_1)$$

Since $x_1 < x_2 \Rightarrow x_2 - x_1 > 0 \Rightarrow$

$$\Rightarrow 3(x_2 - x_1) > 0 \Rightarrow$$
$$\Rightarrow f(x_2) - f(x_1) > 0 \Rightarrow$$
$$\Rightarrow f(x_1) < f(x_2)$$

- Thus: $\forall x_1, x_2 \in \mathbb{R}: (x_1 < x_2 \Rightarrow f(x_1) < f(x_2)) \Rightarrow f \nearrow \mathbb{R}$.

b) Show that $f(x) = \dfrac{2x}{x-1}$ is strictly decreasing in $(1, +\infty)$.

Solution

Let $x_1, x_2 \in (1, +\infty)$ be given with $x_1 < x_2$.
Then:

$$\Delta f(x_1, x_2) = f(x_2) - f(x_1) = \frac{2x_2}{x_2 - 1} - \frac{2x_1}{x_1 - 1} =$$

$$= \frac{2x_2(x_1-1) - 2x_1(x_2-1)}{(x_1-1)(x_2-1)} =$$

$$= \frac{2x_1x_2 - 2x_2 - 2x_1x_2 + 2x_1}{(x_1-1)(x_2-1)} =$$

$$= \frac{-2x_2 + 2x_1}{(x_1-1)(x_2-1)} = \frac{2(x_1 - x_2)}{(x_1-1)(x_2-1)}$$

Since $x_1 < x_2 \Rightarrow x_1 - x_2 < 0$.

$x_1 \in (1, +\infty) \Rightarrow x_1 > 1 \Rightarrow x_1 - 1 > 0$

$x_2 \in (1, +\infty) \Rightarrow x_2 > 1 \Rightarrow x_2 - 1 > 0$

therefore $\Delta f(x_1, x_2) < 0 \Rightarrow f(x_2) - f(x_1) < 0 \Rightarrow$

$\Rightarrow f(x_1) > f(x_2)$

Thus:

$\forall x_1, x_2 \in (1, +\infty) : (x_1 < x_2 \Rightarrow f(x_1) > f(x_2)) \Rightarrow$

$\Rightarrow f \searrow (1, +\infty)$.

c) Show that $f(x) = x^2 + 5x + 6$ is strictly increasing in $(-5/2, +\infty)$.

Solution

Let $x_1, x_2 \in (-5/2, +\infty)$ be given with $x_1 < x_2$

Then

$$\Delta f(x_1, x_2) = f(x_2) - f(x_1) = (x_2^2 + 5x_2 + 6) - (x_1^2 + 5x_1 + 6)$$
$$= (x_2^2 - x_1^2) + 5(x_2 - x_1) =$$
$$= (x_2 - x_1)(x_2 + x_1) + 5(x_2 - x_1) =$$
$$= (x_2 - x_1)(x_2 + x_1 + 5)$$

Since $x_1 < x_2 \Rightarrow x_2 - x_1 > 0 \quad (1)$

$$\left.\begin{array}{l} x_1 \in (-5/2, +\infty) \Rightarrow x_1 > -5/2 \\ x_2 \in (-5/2, +\infty) \Rightarrow x_2 > -5/2 \end{array}\right\} \Rightarrow$$

$\Rightarrow x_1 + x_2 > -5/2 - 5/2 = -5 \Rightarrow x_1 + x_2 + 5 > 0 \quad (2)$

From (1) and (2):

$\Delta f(x_1, x_2) > 0 \Rightarrow f(x_2) - f(x_1) > 0 \Rightarrow \underline{f(x_1) < f(x_2)}$

It follows that:

$\forall x_1, x_2 \in (-5/2, +\infty): (x_1 < x_2 \Rightarrow f(x_1) < f(x_2)) \Rightarrow$

$\Rightarrow f \uparrow (-5/2, +\infty)$.

↳ For quadratics $f(x) = ax^2 + bx + c$, monotonicity changes at the axis of symmetry at $x = -b/2a$.

↳ In addition to the usual properties, it is good to know the following additional properties:

1) We can add two inequalities if they have the same direction:

$$\left.\begin{array}{l} a > b \\ x > y \end{array}\right\} \Rightarrow a + x > b + y$$

2) We can multiply two inequalities if they have the same direction AND all sides are POSITIVE!

$$\left.\begin{array}{l} a > b > 0 \\ x > y > 0 \end{array}\right\} \Rightarrow ax > by$$

3) We can raise an inequality to a positive power if both sides of the inequality are positive

$$\left.\begin{array}{l} a > b > 0 \\ p > 0 \end{array}\right\} \Rightarrow a^p > b^p > 0$$

e.g. $a > b > 0 \Rightarrow \sqrt{a} > \sqrt{b} > 0$ for $p = 1/2$.

4) We can raise an inequality to a negative power if both sides of the inequality are positive but then the direction of the inequality is reversed.

$$\left.\begin{array}{l} a > b > 0 \\ n < 0 \end{array}\right\} \Rightarrow 0 < a^n < b^n$$

e.g. $a > b > 0 \Rightarrow 0 < \dfrac{1}{a} < \dfrac{1}{b}$ for $n = -1$.

