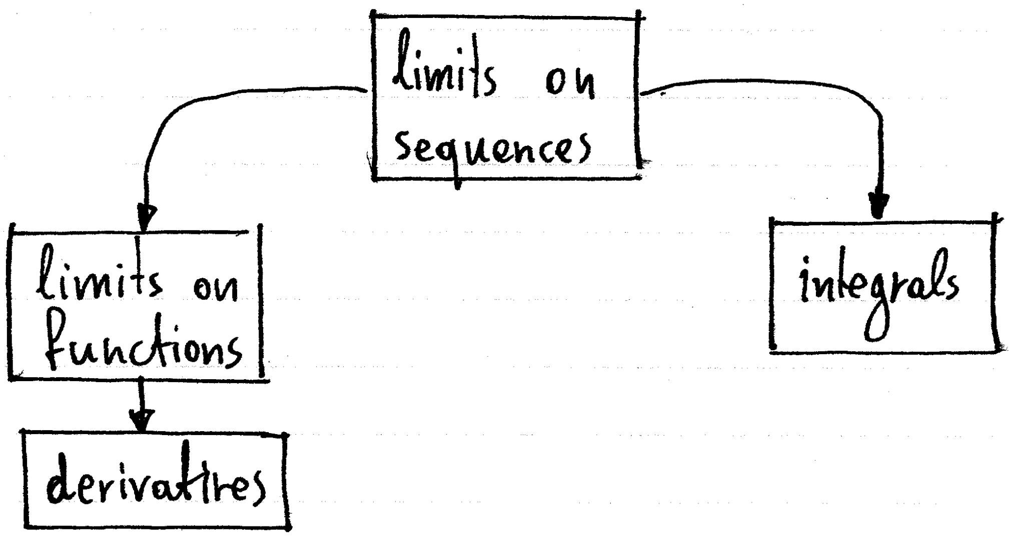
Limits

Outline



Limits on sequences

(1) A sequence a_n converges to ℓ iff for any number $\varepsilon > 0$ there is another number n_0 such that for all $n > n_0$, a_n satisfies

$$|a_n - \ell| < \varepsilon.$$

notation: $\forall$ = for all
$\exists$ = there is (at least one)

$$\boxed{\lim a_n = \ell \iff \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N} : \forall n > n_0 : |a_n - \ell| < \varepsilon}$$

example

$\ell = 1/3 = 0.3333\ldots$

$a_1 = 0.3 \longrightarrow |a_1 - \ell| = 0.0333\ldots$

$a_2 = 0.33 \longrightarrow |a_2 - \ell| = 0.0033\ldots$

$a_3 = 0.333 \longrightarrow |a_{\cancel{2}3} - \ell| = 0.0003\ldots$

- Choose $\varepsilon > 0$: How small you want $|a_n - \ell|$. Eventually, for $n > n_0$, it gets that small and stays that way.

② Limit to $+\infty$

$$\boxed{\lim a_n = +\infty \Leftrightarrow \forall M > 0 : \exists n_0 \in \mathbb{N} : \forall n > n_0 : a_n > M}$$

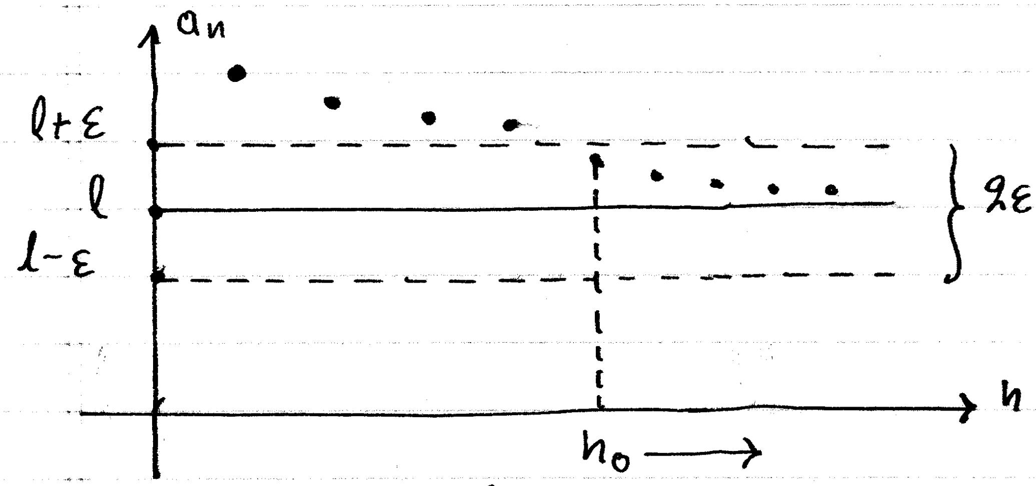
For large enough $n > n_0$, a_n can be as large as we want.

③ Limit to $-\infty$

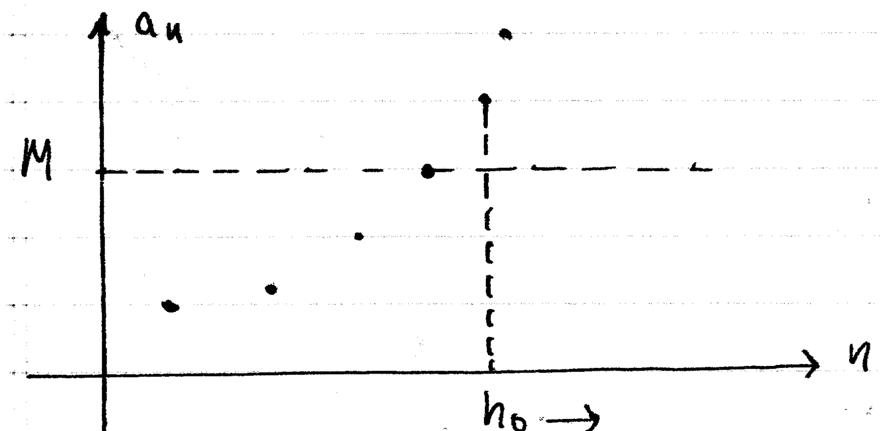
$$\boxed{\lim a_n = -\infty \Leftrightarrow \forall M > 0 : \exists n_0 \in \mathbb{N} : \forall n > n_0 : a_n < -M}$$

- