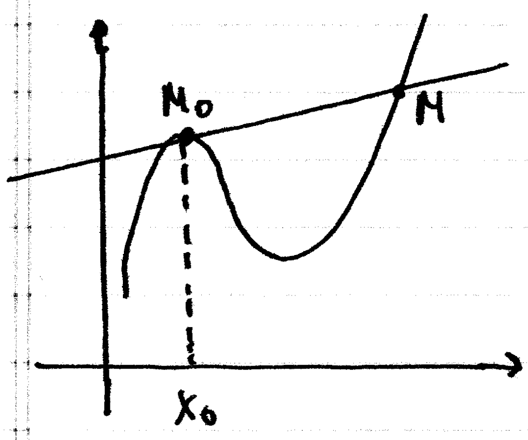
# Differential Calculus

## Tangent problem – Definitions




Let $f: A \to \mathbb{R}$ and $(c): y = f(x)$.
Assume that $f$ continuous at $x_0 \in A_f$.
The <u>tangent line $(\ell)$</u> at $M_0$ is obtained by allowing the point $M$ to approach $M_0$ from <u>BOTH</u> directions. The equation for $(\ell)$ is:

$$\boxed{\begin{aligned}&(\ell): \ y - f(x_0) = a(x - x_0)\\ &a = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}\end{aligned}}$$

### Differentiability

$$\boxed{f \text{ differentiable at } x_0 \in A_f \iff \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \mathbb{R}}$$

$$\boxed{\begin{array}{l} f \text{ differentiable at } I \subseteq A_f \end{array} \Leftrightarrow \forall x \in I : f \text{ differentiable at } x.}$$

If a function $f$ is differentiable at $A$, then we define the derivative $f' : A \to \mathbb{R}$. as

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \ \forall x \in A}$$

If $f'$ is also differentiable at $A_1$, we can define second derivative; etc

$$f''(x) = [f'(x)]'$$
$$f'''(x) = [f''(x)]'$$
$$\text{etc.} \longrightarrow f^{(n)}(x) = n^{th} \text{ derivative}$$

▶ Leibnitz notation

$$f'(x) = \frac{df(x)}{dx}, \quad f''(x) = \frac{d^2 f(x)}{dx^2},$$

$$f^{(n)}(x) = \frac{d^n f(x)}{dx^n}$$

## Differentiability and continuity

Thm : $\boxed{\begin{array}{lcl} f \text{ differentiable at } x_0 \in A_f & \Rightarrow & f \text{ continuous at } x_0 \end{array}}$

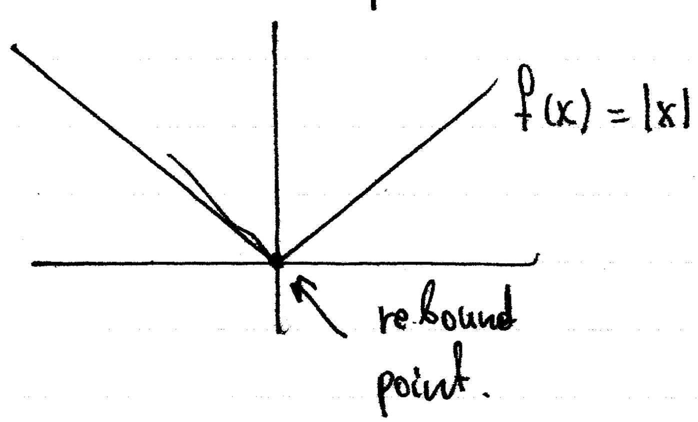
(Proof ...)

If $f$ is continuous at $x_0$ but not differentiable at $x_0$, then $x_0$ is called a <u>rebound point</u>

<u>examples</u>

1) $f(x) = |x| \longrightarrow x=0$ rebound point
2) $f(x) = \sqrt{x} \longrightarrow x=0$ rebound point.

A rebound point sometimes looks like a "kink" is the function graph, so geometrically we cannot conceptualize a unique tangent line at that point



## Calculation of Derivatives

### Differentiation rules

Assume that $f, g$ differentiable at $x$. Then

1) $(f+g)'(x) = f'(x) + g'(x)$ (Proof...)
2) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ (Proof...)
3) $(\lambda f)'(x) = \lambda f'(x)$, $\lambda \in \mathbb{R}$ (Proof...)

4) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ (if $g(x) \neq 0$)
(To be proved by chain rule).

5) $\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{g(x)}$

### Derivatives of basic functions

1) Constant function $u(x) = c$, $\forall x \in \mathbb{R}$
diff. at $\mathbb{R}$ with $u'(x) = 0$.

2) Identity function $f(x) = x$, $\forall x \in \mathbb{R}$
diff. at $\mathbb{R}$ with $f'(x) = 1$

3) <u>Monomial</u>: $f(x) = x^n$, $\forall x \in \mathbb{R}$, $n \in \mathbb{N} - \{0\}$

diff at $\mathbb{R}$ with $f'(x) = nx^{n-1}$.
(Proof)

The more general <u>power-law function</u>

$$f(x) = x^a$$

has domain $A_f$ given by

$$A_f = \begin{cases} \mathbb{R} & , \text{ if } a \in \mathbb{N} - \{0\} \\ \mathbb{R} - \{0\}, & \text{ if } a \in \mathbb{Z} - \mathbb{N} = \{-1, -2, -3, \dots\} \\ [0, +\infty), & \text{ if } a > 0 \text{ and } a \notin \mathbb{N} \\ (0, +\infty), & \text{ if } a < 0 \text{ and } a \notin \mathbb{Z}. \end{cases}$$

$a \in (-\infty, 0] \cup [1, +\infty) \Rightarrow f$ diff. at $A_f$ with $f'(x) = ax^{a-1}$.