We rely on these properties heavily for the synthetic method. We also need the following previously mentioned properties:

5) $x < y \Rightarrow x + a < y + a$

6) $\left.\begin{array}{l} x < y \\ p > 0 \end{array}\right\} \Rightarrow px < py$ 7) $\left.\begin{array}{l} x < y \\ n < 0 \end{array}\right\} \Rightarrow nx > ny$.

to add/multiply a constant to both sides of an inequality.

Synthetic Method

To show that $f \nearrow B$ or $f \searrow B$:

- 1 Let $x_1, x_2 \in B$ be given with $x_1 < x_2$.
- 2 Use a sequence of deductions to show that
  $x_1 < x_2 \Rightarrow \dots \Rightarrow \dots \Rightarrow f(x_1) < f(x_2)$
  or
  $x_1 < x_2 \Rightarrow \dots \Rightarrow \dots \Rightarrow f(x_1) > f(x_2)$
  using the above properties of inequalities.
- 3 Wrap up the argument.

EXAMPLES

a) For $f(x) = 3-(1-2x)^2$ show that $f \searrow (1/2, +\infty)$

Solution

Let $x_1, x_2 \in (1/2, +\infty)$ be given with $x_1 < x_2$. Then:

$x_1 < x_2 \Rightarrow -2x_1 > -2x_2 \Rightarrow 1-2x_1 > 1-2x_2 \overset{*}{\Rightarrow}$

$\Rightarrow \underset{(!)}{\underline{0 <}}\, 2x_1 - 1 < 2x_2 - 1$ [because $x_1 > 1/2 \wedge x_2 > 1/2$]

$\Rightarrow (2x_1-1)^2 < (2x_2-1)^2 \overset{**}{\Rightarrow} (1-2x_1)^2 < (1-2x_2)^2$

$\Rightarrow -(1-2x_1) > -(1-2x_2)^2 \Rightarrow 3-(1-2x_1)^2 > 3-(1-2x_2)^2$

$\Rightarrow f(x_1) > f(x_2)$.

Thus: $\forall x_1, x_2 \in (1/2, +\infty) : (x_1 < x_2 \Rightarrow f(x_1) > f(x_2))$

$\Rightarrow f \searrow (1/2, +\infty)$.

* We multiply inequality with -1 to ensure that both sides are positive before going ahead and squaring it.

** Here we use $x^2 = (-x)^2$.

↳ In the above solution you should be able to identify which inequality property is used at every step.

b) For $f(x) = 3x + 1 + \sqrt{1 - x^2}$, show that $f \nearrow (-1, 0)$

<u>Solution</u>

Let $\underline{x_1, x_2 \in (-1, 0)}$ be given such that $\underline{x_1 < x_2}$. Then

$$x_1 < x_2 \Rightarrow 3x_1 < 3x_2 \Rightarrow 3x_1 + 1 < 3x_2 + 1 \quad (1)$$

Also note that

$$x_1 < x_2 \Rightarrow -x_1 > -x_2 > 0 \Rightarrow (-x_1)^2 > (-x_2)^2 \Rightarrow x_1^2 > x_2^2$$
$$\Rightarrow -x_1^2 < -x_2^2 \Rightarrow 1 - x_1^2 < 1 - x_2^2 \quad (2)$$

and

$$x_1 \in (-1, 0) \Rightarrow -1 < x_1 < 0 \Rightarrow 1 > -x_1 > 0 \Rightarrow 1 > (-x_1)^2 \Rightarrow$$
$$\Rightarrow 1 > x_1^2 \Rightarrow 1 - x_1^2 > 0 \quad (3)$$

and similarly

$$x_2 \in (-1, 0) \Rightarrow \cdots \Rightarrow 1 - x_2^2 > 0. \quad (4)$$

From (2), (3), (4), it follows that

$$0 < 1 - x_1^2 < 1 - x_2^2 \Rightarrow \sqrt{1 - x_1^2} < \sqrt{1 - x_2^2} \quad (5)$$

From (1) and (5), adding the inequalities:

$3x_1+1+\sqrt{1-x_1^2} < 3x_2+1+\sqrt{1-x_2^2} \Rightarrow$

$\Rightarrow \underline{f(x_1) < f(x_2)}$

Thus $\forall x_1,x_2 \in (-1,0): (x_1<x_2 \Rightarrow f(x_1)<f(x_2))$

$\Rightarrow f \nearrow (-1,0)$.

↳ Note that before we raise an inequality to any power we have to ensure/check that both sides of the inequality are positive.
Thus in the above:

$x_1<x_2 \Rightarrow x_1^2<x_2^2$ is WRONG

since $x_1<0$ and $x_2<0$. Be careful!!

