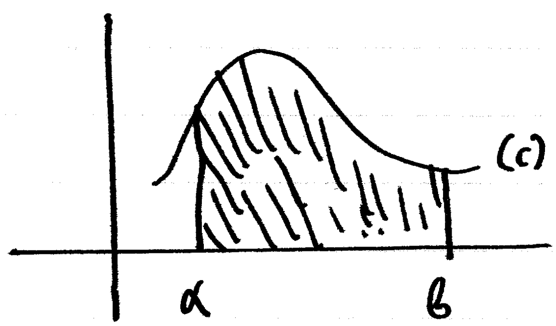
Applications of Integration

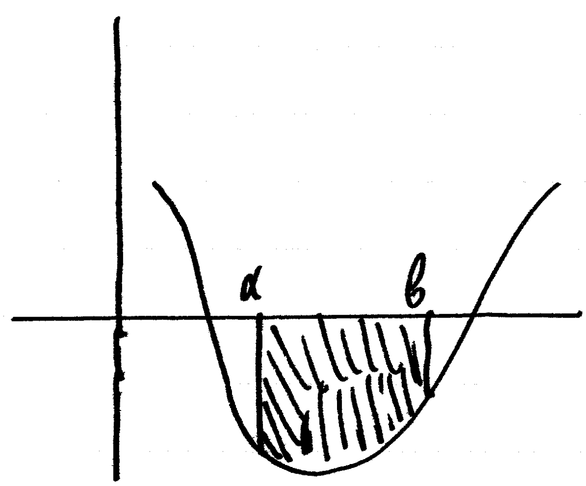
Calculation of areas

Case 1: Area surrounded by
a) The graph $(C): y = f(x)$
b) The x-axis
c) The lines $(l_1): x = a$, $(l_2): x = b$

If $f(x) \geq 0, \forall x \in [a,b] \Rightarrow A = \int_a^b f(x)dx$



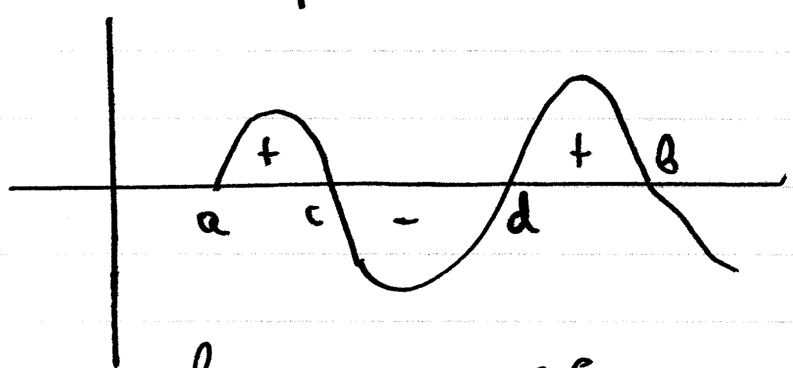
If $f(x) \leq 0, \forall x \in [a,b] \Rightarrow A = -\int_a^b f(x)dx$



$\mapsto$ In general : $A = \int_a^b |f(x)|\, dx$

We break the integral into intervals where f has constant sign.

For example:



$$A = \int_a^b |f(x)|\, dx = \int_a^c f(x)\, dx - \int_c^d f(x)\, dx + \int_d^b f(x)\, dx$$

The points a, b, c, d can be found, if not given, by solving the equation $f(x) = 0$.

The sign of $f(x)$ can be found with a sign table.

examples

1) Find the area between
$(c): y = x^2 - 7x + 10$
and the x-axis.

