EQUATIONS AND INEQUALITIES

Terminology

- An equation is an expression of the form $f(x)=g(x)$ which may or may not be true for some values of x.
- A solution set S of an equation $f(x)=g(x)$ is the set of all real numbers $x\in\mathbb{R}$ for which the equation is true.

▸ example : For the equation $x^2=4$, the solution set is $S=\{-2,2\}$.

- An identity is an equation $f(x)=g(x)$ with solution set $S=\mathbb{R}$. (i.e. the equation is always true for all real numbers $x\in\mathbb{R}$).
- An equation $f(x)=g(x)$ is inconsistent if it's solution set is $S=\emptyset$. (i.e. the equation is always false for all real numbers $x\in\mathbb{R}$. OR equivalently, the equation is never true for any real number $x\in\mathbb{R}$).

▸ examples

a) The equation $(x+1)^2=x^2+2x+1$ is an identity.

b) The equation $x^2+4=0$ is inconsistent.
(because $x^2+4\geqslant 0+4=4>0$ for all $x\in\mathbb{R}$).

Basic Logic Notation

- A predicate $p(x)$ is a statement about x which is either true or false depending on the value of x.

▸ example : Any equation is also a predicate. For example $p(x): 2x+1=3x$ is a predicate.

Logical or/and/implications

$p(x) \wedge q(x)$: $p(x)$ and $q(x)$ are both true.

$p(x) \vee q(x)$: At least one of $p(x)$ or $q(x)$ is true ($p(x)$ OR $q(x)$).

$p(x) \Rightarrow q(x)$: If $p(x)$ is true, then $q(x)$ is true.

$p(x) \Leftrightarrow q(x)$: $p(x)$ is true if and only if $q(x)$ is true. (i.e. if $p(x)$ is true then $q(x)$ is true AND if $q(x)$ is true then $p(x)$ is true).

▸ examples

a) $x>2 \Rightarrow x>1$ TRUE

b) $x>1 \Rightarrow x>2$ FALSE

c) $x>2 \Leftrightarrow x>1$ FALSE

d) $2x=2 \Leftrightarrow x=1$ TRUE.

Basic properties of equations.

1) Let $x, y, a \in \mathbb{R}$. Then
$x = y \iff x + a = y + a$
(i.e.: we can add a number to both sides of an equation).

2) $x + a = y \iff x = y - a$
(i.e.: we can move a term on the other side of the equation but we must change its sign.)

3) Let $x, y, a \in \mathbb{R}$ and assume that $a \neq 0$. Then
$x = y \iff ax = ay$.
(i.e.: we can multiply a <u>non-zero</u> number to both sides of an equation)
▸ Note this property requires that <u>$a \neq 0$.</u> !!

4) Let $x, y \in \mathbb{R}$. Then
$xy = 0 \iff x = 0 \vee y = 0$
(i.e. A product xy is zero if and only if $x = 0$ or $y = 0$).

5) Let $x, y \in \mathbb{R}$. Then
$x^2 = y^2 \iff x = y \vee x = -y$
(i.e. $x^2 = y^2$ if and only if $x = y$ or $x = -y$).
▸ Note that
$x = y \Rightarrow x^2 = y^2$ is TRUE, but
$x^2 = y^2 \Rightarrow x = y$ is FALSE!

6) Let $x, y \in \mathbb{R}$. Then

$$x^2+y^2=0 \Leftrightarrow x=0 \wedge y=0$$

▼ Solving an equation

To solve an equation $f(x)=g(x)$, we use the above properties 1-6 to construct an argument of the form:

$$\begin{aligned} f(x)=g(x) &\Leftrightarrow f_1(x)=g_1(x) \Leftrightarrow \\ &\Leftrightarrow f_2(x)=g_2(x) \Leftrightarrow \\ &\Leftrightarrow \cdots \Leftrightarrow \\ &\Leftrightarrow x=x_1 \vee x=x_2 \vee \cdots \vee x=x_n \end{aligned}$$

It follows that the solution set is

$$S=\{x_1, x_2, \ldots, x_n\}$$

- It is very important that EVERY step in the argument must be valid in BOTH directions (i.e. $\Leftrightarrow$ instead of only $\Rightarrow$ or $\Leftarrow$)
- When $\Rightarrow$ fails (but $\Leftarrow$ works): Every number you find is a solution but you may have more solutions out there that you have failed to find.
- When $\Leftarrow$ fails (but $\Rightarrow$ works): All of your solutions are among the numbers you found, but some of your numbers may not satisfy the equations (extraneous "solutions").

V Polynomial Equations

- A polynomial equation is an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

with $a_0, a_1, \dots, a_n \in \mathbb{R}$.

- n = degree of the equation.

① Linear Equations $\to$ $\boxed{\begin{array}{l} ax+b=0 \\ \text{with } a,b \in \mathbb{R} \end{array}}$

Solution:

- $a \neq 0 \Rightarrow$ unique solution $x = -b/a$
- $a = 0 \wedge b \neq 0 \Rightarrow$ inconsistent $(S = \emptyset)$
- $a = 0 \wedge b = 0 \Rightarrow$ identity $(S = \mathbb{R})$.

EXAMPLES

a) $\dfrac{x-2}{3} - \dfrac{x+1}{2} = \dfrac{1-3x}{6} \Leftrightarrow$

$\Leftrightarrow 6\left[\dfrac{x-2}{3} - \dfrac{x+1}{2}\right] = 6 \cdot \dfrac{1-3x}{6} \Leftrightarrow$

$\Leftrightarrow 2(x-2) - 3(x+1) = 1 - 3x \Leftrightarrow$

$\Leftrightarrow 2x - 4 - 3x - 3 = 1 - 3x \Leftrightarrow$

$\Leftrightarrow (2-3+3)x = 4+3+1 \Leftrightarrow$

$\Leftrightarrow 2x = 8 \Leftrightarrow x = 4 \longleftarrow \underline{S = \{4\}}.$

b) $\dfrac{x+1}{2} = x - \dfrac{2x+3}{4} \Leftrightarrow 2(x+1) = 4x - (2x+3) \Leftrightarrow$

$\Leftrightarrow 2x+2 = 4x-2x-3 \Leftrightarrow (2-4+2)x = -2-3 \Leftrightarrow$

$\Leftrightarrow 0x = -5 \leftarrow$ inconsistent $\rightarrow$ $\underline{S = \emptyset}$

c) $3-5x-2(4-5x) = -4(x-1)+3(3x-2)-3 \Leftrightarrow$

$\Leftrightarrow 3-5x-8+10x = -4x+4+9x-6-3 \Leftrightarrow$

$\Leftrightarrow (-5+10+4-9)x = -3+8+4-6-3 \Leftrightarrow$

$\Leftrightarrow 0x = 0 \leftarrow$ identity $\rightarrow$ $\underline{S = \mathbb{R}}$.

EXERCISE

1) Solve the equations

a) $5x-2(3-x) = -6-3(-x-1)$

b) $-3+2(5+4x) = 9-3(4-2x)$

c) $2x - \dfrac{3-2x}{6} = 1 - \dfrac{5-x}{4}$

d) $\dfrac{2x}{5} - \dfrac{x-3}{15} = -1 - \dfrac{x+1}{10}$

e) $x + \dfrac{3-x}{3} = 1 + \dfrac{2x}{3}$

f) $\dfrac{x-5}{2} + \dfrac{14}{4} = \dfrac{7x}{2} - 3(x-3)$

g) $\dfrac{x+2}{6} - \dfrac{5-x}{2} = \dfrac{2x-7}{6} + \dfrac{x-3}{3}$.

② Completed square equations

$$\boxed{(ax+b)^2 = c}$$

a) If $c>0$, then
$(ax+b)^2 = c \Leftrightarrow ax+b = \sqrt{c} \vee ax+b = -\sqrt{c}$
$\Leftrightarrow \dots$

b) If $c=0$, then
$(ax+b)^2 = 0 \Leftrightarrow ax+b=0 \Leftrightarrow \dots$

c) If $c<0$, then
$(ax+b)^2 = c$ is <u>inconsistent in $\mathbb{R}$.</u>

EXAMPLES

a) $(2x-1)^2 - 5 = 0 \Leftrightarrow (2x-1)^2 = 5 \Leftrightarrow$
$\Leftrightarrow 2x-1 = \sqrt{5} \vee 2x-1 = -\sqrt{5} \Leftrightarrow$
$\Leftrightarrow 2x = 1+\sqrt{5} \vee 2x = 1-\sqrt{5} \Leftrightarrow$
$\Leftrightarrow x = \dfrac{1+\sqrt{5}}{2} \vee x = \dfrac{1-\sqrt{5}}{2}$

b) $(3x+2)^2 = 0 \Leftrightarrow 3x+2 = 0 \Leftrightarrow 3x = -2 \Leftrightarrow x = -\dfrac{2}{3}$

c) $(5x+3)^2 + 2 = 0 \Leftrightarrow (5x+3)^2 = -2 < 0$
thus inconsistent in $\mathbb{R}$.

EXERCISES

② Solve the equations

a) $(x+1)^2 = 5$

b) $(2x+3)^2 - 7 = 0$

c) $(3x-1)^2 + 2 = 5$

d) $(5x-2)^2 + 1 = -2$

e) $(3x+2)^2 = 0$

f) $(3-2x)^2 = 3$

③ Quadratic Equations → $ax^2+bx+c=0$ with $a, b, c \in \mathbb{R}$ and $a \neq 0$.

- 1 Calculate the discriminant
$$\Delta = b^2 - 4ac$$
- 2 Distinguish among the following cases:

a) $\Delta > 0 \Rightarrow$ Two solutions
$$x_1 = \frac{-b+\sqrt{\Delta}}{2a} \vee x_2 = \frac{-b-\sqrt{\Delta}}{2a}$$

b) $\Delta = 0 \Rightarrow$ One solution $x = -\frac{b}{2a} = x_1 = x_2$

c) $\Delta < 0 \Rightarrow$ Equation is inconsistent in $\mathbb{R}$. and has 2 solutions in $\mathbb{C}$: $x_{1,2} = (-b \pm i\sqrt{-\Delta})/2a$

→ If the quadratic has two solutions x_1 and x_2 or one double solution (when $\Delta = 0 \Rightarrow x_1 = x_2 = -b/2a$) then they satisfy:

$$\boxed{x_1 + x_2 = -\frac{b}{a}}$$
$$\boxed{x_1 x_2 = \frac{c}{a}}$$

Proof

$$x_1 + x_2 = \frac{-b+\sqrt{\Delta}}{2a} + \frac{-b-\sqrt{\Delta}}{2a} =$$
$$= \frac{-b+\sqrt{\Delta}-b-\sqrt{\Delta}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

$$x_1 x_2 = \frac{-b+\sqrt{\Delta}}{2a} \cdot \frac{-b-\sqrt{\Delta}}{2a} = \frac{(-b)^2 - (\sqrt{\Delta})^2}{4a^2} =$$
$$= \frac{b^2-\Delta}{4a^2} = \frac{b^2-(b^2-4ac)}{4a^2} =$$
$$= \frac{b^2-b^2+4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a} \quad \square$$

Truncated Forms

1) $ax^2 = 0 \Leftrightarrow x = 0$

2) $ax^2 + bx = 0 \Leftrightarrow x(ax+b) = 0$
$\Leftrightarrow x = 0 \vee ax + b = 0$
(linear equations)

3) $ax^2 + c = 0 \Leftrightarrow x^2 = -\dfrac{c}{a}$

$$\Leftrightarrow \begin{cases} x = \pm\sqrt{-c/a}, & \text{if } -c/a \geq 0 \\ \text{inconsistent in } \mathbb{R}, & \text{if } -c/a < 0. \end{cases}$$

EXAMPLES

a) $2x^2+x-6=0$

Solution

$$\Delta = b^2-4ac = 1^2-4\cdot 2\cdot(-6) = 1+48 = 49 = 7^2 \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{-b\pm\sqrt{\Delta}}{2a} = \frac{-1\pm 7}{2\cdot 2} = \frac{-1\pm 7}{4} =$$

$$= \begin{cases} -8/4 = -2 \\ 6/4 = 3/2. \end{cases}$$

b) $x^2-6x+9=0$

Solution

$$\Delta = b^2-4ac = (-6)^2-4\cdot 1\cdot 9 = 36-36 = 0 \Rightarrow$$

$\Rightarrow$ unique solution $x = \dfrac{-b}{2a} = \dfrac{-(-6)}{2\cdot 1} = 3$

c) $2x^2-5x+4=0$

Solution

$$\Delta = b^2-4ac = (-5)^2-4\cdot 2\cdot 4 = 25-32 = -7 < 0 \Rightarrow$$

$\Rightarrow$ inconsistent in $\mathbb{R}$.

d) $3x^2-2=0 \Leftrightarrow 3x^2=2 \Leftrightarrow x^2=\frac{2}{3} \Leftrightarrow$

$\Leftrightarrow x = \pm\frac{\sqrt{2}}{\sqrt{3}} = \frac{\pm\sqrt{6}}{3}$

e) $2x^2+5x=0 \Leftrightarrow x(2x+5)=0 \Leftrightarrow$
$\Leftrightarrow x=0 \vee 2x+5=0 \Leftrightarrow$
$\Leftrightarrow x=0 \vee 2x=-5 \Leftrightarrow$
$\Leftrightarrow x=0 \vee x=-5/2.$

↳ Fast factorization : $\boxed{x^2+(a+b)x+ab=(x+a)(x+b)}$

f) $x^2+7x+10=0 \Leftrightarrow (x+2)(x+5)=0 \quad [2+5=7 \wedge 2\cdot 5=10]$
$\Leftrightarrow x+2=0 \vee x+5=0 \Leftrightarrow$
$\Leftrightarrow x=-2 \vee x=-5$
thus $S=\{-2,-5\}$.

