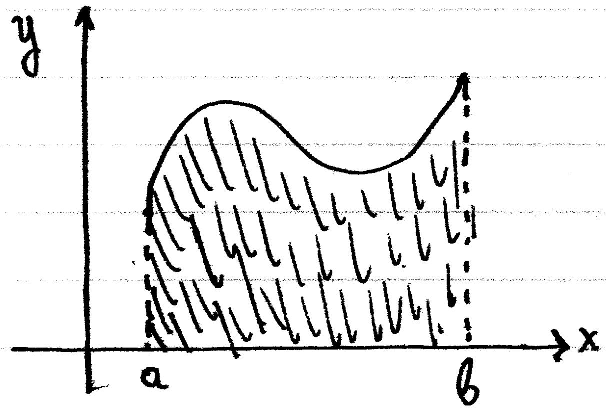
Integral Calculus

Definition of the Riemann integral



The problem is to calculate the area A between the x-axis, the lines $(l_1): x = a$ and $(l_2): x = b$ and the curve $(c): y = f(x)$.

The solution of the problem, according to Riemann is as follows:

- 1 Divide the interval $[a,b]$ to n equal intervals $[x_{k-1}, x_k]$ with
$$x_k = a + (b-a)(k/n), \quad \forall k \in [n]$$
with $[n] = \{0, 1, 2, \dots, n\}$.
- 2 Let m_k and M_k be the min and max value of f in the interval $[x_{k-1}, x_k]$:

$$m_k = \min_{x \in [x_{k-1}, x_k]} f(x)$$
$$M_k = \max_{x \in [x_{k-1}, x_k]} f(x)$$

• 3 We form the Riemann sums

$$L_n = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$$

$$U_n = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$$

Obviously the area A will satisfy

$\forall n \in \mathbb{N} : L_n \leqslant A \leqslant U_n$ (1)

• 4 We prove that $\lim L_n = \lim U_n = \ell$ which combined with (1) implies that

$$\boxed{\lim L_n = \lim U_n = A}$$

If the limits $\lim L_n$ and $\lim U_n$ converge and coincide, we say that

f integrable at $[a, b]$

and write

$$\boxed{\lim L_n = \lim U_n = \int_a^b f(x)\,dx}$$

This definition assumes that $a<b$. For convenience we generalize by defining:

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

From the definition it follows that the integral can be calculated as the limit of the following sequence:

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty}\left[\frac{b-a}{n}\sum_{k=1}^{n} f\left(a+k\,\frac{b-a}{n}\right)\right]$$

▶ Basic Sums

$$S_1(n) = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$S_2(n) = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$S_3(n) = 1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4} = [S_1(n)]^2.$$

example : $\int_0^a x^2\,dx = \frac{a^3}{3}.$

Properties of the integral

① f continuous at $[a,b] \Rightarrow f$ integrable at $[a,b]$

② Let f, g integrable at $[a,b]$
Then

a) $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

b) $\displaystyle\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx \quad , \forall \lambda \in \mathbb{R}$

c) $\displaystyle\gamma \in [a,b] \Rightarrow \int_a^b f(x)\,dx = \int_a^\gamma f(x)\,dx + \int_\gamma^b f(x)\,dx$

③ Let f integrable at $[a,b]$.

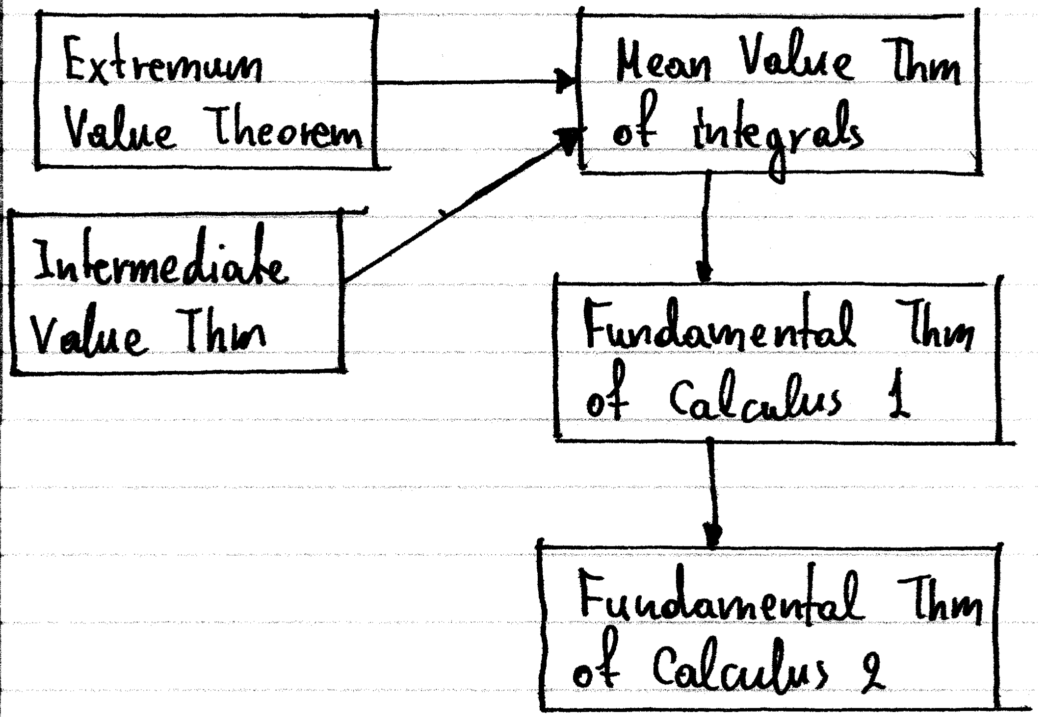
a) $\displaystyle(\forall x \in [a,b] : f(x) \geqslant 0) \Rightarrow \int_a^b f(x)\,dx \geqslant 0$

b) $\displaystyle(\forall x \in [a,b] : f(x) \leqslant g(x)) \Rightarrow \int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx$

c) $\displaystyle\left| \int_a^b f(x)\,dx \right| \leqslant \int_a^b |f(x)|\,dx$

Fundamental Theorems of Calculus

Outline



To bootstrap the theory we first use the definition of the integral to prove that

$$\int_a^b c\,dx = c(b-a), \quad \forall c \in \mathbb{R}.$$

(1) Mean Value Theorem

$$\boxed{\begin{array}{l} f \text{ continuous} \\ \text{at } [a,b] \end{array} \Rightarrow \exists \xi \in [a,b] : \int_a^b f(x)dx = f(\xi)(b-a)}$$

Proof

f continuous at $[a,b] \Rightarrow$

$\Rightarrow \exists \xi_1, \xi_2 \in [a,b] : \forall x \in [a,b] : f(\xi_1) \leq f(x) \leq f(\xi_2)$

$$\Rightarrow \int_a^b f(\xi_1)\,dx \leq \int_a^b f(x)dx \leq \int_a^b f(\xi_2)\,dx$$

$$\Rightarrow f(\xi_1)(b-a) \leq \int_a^b f(x)dx \leq f(\xi_2)(b-a)$$

$$\Rightarrow f(\xi_1) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(\xi_2)$$

From the intermediate value theorem

$$\exists \xi \in [a,b] : f(\xi) = \frac{1}{b-a}\int_a^b f(x)dx$$

$$\Rightarrow \int_a^b f(x)dx = f(\xi)(b-a).$$

② Fundamental theorem of calculus

$$\boxed{\left.\begin{array}{l} f \text{ continuous at } [a,b] \\ F(x) = \displaystyle\int_c^x f(t)\,dt \end{array}\right\} \Rightarrow \left\{\begin{array}{l} F \text{ differentiable at } [a,b] \\ F'(x) = f(x) \end{array}\right.}$$

Proof

Let $\partial(x,h) = \dfrac{F(x+h) - F(x)}{h} =$

$$= \frac{1}{h}\left[\int_c^{x+h} f(t)\,dt - \int_c^x f(t)\,dt\right]$$

$$= \frac{1}{h}\left[\int_c^x f(t)\,dt + \int_x^{x+h} f(t)\,dt - \int_c^x f(t)\,dt\right]$$

$$= \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

Apply MVT for integrals:

$$\forall h : \exists\, \xi(h) : \left\{\begin{array}{l} |x - \xi(h)| \leq h \\ \displaystyle\int_x^{x+h} f(t)\,dt = f(\xi(h))\,h \end{array}\right.$$

Since

$$\left.\begin{array}{l} |x-\xi(h)| \leq h \;, \forall h \in N(0) \\ \lim\limits_{h\to 0} h = 0 \end{array}\right\} \Rightarrow \lim_{h\to 0} (x-\xi(h)) = 0$$

$$\Rightarrow \lim_{h\to 0} \xi(h) = x$$

$$\Rightarrow \lim_{h\to 0} f(\xi(h)) = f(x) \;, \text{ bc. } f \text{ continuous at } x.$$

Thus

$$\lim_{h\to 0} \partial(x,h) = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(x)\,dx$$

$$= \lim_{h\to 0} \frac{1}{h} f(\xi(h))\, h$$

$$= \lim_{h\to 0} f(\xi(h)) = f(x) \qquad \square$$

→ In Leibnitz notation:

$$\boxed{\frac{d}{dx} \int_c^x f(t)\,dt = f(x)}$$

Method : Combining the FTC 1 with the chain rule, we have the following more general differentiation rule.

$$\boxed{\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))\,b'(x) - f(a(x))\,a'(x)}$$

examples

1) $f(x) = \displaystyle\int_2^x \frac{\cos t}{t}\,dt \longrightarrow f'(x)$

2) $f(x) = \displaystyle\int_x^5 \frac{t^2-1}{t^2+1}\,dt \longrightarrow f'(x)$

3) $f(x) = \displaystyle\int_{x^2+x}^{x^2(x+1)^2} \frac{t}{1+t}\,dt \longrightarrow f'(x)$