$a \in (0, 1) \Rightarrow f$ diff. at $\underline{\underline{A_f - \{0\}}}$ with

$f'(x) = ax^{a-1}$. ($x = 0$ is rebound point)

For $a = 1/2$ and $a = -1$ we get:

4) Square root $f(x)=\sqrt{x}$, $\forall x \in [0,+\infty)$
diff. at $(0,+\infty)$ with

$$f'(x)=\frac{1}{2\sqrt{x}}$$

5) Hyperbolic $f(x)=\frac{1}{x}$, $\forall x \in (0,+\infty)$

diff. at $(0,+\infty)$ with

$$f'(x)=-\frac{1}{x^2}.$$

→ Examples

1) $f(x)=x^4+2x^3+3x^2$

2) $f(x)=(x^2+x+1)(2x+1)$

3) $f(x)=\frac{x^2+3x}{x^2-1}$

4) $f(x)=\frac{x+1}{\sqrt{x}}$

5) $f(x)=(x-1)(x-2)(x-3)$

→ $(fgh)'=f'gh+fg'h+fgh'.$

## Chain Rule

$$\left.\begin{array}{l} g \text{ diff. at } x \\ f \text{ diff at } g(x) \end{array}\right\} \Rightarrow f \circ g \text{ diff. at } x \text{ with}$$

$$(f \circ g)'(x) = [f(g(x))]' = f'(g(x))\, g'(x)$$

- The chain rule is a rule that creates new rules when you choose an $f$.

$$f(x) = x^a \Rightarrow [(g(x))^a]' = a(g(x))^{a-1} g'(x)$$

$$f(x) = \sqrt{x} \Rightarrow [\sqrt{g(x)}]' = \frac{1}{2\sqrt{g(x)}} g'(x)$$

$$f(x) = \frac{1}{x} \Rightarrow \left[\frac{1}{g(x)}\right]' = -\frac{1}{g^2(x)} g'(x)$$

- __Application__ : Prove the quotient rule from product rule.

examples

1) $f(x) = (x^4 + 2x^2 + 5x)^3$

2) $f(x) = (2x+1)\sqrt{x^2+x}$

3) $f(x) = \sqrt[3]{(x^2+1)^5}$

# Chain Rule

$$\left.\begin{array}{l} f \text{ diff at } \cancel{x_0}\ g(x_0) \\ g \text{ diff at } x_0 \end{array}\right\} \Rightarrow f\circ g \text{ diff at } x_0 \text{ with } (f(g(x)))' = f'(g(x))\,g'(x).$$

Proof

Define $\lambda(x) = \dfrac{f(g(x)) - f(g(x_0))}{x - x_0}$

and $\mu(x) = \dfrac{g(x) - g(x_0)}{x - x_0}$

and $y_0 = g(x_0)$

and
$$F(y) = \begin{cases} \dfrac{f(y) - f(y_0)}{y - y_0} & , \text{ if } y \neq y_0 \\ f'(y_0) & , \text{ if } y = y_0. \end{cases}$$

Note that
$$\lim_{y \to y_0} F(y) = \lim_{y \to y_0} \frac{f(y) - f(y_0)}{y - y_0} = f'(y_0) = F(y_0).$$

$\Rightarrow F$ continuous at $y = y_0$.

We claim that $\underline{\lambda(x) = F(g(x))\,\mu(x)}$ (1)

Proof of claim:

1) If $g(x) \neq g(x_0)$ then

$$\lambda(x) = \frac{f(g(x)) - f(g(x_0))}{x - x_0}$$

$$= \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \, \frac{g(x) - g(x_0)}{x - x_0}$$

$$= F(g(x))\,\mu(x).$$

2) If $g(x) = g(x_0) \Rightarrow f(g(x)) = f(g(x_0))$

$$\left.\begin{array}{l} \Rightarrow \lambda(x) = 0 \\ \mu(x) = 0 \end{array}\right\} \Rightarrow \lambda(x) = F(g(x))\,\mu(x).$$

End of proof of claim.

$g$ diff at $x_0 \Rightarrow \lim_{x \to x_0} \mu(x) = g'(x_0)$

~~$g$ diff at $x_0 \Rightarrow g$ continuous at $x_0$~~

~~$F$ continuous at $x_0$~~ $\Rightarrow$

~~$\Rightarrow \lim_{\substack{y \to y_0 \\ x \to x_0}} F(g(x)) = \lim_{y \to y_0} F(y)$~~

$$\left.\begin{array}{l}\lim_{x\to x_0} g(x) = g(x_0) = y_0 \\ F \text{ continuous at } x_0\end{array}\right\} \Rightarrow$$

$$\Rightarrow \lim_{x\to x_0} F(g(x)) = F\left(\lim_{x\to x_0} g(x)\right) = F(y_0)$$

$$= f'(g(\cancel{g}x_0)).$$

Thus, $$(f\circ g)' = \lim_{x\to x_0} \partial(x) = \lim_{x\to x_0}\left[F(g(x))\,\mu(x)\right]$$

$$= \lim_{x\to x_0} F(g(x)) \lim_{x\to x_0} \mu(x)$$

$$= f'(g(x_0))\,g'(x_0). \qquad \square$$

## Trigonometric ~~Limits~~ Derivatives

(1) $\boxed{\text{sin diff. at } \mathbb{R} \text{ with } (\sin x)' = \cos x}$

Proof

$$(\sin x)' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} =$$

$$= \lim_{h \to 0} \frac{2\sin\left(\frac{h}{2}\right)\cos\left(x + \frac{h}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(h/2)}{h/2} \lim_{h \to 0} \cos(x + h/2)$$

$$= 1 \cdot \cos x = \cos x. \qquad \square$$

(2) $\boxed{\text{cos diff. at } \mathbb{R} \text{ with } (\cos x)' = -\sin x}$

Proof

$$(\cos x)' = \left[\sin\left(\frac{\pi}{2} - x\right)\right]' = \left[\cos\left(\frac{\pi}{2} - x\right)\right]\left(\frac{\pi}{2} - x\right)'$$

$$= -\cos\left(\frac{\pi}{2} - x\right) = -\sin x.$$

$$\boxed{\begin{array}{l} \text{③ tan diff at } A = \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\} \text{ with} \\ (\tan x)' = \dfrac{1}{\cos^2 x} = 1 + \tan^2 x \end{array}}$$

<u>Proof</u>

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x \cdot (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} =$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \begin{cases} \dfrac{1}{\cos^2 x} \\ 1 + \dfrac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x . \ \square \end{cases}$$

▶ <u>Note</u> : From the chain rule we get

| |
|---|
| $[\sin f(x)]' = f'(x) \cos f(x)$ |
| $[\cos f(x)]' = f'(x) \cos f(x)$ |
| $[\tan f(x)]' = f'(x)[1 + \tan^2 f(x)] = \dfrac{f'(x)}{\cos^2 f(x)}$ |

<u>examples</u> : Differentiate the following functions

1) $f(x) = (x^2+1)\sin x$
2) $f(x) = x - \sin^2 x \cos x$
3) $f(x) = \tan(x^2+1)$
4) $f(x) = \cos(\sin x)$.

## Tangent line problems

For tangent line problems you have to creatively find a way to make use of the following facts:

1) Every line $(\ell)$ has an equation
$$(\ell): Ax + By + C = 0$$
2) Every line $(\ell)$ which IS NOT VERTICAL has an equation
$$(\ell): y = ax + b$$
and $a$ is the <u>slope</u> of the line.
3) For two lines
$$(\ell_1): y = a_1 x + b_1, \quad (\ell_2): y = a_2 x + b_2$$
we have
$$(\ell_1) \parallel (\ell_2) \Leftrightarrow a_1 = a_2$$
$$(\ell_1) \perp (\ell_2) \Leftrightarrow a_1 a_2 = -1$$
4) A line with slope $a$ going through the point $(x_0, y_0)$ has equation
$$(\ell): y - y_0 = a(x - x_0)$$

5) A line going through $(x_1, y_1)$ and $(x_2, y_2)$ has equation
$$(\ell): \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

* 6) The tangent line to the curve $(c): y = f(x)$ at the point $x_0 \in A_{f'}$ is

$$(\ell): y - f(x_0) = f'(x_0)(x - x_0).$$

slope $f'(x_0)$

Goes through $(x_0, f(x_0))$.

<u>examples</u>

1) Find all the tangent lines to the curve $(c): y = x^2 + 3x + 2$ from the point $(1, -5)$.

2) Find all tangent lines to the curve $(c): y = x^2$ from the point $(a, b)$ with $a, b \in \mathbb{R}$.

3) Find all tangent lines to the ~~curve~~ curve

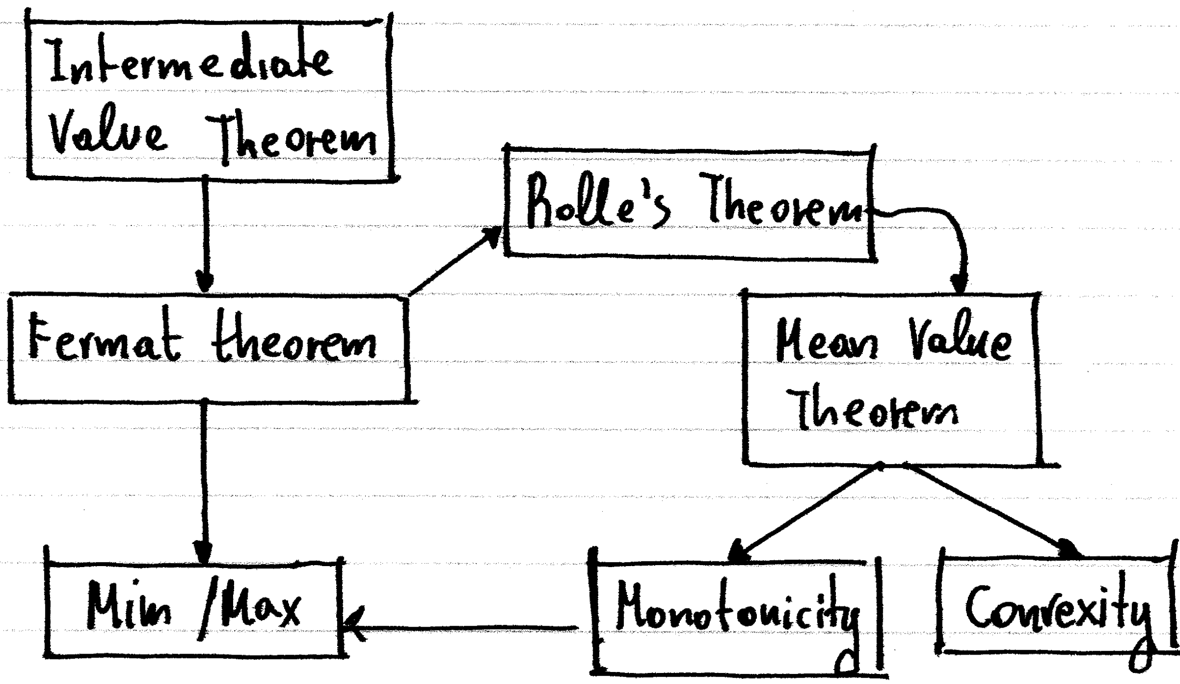
$$(c): y = \frac{2x+1}{2x-1}$$

that are perpendicular to $(\ell): y = 3x + 1$.

4) For what values of $x$ does the curve $(c): y = x + 2\sin x$ has a tangent parallel to $(\ell): y = 3x - 1$?

# ▼ Foundation of Differential Calculus

## Outline




## Definitions : Monotonicity and min/max

Let $f : A_f \longrightarrow \mathbb{R}$ be a function. Let $I \subseteq A_f$.

### ① Monotonicity : Increasing / Decreasing

$$\boxed{\begin{array}{l} f \nearrow I \iff \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2)) \\ f \searrow I \iff \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow f(x_1) > f(x_2)) \end{array}}$$

Read: $f \nearrow I$   $f$ increasing in $I$
$f \searrow I$   $f$ decreasing in $I$

② Monotonicity : Weakly increasing/decreasing.

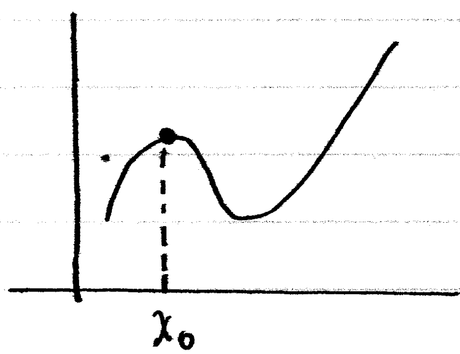
$$\boxed{\begin{array}{l} f \nearrow I \Leftrightarrow \forall x_1, x_2 \in I : x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \\ f \searrow I \Leftrightarrow \forall x_1, x_2 \in I : x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \end{array}}$$

Read $f \nearrow I$ : $f$ weakly increasing
$f \searrow I$ : $f$ weakly decreasing

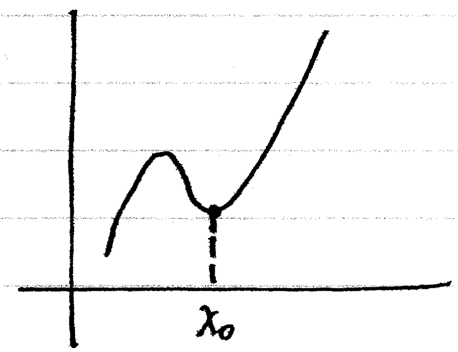
## ③ Local min/max

f has local max at $x_0 \in A_f$ $\iff \exists \delta > 0 : \forall x \in (x_0-\delta, x_0+\delta) \cap A_f : f(x) \leq f(x_0)$

f has local min at $x_0 \in A_f$ $\iff \exists \delta > 0 : \forall x \in (x_0-\delta, x_0+\delta) \cap A_f : f(x) \geq f(x_0)$




"local max"




"local min"

A point $x_0$ is interior iff $(x_0-\delta, x_0+\delta) \subseteq A_f$.
For an interior point

f has local max at $x_0 \in A_f$ $\iff \begin{cases} f \nearrow & (x_0-\delta, x_0) \\ f \searrow & (x_0, x_0+\delta) \end{cases}$

f has local min at $x_0 \in A_f$ $\iff \begin{cases} f \searrow & (x_0-\delta, x_0) \\ f \nearrow & (x_0, x_0+\delta) \end{cases}$

## Fermat theorem

Let $f: A_f \to \mathbb{R}$ and assume that

$$\boxed{\left.\begin{array}{l} \bullet_1\ x_0 \in A_f \text{ local max or min} \\ \bullet_2\ \exists\, a, b \in A_f : x_0 \in (a,b) \subseteq A_f \\ \bullet_3\ f \text{ differentiable at } x_0 \end{array}\right\} \Rightarrow f'(x_0) = 0}$$

**Proof**

Assume that $x_0$ local max without loss of generality. Define

$$\lambda(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

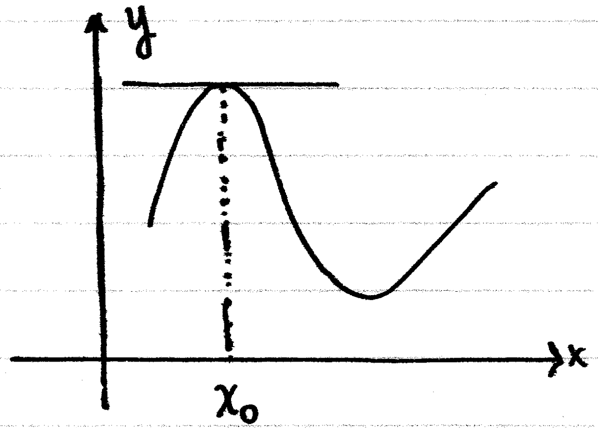
$$f \text{ local max} \Rightarrow \exists\, \delta > 0 : \begin{cases} f \nearrow (x_0 - \delta, x_0) \\ f \searrow (x_0, x_0 + \delta) \end{cases}$$

$$\Rightarrow \begin{cases} \lambda(x, x_0) \geqslant 0 \quad , \forall x \in (x_0 - \delta, x_0) & (1) \\ \lambda(x, x_0) \leqslant 0 \quad , \forall x \in (x_0, x_0 + \delta) & (2) \end{cases}$$

Since ~~$x_0$~~ $f$ differentiable at $x_0$:

$$\left.\begin{array}{l} f'(x_0) = \lim\limits_{x \to x_0^+} \lambda(x, x_0) \underset{(1)}{\geqslant} 0 \\ \\ f'(x_0) = \lim\limits_{x \to x_0^-} \lambda(x, x_0) \underset{(2)}{\leqslant} 0 \end{array}\right\} \Rightarrow f'(x_0) = 0. \qquad \square$$

## Geometrical interpretation




### Rolle Theorem

$$\boxed{\left.\begin{array}{l} f \text{ continous at } [a,b] \\ f \text{ diff. at } (a,b) \\ f(a) = f(b) \end{array}\right\} \Rightarrow \exists \xi \in (a,b) : f'(\xi) = 0}$$

**Proof**

$f$ continuous at $[a,b] \Rightarrow$

$\Rightarrow \exists \xi_1, \xi_2 \in [a,b] : \forall x \in [a,b] : f(\xi_1) \leq f(x) \leq f(\xi_2)$

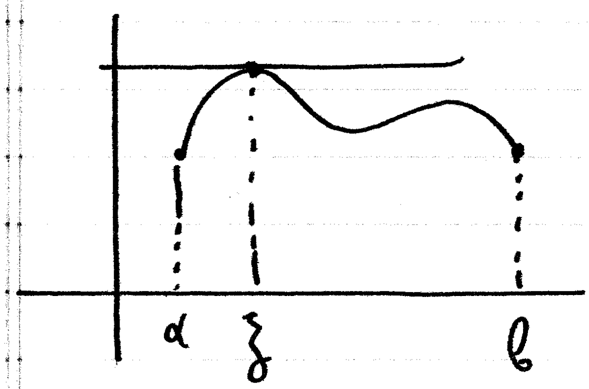
Case 1: If $\xi_1 \in (a,b)$ or $\xi_2 \in (a,b)$, let $\xi \in \{\xi_1, \xi_2\}$ such that $\xi \in (a,b)$. Then

$$\left.\begin{array}{l} \xi \in (a,b) \text{ local min or max} \\ f \text{ diff. at } (a,b) \end{array}\right\} \Rightarrow \text{Fermat theorem applies at } \xi \Rightarrow f'(\xi) = 0$$

Case 2: If $\xi_1 = a$ and $\xi_2 = b$ then

$$f(a) = f(b) = c \Rightarrow \forall x \in [a,b] : c \leq f(x) \leq c$$
$$\Rightarrow f(x) = c, \forall x \in [a,b]$$
$$\Rightarrow f'(x) = 0, \forall x \in [a,b]$$
$$\Rightarrow \exists \xi \in (a,b) : f'(\xi) = 0 \qquad \square.$$