- Fast factorization circumvents the application of the quadratic formula. However, do not spend too much time looking for the fast factorization. The quadratic formula is also very efficient technique.

EXERCISES

③ Solve the equations

a) $2x^2-3x+1=0$

b) $x^2-4x+4=0$

c) $x^2+2x+4=0$

d) $2x^2-x-3=0$

e) $x^2=3x$

f) $x(2x+1)=x+4$

g) $x(x+1)=4$

h) $x^2-6x+9=0$

i) $x(2x+1)-x^2=0$

j) $(x-1)(x+2)=4$

④ Polynomial equations of high order

Form: $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$
with $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$,
$a_n \neq 0$, $n \in \mathbb{N}$ with $n \geq 3$.

Such equations can be solved by factoring:

- •1 Factor the equation to 1st and 2nd order factors.
- •2 Use the property:

$$a_1 a_2 a_3 \cdots a_n = 0 \Leftrightarrow a_1 = 0 \vee a_2 = 0 \vee \cdots \vee a_n = 0$$

- •3 Solve the resulting equations.

EXAMPLES

a) $(x+3)^2(x^2-4)^3(x^2-1)=0 \Leftrightarrow$

$\Leftrightarrow x+3=0 \lor x^2-4=0 \lor x^2-1=0 \Leftrightarrow$

$\Leftrightarrow x=-3 \lor x^2=4 \lor x^2=1 \Leftrightarrow$

$\Leftrightarrow x=-3 \lor x=2 \lor x=-2 \lor x=1 \lor x=-1.$

b) $x^3=10-2(x-1)^2 \Leftrightarrow$

$\Leftrightarrow x^3=10-2(x^2-2x+1) \Leftrightarrow$

$\Leftrightarrow x^3=10-2x^2+4x-2 \Leftrightarrow$

$\Leftrightarrow x^3+2x^2-4x-8=0 \Leftrightarrow$

$\Leftrightarrow x^2(x+2)-4(x+2)=0 \Leftrightarrow$

$\Leftrightarrow (x^2-4)(x+2)=0 \Leftrightarrow x^2-4=0 \lor x+2=0 \Leftrightarrow$

$\Leftrightarrow x=2 \lor x=-2 \lor x=-2 \Leftrightarrow x=2 \lor x=-2.$

c) $(x+2)(x^2-4)+(x-2)(x^2+5x+6)=0 \Leftrightarrow$

$\Leftrightarrow (x+2)(x+2)(x-2)+(x-2)(x+2)(x+3)=0 \Leftrightarrow$

$\Leftrightarrow (x+2)(x-2)[(x+2)+(x+3)]=0 \Leftrightarrow$

$\Leftrightarrow (x+2)(x-2)(2x+5)=0 \Leftrightarrow$

$\Leftrightarrow x+2=0 \lor x-2=0 \lor 2x+5=0 \Leftrightarrow$

$\Leftrightarrow x=-2 \lor x=2 \lor x=-5/2.$

d) $(x^3-8)(x^2+4x+4)+(x^2-4)(x^2+5x+6)=0 \Leftrightarrow$

$\Leftrightarrow (x-2)(x^2+2x+4)(x+2)^2+(x-2)(x+2)(x+2)(x+3)=0$

$\Leftrightarrow (x-2)(x+2)^2[(x^2+2x+4)+(x+3)]=0 \Leftrightarrow$

$\Leftrightarrow (x-2)(x+2)^2(x^2+2x+4+x+3)=0 \Leftrightarrow$

$\Leftrightarrow (x-2)(x+2)^2(x^2+3x+7)=0 \Leftrightarrow$

$\Leftrightarrow x-2=0 \vee x+2=0 \vee x^2+3x+7=0 \qquad (1)$

To solve $x^2+3x+7=0$:

$\Delta = b^2-4ac = 3^2-4\cdot 1\cdot 7 = 9-28<0 \Rightarrow$

$\Rightarrow$ no real solutions

It follows that

$(1) \Leftrightarrow x-2=0 \vee x+2=0 \Leftrightarrow$

$\Leftrightarrow x=2 \vee x=-2$

thus $S=\{-2,2\}$.

$\hookrightarrow$ Note that we use equation labelling to interrupt the main line of our argument, to solve all quadratic factors, and then restart and finish it.

EXERCISES

4. Solve the equations

a) $(2x+1)(x-2)(x+3)=0$

b) $(x-1)(x+1)^2(1-3x)=0$

c) $(3x-1)(x+2)(x^2+1)=0$

d) $(2x-1)^3(2x^2+1)(x^2-1)=0$

5. Solve the equations

a) $5x^3-20x=0$

b) $x^3=x^2+6x$

c) $(3x-1)(x-2)^2=9(3x-1)$

d) $x^3-x^2-x+1=0$

e) $(x^2-4)^2-(x+2)^2(5x-4)=0$

f) $3(x-1)^2-2(x-1)(x+1)=(x+1)^2$

g) $(x-3)(2x+1)^2-(x^2-9)(x+3)=0$

h) $x^5+x^4+x^3+x^2+x+1=0$

i) $(x+1)^4-x^4=4x^3$.

Special cases / tricks

1 Binomial Equations

Let $k \in \mathbb{Z}$ be an integer, let $p \in (0, +\infty)$ be a positive number and $n \in (-\infty, 0)$ be a negative number. Then

Odd Binomial: $[f(x)]^{2k+1} = a \Leftrightarrow f(x) = \sqrt[2k+1]{a}$

Even Binomial: $[f(x)]^{2k} = p \Leftrightarrow f(x) = \sqrt[2k]{p} \vee f(x) = -\sqrt[2k]{p}$

$[f(x)]^{2k} = 0 \Leftrightarrow f(x) = 0$

$[f(x)]^{2k} = n \leftarrow$ inconsistent.

EXAMPLES

a) $(2x+1)^3 - 8 = 0$

Solution

$(2x+1)^3 - 8 = 0 \Leftrightarrow (2x+1)^3 = 8 \Leftrightarrow 2x+1 = \sqrt[3]{8} \Leftrightarrow 2x+1 = 2$

$\Leftrightarrow 2x = 2 - 1 \Leftrightarrow 2x = 1 \Leftrightarrow x = 1/2$

thus $S = \{1/2\}$.

b) $(1-3x)^4 - 16 = 0$

Solution

$(1-3x)^4 - 16 = 0 \Leftrightarrow (1-3x)^4 = 16 \Leftrightarrow$

$\Leftrightarrow 1-3x = \sqrt[4]{16} \vee 1-3x = -\sqrt[4]{16} \Leftrightarrow$

$\Leftrightarrow 1-3x = 2 \vee 1-3x = -2 \Leftrightarrow$

$\Leftrightarrow -3x = -1+2 \vee -3x = -1-2 \Leftrightarrow$

$\Leftrightarrow -3x = 1 \vee -3x = -3 \Leftrightarrow x = -1/3 \vee x = 1$

thus $S = \{-1/3, 1\}$.

c) $(x^2+x)^4 = 0$

Solution

$(x^2+x)^4 = 0 \Leftrightarrow x^2+x = 0 \Leftrightarrow x(x+1) = 0 \Leftrightarrow$

$\Leftrightarrow x = 0 \vee x+1 = 0 \Leftrightarrow x = 0 \vee x = -1$

thus $S = \{0, -1\}$.

d) $(x-3)^6 + 2 = 0$

Solution

$(x-3)^6 + 2 = 0 \Leftrightarrow (x-3)^6 = -2 \leftarrow$ inconsistent

thus $S = \emptyset$.

2 $\rightarrow$ Auxilliary Substitution

Sometimes, equations can be solved only via auxilliary substitution, as in the following example:

EXAMPLE

a) $(x+1)^4 - 4(x+1)^2 + 3 = 0$

Solution

Let $y = (x+1)^2$. Then the equation yields:

$$y^2 - 4y + 3 = 0 \Leftrightarrow (y-3)(y-1) = 0 \Leftrightarrow y-3=0 \vee y-1=0$$
$$\Leftrightarrow y = 3 \vee y = 1 \Leftrightarrow$$
$$\Leftrightarrow (x+1)^2 = 3 \vee (x+1)^2 = 1 \Leftrightarrow$$
$$\Leftrightarrow x+1 = \sqrt{3} \vee x+1 = -\sqrt{3} \vee x+1 = 1 \vee x+1 = -1$$
$$\Leftrightarrow x = -1+\sqrt{3} \vee x = -1-\sqrt{3} \vee x = 0 \vee x = -2$$

and therefore

$$S = \{-1+\sqrt{3}, -1-\sqrt{3}, 0, 2\}.$$

b) $(x^2+2x)^2 - 5(x^2+2x) + 4 = 0$

Solution

Let $y = x^2 + 2x$. Then the equation yields:

$$y^2 - 5y + 4 = 0 \Leftrightarrow (y-4)(y-1) = 0 \Leftrightarrow y-4=0 \vee y-1=0 \Leftrightarrow$$
$$\Leftrightarrow x^2+2x-4 = 0 \vee x^2+2x-1 = 0 \quad (1).$$

Solve: $x^2 + 2x - 4 = 0$.

$$\Delta = b^2 - 4ac = 2^2 - 4\cdot 1\cdot(-4) = 4 + 16 = 20 = 5\cdot 4 \Rightarrow$$
$$\Rightarrow x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-2 \pm 2\sqrt{5}}{2\cdot 1} = -1 \pm \sqrt{5}$$

Solve: $x^2+2x-1=0$

$\Delta = b^2-4ac = 2^2-4\cdot 1\cdot(-1) = 4+4 = 8 = (2\sqrt{2})^2 \Rightarrow$

$\Rightarrow x_{1,2} = \dfrac{-b\pm\sqrt{\Delta}}{2a} = \dfrac{-2\pm 2\sqrt{2}}{2} = -1\pm\sqrt{2}$

Therefore:

$(1) \Leftrightarrow x=-1+\sqrt{5} \vee x=-1+\sqrt{5} \vee x=-1-\sqrt{2} \vee x=-1+\sqrt{2}$.

so $S=\{-1+\sqrt{5}, -1-\sqrt{5}, -1-\sqrt{2}, -1+\sqrt{2}\}$.

↳ Note that it is not convenient to solve these 2 quadratics equations simultaneously, so we stop the main argument, label the last step as equation (1), solve the two quadratic equations separately and then we use the equation label to restart the argument.