③ Fundamental theorem of calculus II

$$\boxed{\left.\begin{array}{l} F \text{ differentiable at } [a,b] \\ F'(x) = f(x),\ \forall x \in [a,b] \\ f \text{ continuous at } [a,b] \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)}$$

Proof

$$\forall x \in [a,b] : \frac{d}{dx}\int_a^x f(t)\,dt = f(x) = \frac{dF(x)}{dx} \Rightarrow$$

$$\Rightarrow \exists\, c \in \mathbb{R} : \int_a^x f(t)\,dt = F(x) + c \quad , \quad \forall x \in [a,b]$$

For $x = a$:

$$F(a) + c = \int_a^a f(t)\,dt = 0 \Rightarrow c = -F(a)$$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a). \quad \square$$

Equivalently

$$\boxed{\int_a^b f'(x)\,dx = f(b) - f(a)}$$

↳ The FTC II motivates the definition of the indefinite integral

$$\int f'(x)\,dx = f(x) + c$$

↳ <u>Integration formulas</u>

1) $$\int x^{a}\,dx = \begin{cases} \dfrac{x^{a+1}}{a+1} + c &, \text{ if } a \neq -1 \\ \ln|x| + c &, \text{ if } a = -1 \end{cases}$$

2) Special cases:

a) $\int dx = x + c$ b) $\int \dfrac{dx}{\sqrt{x}} = 2\sqrt{x} + c$

2) $\int \sin x\,dx = -\cos x + c$

3) $\int \cos x\,dx = \sin x + c$

4) $\int \dfrac{dx}{\cos^2 x} = \tan x + c$

5) $\int \dfrac{dx}{\sin^2 x} = -\cot x + c$

examples

a) $I = \int_1^2 \frac{x+1}{x^3}\,dx$

b) $I = \int_0^{\pi/4} \frac{1+\cos^3 x}{\cos^2 x}\,dx$

c) $I = \int_{-1}^{1} \frac{dx}{x^2}$ ← CAUTION!

Method of substitution

Thm : Assume that

- 1 g' continuous at $[a,b]$
- 2 f continuous at $g([a,b])$

Then

$$\boxed{\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy}$$

Proof

Let $F(x) = \int_{g(a)}^{x} f(t)\,dt \,,\ \forall x \in g([a,b])$

Then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_a^b [F(g(x))]'\,dx$$
$$= F(g(b)) - F(g(a))$$
$$= \int_{g(a)}^{g(b)} f(t)\,dt. \qquad \square$$

<u>example</u> : Watch out the notation!

$$I = \int_1^2 \sqrt[3]{2x+3}\, dx.$$

Let $y = g(x) = 2x+3 \Rightarrow \begin{cases} dy = g'(x)dx = 2dx \Rightarrow dx = dy/2 \\ g(1) = 2+3 = 5 \\ g(2) = 2\cdot 2 + 3 = 7 \end{cases}$

$$\Rightarrow I = \int_5^7 \sqrt[3]{y}\ \frac{1}{2}\, dy = \int_5^7 y^{1/3}\cdot(1/2)\, dy =$$

$$= \left[\frac{y^{4/3}}{4/3}\ \frac{1}{2}\right]_5^7 = \frac{3}{8}\left(7^{4/3} - 5^{4/3}\right)$$

$$= \frac{3(7\sqrt[3]{7} - 5\sqrt[3]{5})}{8}$$

↳ We see that the substitution method theorem can be understood loosely as a transformation of the differential:

$$y = g(x) \Rightarrow dy = g'(x)dx$$

<u>Another example</u>: $I = \displaystyle\int_0^1 \frac{3x^2}{\sqrt{x^3+1}}\, dx$

Methodology

① $I = \int \frac{f'(x)}{\sqrt{f(x)}}\,dx \longrightarrow$ set $y = f(x)$

<u>example</u> : $I = \int_2^3 \frac{6x+3}{\sqrt{x^2+x}}\,dx$

② $I = \int f(ax+b)\,dx \longrightarrow$ set $y = ax+b$.

<u>example</u> : $I = \int_0^1 (5x-3)^7\,dx$

$I = \int_0^2 \frac{dx}{(2x+1)^4}$

③ $I = \int F(x, \sqrt{ax+b})\,dx$

Let $y = \sqrt{ax+b}$ and solve for x. Then calculate dx and proceed.

<u>example</u> $I = \int_1^2 x\sqrt{3x+2}\,dx$

__example:__ $I = \int x^5 \sqrt{1-x^2}\, dx$

Let $u = 1-x^2 \Rightarrow du = -2x\,dx$

and $x^2 = 1-u^2 \Rightarrow$

$\Rightarrow x^4 = (1-u^2)^2$ etc...