↳ Note that it was necessary to interrupt the main line of the argument:

$x_1<x_2 \Rightarrow \cdots \Rightarrow \sqrt{1-x_1^2} < \sqrt{1-x_2^2}$

to show that $1-x_1^2>0$ and $1-x_2^2>0$.
Note the careful use of equation labels to interrupt and restart our main argument.

c) For $f(x) = \dfrac{1}{x^2-2}$, show that $f \nearrow (-\infty,-\sqrt{2})$

<u>Solution</u>

Let $\underline{x_1,x_2 \in (-\infty,-\sqrt{2})}$ be given with $\underline{x_1<x_2}$.
Then

$x_1 < x_2 \Rightarrow -x_1 > -x_2 > 0 \Rightarrow (-x_1)^2 > (-x_2)^2 \Rightarrow x_1^2 > x_2^2$

$\Rightarrow x_1^2 - 2 > x_2^2 - 2 \qquad (1)$

Also note that

$x_1 \in (-\infty, -\sqrt{2}) \Rightarrow x_1 < -\sqrt{2} \Rightarrow -x_1 > \sqrt{2} \Rightarrow (-x_1)^2 > 2 \Rightarrow$

$\Rightarrow x_1^2 > 2 \Rightarrow x_1^2 - 2 > 0. \quad (2)$

and similarly $x_2 \in (-\infty, -\sqrt{2}) \Rightarrow x_2^2 - 2 > 0 \quad (3)$.

From (1), (2), and (3):

$$x_1^2 - 2 > x_2^2 - 2 > 0 \Rightarrow \frac{1}{x_1^2-2} < \frac{1}{x_2^2-2} \Rightarrow \underline{f(x_1) < f(x_2)}$$

It follows that

$\forall x_1, x_2 \in (-\infty, -\sqrt{2}) : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2))$

$\Rightarrow f \nearrow (-\infty, -\sqrt{2})$.

EXERCISES

18) Use the analytic method to determine the monotonicity of the following functions.

a) $f(x) = 3x+2$ on $\mathbb{R}$

b) $f(x) = 5-4x$ on $\mathbb{R}$

c) $f(x) = x^2-4x+5$ on $(-\infty, 2)$

d) $f(x) = \dfrac{3x+1}{x+2}$ on $(-2, +\infty)$

e) $f(x) = \dfrac{x+8}{3x+1}$ on $(-\infty, -1/3)$

f) $f(x) = (2x+5)^2-3$ on $(-\infty, -5/2)$

g) $f(x) = (x-1)(2x+1)$ on $(1, +\infty)$

19) Use the synthetic method to determine the monotonicity of the following functions

a) $f(x) = 5x-3$ on $\mathbb{R}$

b) $f(x) = 2-7x$ on $\mathbb{R}$

c) $f(x) = (2x+3)^2+1$ on $(0, +\infty)$

d) $f(x) = (2-5x)^3-2$ on $(0, +\infty)$

e) $f(x) = \dfrac{-2}{2x^2+3}$ on $(0, +\infty)$

f) $f(x) = \sqrt{2x-1}$ on $(1,+\infty)$

g) $f(x) = 2 - 3\sqrt{4-x^2}$ on $(0,2)$

h) $f(x) = -1 + 2\sqrt{9-(x+1)^2}$ on $(-4,-1)$

i) $f(x) = 3x+2+\sqrt{x+1}$ on $(0,+\infty)$

j) $f(x) = (2x-1)\sqrt{2x+1}$ on $(1,+\infty)$

(20) Let $f(x) = -1/x$.

a) Show that $f \uparrow (-\infty,0)$ and $f \uparrow (-\infty,0)$

b) Now, show that the statement $f \uparrow (-\infty,0) \cup (0,+\infty)$ is FALSE.

↳ To show that $f \uparrow A$ is FALSE, it is sufficient to find a counterexample, that is, to find some $x_1, x_2 \in A$ with $x_1 < x_2$ and $f(x_1) \geq f(x_2)$. In other words:

$f \uparrow A$ is false $\Leftrightarrow \exists x_1, x_2 \in A : (x_1 < x_2 \wedge f(x_1) \geq f(x_2))$

$f \downarrow A$ is false $\Leftrightarrow \exists x_1, x_2 \in A : (x_1 < x_2 \wedge f(x_1) \leq f(x_2))$

This exercise shows that the general claim

$$f \uparrow A_1 \wedge f \uparrow A_2 \Rightarrow f \uparrow A_1 \cup A_2$$

is not always true, by demonstrating a counterexample.

(21) Let $f(x) = \dfrac{ax+b}{cx+d}$. Show that given $D = ad-bc$,

a) If $D > 0$, then $f \uparrow (-\infty, -d/c)$ and $f \uparrow (-d/c, +\infty)$

b) If $D < 0$, then $f \downarrow (-\infty, -d/c)$ and $f \downarrow (-d/c, +\infty)$

(22) Given the functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$, show that

a) $f\nearrow\mathbb{R}$ and $g\nearrow\mathbb{R} \Rightarrow f+g\nearrow\mathbb{R}$

b) $f\nearrow\mathbb{R}$ and $g\searrow\mathbb{R} \Rightarrow f\circ g\searrow\mathbb{R}$

c) $f\searrow\mathbb{R}$ and $g\searrow\mathbb{R} \Rightarrow g\circ f\nearrow\mathbb{R}$

d) f odd and $f\nearrow[0,+\infty) \Rightarrow f\nearrow\mathbb{R}$

e) f even and $f\nearrow(0,+\infty) \Rightarrow f\searrow(-\infty,0)$.