## Interpretation




## Mean Value Theorem (Lagrange)

$$\boxed{\left.\begin{array}{l} f \text{ continuous at } [a,b] \\ f \text{ diff. at } (a,b) \end{array}\right\} \Rightarrow \exists \xi \in (a,b) : f'(\xi) = \frac{f(b)-f(a)}{b-a}}$$

### Proof

Let $F(x) = (a-b)f(x) + [f(b)-f(a)]x + [bf(a) - af(b)]$

$F(a) = (a-b)f(a) + [f(b) - f(a)]a + [bf(a) - af(b)]$

$= (a - b - a + b)f(a) + (a-a)f(b) = 0.$

$$F(b) = f(b)(a-b) + b[f(b)-f(a)] + [b f(a) - a f(b)]$$
$$= (a-b+b-a) f(b) + (-b+b) f(a)$$
$$= 0.$$

Thus $F(a) = F(b) = 0.$ (1)

$f$ continuous at $[a,b] \Rightarrow F$ continuous at $[a,b]$ (2)

$f$ diff. at $(a,b) \Rightarrow F$ diff. at $(a,b)$ with

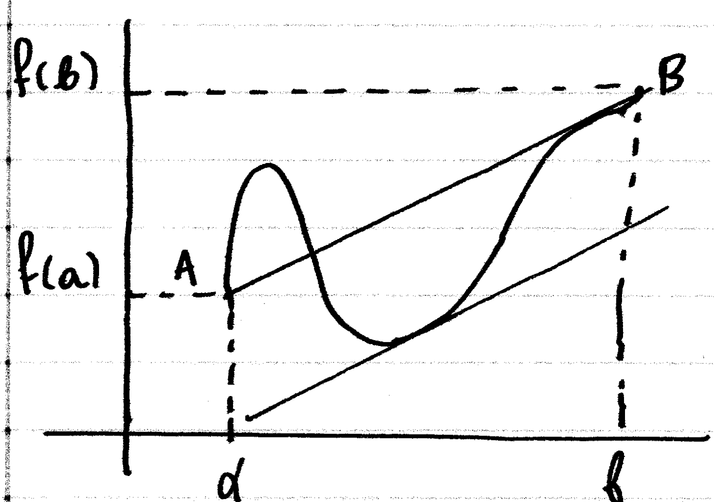
$$F'(x) = f'(x)(a-b) - [f(a)-f(b)] \quad (3)$$

(1),(2),(3) $\Rightarrow$ Rolle Thm applies on $F \Rightarrow$

$$\Rightarrow \exists\, \xi \in (a,b) : F'(\xi) = 0$$

$$\Rightarrow \exists\, \xi \in (a,b) : f'(\xi) = \frac{f(b)-f(a)}{b-a} \qquad \square$$

## Interpretation



## Immediate corollaries of MVT

(1) $\boxed{\left.\begin{array}{l} f \text{ differentiable at } I \subseteq A_f \\ f'(x)=0,\ \forall x \in I \end{array}\right\} \Rightarrow f \text{ constant in } I}$

### Proof

Sufficient: $\forall x_1, x_2 \in I : f(x_1) = f(x_2)$.

Let $x_1, x_2 \in I$ be given and assume $x_1 < x_2$.

$f$ diff. at $I \Rightarrow f$ diff. at $[x_1, x_2] \Rightarrow$

$\Rightarrow$ MVT applies at $[x_1, x_2]$

$\Rightarrow \exists\, x_0 \in (x_1, x_2) : f'(x_0) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Since $x_0 \in I \to f'(x_0) = 0$

$$\begin{aligned} \Rightarrow f(x_2) - f(x_1) &= f'(x_0)(x_2 - x_1) \\ &= 0 \cdot (x_2 - x_1) \\ &= 0 \end{aligned}$$

$\Rightarrow f(x_1) = f(x_2)$ $\square$.

(2) $\boxed{\left.\begin{array}{l} f, g \text{ diff. at } I \subseteq A_f \\ f'(x) = g'(x),\ \forall x \in I \end{array}\right\} \Rightarrow \exists\, c \in \mathbb{R} : f(x) = g(x) + c,\ \forall x \in I}$

Let $h = f - g$. Then

$$\left.\begin{array}{l} h'(x) = (f-g)'(x) = f'(x) - g'(x) = 0, \; \forall x \in I \\ f, g \text{ diff. at } I \Rightarrow h \text{ diff at } I \end{array}\right\} \Rightarrow$$

$\Rightarrow h$ constant in $I \Rightarrow$

$\Rightarrow \exists\, c \in \mathbb{R} : \forall x \in I : h(x) = f(x) - g(x) = c$

$\Rightarrow \exists\, c \in \mathbb{R} : \forall x \in I : f(x) = g(x) + c.$

## Method - Examples

① To show an equation has a unique solution (e.g. $f(x)=0$)

- •1 Use Bolzano Thm to establish existence of a solution $x_0 \in (a,b)$
- •2 Show that $f'(x) \neq 0$, $\forall x \in (a,b)$
- •3 Use Rolle Thm to establish uniqueness by proof by contradiction.

examples

a) Show that $x^3-3x+1=0$ has a unique solution at $(-1,1)$

b) Show that $x^5+2x^3+7x+12=0$ has a unique solution in $\mathbb{R}$.

② Inequalities : Using the MVT, an inequality satisfied by $f'(x)$ implies an inequality satisfied by $f(x)$.

examples

a) If $3 \leq f'(x) \leq 5$, $\forall x \in \mathbb{R}$, show that $18 \leq f(8)-f(2) \leq 30$

b) If $f,g$ continuous at $[a,b]$ and diff. at $(a,b)$, and $f(a)=g(a)$ and $f'(x)<g'(x)$, $\forall x \in (a,b)$, show that $f(b)<g(b)$.

## Monotonicity and Min/Max

Let $I \subseteq A_f$ be an interval and $f: A_f \to \mathbb{R}$ a function which is differentiable at $I$.

### (1) Monotonicity Theorem

a) $f \nearrow I \iff \forall x \in I : f'(x) \geq 0$
b) $f \searrow I \iff \forall x \in I : f'(x) \leq 0$
c) $\forall x \in I : f'(x) > 0 \Rightarrow f \uparrow I$
d) $\forall x \in I : f'(x) < 0 \Rightarrow f \downarrow I$

↳ The converse of statements (c) and (d) is not necessarily true.