3 → Sum of squares

Some special equations can be solved via using the following properties:

$$a^2+b^2=0 \Leftrightarrow a=0 \wedge b=0$$
$$a^2+b^2+c^2=0 \Leftrightarrow a=0 \wedge b=0 \wedge c=0$$

EXAMPLES

a) $(x+2)^6+(x^3-4x)^4=0$

Solution

$(x+2)^6+(x^3-4x)^4=0 \Leftrightarrow [(x+2)^3]^2+[(x^3-4x)^2]^2=0$

$\Leftrightarrow (x+2)^3=0 \wedge (x^3-4x)^2=0 \Leftrightarrow x+2=0 \wedge x^3-4x=0$

$\Leftrightarrow x+2=0 \wedge x(x^2-4)=0 \Leftrightarrow$

$\Leftrightarrow x+2=0 \wedge x(x-2)(x+2)=0 \Leftrightarrow$

$\Leftrightarrow x=-2 \wedge (x=0 \vee x-2=0 \vee x+2=0) \Leftrightarrow$

$\Leftrightarrow x=-2 \wedge (x=0 \vee x=2 \vee x=-2) \Leftrightarrow$

$\Leftrightarrow x=-2.$

b) $(x^2-9)^2+(x-3)^2(x^2+4x+3)^2=0$

Solution

$$(x^2-9)^2+(x-3)^2(x^2+4x+3)^2=0 \Leftrightarrow$$

$$\Leftrightarrow (x^2-9)^2+[(x-3)(x^2+4x+3)]^2=0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x^2-9=0 \\ (x-3)(x^2+4x+3)=0 \end{cases} \quad (1).$$

Solve: $x^2-9=0 \Leftrightarrow x^2=9 \Leftrightarrow x=3 \vee x=-3$ (2)

Solve: $(x-3)(x^2+4x+3)=0 \Leftrightarrow (x-3)(x+1)(x+3)=0$

$\Leftrightarrow x-3=0 \vee x+1=0 \vee x+3=0 \Leftrightarrow$

$\Leftrightarrow x=3 \vee x=-1 \vee x=-3$ (3)

From (1), (2), and (3):

$$(1) \Leftrightarrow \begin{cases} x=3 \vee x=-3 \\ x=3 \vee x=-1 \vee x=-3 \end{cases} \Leftrightarrow x=3 \vee x=-3$$

Thus $S=\{3,-3\}$.

↳ Note that braces are used as an abbreviation for AND ($\wedge$):

$$\begin{cases} f_1(x)=f_2(x) \\ g_1(x)=g_2(x) \end{cases}$$

is the same thing as:

$f_1(x)=f_2(x) \wedge g_1(x)=g_2(x)$.

EXERCISES

(6) Solve the equations:

a) $(3x+2)^4 = 16$

b) $(x^2+3x)^3 = -8$

c) $(2x+5)^5 = 32$

d) $(2x-1)^5 + 3 = 0$

e) $(x+2)^6 - 2 = 0$

f) $(x^2-2x)^7 + 1 = 0$

g) $16(5x+3)^4 - 81 = 0$

h) $2(x^2-1)^3 - 54 = 0$

(7) Solve the equations

a) $x^4 - 5x^2 + 6 = 0$

b) $3x^6 + 5x^3 + 2 = 0$

c) $2x^4 - 7x^2 - 4 = 0$

(8) Solve the following equations

a) $(x-1)^6 - 9(x-1)^3 + 8 = 0$

b) $(x^2+x)^2 - 3(x^2+x) + 2 = 0$

c) $(x^4-1)^2 + 2(x^4-1) + 1 = 0$

d) $(x^2-3x)^2 + 5(x^2-3x) + 6 = 0$

↳ Use an auxilliary substitution

(9) Solve the equations:

a) $(3x-9)^2 + (2x-6)^2 = 0$

b) $(x^2+7x+10)^2 + (x^2-5x+6)^2 = 0$

c) $(x+1)^2 + (x^2-1)^4 = 0$

↳ Use "sum of squares" technique.

Rational Equations

- A rational equation is an equation that has an unknown x in the denominator of at least one fraction.

Solution

- 1 Find the LCM (Least Common Multiple) of the denominators
- 2 From the condition $LCM(x) \neq 0$ find the domain $A \subseteq \mathbb{R}$ of the equation.
- 3 Multiply both sides of the equation with the LCM and solve the resulting polynomial equation
- 4 Accept the solutions that belong to the domain A and reject any solutions that do not belong to the domain A

EXAMPLES

a) $\dfrac{4x}{x^2-x} = \dfrac{4}{x^2-1} - \dfrac{x}{x+1}$ (1)

Require:

$$\begin{cases} x^2-x \neq 0 \\ x^2-1 \neq 0 \\ x+1 \neq 0 \end{cases} \Leftrightarrow \begin{cases} x(x-1) \neq 0 \\ (x-1)(x+1) \neq 0 \\ x+1 \neq 0 \end{cases} \Leftrightarrow \begin{cases} x \neq 0 \\ x \neq 1 \\ x \neq -1 \end{cases}$$

thus domain: $A = \mathbb{R} - \{0, -1, 1\}$.

$$(1) \Leftrightarrow \frac{4}{x-1} = \frac{4}{x^2-1} - \frac{x}{x+1} \leftarrow LCM = x^2-1 = (x-1)(x+1)$$

$\Leftrightarrow 4(x+1) = 4 - x(x-1) \Leftrightarrow$

$\Leftrightarrow 4x+4 = 4 - x^2 + x \Leftrightarrow 4x = -x^2 + x \Leftrightarrow$

$\Leftrightarrow x^2 + (4-1)x = 0 \Leftrightarrow x^2 + 3x = 0 \Leftrightarrow$

$\Leftrightarrow x(x+3) = 0 \Leftrightarrow x = 0 \vee x+3 = 0 \Leftrightarrow$

$\Leftrightarrow x = 0 \vee x = -3$

Reject $x = 0$ since $0 \notin A$

Accept $x = -3$ since $-3 \in A$.

Solution set: $S = \{-3\}$.

b) $\dfrac{x-14}{x^2-4} + \dfrac{3}{x-2} = \dfrac{4}{x+2}$ (1)

Require:

$$\begin{cases} x^2-4 \neq 0 \\ x-2 \neq 0 \\ x+2 \neq 0 \end{cases} \Leftrightarrow \begin{cases} (x-2)(x+2) \neq 0 \\ x-2 \neq 0 \\ x+2 \neq 0 \end{cases} \Leftrightarrow \begin{cases} x \neq 2 \\ x \neq -2 \end{cases}$$

thus domain : $A = \mathbb{R} - \{2, -2\}$.

$LCM = x^2 - 4 = (x-2)(x+2)$

$(1) \Leftrightarrow (x-14) + 3(x+2) = 4(x-2) \Leftrightarrow$

$\Leftrightarrow x - 14 + 3x + 6 = 4x - 8 \Leftrightarrow$

$\Leftrightarrow 4x - 8 = 4x - 8 \Leftrightarrow 0x = 0 \leftarrow$ identity.

Solution set: $\underline{S = \mathbb{R} - \{-2, 2\}}$.

EXERCISES

(10) Solve the equations

a) $\dfrac{x}{x-3} + 3 = \dfrac{3}{x-3}$

b) $\dfrac{1}{x+1} + \dfrac{1}{x-1} = \dfrac{2x}{x^2-1}$

c) $\dfrac{1}{2-x} + \dfrac{2}{x+1} + \dfrac{3}{x^2-x-2} = 0$

d) $\dfrac{1}{x} - \dfrac{x}{1-x} = \dfrac{6x+5}{x^2-x}$

e) $\dfrac{13}{x+1} - \dfrac{1}{1-x} = \dfrac{5x-3}{x^2-1}$

f) $\dfrac{1}{x+1} + \dfrac{1}{x+2} + \dfrac{1}{x-1} + \dfrac{1}{x-2} = 0$

g) $\dfrac{2}{x(x+2)} = \dfrac{-1}{x^2+5x+6}$

h) $\dfrac{x+1}{x-2} + \dfrac{x-1}{x+2} = \dfrac{2x^2+4}{x^2-4}$

Parametric Linear Equations

These are equations where in addition to the unknown x, there is another <u>parameter</u> a. The goal is to find x in terms of a. In doing so, it is necessary to distinguish the values of a for which the equation has a unique solution from the values of a for which the equation is either inconsistent or an identity.

<u>Solution</u>

- 1 Simplify equation to
$$A(a)\,x = B(a).$$
- 2 For $A(a) \neq 0$, unique solution $x = \dfrac{B(a)}{A(a)}$
- 3 For $A(a) = 0$, consider what happens on a case by case basis.
(i.e. equation is either identity or inconsistent).

EXAMPLE

a) $a^2(x-1) = 4(x-a+1) \Leftrightarrow$

$\Leftrightarrow a^2x - a^2 = 4x - 4a + 4 \Leftrightarrow$

$\Leftrightarrow (a^2-4)x = a^2 - 4a + 4 \Leftrightarrow$

$\Leftrightarrow (a-2)(a+2)x = (a-2)^2 \quad (1)$

Distinguish cases:

▸ Case 1: $a \in \mathbb{R} - \{2, -2\}$

(1) has a unique solution

$$x = \frac{(a-2)^2}{(a-2)(a+2)} = \frac{a-2}{a+2}$$

▸ Case 2: $a = 2$

$(1) \Leftrightarrow 0x = 0 \leftarrow$ identity

▸ Case 3: $a = -2$

$(1) \Leftrightarrow 0x = (-2-2)^2 \Leftrightarrow 0x = 16 \leftarrow$ inconsistent.

Solution set:

$$S = \begin{cases} \{(a-2)/(a+2)\} & , a \in \mathbb{R} - \{2, -2\} \\ \mathbb{R} & , a = 2 \\ \varnothing & , a = -2. \end{cases}$$

EXERCISES

(11) Solve the equations with respect to x:

a) $a^2x + 2 = 2ax + x + a$
b) $2a + 3x = a^2x + 1$
c) $2a^2x - 5 = 4a - x$
d) $4ax + a^2x = 3x + 2$
e) $a^2(x-1) + a(x+2) - 6x + 15 = 0$
f) $a^3 + a^2x + a^2 + ax + a + x = 0$

(12) Solve the equations with respect to x:

a) $\dfrac{x+a}{a+1} + \dfrac{a+1}{a-1} = \dfrac{(a+1)^2}{a^2-1}$

b) $\dfrac{x-2}{a-2} + \dfrac{x-2}{a+2} = 1$

c) $x - 2 = \dfrac{3}{a} + \dfrac{x+1}{a^2}$

d) $\dfrac{x+2}{3a} - \dfrac{1}{6a} + \dfrac{a}{6} + \dfrac{x}{2a} = 0$

Inequalities – Terminology.

- An <u>inequality</u> is an expression of the form $f(x) < g(x)$ or $f(x) \leq g(x)$ or $f(x) > g(x)$ or $f(x) \geq g(x)$. which is either true or false depending on the value of the variable x.
- The <u>solution set S</u> of an inequality is the set of all real numbers $x \in \mathbb{R}$ for which the inequality is true.

Intervals

The solution sets of inequalities are written as unions of intervals, which are defined as follows:

$x \in [a, b] \Leftrightarrow a \leq x \leq b$	$x \in [a, +\infty) \Leftrightarrow a \leq x$
$x \in (a, b] \Leftrightarrow a < x \leq b$	$x \in (a, +\infty) \Leftrightarrow a < x$
$x \in [a, b) \Leftrightarrow a \leq x < b$	$x \in (-\infty, b] \Leftrightarrow x \leq b$
$x \in (a, b) \Leftrightarrow a < x < b$	$x \in (-\infty, b) \Leftrightarrow x < b$

The set of all real numbers can also be written as
$$\mathbb{R} = (-\infty, +\infty).$$

Basic properties of inequalities

1) Let $x, y, a \in \mathbb{R}$. Then

$x > y \Leftrightarrow x+a > y+a$

$x \geq y \Leftrightarrow x+a \geq y+a$

$x < y \Leftrightarrow x+a < y+a$

$x \leq y \Leftrightarrow x+a \leq y+a$

(i.e.: We can add any number to both sides of an inequality)

2) Let $x, y \in \mathbb{R}$ and $p \in (0, +\infty)$. Then

$x > y \Leftrightarrow px > py$	$x < y \Leftrightarrow px < py$
$x \geq y \Leftrightarrow px \geq py$	$x \leq y \Leftrightarrow px \leq py$

(i.e: We can multiply a positive number to both sides of an inequality.)