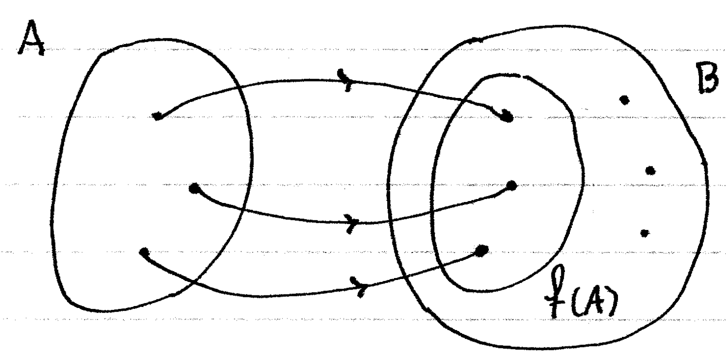
Inverse Functions

- Let $f: A \longrightarrow \mathbb{R}$ be a function with range $f(A)$. In order for f to have an inverse function, it has to satisfy the "one-to-one" property.

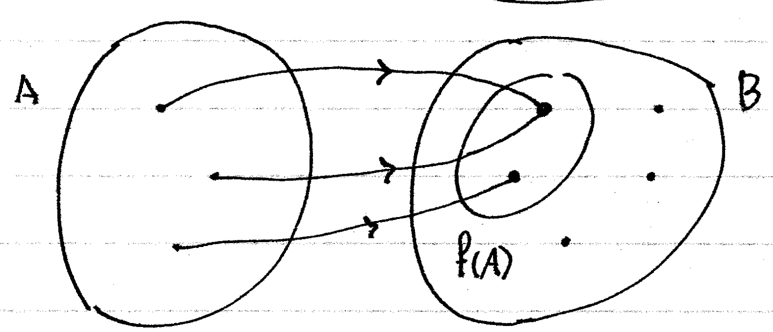
One-to-one functions

- $$\boxed{f \text{ one-to-one} \Leftrightarrow \forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)}$$

▸ Venn diagram interpretation: In a one-to-one function, every point in the range $f(A)$ of f receives only one arrow.



f is one-to-one



f is NOT one-to-one.

- Negated definition:

f NOT one-to-one $\Leftrightarrow \exists x_1, x_2 \in A : (f(x_1) = f(x_2) \wedge x_1 \neq x_2)$

EXAMPLES

a) Show that $f(x) = \dfrac{2x-1}{3x+2}$ is one-to-one.

Solution

- Domain: $A = \mathbb{R} - \{-2/3\}$ (Require $3x+2 \neq 0$).
- Let $x_1, x_2 \in A$ be given such that $f(x_1) = f(x_2)$.

  Then:

$$f(x_1) = f(x_2) \Rightarrow \frac{2x_1-1}{3x_1-1} = \frac{2x_2-1}{3x_2-1} \Rightarrow$$

$$\Rightarrow (2x_1-1)(3x_2-1) = (3x_1-1)(2x_2-1) \Rightarrow$$

$$\Rightarrow 6x_1x_2 - 2x_1 - 3x_2 + 1 = 6x_1x_2 - 3x_1 - 2x_2 + 1 \Rightarrow$$

$$\Rightarrow -2x_1 - 3x_2 = -3x_1 - 2x_2 \Rightarrow 3x_1 - 2x_1 = 3x_2 - 2x_2$$

$$\Rightarrow x_1 = x_2.$$

Thus $\forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2) \Rightarrow$

$\Rightarrow f$ one-to-one.

↳ To show that a function f is one-to-one

1. Let $x_1, x_2 \in A$ be given with $f(x_1) = f(x_2)$.
2. Show: $f(x_1) = f(x_2) \Rightarrow \dots \Rightarrow \dots \Rightarrow x_1 = x_2$
3. Conclude argument.

b) Show that $f(x) = 2x^2 + 6x - 7$ is NOT one-to-one.

Solution

• Domain: $A = \mathbb{R}$ (no restrictions)

▸ Solve $f(x) = -7 \Leftrightarrow 2x^2 + 6x - 7 = -7 \Leftrightarrow 2x^2 + 6x = 0$

$$\Leftrightarrow 2x(x+3) = 0 \Leftrightarrow$$

$$\Leftrightarrow 2x = 0 \vee x+3 = 0 \Leftrightarrow$$

$$\Leftrightarrow x = 0 \vee x = -3$$

It follows that for $x_1 = 0$ and $x_2 = -3$:

$$\begin{cases} f(x_1) = f(x_2) = -7 \\ x_1 \neq x_2 \end{cases} \Rightarrow f \text{ not one-to-one.}$$

↳ To show that a function f is NOT one-to-one, it is enough to find just one counterexample $x_1, x_2 \in A$ (i.e. specific choices for x_1 and x_2) such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$.