### (2) Min/Max Theorem

If $f$ differentiable at $(a,b) \subseteq A_f$ and $x_0 \in (a,b)$ with $f'(x_0) = 0$, then

a) $\left.\begin{array}{l} \forall x \in (a, x_0] : f'(x) \geq 0 \\ \forall x \in [x_0, b) : f'(x) \leq 0 \end{array}\right\} \Rightarrow x_0$ local max of $f$

b) $\left.\begin{array}{l} \forall x \in (a, x_0] : f'(x) \leq 0 \\ \forall x \in [x_0, b) : f'(x) \geq 0 \end{array}\right\} \Rightarrow x_0$ local min of $f$.

③ **2nd derivative test**

Let $f$ twice differentiable at $(a,b)$ and $x_0 \in (a,b)$ with $f'(x_0)=0$.

a) $f''(x_0) < 0 \Rightarrow x_0$ local max

b) $f''(x_0) > 0 \Rightarrow x_0$ local min.

**Proof of ①**

1a) $f \nearrow I \Leftrightarrow \forall x \in I : f'(x) \geq 0$.

($\Rightarrow$): Assume $f \nearrow I$

$f \nearrow I \Rightarrow \forall\, x, x_0 \in I : \dfrac{f(x)-f(x_0)}{x-x_0} \geq 0 \Rightarrow$

$$\Rightarrow f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \geq 0.$$

($\Leftarrow$): Assume $\forall x \in I : f'(x) \geq 0$.

Let $x_1, x_2 \in I$ with $x_1 < x_2$.

Sufficient to show that $f(x_1) \leq f(x_2)$.

Apply the mean-value theorem on the $[x_1, x_2]$:

$$\exists \xi \in (x_1, x_2) : f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2).$$

Since $\left.\begin{array}{l} x_1 - x_2 < 0 \\ f'(\xi) \geq 0 \end{array}\right\} \Rightarrow \begin{array}{l} f(x_1) - f(x_2) \leq 0 \\ \Rightarrow f(x_1) \leq f(x_2) \end{array}$

Thus: $\forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)) \Rightarrow$

$\Rightarrow f \nearrow I$.

1b) Similar to 1a

1cd) Similar to 1a ($\Leftarrow$)

Proof of ②

2a) $\begin{cases} f'(x) \geq 0, \forall x \in [a, x_0] \\ f'(x) \leq 0, \forall x \in [x_0, b) \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} f \nearrow (a, x_0] \\ f \searrow [x_0, b) \end{cases}$

$\Rightarrow \begin{cases} f(x) \leq f(x_0), \forall x \in (a, x_0] & \text{(because } x < x_0) \\ f(x) \leq f(x_0), \forall x \in [x_0, b) & \text{(because } x > x_0) \end{cases}$

$\Rightarrow f(x) \leq f(x_0), \forall x \in (a, b)$

$\Rightarrow$ $x_0$ local max of $f$.

2b) Similar to 2a

Proof of ③

3b) Assume that $f''(x_0) > 0$. and $f'(x_0) = 0$

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f'(x)}{x - x_0} > 0$$

$$\Rightarrow \exists \delta > 0 : \forall x \in N(x_0, \delta) = (x_0 - \delta, x_0 + \delta) - \{x_0\} : \frac{f'(x)}{x - x_0} > 0$$

For $x \in (x_0-\delta, x_0) \Rightarrow x - x_0 < 0 \Rightarrow f'(x) < 0, \forall x \in (x_0-\delta, x_0)$
$\Rightarrow f \searrow (x_0-\delta, x_0)$ (1)
For $x \in (x_0, x_0+\delta) \Rightarrow x - x_0 > 0 \Rightarrow f'(x) > 0, \forall x \in (x_0, x_0+\delta)$
$\Rightarrow f \nearrow (x_0, x_0+\delta)$ (2)
From (1) and (2): $x_0$ local min of $f$.

3a) Similar argument.

- <u>Methodology:</u> In general to find monotonicity and min/max.

1) Find domain $A_f$.
2) Find $f'(x)$ and FACTOR it.
   Check if $A_{f'} = A_f$ or $A_{f'} \subseteq A_f$.
3) Make a table of signs for $f'$.
4) Use sign table to deduce monotonicity.
5) Use monotonicity to deduce min/max.

→ Location of possible min/max

1) Points $x_0$ with $f'(x_0) = 0$.
2) Endpoints of closed intervals in $A_f$.
   e.g. $A_f = [1,2) \cup [3,+\infty)$
   possible min/max at $x_0 = 1$ and $x_0 = 3$.
3) Points $x_0 \in A_f$ such that $x_0 \notin A_{f'}$.
   e.g: functions with square roots.

The above is a consequence of Fermat thm.

<u>example</u> : $f(x) = (x-1)^2(x+2)^3$

- 1 $A_f = \mathbb{R}$.
- 2 $f'(x) = \dots = (x-1)(x+2)^2(5x+1)$

  with $A_{f'} = \mathbb{R}$.

| $x$ | | $-2$ | | $-1/5$ | | $1$ | |
|---|---|---|---|---|---|---|---|
| $x-1$ | $-$ | | $-$ | | $-$ | 0 | $+$ |
| $(x+2)^2$ | $+$ | 0 | $+$ | | $+$ | | $+$ |
| $(5x+1)$ | $-$ | | $-$ | 0 | $+$ | | $+$ |
| $f'$ | $+$ | 0 | $+$ | 0 | $-$ | 0 | $+$ |
| $f$ | $\nearrow$ | | $\nearrow$ | max | $\searrow$ | min | $\nearrow$ |

$f \nearrow$ at $(-\infty,-2)$, $(-2,-1/5)$, $(1,+\infty)$

$f \searrow$ at $(-1/5,1)$

max at $x=-1/5$

min at $x=1$

▶ $x=-2$ is not a min or max!

example : $f(x) = x\sqrt{1-2x}$

•1 Domain:

Need $1-2x \geq 0 \Leftrightarrow x \leq 1/2$ thus

$$A_f = (-\infty, 1/2].$$

•2 Derivative:

$$f'(x) = (x)'\sqrt{1-2x} + x\left(\sqrt{1-2x}\right)' =$$

$$= \sqrt{1-2x} + x\,\frac{(1-2x)'}{2\sqrt{1-2x}} =$$

$$= \sqrt{1-2x} + \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-2x}} =$$

$$= \frac{(1-2x)-x}{\sqrt{1-2x}} = \frac{1-3x}{\sqrt{1-2x}}$$

with

$$A_{f'} = (-\infty, 1/2) \subseteq A_f \quad (!).$$

| $x$ | | $1/3$ | | $1/2$ |
|---|---|---|---|---|
| $1-3x$ | + | 0 | − | |
| $\sqrt{1-2x}$ | + | | + | 0 |
| $f'$ | + | 0 | − | \|\| |
| $f$ | $\nearrow$ | max | $\searrow$ | min !!! |

$f \nearrow (-\infty, 1/3)$

$f \searrow (1/3, 1/2]$

$x = 1/3$ max

$x = 1/2$ min

!!!