3) Let $x, y \in \mathbb{R}$ and $n \in (-\infty, 0)$. Then

$x > y \Leftrightarrow nx < ny$	$x < y \Leftrightarrow nx > ny$
$x \geq y \Leftrightarrow nx \leq ny$	$x \leq y \Leftrightarrow nx \geq ny$

(i.e: We can multiply a negative number to both sides of an inequality, but then the direction of the inequality must be reversed).

↳ The general strategy for solving inequalities is to first move every term to the same side, then <u>simplify</u> and <u>factor</u> the resulting expression.

▼ Polynomial inequalities

1) Linear Inequalities $\longrightarrow$ $ax+b \gtrless 0$

Consider, for example, the inequality $ax+b>0$.

a) For $a>0$:

$ax+b>0 \Leftrightarrow ax>-b \Leftrightarrow x>-b/a$.

thus $S=(-b/a, +\infty)$.

b) For $a<0$:

$ax+b>0 \Leftrightarrow ax>-b \Leftrightarrow \underline{\underline{x<-b/a}}$

thus $S=(-\infty, -b/a)$.

c) For $a=0$: the inequality is an identity or it is inconsistent, which we determine on a case by case basis.

EXAMPLES

a) $3(x-2)-5(x+1) \geqslant 3-2(3-x) \Leftrightarrow$

$\Leftrightarrow 3x-6-5x-5 \geqslant 3-6+2x \Leftrightarrow$

$\Leftrightarrow -2x-11 \geqslant -3+2x \Leftrightarrow -2x-2x \geqslant 11-3 \Leftrightarrow$

$\Leftrightarrow -4x \geqslant 8 \underset{(!!)}{\Leftrightarrow} x \leqslant \dfrac{8}{-4} \Leftrightarrow x \leqslant -2$

therefore $S=(-\infty, -2]$

▸ Note that because of $\leqslant$, -2 is included in S.

b) $x+3-\frac{3x-5}{2} > 2-\frac{x}{2} \Leftrightarrow$

$\Leftrightarrow 2(x+3)-(3x-5) > 4-x \Leftrightarrow$

$\Leftrightarrow 2x+6-3x+5 > 4-x \Leftrightarrow -x+11 > 4-x$

$\Leftrightarrow 0x > 4-11 \Leftrightarrow 0x > -7 \leftarrow$ always true.

therefore $S = \mathbb{R}$.

(i.e. the inequality is an identity; it is satisfied by all real numbers $x \in \mathbb{R}$).

c) $\frac{x-3}{4} - \frac{x+5}{2} < -1 - \frac{10+x}{4} \Leftrightarrow$

$\Leftrightarrow (x-3) - 2(x+5) < -4 - (10+x) \Leftrightarrow$

$\Leftrightarrow x-3-2x-10 < -4-10-x \Leftrightarrow$

$\Leftrightarrow -x-13 < -14-x \Leftrightarrow 0x < 13-14 \Leftrightarrow 0x < -1. \leftarrow$ always false

therefore $S = \emptyset$.

(i.e. the inequality is inconsistent. It can never be satisfied).

↳ Note that when the inequality has fractions, we first eliminate all fractions by multiplying both sides of the inequality with a large enough positive number.

EXERCISES

(13) Solve the following inequalities

1) $-3x+1>0$
2) $0x>4$
3) $0x<4$
4) $0x\geqslant 2$
5) $0x>-4$
6) $0x<-4$
7) $0x>0$
8) $0x\geqslant 0$
9) $0x\leqslant 2$
10) $0x\leqslant 0$
11) $-x-2<0$
12) $1>3x$
13) $4(2x-1)\leqslant x-2$
14) $3(2x+7)-4(15-x)\leqslant 29+12x$
15) $2(4x+5)-3(x+3)\leqslant -5x-9(1-x)$
16) $-6(x-2)-(5-3x)<9(x+3)-2x$
17) $2(x+1)\geqslant 4-(x+3)-3(2-x)$
18) $13-3(x-2)<4(x+3)-7(x-3)$
19) $1-\dfrac{3-x}{3}\geqslant\dfrac{19}{21}-\dfrac{1-x}{7}$
20) $\dfrac{x-3}{2}-\dfrac{x-5}{4}>1-\dfrac{4-x}{3}$
21) $\dfrac{x+1}{3}-\dfrac{5x-16}{6}\geqslant\dfrac{x+8}{12}$
22) $\dfrac{10x-1}{24}-\dfrac{2x-1}{8}<\dfrac{2x+5}{4}-\dfrac{x+3}{2}$
23) $\dfrac{2}{5}-\dfrac{3-x}{2}<\dfrac{x-1}{10}-\dfrac{3-2x}{5}$
24) $\dfrac{x+1}{16}-\dfrac{1+x}{2}\geqslant\dfrac{x-1}{16}-\dfrac{2x+1}{4}$

2) Quadratic Inequalities → $\boxed{ax^2+bx+c \gtrless 0}$

- •1 Calculate the discriminant $\Delta = b^2-4ac$ and the two zeroes x_1 and x_2 (if they exist) given by: $x_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

- •2 The expression $f(x) = ax^2+bx+c$ has the same sign as the coefficient "a" for all values of x EXCEPT when $\Delta > 0$ and $x_1 < x < x_2$ (i.e. when x is located between the zeroes x_1 and x_2). We use this rule to construct a sign chart.
- •3 From the sign chart we deduce the solution set.

→ Sign charts

x		x_1		x_2	
ax^2+bx+c	+	0	−	0	+

($a>0$ and $\Delta>0$)

x		x_1		x_2	
ax^2+bx+c	−	0	+	0	−

($a<0$ and $\Delta>0$).

x		$x_1=x_2$	
ax^2+bx+c	+	0	+

($a>0$ and $\Delta=0$)

x		$x_1=x_2$	
ax^2+bx+c	−	0	−

($a<0$ and $\Delta=0$)

x	
ax^2+bx+c	+

($a>0$ and $\Delta<0$)

x	
ax^2+bx+c	−

($a<0$ and $\Delta<0$).

EXAMPLES

a) Solve $-3x^2+6x+2 \geq 0$.

Solution

$$\Delta = b^2 - 4ac = 6^2 - 4(-3)\cdot 2 = 36+24 = 60 = 2^2\cdot 3\cdot 5 =$$
$$= 2^2\cdot 15 \Rightarrow \sqrt{\Delta} = 2\sqrt{15} \Rightarrow$$
$$\Rightarrow x_{1,2} = \frac{-b\pm\sqrt{\Delta}}{2a} = \frac{-6\pm 2\sqrt{15}}{2(-3)} = \frac{-3\pm\sqrt{15}}{-3} =$$
$$= 1 \pm \frac{\sqrt{15}}{3}.$$

x		$1-(\sqrt{15})/3$		$1+(\sqrt{15})/3$	
$-3x^2+6x+2$	$-$	0	$+$	0	$-$

and therefore $S = \left[1-\frac{\sqrt{15}}{3}, 1+\frac{\sqrt{15}}{3}\right]$

b) $-3x^2+x-2>0$

Solution

$$\Delta = b^2 - 4ac = 1^2 - 4(-3)(-2) = 1-24 = -23 < 0$$

x	
$-3x^2+x-2$	$-$

Therefore $S = \emptyset$ (i.e. the equation is inconsistent).

c) $x^2+x+3>0$

<u>Solution</u>

$\Delta = b^2-4ac = 1^2-4\cdot 1\cdot 3 = 1-12 = -11 < 0$

x	
x^2+x+3	+

therefore $S=\mathbb{R}$. (i.e. equation is an identity).

d) $x^2+4x+4 \leqslant 0$

<u>Solution</u>

$\Delta = b^2-4ac = 4^2-4\cdot 1\cdot 4 = 16-16=0 \Rightarrow$

$\Rightarrow x_1 = x_2 = \dfrac{-b}{2a} = \dfrac{-4}{2\cdot 1} = -2$

x		-2	
x^2+4x+4	+	0	+

therefore $S=\{2\}$ (!)

e) $x^2+6x+9>0$

<u>Solution</u>

$\Delta = b^2-4ac = 6^2-4\cdot 1\cdot 9 = 36-36=0 \Rightarrow$

$\Rightarrow x_1 = x_2 = \dfrac{-b}{2a} = \dfrac{-6}{2\cdot 1} = -3$

x		-3	
x^2+6x+9	$+$	0	$+$

therefore $S = (-\infty, -3) \cup (-3, +\infty) = \mathbb{R} - \{-3\}$.

Factorizable quadratic inequalities

An alternative technique is available if the quadratic has an obvious factorization. Recall the following obvious factorizations:

$x^2 - a^2 = (x-a)(x+a)$

$ax^2 + bx = x(ax+b)$.

$x^2 + (a+b)x + ab = (x+\alpha)(x+\beta)$.

To use this method we require the following templates for the sign of the linear factor $ax+b$.

x		$-b/a$	
$ax+b$	$-$	0	$+$

$(a>0) \leftrightarrow$ increasing

x		$-b/\alpha$	
$ax+b$	$+$	0	$-$

$(a<0) \leftrightarrow$ decreasing

EXAMPLES

a) $x^2+5x+6>0$.

Solution

$x^2+5x+6>0 \Leftrightarrow (x+2)(x+3)>0$

x		-3		-2	
$x+2$	$-$		$-$	0	$+$
$x+3$	$-$	0	$+$		$+$
ineq	$+$	0	$-$	0	$+$

and therefore
$S = (-\infty, -3) \cup (-2, +\infty)$.

- 1 Identify the zeroes of every factor and sort them from smallest to largest.
- 2 Write the zeroes and signs for each factor.
- 3 Multiply signs of all factors to determine the sign of the inequality.

b) $3x - 2x^2 \leq 0$

Solution

$3x - 2x^2 \leq 0 \Leftrightarrow x(3-2x) \leq 0 \leftarrow$ Zeroes: 0, $3/2$

x		0		$3/2$	
x	$-$	0	$+$		$+$
$3-2x$	$+$		$+$	0	$-$
ineq	$-$	0	$+$	0	$-$

and therefore:
$S = (-\infty, 0] \cup [3/2, +\infty)$.

3) Higher-order inequalities

These are inequalities of the form

$$a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \gtrless 0$$

- •1 Move everything to the left-hand side and factor to linear and quadratic factors.
- •2 Find the zeroes of every factor.
- •3 Make a sign chart for each factor and for their product.
- •4 See where the inequality is satisfied.

EXAMPLES

a) Solve: $x^2(x-2) \leq 2x(x-2)^2$.

Solution

$$x^2(x-2) \leq 2x(x-2)^2 \Leftrightarrow x^2(x-2) - 2x(x-2)^2 \leq 0 \Leftrightarrow$$

$$\Leftrightarrow x(x-2)(x-2(x-2)) \leq 0 \Leftrightarrow$$

$$\Leftrightarrow x(x-2)(x-2x+4) \leq 0 \Leftrightarrow x(x-2)(-x+4) \leq 0. \qquad (1)$$

Zeroes: $0, 2, 4$

x		0		2		4	
x	$-$	0	$+$		$+$		$+$
$x-2$	$-$		$-$	0	$+$		$+$
$4-x$	$+$		$+$		$+$	0	$-$
ineq	$+$	0	$-$	0	$+$	0	$-$

thus

$(1) \Leftrightarrow x \in [0,2] \cup [4, +\infty)$

and therefore

$S = [0,2] \cup [4, +\infty)$.

→ Let $k \in \mathbb{N}$. Then:

a) <u>Even powers:</u> $(ax+b)^{2k}$ and $(ax^2+bx+c)^{2k}$ are ALWAYS positive.

b) <u>Odd powers:</u> $(ax+b)^{2k+1}$ and $(ax^2+bx+c)^{2k+1}$ have the same sign they would have had without the odd power. Therefore:

$(ax+b)^{2k+1}$ has the same sign as $(ax+b)$.