EXERCISES

(23) Show that the following functions are one-to-one

a) $f(x) = ax+b$ with $a, b \in \mathbb{R}$ and $a \neq 0$.

b) $f(x) = a/x$ with $a \in \mathbb{R}$ and $a \neq 0$

c) $f(x) = \dfrac{ax+b}{cx+d}$ with $a, b, c, d \in \mathbb{R}$ and $D = ad - bc \neq 0$.

(24) Show that $f(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$ and $a \neq 0$ is NOT one-to-one.

(Hint: Solve $f(x) = c$)

(25) Let $f : A \longrightarrow \mathbb{R}$ be a function. Show that

a) $f \nearrow A \Rightarrow f$ one-to-one

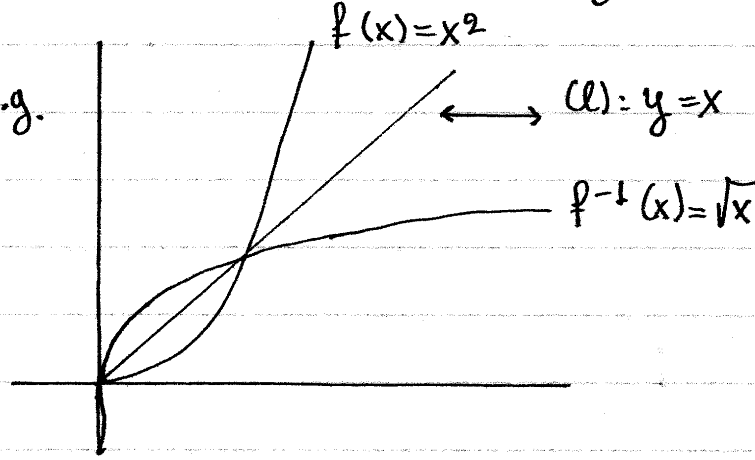
b) $f \searrow A \Rightarrow f$ one-to-one

c) f even $\Rightarrow f$ not one-to-one.

Definition of the inverse function

- Let $f: A \to \mathbb{R}$ be a one-to-one function with range $f(A)$. Then there is a unique function $f^{-1}: f(A) \to \mathbb{R}$ such that
$$f^{-1}(x) = y \Leftrightarrow f(y) = x$$
We call f^{-1} the <u>inverse</u> of f.
- Note that the range $f(A)$ of f is the domain of its inverse f^{-1}.
- It can be shown that
$$\forall x \in A : f^{-1}(f(x)) = x$$
$$\forall x \in f(A) : f(f^{-1}(x)) = x$$
- The graph of f^{-1} is the reflection of the graph of f accross the line $(\ell): y = x$.

e.g.



Method : To find the inverse of a function $f: A \to \mathbb{R}$ we work as follows:

- 1 We setup the equation
$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow \cdots$
- 2 It may be necessary to require restrictions on y to evaluate $f(y)$. If that is the case, then do so.
- 3 Solve for y. During the process, it may be necessary to require restrictions on x to ensure that at least one solution exists. These restrictions define the domain of the inverse function f^{-1}.
- 4 When you show that, under possible restrictions on x, that your equation has a unique solution $y = y_0(x)$, you implicitly prove that both f is one-to-one and that $f^{-1}(x) = y_0(x)$. Thus you have the formula of the inverse function.
- 5 If applicable, check the constraints on y from step 2. They may or may not introduce further restrictions on the variable x and therefore on the domain of the inverse function.

EXAMPLES

a) Find the inverse function of $f(x) = \dfrac{x+3}{2x-5}$

Solution

$$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow \frac{y+3}{2y-5} = x \quad (\text{Require } \underline{2y-5 \neq 0})$$

$$\Leftrightarrow y+3 = x(2y-5) \Leftrightarrow y+3 = 2xy - 5x \Leftrightarrow (1-2x)y = -3-5x \quad (1)$$

For $1-2x = 0$: $x = 1/2$, and therefore

$(1) \Leftrightarrow 0y = -3-5\cdot(1/2) \Leftrightarrow 0y = -3-5/2 \leftarrow$ inconsistent

thus $\underline{x = 1/2 \notin dom(f^{-1})}$.

For $1-2x \neq 0$:

$$(1) \Leftrightarrow y = \frac{-3-5x}{1-2x}$$

Now we must check the requirement $2y-5 \neq 0$.

We note that:

$$2y-5 = 2\cdot\left(\frac{-3-5x}{1-2x}\right) - 5 = \frac{2(-3-5x)-5}{1-2x} =$$

$$= \frac{-6-10x-5(1-2x)}{1-2x} = \frac{-6-10x-5+10x}{1-2x} =$$

$$= \frac{-11}{1-2x} \neq 0$$

thus $2y-5 \neq 0$ is satisfied.

Thus $f^{-1}(x) = \dfrac{-3-5x}{1-2x}$ with $\mathrm{dom}(f^{-1}) = \mathbb{R} - \{1/2\}$.