2) Find the area between
$(c): y = -x^2 + 2x$,
the x-axis,
the line $(\ell): x = -2$

Case 2: Area between the curves

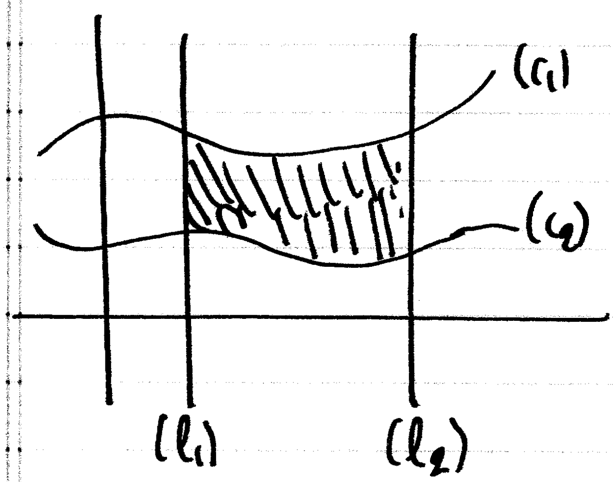
$(C_1): y = f(x)$

$(C_2): y = g(x)$

and the lines

$(l_1): x = a$

$(l_2): x = b$



$$\boxed{A = \int_a^b |f(x) - g(x)|\,dx}$$

- Thus our first job is to find the sign of $f(x) - g(x)$.

- If a, b are not given, i.e. we want the area between (C_1) and (C_2), the we must find the points of intersection by solving

$$\begin{cases} y = f(x) \\ y = g(x) \end{cases} \Leftrightarrow f(x) = g(x) \Leftrightarrow \ldots$$

examples

a) Find area between

$(c_1): y = x^2 \quad , \quad (l_1): x = 1/2$

$(c_2): y = x \quad , \quad (l_2): x = 2$

b) Find area between

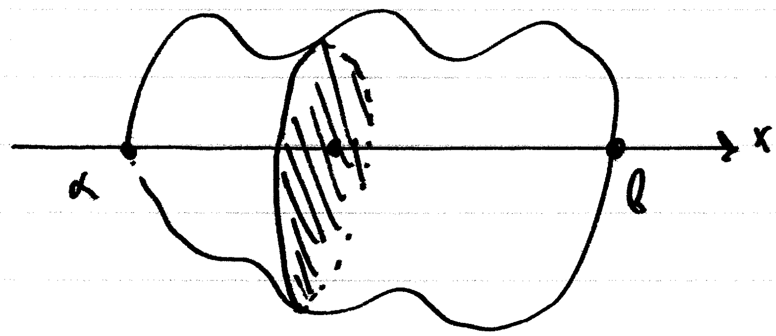
$(c_1): y = x^3 - x$

$(c_2): y = 3x$

Calculation of Volumes

Case 1 : Cross-section Method

Consider a solid "pierced" by the x-axis:



- Cut the solid with a plane perpendicular to the x-axis and find the area $A(x)$ of the cross-section.
- a) If the cross-section is a <u>disc</u> with radius $R(x)$ then

  $$A(x) = \pi R(x)^2 .$$

  b) If the cross-section is a <u>ring</u> with small radius $r(x)$ and big radius $R(x)$, then

$$A(x) = \pi\left[R^2(x) - r^2(x)\right]$$

- 2 Locate the points a, b
- 3 The volume of a slice is

$$dV = A(x)\,dx$$

so the total volume is:

$$V = \int_a^b A(x)\,dx$$

<u>examples</u>

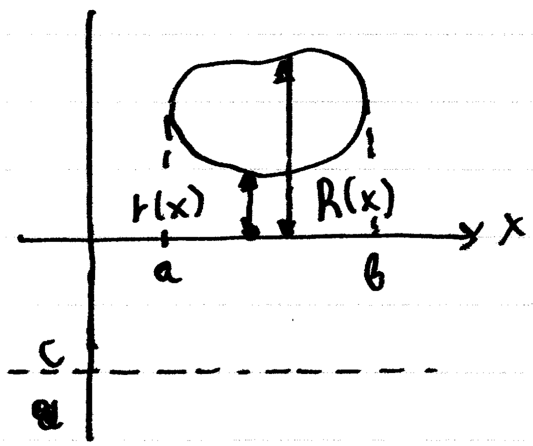
a) Volume of a sphere $\longrightarrow V = \frac{4}{3}\pi R^3$.

b) Volume of solid by rotating area between $(c_1): y = x$ $(c_2): y = x^4$ around $(\ell): y = -1$.