$(ax^2+bx+c)^{2k+1}$ has the same sign as (ax^2+bx+c).

b) Solve: $(x^2-1)^2(x^2+x-1)^3 > 0$

<u>Solution</u>

Zeroes of $f_1(x) = x^2-1 = (x-1)(x+1)$: -1 and $+1$.

Zeroes of $f_2(x) = x^2+x-1$.

$\Delta = b^2-4ac = 1^2-4\cdot 1\cdot(-1) = 1+4 = 5 \Rightarrow$

$\Rightarrow x_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a} = \dfrac{-1\pm\sqrt{5}}{2} \longrightarrow \dfrac{-1-\sqrt{5}}{2}$ and $\dfrac{-1+\sqrt{5}}{2}$.

- Now, we must sort the zeroes.

We claim that: $\dfrac{-1-\sqrt{5}}{2} < -1 < \dfrac{-1+\sqrt{5}}{2} < 1$.

We can show this easily with a calculator or via the following argument:

$$\frac{-1-\sqrt{5}}{2} < \frac{-1-\sqrt{4}}{2} = \frac{-1-2}{2} = \frac{-3}{2} < -1$$

$$\frac{-1+\sqrt{5}}{2} > \frac{-1+\sqrt{4}}{2} = \frac{-1+2}{2} = \frac{1}{2} > -1$$

$$\frac{-1+\sqrt{5}}{2} < \frac{-1+\sqrt{9}}{2} = \frac{-1+3}{2} = \frac{2}{2} = 1.$$

From the above it follows that

$$\frac{-1-\sqrt{5}}{2} < -1 < \frac{-1+\sqrt{5}}{2} < 1$$

• Now we construct the sign table:

x		$(-1-\sqrt{5})/2$		-1		$(-1+\sqrt{5})/2$		1	
$(x^2-1)^2$	+		+	0	+		+	0	+
$(x^2+x-1)^3$	+	0	−		−	0	+		+
ineq	+	0	−	0	−	0	+	0	+

It follows that

$$S = \left(-\infty, \frac{-1-\sqrt{5}}{2}\right) \cup \left(\frac{-1+\sqrt{5}}{2}, 1\right) \cup (1, +\infty).$$

c) Solve: $x^7 < x^3$.

Solution

$$x^7 < x^3 \Leftrightarrow x^7 - x^3 < 0 \Leftrightarrow x^3(x^4-1) < 0 \Leftrightarrow$$

$$\Leftrightarrow x^3(x^2-1)(x^2+1) < 0 \Leftrightarrow$$

$$\Leftrightarrow x^3(x-1)(x+1)(x^2+1) < 0.$$

Zeroes: $0, -1, +1$.

x		-1		0		$+1$	
x^3	$-$		$-$	0	$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$		$+$
x^2+1	$+$		$+$		$+$		$+$
ineq	$-$	0	$+$	0	$-$	0	$+$

therefore: $S = (-\infty, -1) \cup (0, 1)$.

↳ Note that incomplete quadratic factors of the form ax^2+b with $a>0$ and $b>0$ are always positive. (Because $\Delta<0$ and $a>0$).

EXERCISES

(14) Solve the inequalities:

a) $x^2+3x+2\leq 0$
b) $x^2+5x+6>0$
c) $x^2+8x-33\geq 0$
d) $2x^2-20x+50\leq 0$
e) $-2x^2+x-1\geq 0$
f) $x^2+x+2<0$
g) $x^2-4x+8>0$
h) $x^2+28x+196\leq 0$
i) $x^2-22x+121>0$
j) $x^2-26x+169<0$
k) $x^2-4<0$
l) $x^2-5\leq 0$
m) $x^2+3>0$
n) $x^2+2\leq 0$
o) $x^2+3x>0$
p) $2x^2-5x<0$
q) $x^2+x-3\geq 0$
r) $x^2+2x-7<0$

(15) Solve the inequalities

a) $(3x-1)(x-1)^3(2-x)(2x+5)^4\geq 0$
b) $5x(x^2-4x+3)(x^2-10x+25)(x^2+x+1)\leq 0$
c) $(2x^4-x^2)(x^2-3)^2(2-x)^3<0$
d) $(2x^2-5x-3)^2(x^3-x^2-x)\geq 0$

(16) Solve the inequalities:

a) $x^3+4x>5x^2$
b) $x^3+x\leq x^2+1$
c) $x^3<8$
d) $x^8>x^2$
e) $x^9\leq 4x^5$
f) $x^8+64x^2\leq 18x^5$

g) $2x(x+1)^2 - 3x^2(x+1) \leq 0$

h) $(2x+1)^4(2x-1)^3 > (2x+1)^3(2x-1)^4$

i) $x(x+1)^2(x+2) \geq (x^2+2x)(x^2+3x+2)$

j) $3(2x+3)^2(x-1)^3 \leq 2(2x+3)^3(x-1)^2$

k) $x(x+1)^5 < x(x+1)^3$

l) $(x+3)^5(2x-1)^3 \geq (x+3)^3(2x-1)^5$

m) $x^3(x^2+3x)^7 \leq x^5(x^2+3x)^5$.

Rational Inequalities

Form: $\dfrac{P(x)}{Q(x)} \gtrless 0$

with P, Q polynomials.

Method: The method entails the same steps as with polynomial inequalities. However, the zeroes of numerator factors must be distinguished from the zeroes of denominator factors.

- Denominator zeroes are shown with the $\nparallel$ symbol instead of ϕ in the last entry of your sign table because at these zeroes, the expression is undefined.
- Denominator zeroes are to be excluded from the solution set.

examples

1) $\dfrac{x-5}{x-3} \geq \dfrac{x-2}{x-1}$ $\quad (1)$

Solution:

$$(1) \Leftrightarrow \frac{x-5}{x-3} - \frac{x-2}{x-1} \geq 0 \Leftrightarrow \frac{(x-5)(x-1)-(x-2)(x-3)}{(x-3)(x-1)} \geq 0$$

$$\Leftrightarrow \frac{(x^2-6x+5)-(x^2-5x+6)}{(x-3)(x-1)} \geq 0$$

$$\Leftrightarrow \frac{(-6+5)x+(5-6)}{(x-3)(x-1)} \geq 0$$

$$\Leftrightarrow \frac{-x-1}{(x-3)(x-1)} \geq 0 . \quad (2)$$

Zeroes: $-1, 3, 1$

x		-1		1		3	
$-x-1$	$+$	0	$-$		$-$		$-$
$x-3$	$-$		$-$		$-$	0	$+$
$x-1$	$-$		$-$	0	$+$		$+$
$f(x)$	$+$	0	$-$	$\parallel$	$+$	$\parallel$	$-$

$(2) \Leftrightarrow x \in (-\infty, -1] \cup (1, 3)$

Note that -1 is a zero of $f(x)$ but 1 and 3 are not, so they are not included in the solution.

→ <u>CAUTION</u> : If the fraction has cancellations then you must find the domain of the inequality before solving it :

<u>example</u> : $\dfrac{(x+1)(x^2+4x+4)}{(x^2+5x+6)} \geq 0. \quad (1)$

$$(1) \Leftrightarrow \frac{(x+1)(x+2)^2}{(x+2)(x+3)} \geq 0 \Leftrightarrow \frac{(x+1)(x+2)}{x+3} \geq 0$$

▶ Domain:

$x^2+5x+6 \neq 0 \Leftrightarrow \underline{x \in \mathbb{R} - \{-2, -3\} = A}$

x		-3		-2		-1	
$x+1$	$-$		$-$		$-$	0	$+$
$x+2$	$-$		$-$	0	$+$		$+$
$x+3$	$-$	0	$+$		$+$		$+$
$f(x)$	$-$	$\Vert$	$+$	$\Vert$	$-$	0	$+$

thus

$(1) \Leftrightarrow \underline{x \in (-3, -2) \cup [-1, +\infty)}$

-2 looks like a numerator zero but it cannot solve the original inequality because the domain

$$A = \mathbb{R} - \{-2, -3\}$$

of the inequality EXCLUDES -2 !!

EXERCISES

(17) Solve the inequalities:

a) $\dfrac{2-x}{3x+1} \geq 0$ b) $\dfrac{-(1-x)(3+x)(-3+x)}{(x+2)^2(x+1)^3} \geq 0$

c) $\dfrac{-x^2(3-x)(x^2+3x+2)(x^2-3)}{3(x+1)} \geq 0$

(18) Solve the inequalities

a) $\dfrac{2x-1}{x^2+4x+3} \leq \dfrac{1}{5}$

b) $\dfrac{x+1}{x-1} < \dfrac{2x+3}{x+1}$

c) $\dfrac{x^2+14}{x^2+6x+8} \leq 1$

d) $\dfrac{x+1}{x^2+x-2} \leq \dfrac{x}{x^2-1}$

e) $\dfrac{(x+1)^3-1}{(x-1)^3-1} \leq 1$

f) $\dfrac{x-10}{x^2+5} < \dfrac{1}{2}$

g) $\dfrac{6x^2-3x+8}{x^2-5x+6} \leq 6$

h) $\dfrac{x}{1+x^2} > 10$

System of inequalities

$$\begin{cases} f_1(x) \gtrless g_1(x) \\ f_2(x) \gtrless g_2(x) \\ \cdots \\ f_n(x) \gtrless g_n(x) \end{cases}$$

- 1 Find the solution sets $S_1, S_2, \ldots, S_n$ for each inequality separately.
- 2 The solution set S for the system is the intersection $S = S_1 \cap S_2 \cap \cdots \cap S_n$

EXAMPLE

$x+1 \leq (2x+1)^2 < 10x+5$

Solution

$x+1 \leq (2x+1)^2 \Leftrightarrow x+1 \leq 4x^2+4x+1 \Leftrightarrow$

$\Leftrightarrow 4x^2+3x \geq 0 \Leftrightarrow x(4x+3) \geq 0.$ (1)

x		$-3/4$		0	
x	$-$		$-$	0	$+$
$4x+3$	$-$	0	$+$		$+$
	$+$	0	$-$	0	$+$

(1) $\Leftrightarrow x \in (-\infty, -3/4] \cup [0, +\infty)$.

and

$(2x+1)^2 < 10x+5 \Leftrightarrow (2x+1)^2 - 5(2x+1) < 0 \Leftrightarrow$

$\Leftrightarrow (2x+1)[(2x+1)-5] < 0 \Leftrightarrow (2x+1)(2x-4) < 0$ (2)

$\Leftrightarrow (2x+1)(x-2) < 0$ (2)

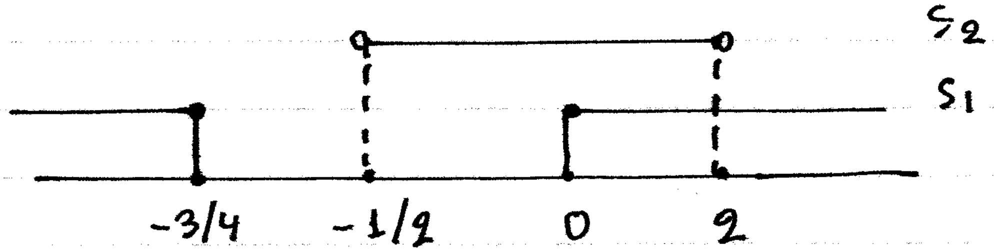
x		$-1/2$		2	
$2x+1$	$-$		$+$		$+$
$x-2$	$-$		$-$		$+$
	$+$		$-$		$+$

$(2) \Leftrightarrow x \in (-\infty, -1/2)$

$(2) \Leftrightarrow x \in (-1/2, 2)$.

Thus $S_1 = (-\infty, -3/4] \cup [0, +\infty)$

$S_2 = (-1/2, 2)$



It follows that the solution set is

$$S = S_1 \cap S_2 = $$
$$= [(-\infty, -3/4] \cup [0, +\infty)] \cap [(-1/2, 2)]$$
$$= [0, 2)$$

↳ In general, to calculate intersections of solution sets we use a geometrical construction as shown in the above example.