In the above example we see that the domain of f^{-1} coincides with the widest possible domain. However, this is not always true, as seen in the next example.

b) Find the inverse function of $f(x) = 2 + \dfrac{\sqrt{3x+1}}{3}$

Solution

$$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow 2 + \frac{\sqrt{3y+1}}{3} = x \Leftrightarrow$$

$$\Leftrightarrow 6 + \sqrt{3y+1} = 3x \Leftrightarrow \sqrt{3y+1} = 3x-6 \Leftrightarrow \sqrt{3y+1} = 3(x-2) \quad (1)$$

Require $3(x-2) \geq 0 \Leftrightarrow x \geq 2$, otherwise equation (1) is inconsistent. For $x \geq 2$:

$$(1) \Leftrightarrow 3y+1 = 9(x-2)^2 \Leftrightarrow 3y = 9(x-2)^2 - 1 \Leftrightarrow$$

$$\Leftrightarrow y = 3(x-2)^2 - \frac{1}{3}$$

It follows that

$$f^{-1}(x) = 3(x-2)^2 - 1/3 \text{ with } \mathrm{dom}(f^{-1}) = [2, +\infty)$$

In this example we see that the domain $\mathrm{dom}(f^{-1})$ is restricted from the widest possible domain of the polynomial formula for $f^{-1}(x)$ which is $\mathbb{R}$.

Thus, to determine the domain of the inverse function f^{-1}, it is necessary to keep track of all constraints, as I suggested in the methodology.

c) Find the inverse function of $f(x) = 4x - 3$.

Solution

$$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow 4y - 3 = x \Leftrightarrow 4y = x + 3 \Leftrightarrow$$
$$\Leftrightarrow y = \frac{x+3}{4}$$

It follows that:

$$f^{-1}(x) = \frac{x+3}{4} \quad \text{with } \mathrm{dom}(f^{-1}) = \mathbb{R} \text{ (no constraints).}$$

EXERCISES

(26) Find the inverse function f^{-1} for the following functions:

a) $f(x) = 3x+2$

b) $f(x) = 1-2x$

c) $f(x) = 2(1-x)+3(2x+1)$

d) $f(x) = x(x+2)-(x^2-5)$

e) $f(x) = \dfrac{2}{3x}$

f) $f(x) = \dfrac{3}{2x-1}$

g) $f(x) = \dfrac{2}{x-4}$

h) $f(x) = \dfrac{2x+3}{x-1}$

i) $f(x) = \dfrac{x+4}{3x-1}$

j) $f(x) = 3+\sqrt{x-1}$

k) $f(x) = -1-2\sqrt{2-3x}$

l) $f(x) = 2+\dfrac{\sqrt{1-3x}}{4}$

m) $f(x) = \dfrac{2-\sqrt{3x+2}}{5}$

n) $f(x) = \sqrt{\dfrac{2x+1}{3x-2}}$

o) $f(x) = \sqrt{\dfrac{x-3}{4x+5}}$

$\hookrightarrow$ To confirm $f^{-1}(x)$ with computer algebra, either simplify $f^{-1}(f(x))$ or evaluate it for a few values of x and confirm that $f^{-1}(f(x)) = x$

Calculating the range $f(A)$

- Let $f: A \to \mathbb{R}$ be a function. Recall that the range $f(A)$ of f is given by

$$f(A) = \{f(x) \mid x \in A\}$$

It follows that $y \in f(A)$ if and only if the equation $y = f(x)$ has at least one solution $x = x_0$ with $x_0 \in A$. (if there are more solutions it is not necessary for ALL to belong to A. One is sufficient).

Method : To find the range of a function we work as follows:

- 1 Find the ~~ko~~ domain A.
- 2 Solve the equation $y = f(x)$ with respect to x until we obtain a parametric equation of the form

$$a(y)x + b(y) = 0 \qquad \text{OR} \qquad (\text{Case } 1)$$
$$a(y)x^2 + b(y)x + c(y) = 0. \qquad (\text{Case } 2)$$

- 3 Find the solvability set S for which the simplified equation has a solution.

Case 1: $a(y)x + b(y) = 0$

a) For $a(y) = 0$, check on a case by case basis whether the equation is inconsistent.

b) For $a(y) \neq 0$, $y \in \mathcal{S}$.

Case 2: $a(y)x^2 + b(y)x + c(y) = 0$

a) For $a(y) = 0$, check on a case by case basis whether the equation has a solution.

b) For $a(y) \neq 0$, require that
$$b^2(y) - 4a(y)c(y) \geq 0.$$

- 4 Find which elements of $\mathcal{S}$ also belong to $f(A)$.

Case 1: If $A = \mathbb{R}$

then, $f(A) = \mathcal{S} \leftarrow$ you are done

Case 2: If $A = \mathbb{R} - \{x_0, x_1, x_2, \dots, x_n\}$

then from the simplified equation
$$a(y)x + b(y) = 0 \quad \text{OR}$$
$$a(y)x^2 + b(y)x + c(y) = 0$$
we set $x = x_0$, $x = x_1, \dots, x = x_n$, solve for y, and find $y = y_0$, $y = y_1, \dots, y = y_n$. These y values give solutions x that do not belong to A and must be therefore excluded. It follows that
$$f(A) = \mathcal{S} - \{y_0, y_1, y_2, \dots, y_n\}$$

Case 3: If A is a union of intervals then solve for x all the way and then solve

$x \in A \Leftrightarrow$ system of inequalities in terms of y

$\Leftrightarrow y \in f(A)$

(bite the bullet case).