▶ $f$ is defined at $x = 1/2$ even though $f'$ isn't!

↳ The 2nd derivative test is usually avoided because

1) Calculating $f''(x)$ may be tedious
2) If $f''(x_0) = 0$, then the test is inconclusive (i.e. worthless, useless).

An EXCEPTION is with trig functions where we have infinite min/max.

<u>example</u> : $f(x) = x - \sin 2x \leftarrow$ min/max.

- 1 $A_f = \mathbb{R}$
- 2 $f'(x) = 1 - 2\cos 2x$ , $A_{f'} = \mathbb{R}$
  $f''(x) = 4\sin 2x$ , $A_{f''} = \mathbb{R}$.

▸ All possible min/max are zeroes of $f'$.

$$f'(x) = 0 \Leftrightarrow 1 - 2\cos 2x = 0 \Leftrightarrow \cos 2x = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\Leftrightarrow 2x = \underline{2k\pi} \pm \pi/3 \Leftrightarrow x = k\pi \pm \pi/6,\ k \in \mathbb{Z}.$$

For $x_0 = k\pi \pm \pi/6$ :

$$f''(x_0) = 4\sin\left[2(k\pi \pm \pi/6)\right] = 4\sin(2k\pi \pm \pi/3) =$$

$$= 4\sin(\pm\pi/3) = \pm 4\sin(\pi/3) = \pm 4\sqrt{3}/2$$
$$= \pm 2\sqrt{3}.$$

For $x_0 = k\pi + \pi/6 \Rightarrow f''(x_0) > 0$

$\Rightarrow x_0$ min

$x_0 = k\pi - \pi/6 \Rightarrow f''(x_0) < 0$

$\Rightarrow x_0$ max.

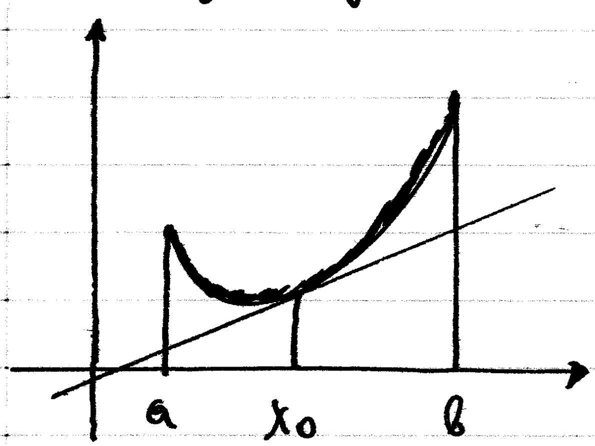
## Convexity

Let $f: A_f \to \mathbb{R}$ be a function.

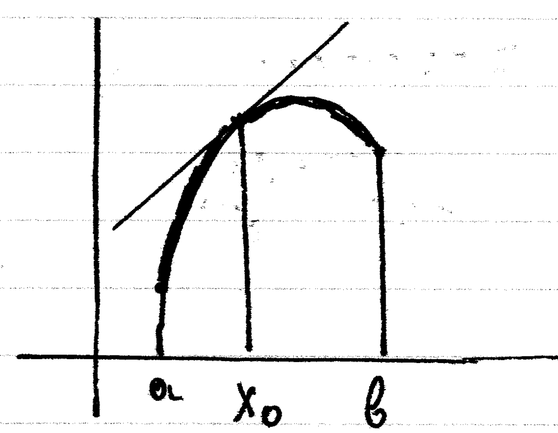
### Geometric Definition

a) $f$ convex up at $(a,b)$ iff
the curve $(c): y = f(x)$ is ABOVE
every tangent line for all points $x_0 \in (a,b)$

b) $f$ convex down at $(a,b)$ iff
the curve $(c): y = f(x)$ is BELLOW
every tangent line for all points $x_0 \in (a,b)$




convex up




convex down.

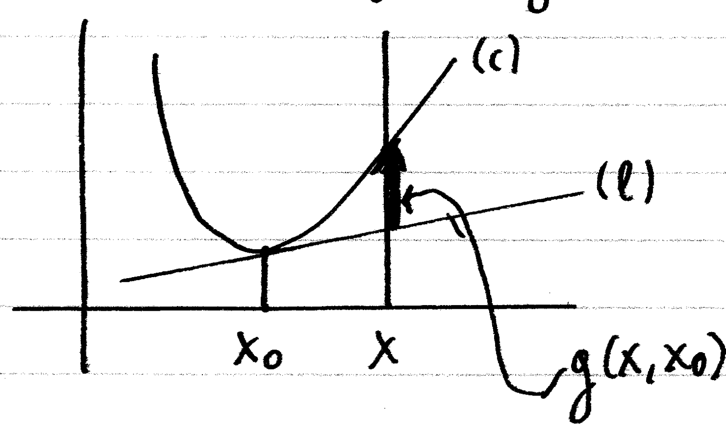
## Algebraic Definition

Recall that the tangent line $(\ell)$ at $x_0$ of the curve $(c): y = f(x)$ is:

$$(\ell): y = f'(x_0)(x - x_0) + f(x_0)$$

Define $g(x, x_0) = f(x) - [f'(x_0)(x - x_0) + f(x_0)]$

→ Meaning of $g(x, x_0)$ : How far ABOVE is $(c)$ relative to the line $(\ell)$ at coordinate $x$?




a) $\left.\begin{array}{l} f \text{ convex up} \\ \text{at } (a,b) \end{array}\right\} \Leftrightarrow \forall x, x_0 \in (a,b) : g(x, x_0) > 0$

b) $\left.\begin{array}{l} f \text{ convex down} \\ \text{at } (a,b) \end{array}\right\} \Leftrightarrow \forall x, x_0 \in (a,b) : g(x, x_0) < 0$

- ## Convexity Criterion

$$\left.\begin{array}{l} f \text{ convex up} \\ \text{at } (a,b) \end{array}\right\} \Leftrightarrow \forall x \in (a,b) : f''(x) \geq 0$$

$$\left.\begin{array}{l} f \text{ convex down} \\ \text{at } (a,b) \end{array}\right\} \Leftrightarrow \forall x \in (a,b) : f''(x) \leq 0.$$

### Proof

($\Rightarrow$): Assume $f$ convex up at $(a,b)$ (1)

- Sufficient to show that $f' \uparrow (a,b)$.