Applic

Application: Consider an area A above the x-axis that generates a solid with volume V_0 when rotated around the x-axis.
Find the volume $V(c)$ if the same area is rotated around $(\ell): y = -c$

Solution



Let $r(x)$ and $R(x)$ be the small / large radius of the ring cross-section when area is rotated around the x-axis. Then:
Let $h = b - a$ be the "height"

$$A = \int_a^b [R(x) - r(x)]\,dx$$

and

$$V_0 = \int_a^b \pi [R^2(x) - r^2(x)]\,dx$$

When rotated around $(\ell): y = -c$, the cross-section is a ring with small radius $r(x) + c$ and big radius $R(x) + c$, with area

$$A(x) = \pi\left[(R(x)+c)^2 - (r(x)+c)^2\right]$$
$$= \pi\left[R^2(x) + 2cR(x) + c^2 - r^2(x) - 2cr(x) - c^2\right]$$
$$= \pi\left[R^2(x) - r^2(x)\right] + 2\pi c\left[R(x) - r(x)\right] \Rightarrow$$

$$\Rightarrow V = \int_a^b A(x)\,dx =$$

$$= \int_a^b \pi\left[R^2(x) - r^2(x)\right]dx + 2\pi c\int_a^b \left[R(x) - r(x)\right]dx$$

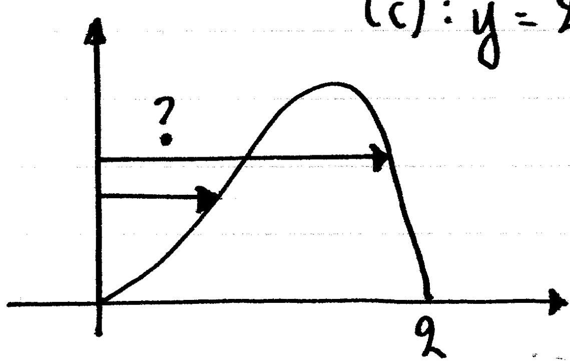
$$= V_0 + 2\pi c A.$$

So, we find $\boxed{V = V_0 + 2\pi c A}$

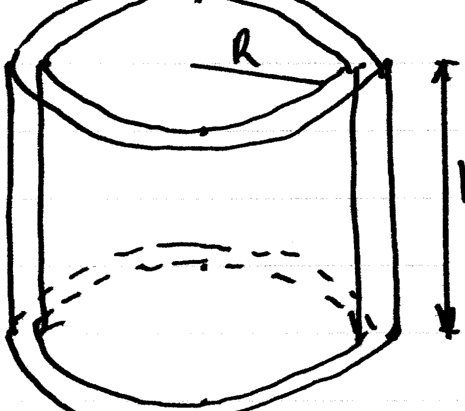
Case 2 : Cylindrical shell method

→ This method is practical when calculating the radius $r(x)$ and/or $R(x)$ requires the solution of a "difficult" equation $y = f(x)$ with respect to x.

example : The volume V by rotating around y-axis the area bound by
(l) : $y = 6$
(c) : $y = 2x^2 - x^3$.



▶ The volume of a cylinder with radius R, height h, and thickness dR is :



$$dV = \pi(R+dR)^2 h - \pi R^2 h$$
$$= \pi h[R^2 + 2RdR + (dR)^2 - R^2]$$
$$= 2\pi R h\, dR + O(dR^2)$$

We neglect the dR^2 term.

▸ Consequently, the volume of a solid that consists of cylindrical shells with radius $R(x)$ and height $h(x)$ where $x \in [a, b]$ is given by

$$\boxed{V = \int_a^b 2\pi R(x) h(x)\, dx}$$

Method: • 1 First calculate $R(x)$ and $h(x)$
• 2 Find a and b
• 3 Calculate the integral.

examples

a) Area between
$(c_1): y = x$ and $(c_2): y = 4x - x^2$
around ~~x=7~~ . $(l): x = 7$.

b) Area between
$(c_1): y = x^2 - 3x + 2$
$(c_2): y = 0$
about the y-axis.