Functions whose range is obvious

1) For $f(x) = ax + b$ with $a \neq 0$
$A = \text{dom}(f) = \mathbb{R}$
$f(A) = \mathbb{R}$.

2) For $f(x) = \dfrac{ax+b}{cx+d}$ with $D = ad - bc \neq 0$, $c \neq 0$.

$A = \text{dom}(f) = \mathbb{R} - \{-d/c\}$
$f(A) = \mathbb{R} - \{\frac{a}{c}\}$.

examples

a) For $f(x) = x^2 + 2x + 3$, $A = \text{dom}(f) = \mathbb{R}$.
Solve $y = f(x) \Leftrightarrow y = x^2 + 2x + 3 \Leftrightarrow$
$\Leftrightarrow \underline{x^2 + 2x + (3-y) = 0}$ (1)

Solvability set:
(1) has a solution $\Leftrightarrow \Delta(y) \geq 0 \Leftrightarrow$
$\Leftrightarrow 2^2 - 4 \cdot 1 \cdot (3-y) \geq 0 \Leftrightarrow 4 - 12 + 4y \geq 0$
$\Leftrightarrow 4y \geq 8 \Leftrightarrow y \geq 2 \Leftrightarrow y \in [2, +\infty) = S$

Since $A = \mathbb{R} \Rightarrow f(A) = S = [2, +\infty)$.

B) For $f(x) = \dfrac{x^2+x+1}{x^2+5x+6}$

Domain: Require $x^2+5x+6 \neq 0 \Leftrightarrow (x+2)(x+3) \neq 0$

$\Leftrightarrow x \in \mathbb{R} - \{-2,-3\}$

thus $A = \mathrm{dom}(f) = \mathbb{R} - \{-2,-3\}$.

Solve:

$y = f(x) \Leftrightarrow y = \dfrac{x^2+x+1}{x^2+5x+6} \Leftrightarrow \cdots \Leftrightarrow$

$\Leftrightarrow (y-1)x^2 + (5y-1)x + (6y-1) = 0 \quad (1)$

Solvability condition:

For $y-1=0 \Leftrightarrow y=1$, eq. (1) gives $4x+5=0 \Leftrightarrow$

$\Leftrightarrow x = -5/4 \in A$

thus $\underline{\underline{1 \in f(A)}} \quad (2)$

For $y-1 \neq 0$, eq (1) has a solution $\Leftrightarrow$

$\Leftrightarrow \Delta(y) \geq 0 \Leftrightarrow (5y-1)^2 - 4(y-1)(6y-1) \geq 0 \Leftrightarrow \cdots \Leftrightarrow$

$\Leftrightarrow y^2 + 18y - 3 \geq 0 \Leftrightarrow \cdots \Leftrightarrow$

$\Leftrightarrow \underline{y \in (-\infty, -9-2\sqrt{21}] \cup [-9+2\sqrt{21}, +\infty)} \quad (3)$

Possible exclusions:

For $x=-2$, eq. (1) gives

$4(y-1) - 2(5y-1) + (6y-1) = 0 \Leftrightarrow \cdots \Leftrightarrow$

$\Leftrightarrow 0y = 3 \leftarrow$ inconsistent, so no exclusion.

For $x=-3$, eq. (1) gives:

$$9(y-1)-3(5y-1)+(6y-1)=0 \Leftrightarrow \cdots \Leftrightarrow$$

$$\Leftrightarrow 0y=7 \leftarrow \text{inconsistent, so no exclusions.}$$

From (2), (3), as there are no exclusions, it follows that

$$f(A)=(-\infty,-9-2\sqrt{21}\,]\cup[-9+2\sqrt{21},+\infty)\cup\{1\}$$

$$=(-\infty,-9-2\sqrt{21}\,]\cup[-9+2\sqrt{21},+\infty),$$

because $-9+2\sqrt{21}<1$.

c) $f(x)=3-(2x+1)^2$

Domain: $A=\text{dom}(f)=\mathbb{R}$.

Solve: $y=f(x) \Leftrightarrow y=3-(2x+1)^2 \Leftrightarrow$

$$\Leftrightarrow \underline{(2x+1)^2=3-y} \quad (1)$$

Solvability:

Eq. (1) has a solution $\Leftrightarrow 3-y\geq 0 \Leftrightarrow y\leq 3$

$$\Leftrightarrow y\in(-\infty,3]$$

Since $A=\mathbb{R} \Rightarrow f(A)=(-\infty,3]$. (no exclusions)

$\mapsto$ For functions with roots, we follow the general methodology and also note that

1) The equation $y = f(x)$ usually simplifies to

$$\sqrt{a(x)} = b(y)$$

At this step we require $\underline{b(y) \geq 0}$. Then we continue solving for x to get

$$a(y)x + b(y) = 0$$

OR $a(y)x^2 + b(y)x + c(y) = 0$.

and continue as usual.