EXERCISES

(20) Solve the following systems of inequalities.

a) $-3 \leq \dfrac{x+5}{2-x} < 4$

b) $-2x < \dfrac{3x-1}{4} \leq x^2-1$

c) $5x-1 < (x+1)^2 \leq 7x-3$

d) $\begin{cases} 3x^3+2x > 5x^2 \\ x^3+4x > x^2 \end{cases}$

e) $\begin{cases} 2x-5 < 3x-7 \\ 2x^2 \leq 9 \end{cases}$

f) $\begin{cases} \dfrac{x-1}{3x+2} > 0 \\ (x^2-9)(x^2+x+5) \leq 0 \end{cases}$

g) $\begin{cases} \dfrac{x-1}{x+1} < \dfrac{1}{2} \\ \dfrac{(x-1)(x-2)}{(x+1)(x+2)} > 2 \\ \dfrac{x^2-4x+1}{x^2+x-1} \leq 2 \end{cases}$

Absolute Values

- Let $x \in \mathbb{R}$ be given. We define the <u>absolute value</u> $|x|$ of x as:

$$\boxed{|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}}$$

► examples : $|3| = 3$, $|-7| = 7$, $|0| = 0$

Properties of absolute value

$\lvert x\rvert \geq 0$	$\lvert x\rvert - \lvert y\rvert \leq \lvert x+y\rvert \leq \lvert x\rvert + \lvert y\rvert$
$\lvert -x\rvert = \lvert x\rvert$	$\lvert x\rvert - \lvert y\rvert \leq \lvert x-y\rvert \leq \lvert x\rvert + \lvert y\rvert$
$-\lvert x\rvert \leq x \leq \lvert x\rvert$	$\lvert xy\rvert = \lvert x\rvert \lvert y\rvert$, $\left\lvert \frac{x}{y} \right\rvert = \frac{\lvert x\rvert}{\lvert y\rvert}$
$\lvert x\rvert^2 = x^2$	

Equations with absolute values

Let $a, x \in \mathbb{R}$, and $p \in [0, +\infty)$, and $n \in (-\infty, 0)$. Then:

1) $|x| = |a| \Leftrightarrow x = a \vee x = -a$
2) $|x| = p \Leftrightarrow x = p \vee x = -p$
3) $|x| = n$ is inconsistant.

We use the above 3 properties to solve equations with absolute values as in the following examples.

EXERCISES

(23) If $a<3<b$, show that

a) $A=|3-a|+|3-b|-|a-b|$
b) $B=|a-3|+|b|+|a-b|$
c) $C=|a-4|-|b-2|$
d) $D=|a-b|-|5-a|-|1-b|$
e) $E=|a^3-5a^2|-|b^2-ab|+|2b+a-1|$
f) $F=|b^2-9|+|2b+1|-|3b-ab|$.
g) $G=|b^3-4b|+|2a^3b-2a^2b^2|+|b+2ab-3b^2|$

(24) Simplify the following expressions.

a) $A=\dfrac{x^2+2|x|}{|x|+2}$ b) $B=\dfrac{|x|^3+3x^2}{2|x|+6}$

c) $C=\dfrac{x^2+4|x|+4}{|x|+2}$ d) $D=\dfrac{x^2-1}{|x|+1}$

EXAMPLES

a) $(x^2 - |x|)(3|x|+1) = 0 \quad (1)$

Solution

Let $y = |x| \Rightarrow x^2 = |x|^2 = y^2$, thus

$(1) \Leftrightarrow (y^2 - y)(3y+1) = 0 \Leftrightarrow y(y-1)(3y+1) = 0 \Leftrightarrow$

$\Leftrightarrow y = 0 \vee y-1 = 0 \vee 3y+1 = 0 \Leftrightarrow$

$\Leftrightarrow y = 0 \vee y = 1 \vee y = -1/3 \Leftrightarrow$

$\Leftrightarrow |x| = 0 \vee |x| = 1 \vee |x| = -1/3 \Leftrightarrow$

$\Leftrightarrow x = 0 \vee x = 1 \vee x = -1$

thus $S = \{0, 1, -1\}$.

Note that $|x| = -1/3$ has no solutions since $-1/3 < 0$.

b) $|x+3| + 2 = 0 \quad (1)$

Solution

$(1) \Leftrightarrow |x+3| = -2 < 0 \leftarrow$ inconsistent.

Thus $S = \emptyset$.

c) $|2x-1| - 5 = 0 \quad (1)$

Solution

$(1) \Leftrightarrow |2x-1| = 5 \Leftrightarrow 2x - 1 = \pm 5 \Leftrightarrow 2x = 1 \pm 5$

$\Leftrightarrow x = \dfrac{1 \pm 5}{2} = \begin{cases} 6/2 = 3 \\ -4/2 = -2 \end{cases}$, thus $S = \{3, -2\}$.

d) $|2x+3| = |x+9|$

Solution

$|2x+3| = |x+9| \Leftrightarrow 2x+3 = x+9 \vee 2x+3 = -(x+9) \Leftrightarrow$

$\Leftrightarrow (2-1)x = 9-3 \vee 2x+3 = -x-9 \Leftrightarrow$

$\Leftrightarrow x = 6 \vee 2x+x = -3-9 \Leftrightarrow x = 6 \vee 3x = -12$

$\Leftrightarrow x = 6 \vee x = -4.$

thus $S = \{6, -4\}$.

e) $|x-4| = 5-2x \quad (1)$

Solution

Require $5-2x \geq 0 \Leftrightarrow 5 \geq 2x \Leftrightarrow x \leq 5/2$

thus domain: $A = (-\infty, 5/2]$.

$(1) \Leftrightarrow x-4 = 5-2x \vee x-4 = -(5-2x) \Leftrightarrow$

$\Leftrightarrow x+2x = 4+5 \vee x-4 = -5+2x \Leftrightarrow$

$\Leftrightarrow 3x = 9 \vee x-2x = 4-5 \Leftrightarrow$

$\Leftrightarrow x = 3 \vee -x = -1 \Leftrightarrow$

$\Leftrightarrow x = 3 \vee x = 1.$ ← accept $x=1$, reject $x=3$

→ For equations of the form $|f(x)| = g(x)$ we require

$g(x) \geq 0 \Leftrightarrow x \in A$

and reject solutions that do not belong to A.

f) $|x+3|-|2-x|=x+5 \quad (1)$

Solution

x		-3		2	
$x+3$	$-$	0	$+$		$+$
$2-x$	$+$		$+$	0	$-$

Distinguish 3 cases:

Case 1: If $x \in (-\infty, -3)$ then

$|x+3| = -(x+3)$ and $|2-x| = 2-x$.

$(1) \Leftrightarrow -(x+3)-(2-x) = x+5 \Leftrightarrow$

$\Leftrightarrow -x-3-2+x = x+5 \Leftrightarrow$

$\Leftrightarrow -x-5 = 5 \Leftrightarrow x = -5-5 = -10 \leftarrow$ accepted

$-10 \in (-\infty, -3)$.

Case 2: If $x \in [-3, 2)$ then

$|x+3| = x+3$ and $|2-x| = 2-x$.

$(1) \Leftrightarrow (x+3)-(2-x) = x+5 \Leftrightarrow$

$\Leftrightarrow x+3-2+x = x+5 \Leftrightarrow 2x+1 = x+5 \Leftrightarrow$

$\Leftrightarrow 2x-x = 5-1 \Leftrightarrow x = 4 \leftarrow$ rejected

$4 \notin [-3, 2)$.

Case 3: If $x \in [2, +\infty)$ then

$|x+3| = x+3$ and $|2-x| = -(2-x)$

$(1) \Leftrightarrow (x+3)+(2-x) = x+5 \Leftrightarrow$

$\Leftrightarrow x+3+2-x = x+5 \Leftrightarrow 5 = x+5 \Leftrightarrow x = 0$

rejected

$0 \notin [2, +\infty)$.

Thus $S = \{-10\}$.

EXERCISES

25 Solve the equations

a) $\dfrac{2+|-5x|}{|x|-1}=3$

b) $\dfrac{3+|x|}{|2x|+1}=4$

c) $2x^2+5|x|+7=0$

d) $(|2x|-3)(|x^3|-x^2)=0$

26 Solve the equations

a) $|2x|=|x-1|$

b) $|3x-2|=|2-x|$

c) $|x^2-1|=|2-x|$

d) $|2x-3|=x$

e) $|2x-1|=4$

f) $|2x^2-5x-1|=4x$

g) $|x^2-1|=2x+1$

h) $x^3-x^2+|x-1|=0$

27 Solve the equations

a) $|3x|+|2-x|-x+1=0$

b) $|x-3|-3|x-1|+|x|=5$

c) $2|x+1|-3|x-1|=1$

d) $|x^2-4x+3|-2|3-x^2|=1$

Inequalities with absolute values

The solution of inequalities with absolute values is based on the following properties:

1) If $a>0$, then

$|x| \leq a \Leftrightarrow -a \leq x \leq a$

$|x| < a \Leftrightarrow -a < x < a$

$|x| \geq a \Leftrightarrow x \geq a \vee x \leq -a$

$|x| > a \Leftrightarrow x > a \vee x < -a$

2) If $a<0$, then

$$\left.\begin{array}{l} |x| \leq a \longleftarrow \text{Inconsistent} \\ |x| < a \longleftarrow \text{Inconsistent} \\ |x| \geq a \longleftarrow \text{Identity} \\ |x| > a \longleftarrow \text{Identify} \end{array}\right\} \begin{array}{l} \text{because} \\ |x| \geq 0 \end{array}$$

3) For the case $a=0$:

$|x| \leq 0 \Leftrightarrow x=0$

$|x| < 0 \longleftarrow$ Inconsistent

$|x| \geq 0 \longleftarrow$ Identity

$|x| > 0 \Leftrightarrow x \neq 0$

We apply these properties as in the following examples.

EXAMPLES

a) $|2x-3| \leq 5$

Solution

$$|2x-3| \leq 5 \Leftrightarrow -5 \leq 2x-3 \leq 5 \Leftrightarrow \begin{cases} 2x-3 \leq 5 \\ -5 \leq 2x-3 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2x \leq 8 \\ 2x \geq -5+3 = -2 \end{cases} \Leftrightarrow \begin{cases} x \leq 4 \\ x \geq -1 \end{cases} \Leftrightarrow x \in [-1,4]$$

thus $S = [-1,4]$

b) $|1-2x| > 7$

Solution

$$|1-2x| > 7 \Leftrightarrow 1-2x > 7 \vee 1-2x < -7 \Leftrightarrow$$

$$\Leftrightarrow 1-7 > 2x \vee 1+7 < 2x \Leftrightarrow -6 > 2x \vee 8 < 2x$$

$$\Leftrightarrow x < -3 \vee x > 4$$

thus $S = (-\infty, -3) \cup (4, +\infty)$.

c) $|x-5| < -2$

Solution

Since $\forall x \in \mathbb{R}: |x-5| \geq 0$, it follows that $|x-5| < -2$ is inconsistent. Thus $S = \emptyset$.

d) $|21-4x| \geq -3$

Solution

$\forall x \in \mathbb{R}: |21-4x| \geq 0$ thus
$\forall x \in \mathbb{R}: |21-4x| \geq -3$ thus
solution set $S = \mathbb{R}$.

(!) e) $|x-2| \geq |x+3|$

Solution

$|x-2| \geq |x+3| \Leftrightarrow (x-2)^2 \geq (x+3)^2 \Leftrightarrow$
$\Leftrightarrow x^2-4x+4 \geq x^2+6x+9 \Leftrightarrow$
$\Leftrightarrow -4x+4 \geq 6x+9 \Leftrightarrow -4x-6x \geq -4+9 \Leftrightarrow$
$\Leftrightarrow -10x \geq 5 \Leftrightarrow 10x \leq -5 \Leftrightarrow x \leq -\frac{1}{2}$
thus $S = (-\infty, -1/2]$.

↳ For inequalities of the form $|f(x)| \lesseqgtr |g(x)|$ we can raise squares because BOTH sides of the inequality are guaranteed to be positive.

f) $|x-3| > 2x+1$ (1)

↳ We CANNOT square both sides because we do NOT know whether $2x+1$ is positive or negative.