Let $x_1, x_2 \in (a,b)$ with $\underline{x_1 < x_2}$.

From (1)

$$g(x_1,x_2) \geq 0 \Rightarrow f(x_1) - [f'(x_2)(x_1-x_2) + f(x_2)] \geq 0$$
$$\Rightarrow f'(x_2)(x_1-x_2) \leq f(x_1) - f(x_2) \quad (2)$$

Similarly

$$g(x_2,x_1) \geq 0 \Rightarrow f'(x_1)(x_2-x_1) \leq f(x_2) - f(x_1) \quad (3)$$

Add inequalities (2) and (3):

$$[f'(x_1) - f'(x_2)](x_2 - x_1) \leq 0 \Rightarrow$$
$$\Rightarrow f'(x_1) - f'(x_2) \leq 0 \quad , \text{because } x_2 - x_1 < 0$$
$$\Rightarrow \underline{f'(x_1) \leq f'(x_2)}$$

Thus we showed that

$$f' \nearrow (a,b) \Rightarrow \forall x \in (a,b) : f''(x) \geq 0. \quad (1)$$

$(\Leftarrow)$: Assume $\forall x \in (a,b) : f''(x) \geq 0$.

Choose an $x_0 \in (a,b)$ and define

$$\begin{aligned} h(x) &= g(x, x_0) \\ &= f(x) - [f'(x_0)(x-x_0) + f(x_0)] \end{aligned}$$

We need to show $\forall x \in (a,b) : h(x) \geq 0$.

Note that

$$h(x_0) = g(x_0, x_0) = f(x_0) - 0 - f(x_0) = 0.$$

and

$$\begin{aligned} h'(x) &= f'(x) - [f'(x_0)(x-x_0)]' \\ &= f'(x) - f'(x_0)[(x-x_0)]' \\ &= f'(x) - f'(x_0) \\ &\overset{*}{=} f''(\xi(x))(x-x_0) \end{aligned}$$

* Mean Value Thm on $f'$ with $\xi(x)$ between $x, x_0$ and consequently $\xi(h) \in (a,b)$.

From (1)

$$f''(\xi(x)) \geq 0 \Rightarrow \begin{cases} \forall x \in (a, x_0) : h'(x) \leq 0 \\ \forall x \in (x_0, b) : h'(x) \geq 0 \end{cases}$$

$\Rightarrow x_0$ local min of $h(x)$

$\Rightarrow \forall x \in (a,b) : h(x) \geq h(x_0) = 0$

$\Rightarrow \forall x \in (a,b) : h(x) \geq 0$ ✓

## Method for convexity

- 1 Calculate and FACTOR $f''(x)$
- 2 Make sign table for $f''(x)$.
- 3 Indicate convexity on table
- 4 Inflection points arises when convexity changes.
  (i.e. not ALL zeroes of $f''$ have to be inflection points).

## Variation Table : Indicates curve shape in more detail.

- 1 Make a monotonicity and convexity table separately
- 2 Merge two tables into a table of the form:

| |
|---|
| $x$ |
| $f'$ |
| $f''$ |
| $f$ |

Here $f$ :

Also may indicate min/max / inflection points.

**example** : $f(x) = \dfrac{x^3}{x^2 - 1} \longleftarrow \begin{cases} \text{Monotonicity} \\ \text{Convexity} \\ \text{Variation} \end{cases}$

$$f'(x) = \dots = \frac{x^2(x^2-3)}{(x-1)^2(x+1)^2}$$

$$f''(x) = \dots = \frac{2x\,(x^2+3)}{(x-1)^3(x+1)^3}$$

- Monotonicity

| $x$ | | $-\sqrt{3}$ | | $-1$ | | $0$ | | $1$ | | $+\sqrt{3}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x^2$ | + | | + | | + | 0 | + | | + | | + |
| $x^2-3$ | + | 0 | − | | − | | − | | − | 0 | + |
| $(x-1)^2$ | + | | + | | + | | + | 0 | + | | + |
| $(x+1)^2$ | + | | + | 0 | + | | + | | + | | + |
| $f'(x)$ | + | 0 | − | $\parallel$ | − | 0 | − | $\parallel$ | − | 0 | + |
| $f(x)$ | $\nearrow$ | | $\searrow$ | | $\searrow$ | | $\searrow$ | | $\searrow$ | | $\nearrow$ |

- Convexity

| $x$ | | $-1$ | | $0$ | | $1$ | |
|---|---|---|---|---|---|---|---|
| $2x$ | − | | − | 0 | + | | + |
| $x^2+3$ | + | | + | | + | | + |
| $(x-1)^3$ | − | | − | | − | 0 | + |
| $(x+1)^3$ | − | 0 | + | | + | | + |
| $f''$ | − | $\parallel$ | + | 0 | − | $\parallel$ | + |
| $f$ | $\cap$ | | $\cup$ | | $\cap$ | | $\cup$ |

- Variation Table

| $x$ | | $-\sqrt{3}$ | | $-1$ | | $0$ | | $1$ | | $\sqrt{3}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f'$ | $+$ | $0$ | $-$ | $\Vert$ | $-$ | $0$ | $-$ | $\Vert$ | $-$ | $0$ | $+$ |
| $f''$ | $-$ | | $-$ | $\Vert$ | $+$ | $0$ | $-$ | $\Vert$ | $+$ | | $+$ |
| $f$ | $\nearrow$ | | $\searrow$ | $\Vert$ | $\searrow$ | | $\searrow$ | $\Vert$ | $\searrow$ | | $\nearrow$ |
| | | max | | vertical asymptote | | infl. | | vertical asymptote | | min | |