2) The domain A is usually complicated so we may have to solve for x all the way e.g.

$$y = f(x) \Leftrightarrow \cdots \Leftrightarrow x = g_1(y) \vee x = g_2(y).$$

Then in addition to solvability conditions we also require

$$g_1(y) \in A \quad \vee \quad g_2(y) \in A$$

system of inequalities — another system of inequalities

Take $\underline{\text{union}}$ of solution sets

examples

a) For $f(x) = \sqrt{x+3} + 2$

Domain: Require $x+3 \geq 0 \Leftrightarrow x \in [-3, +\infty)$
thus $A = dom(f) = [-3, +\infty)$

Solve:

$y = f(x) \Leftrightarrow y = \sqrt{x+3} + 2 \Leftrightarrow$

$\Leftrightarrow \sqrt{x+3} = y - 2 \quad (1)$

▸ Require $y - 2 \geq 0 \Leftrightarrow y \in [2, +\infty)$

$(1) \Leftrightarrow x+3 = (y-2)^2 \Leftrightarrow x = -3 + (y-2)^2$

Require

$x \in A \Leftrightarrow -3 + (y-2)^2 \in [-3, +\infty) \Leftrightarrow$

$\Leftrightarrow -3 + (y-2)^2 \geq -3$

$\Leftrightarrow (y-2)^2 \geq 0 \leftarrow$ Always true

Thus

$y \in f(A) \Leftrightarrow y \in [2, +\infty)$, so $f(A) = [2, +\infty)$.

b) For $f(x) = 2 - \sqrt{1-x^2}$

Domain: Require $1 - x^2 \geq 0 \Leftrightarrow \cdots \Leftrightarrow x \in [-1, 1]$
thus $A = dom(f) = [-1, 1]$

Solve:

$y = f(x) \Leftrightarrow y = 2 - \sqrt{1-x^2} \Leftrightarrow \sqrt{1-x^2} = 2 - y \quad (1)$

▸ Require $2 - y \geq 0$

$(1) \Leftrightarrow 1 - x^2 = (2-y)^2 \Leftrightarrow x^2 = 1-(2-y)^2$. (2)

Eq. (2) has a solution $\Leftrightarrow$
$1-(2-y)^2 \geq 0 \Leftrightarrow \cdots \Leftrightarrow \underline{y \in [1,3]}$

Trick $\left\{ \right.$ ▸ Since $1-x^2 = (2-y)^2 \geq 0$, this solution is in the domain A so there are no exclusions!! ◂

Thus:

$$y \in f(A) \Leftrightarrow \begin{cases} 2-y \geq 0 \\ y \in [1,3] \end{cases} \Leftrightarrow \begin{cases} y \in (-\infty, 2] \\ y \in [1,3] \end{cases}$$

so

$$f(A) = (-\infty, 2] \cap [1,3] = [1,2].$$

c) For $f(x) = \sqrt{x^2-9} - 2$

Domain: Require $x^2 - 9 \geq 0 \Leftrightarrow \cdots \Leftrightarrow x \in (-\infty, -3] \cup [3, +\infty)$
thus $A = \mathrm{dom}(f) = (-\infty, -3] \cup [3, +\infty)$

Solve

$$y = f(x) \Leftrightarrow y = \sqrt{x^2-9} - 2 \Leftrightarrow y + 2 = \sqrt{x^2-9} \quad (1)$$

Require $y+2 \geq 0 \Leftrightarrow \underline{y \in [-2, +\infty)}$

$(1) \Leftrightarrow (y+2)^2 = x^2 - 9 \Leftrightarrow x^2 = 9 + (y+2)^2$ (2)

Eq. (2) has a solution $\Leftrightarrow \underline{9 + (y+2)^2 \geq 0}$

Since $x^2 - 9 = (y+2)^2 \geq 0$, the solutions of (2) will belong to A so there are no exclusions.

Thus

$$y \in f(A) \Leftrightarrow \begin{cases} y \in [-2, +\infty) \\ 9 + (y+2)^2 \geq 0 \leftarrow \text{identity} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow y \in [-2, +\infty)$$

thus

$$f(A) = [-2, +\infty).$$

EXERCISES

(27) Find the range and domain of the following functions

a) $f(x) = 3x-1$

b) $f(x) = 1-(2x+3)^2$

c) $f(x) = x^2+5x+6$

d) $f(x) = x^2-10x+9$

e) $f(x) = \dfrac{2x+5}{x-3}$

f) $f(x) = 2-\sqrt{3x+2}$

g) $f(x) = 2+\sqrt{1-2x}$

h) $f(x) = 1-2\sqrt{4-x^2}$

i) $f(x) = \sqrt{(x+1)^2-9}$

j) $f(x) = \sqrt{x^2+3x+2}$

(28) Same with the following functions

a) $f(x) = \dfrac{x^2+x-2}{x^2+1}$

b) $f(x) = \dfrac{x^2-1}{2x+1}$

c) $f(x) = \dfrac{(x+1)^2}{x^2+3x+2}$

d) $f(x) = \sqrt{\dfrac{x-1}{x+2}}$

e) $f(x) = \dfrac{\sqrt{x}}{\sqrt{x}-1}$