Solution

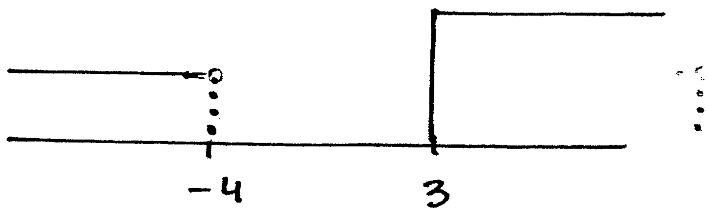
Distinguish two cases:

Case 1: $x-3 \geq 0 \Leftrightarrow x \geq 3 \Leftrightarrow x \in [3,+\infty)$

Then $|x-3| = x-3$.

$(1) \Leftrightarrow x-3 > 2x+1 \Leftrightarrow x-2x > 3+1 \Leftrightarrow -x > 4$

$\Leftrightarrow x < -4 \Leftrightarrow x \in (-\infty,-4)$.



thus $S_1 = (-\infty,-4) \cap [3,+\infty) = \emptyset$.

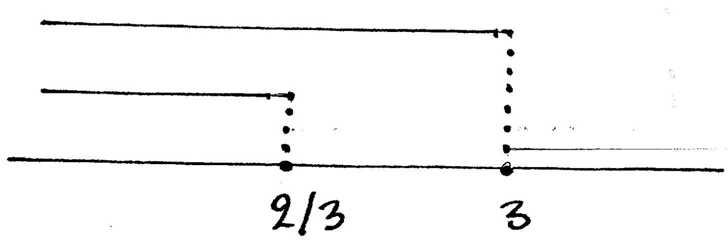
Case 2: $x-3 < 0 \Leftrightarrow x < 3 \Leftrightarrow x \in (-\infty,3)$

then $|x-3| = -(x-3)$.

$(1) \Leftrightarrow -(x-3) > 2x+1 \Leftrightarrow -x+3 > 2x+1 \Leftrightarrow$

$\Leftrightarrow -x-2x > 1-3 \Leftrightarrow -3x > -2 \Leftrightarrow 3x < 2 \Leftrightarrow$

$\Leftrightarrow x < 2/3 \Leftrightarrow x \in (-\infty, 2/3)$.



thus $S_2 = (-\infty, 2/3) \cap (-\infty,3) = (-\infty, 2/3)$.

It follows that the solution set is:

$S = S_1 \cup S_2 = \emptyset \cup (-\infty, 2/3)$

$= (-\infty, 2/3)$.

g) $|x+3| - |1-x| - 2x > 7 \quad (1)$

Solution

x		-3		1	
$x+3$	$-$	0	$+$		$+$
$1-x$	$+$		$+$	0	$-$

Distinguish three cases:

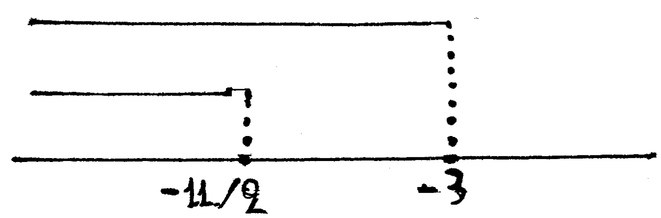
Case 1 : For $x \in (-\infty, -3)$, we have $|x+3| = -(x+3)$ and $|1-x| = 1-x$

$(1) \Leftrightarrow -(x+3) - (1-x) - 2x > 7 \Leftrightarrow$

$\Leftrightarrow -x - 3 - 1 + x \quad -2x > 7 \Leftrightarrow$

$\Leftrightarrow -2x - 4 > 7 \Leftrightarrow -2x > 7+4 \Leftrightarrow -2x > 11$

$\Leftrightarrow x < -11/2.$



thus $S_1 = (-\infty, -11/2) \cap (-\infty, -3) = (-\infty, -11/2)$

Case 2 : For $x \in [-3, 1)$, we have $|x+3| = x+3$ and $|1-x| = 1-x$

$(1) \Leftrightarrow (x+3) - (1-x) - 2x > 7 \Leftrightarrow$

$\Leftrightarrow x + 3 - 1 + x - 2x > 7 \Leftrightarrow$

$\Leftrightarrow 0x + 2 > 7 \Leftrightarrow 0x > 7-2 \Leftrightarrow 0x > 5 \leftarrow$ inconsistent

Thus: $S_2 = \emptyset \cap [-3, 1) = \emptyset$.

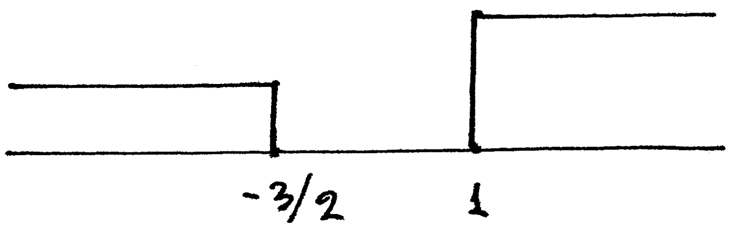
Case 3: For $x \in [1, +\infty)$, we have
$|x+3| = x+3$ and $|1-x| = -(1-x)$

$(1) \Leftrightarrow (x+3) + (1-x) - 2x > 7 \Leftrightarrow$

$\Leftrightarrow x+3+1-x-2x > 7 \Leftrightarrow$

$\Leftrightarrow -2x+4 > 7 \Leftrightarrow -2x > 7-4 \Leftrightarrow -2x > 3 \Leftrightarrow$

$\Leftrightarrow x < -3/2$



thus $S_3 = (-\infty, -3/2) \cap [1, +\infty) = \emptyset$.
It follows that the solution set is
$S = S_1 \cup S_2 \cup S_3 = (-\infty, -11/2) \cup \emptyset \cup \emptyset = (-\infty, -11/2)$

EXERCISES

28 Solve the inequalities

a) $|x| < 5$

b) $|x| \geq 3$

c) $|x| < -1$

d) $|x| \geq 0$

e) $|x| < 0$

f) $|x| > 0$

g) $|x| > -2$

h) $|x| \geq -3$

i) $3|x| - 2 > |x| + 8$

j) $2(|x| - 1) \geq 3|x| - 2$

k) $2|x| - |-2x| \geq 3 - |3x|$

l) $\dfrac{2+|x|}{|x|+1} > 2$

29 Solve the inequalities

a) $|-3x| \geq -2$

b) $|-2x| \geq |2x|$

c) $|1+2x| \leq 3$

d) $|x-3| > 1$

e) $|x-30| - 6 > 0$

f) $|3x+2| - 4 \leq 0$

g) $|x-5| - 4 \leq 0$

h) $|2x-3| + 3 < 0$

i) $-3|7-x| + 5 \leq 0$

j) $|x^2+3| \geq 4x$

30 Solve the inequalities

a) $|2x| + |x+1| < 3$

b) $|x+1| - |x-1| \geq 1/2$

c) $|2x-1| + |x-3| + 3 < -|x+1|$

d) $|x^2-1| - 3x \geq 0.$

e) $|2x+3| - |x+5| < 0$

f) $|x+1| - 3x > 0$

g) $|x^2-1| \leq |2x+1|.$

Quadratic parametric equations

- 1 Simplify to the form
  $A(a)x^2 + B(a)x + C(a) = 0$
- 2 Make sign chart for the discriminant
  $\Delta(a) = B^2(a) - 4A(a)C(a)$
- 3 Distinguish the cases:
  $A(a) = 0 \leftarrow$ Linear equation
  $\Delta(a) > 0 \leftarrow$ 2 solutions
  $\Delta(a) = 0 \leftarrow$ 1 solution
  $\Delta(a) < 0 \leftarrow$ no real solutions.

EXAMPLE

$(a+4)x^2 + (a+2)x - 2 = 0 \qquad (1)$

Solution

For $a+4=0 \Leftrightarrow a=-4$

$(1) \Leftrightarrow (-4+2)x - 2 = 0 \Leftrightarrow -2x - 2 = 0 \Leftrightarrow 2x = -2$

$\Leftrightarrow x = -1.$

Assume $a \neq -4.$

$$\begin{aligned}
\Delta(a) &= B^2(a) - 4A(a)C(a) = \\
&= (a+2)^2 - 4(a+4)\cdot(-2) = \\
&= a^2 + 4a + 4 + 8(a+4) = \\
&= a^2 + 4a + 4 + 8a + 32 = a^2 + 12a + 36 = \\
&= (a+6)^2
\end{aligned}$$

a		-6	
$(a+6)^2$	$+$	0	$+$
$\Delta(a)$	$+$	0	$+$

For $a \in \mathbb{R} - \{-6, -4\} \Rightarrow \Delta(a) > 0 \Rightarrow$

$\Rightarrow$ 2 solutions:

$$x_{1,2} = \frac{-B(a) \pm \sqrt{\Delta(a)}}{2A(a)} = \frac{-(a+2) \pm \sqrt{(a+6)^2}}{2(a+4)} =$$

$$= \frac{-(a+2) \pm |a+6|}{2(a+4)} = \frac{-(a+2) \pm (a+6)}{2(a+4)}$$

with

$$x_1 = \frac{-(a+2)-(a+6)}{2(a+4)} = \frac{-a-2-a-6}{2(a+4)} = \frac{-2a-8}{2(a+4)}$$

$$= \frac{-2(a+4)}{2(a+4)} = -1$$

$$x_2 = \frac{-(a+2)+(a+6)}{2(a+4)} = \frac{-a-2+a+6}{2(a+4)} = \frac{4}{2(a+4)} =$$

$$= \frac{2}{a+4}$$

For $a = -6$:

$(1) \Leftrightarrow (-6+4)x^2 + (-6+2)x - 2 = 0 \Leftrightarrow$

$\Leftrightarrow -2x^2 - 4x - 2 = 0 \Leftrightarrow -2(x^2+2x+1) = 0$

$\Leftrightarrow -2(x+1)^2 = 0 \Leftrightarrow x+1 = 0 \Leftrightarrow x = -1.$

Solution set:

$$S = \begin{cases} \{-1, 2/(a+4)\} & , a \in \mathbb{R} - \{-6, -4\} \\ \{-1\} & , a \in \{-6, -4\} \end{cases}$$

b) $(a+4)x^2 + (a+1)x + 1 = 0 \quad (1)$

Solution

For $a+4=0 \Leftrightarrow a=-4$

$(1) \Leftrightarrow (-4+1)x+1=0 \Leftrightarrow -3x+1=0 \Leftrightarrow 3x=1 \Leftrightarrow$

$\Leftrightarrow x = 1/3.$

Assume that $a \neq -4$.

$\Delta(a) = (a+1)^2 - 4(a+4) \cdot 1 =$

$= a^2+2a+1-4a-16 =$

$= a^2-2a-15 = (a+3)(a-5)$

a		-3		5	
$a+3$	$-$	0	$+$		$+$
$a-5$	$-$		$-$	0	$+$
$\Delta(a)$	$+$	0	$-$	0	$+$

For $a \in (-\infty, -3) \cup (5, +\infty)$, two solutions:

$$x_{1,2} = \frac{-B(a) \pm \sqrt{\Delta(a)}}{2A(a)} =$$

$$= \frac{-(a+1) \pm \sqrt{a^2-2a-15}}{2(a+4)}$$

For $a=-3$:

$(-3+4)x^2+(-3+1)x+1=0 \Leftrightarrow$

$\Leftrightarrow x^2-2x+1=0 \Leftrightarrow (x-1)^2=0 \Leftrightarrow x-1=0$

$\Leftrightarrow x=1$.

For $a=5$:

$(5+4)x^2+(5+1)x+1=0 \Leftrightarrow$

$\Leftrightarrow 9x^2+6x+1=0 \Leftrightarrow (3x+1)^2=0 \Leftrightarrow$

$\Leftrightarrow 3x+1=0 \Leftrightarrow 3x=-1 \Leftrightarrow x=-1/3$.

For $a \in (-3,5) \Rightarrow \Delta(a)<0 \Rightarrow$ no real solutions.

Solution set:

$$S=\begin{cases} \left\{ \dfrac{-(a+1)+\sqrt{a^2-2a-15}}{2(a+4)}, \dfrac{-(a+1)-\sqrt{a^2-2a-15}}{2(a+4)} \right\} \\ \text{if } a \in (-\infty,-4) \cup (-4,-3) \cup (-3,+\infty) \\ \{1/3\}, \text{ if } a=-4 \\ \{1\}, \text{ if } a=-3 \\ \{-1/3\}, \text{ if } a=5 \\ \emptyset, \text{ if } a \in (-3,5) \end{cases}$$

EXERCISES

(19) Solve the following equations with respect to x

a) $(3a-1)x^2+2x+4a-1=0$

b) $3ax^2-(a+1)x+3=0$

c) $(1-a)^2x^2+(a-1)x-a(a+1)=0$

d) $(a-2)x^2+2(a+3)x+(2a-18)=0$

e) $(2a-1)x^2-2(a+1)x+a+1=0$

f) $(a^2-1)x^2-4(a+2)x+3=0$

Equations with radicals

- These are equations wherein the unknown x appears at least under one square root.
- The square roots can be eliminated by squaring both sides of the equation using the property

$$\boxed{\forall a, b \in [0, +\infty) : (a = b \Leftrightarrow a^2 = b^2)}$$

We therefore have to satisfy the constraint that both sides of the equation must be positive or zero before raising the square. This constraint imposes a <u>domain A</u> that must be used to <u>accept</u> or <u>reject</u> the solutions found.

- We distinguish the following cases:

① $\longrightarrow$ $\boxed{\sqrt{f(x)} = \sqrt{g(x)}}$

•1 We require:

$$\begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \end{cases} \Leftrightarrow \dots \Leftrightarrow x \in A \longleftarrow \text{Domain}$$

•2 Solve the equation:

$$\sqrt{f(x)} = \sqrt{g(x)} \Leftrightarrow f(x) = g(x) \Leftrightarrow \dots \Leftrightarrow x \in S_0$$

•3 Accept / reject solutions: $S = S_0 \cap A$.

EXAMPLE

Solve: $\sqrt{3x+1} = \sqrt{2-x}$

Solution

Require: $\begin{cases} 3x+1 \geq 0 \\ 2-x \geq 0 \end{cases} \Leftrightarrow \begin{cases} 3x \geq -1 \\ 2 \geq x \end{cases} \Leftrightarrow \begin{cases} x \geq -1/3 \\ x \leq 2 \end{cases}$

$\Leftrightarrow x \in [-1/3, 2]$ thus: $A = [-1/3, 2]$.

Note that

$\sqrt{3x+1} = \sqrt{2-x} \Leftrightarrow 3x+1 = 2-x \Leftrightarrow 3x+x = 2-1 \Leftrightarrow$

$\Leftrightarrow 4x = 1 \Leftrightarrow x = 1/4 \leftarrow$ accepted (because $1/4 \in A$).

It follows that $S = \{1/4\}$.

2 $\rightarrow$ $\boxed{\sqrt{f(x)} = g(x)}$

Note that we need $f(x) \geq 0$. Furthermore, the equation has no solutions with $g(x) < 0$ since $f(x) \in \mathbb{R} \Rightarrow \sqrt{f(x)} \geq 0$, so we can go ahead and require $g(x) \geq 0$ to justify squaring both sides of the equation. On the other hand, squaring the equation gives $f(x) = [g(x)]^2$ and since for any $x \in S$ $[g(x)]^2 \geq 0 \Rightarrow f(x) \geq 0$, it follows that the resulting solutions are guaranteed to satisfy $f(x) \geq 0$. Consequently, it is enough to require <u>only</u> $g(x) \geq 0$.

Solution Method

- 1 Require $g(x) \geq 0 \Leftrightarrow \cdots \Leftrightarrow x \in A$.
- 2 Solve:
  $\sqrt{f(x)} = g(x) \Leftrightarrow f(x) = [g(x)]^2 \Leftrightarrow \cdots \Leftrightarrow x \in S_0$
- 3 Solution set: $S = S_0 \cap A$.

EXAMPLE

Solve $\sqrt{x^2-2x+6} + 3 = 2x$.

Solution

$\sqrt{x^2-2x+6} + 3 = 2x \Leftrightarrow \sqrt{x^2-2x+6} = 2x - 3$. (1)

Require: $2x-3 \geq 0 \Leftrightarrow 2x \geq 3 \Leftrightarrow x \in [3/2, +\infty)$

thus domain: $A = [3/2, +\infty)$.

$(1) \Leftrightarrow x^2-2x+6 = (2x-3)^2 \Leftrightarrow x^2-2x+6 = 4x^2-12x+9$

$\Leftrightarrow (4-1)x^2 + (-12+2)x + (9-6) = 0 \Leftrightarrow$

$\Leftrightarrow 3x^2 - 10x + 3 = 0$ (2)

$\Delta = b^2 - 4ac = (-10)^2 - 4 \cdot 3 \cdot 3 = 100 - 36 = 64 = 8^2 \Rightarrow$

$$\Rightarrow x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-10) \pm 8}{2 \cdot 3} = \frac{10 \pm 8}{6} =$$

$$= \begin{cases} 18/6 = 3 \in A & \leftarrow \text{accept} \\ 2/6 = 1/3 \notin A & \leftarrow \text{reject.} \end{cases}$$

It follows that $S = \{3\}$.

3 →

$$\boxed{\begin{array}{l} \sqrt{f(x)} + \sqrt{g(x)} = 0 \\ \sqrt{f(x)} + \sqrt{g(x)} + \sqrt{h(x)} = 0 \end{array}}$$

We use the following property:
$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} = 0 \Leftrightarrow a_1 = 0 \wedge a_2 = 0 \wedge \cdots \wedge a_n = 0$.
Applying this property does NOT require us to impose a <u>domain A</u> on the unknown x.

EXAMPLE

Solve: $\sqrt{x^2-9} + \sqrt{x^2+5x+6} = 0$

<u>Solution</u>

$$\sqrt{x^2-9} + \sqrt{x^2+5x+6} = 0 \Leftrightarrow \begin{cases} x^2-9=0 \\ x^2+5x+6=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (x-3)(x+3)=0 \\ (x+2)(x+3)=0 \end{cases} \Leftrightarrow \begin{cases} x-3=0 \vee x+3=0 \\ x+2=0 \vee x+3=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=3 \vee x=-3 \\ x=-2 \vee x=-3 \end{cases} \Leftrightarrow x \in \{3,-3\} \cap \{-2,-3\} = \{-3\}.$$

thus $S = \{-3\}$.

(4) $\rightarrow$ $$\boxed{\begin{array}{l} \sqrt{f(x)}+\sqrt{g(x)}=h(x) \\ \sqrt{f(x)}+\sqrt{g(x)}=\sqrt{h(x)} \end{array}}$$

The solution method is the same for both types of equations. Without loss of generality consider the equation:

$$\sqrt{f(x)}+\sqrt{g(x)}=h(x).$$

▸ <u>Method</u>

•1 Require $\begin{cases} f(x)\geqslant 0 \\ g(x)\geqslant 0 \Leftrightarrow \cdots \Leftrightarrow x\in A_1 \\ h(x)\geqslant 0 \end{cases}$

•2 Solve:

$$\sqrt{f(x)}+\sqrt{g(x)}=h(x)\Leftrightarrow \left(\sqrt{f(x)}+\sqrt{g(x)}\right)^2=[h(x)]^2$$

$$\Leftrightarrow f(x)+2\sqrt{f(x)g(x)}+g(x)=[h(x)]^2\Leftrightarrow$$

$$\Leftrightarrow 2\sqrt{f(x)g(x)}=[h(x)]^2-f(x)-g(x) \qquad (1).$$

(type 2 equation)

•3 Require: $[h(x)]^2-f(x)-g(x)\geqslant 0\Leftrightarrow \cdots \Leftrightarrow x\in A_2$.

•4 Solve:

$$(1)\Leftrightarrow 4f(x)g(x)=\left([h(x)]^2-f(x)-g(x)\right)^2\Leftrightarrow$$

$$\Leftrightarrow \cdots \Leftrightarrow x\in S_0.$$

•5 For the solution set:

$$S=S_0\cap A_1\cap A_2.$$

EXAMPLE

Solve: $\sqrt{x+6} = \sqrt{5(x+2)} - \sqrt{x+1}$

Solution

$$\sqrt{x+6} = \sqrt{5(x+2)} - \sqrt{x+1} \Leftrightarrow$$

$$\Leftrightarrow \sqrt{x+6} + \sqrt{x+1} = \sqrt{5(x+2)} \qquad (1)$$

▸ Require: $\begin{cases} x+6 \geq 0 \\ x+1 \geq 0 \\ 5(x+2) \geq 0 \end{cases} \Leftrightarrow \begin{cases} x \geq -6 \\ x \geq -1 \\ x+2 \geq 0 \end{cases} \Leftrightarrow \begin{cases} x \geq -6 \\ x \geq -1 \\ x \geq -2 \end{cases} \Leftrightarrow x \geq -1$

thus $A_1 = [-1, +\infty)$.

$$(1) \Leftrightarrow \left(\sqrt{x+6} + \sqrt{x+1}\right)^2 = 5(x+2) \Leftrightarrow$$

$$\Leftrightarrow (x+6) + 2\sqrt{(x+6)(x+1)} + (x+1) = 5(x+2) \Leftrightarrow$$

$$\Leftrightarrow 2\sqrt{(x+6)(x+1)} = 5(x+2) - (x+6) - (x+1) \Leftrightarrow$$

$$\Leftrightarrow 2\sqrt{(x+6)(x+1)} = 5x+10-x-6-x-1 \Leftrightarrow$$

$$\Leftrightarrow 2\sqrt{(x+6)(x+1)} = 3x+3 \qquad (2)$$

▸ Require $3x+3 \geq 0 \Leftrightarrow x+1 \geq 0 \Leftrightarrow x \geq -1$

thus $A_2 = [-1, +\infty)$.

$$(2) \Leftrightarrow 4(x+6)(x+1) = (3x+3)^2 \Leftrightarrow 4(x^2+7x+6) = 9(x+1)^2$$

$$\Leftrightarrow 4x^2+28x+24 = 9(x^2+2x+1) \Leftrightarrow$$

$$\Leftrightarrow 4x^2+28x+24 = 9x^2+18x+9 \Leftrightarrow$$

$$\Leftrightarrow (9-4)x^2 + (18-28)x + (9-24) = 0 \Leftrightarrow$$

$$\Leftrightarrow 5x^2-10x-15=0 \Leftrightarrow x^2-2x-3=0$$

$$\Leftrightarrow (x+1)(x-3)=0 \Leftrightarrow x+1=0 \vee x-3=0 \Leftrightarrow$$

$$\Leftrightarrow x=-1 \vee x=3.$$

It follows that the solution set reads:

$S = \{-1, 3\} \cap A_1 \cap A_2 = \{-1, 3\} \cap [-1, +\infty) \cap [-1, +\infty)$

$= \{-1, 3\}$.

thus, both solutions are accepted.

EXERCISES

(21) Solve the equations

a) $\sqrt{3x^2+2x+1} - 13 = 5x$
b) $x - \sqrt{x^2-7} = 7$
c) $x - \sqrt{4-x^2} = 1$
d) $x - 2\sqrt{x^2+x+3} = -x-2$
e) $13 - \sqrt{4x^2+7x-8} = 2x$
f) $\sqrt{x-2} + \sqrt{x^2-2x} = 0$
g) $3\sqrt{x-1} + \sqrt{x^2-2x+1} + \sqrt{x^3-x^2} = 0$
h) $\sqrt{2x+1} + \sqrt{x+1} = 1$
i) $\sqrt{x-2} - \sqrt{3x} = -\sqrt{7-x}$
j) $\sqrt{x+1} - \sqrt{2x+9} = \sqrt{4-x}$
k) $\sqrt{2x+1} = 1 - \sqrt{x+1}$

(22) Solve the equations

a) $2x^2-7x = 3\sqrt{2x^2-7x+7} - 3$ } substitution
b) $\sqrt{2+\sqrt{x-5}} = \sqrt{13-x}$

c) $\sqrt{x^2-6x+5} = x-a$
d) $\sqrt{x^2+1} = x-a$
e) $\sqrt{x+a} = a-\sqrt{x}$

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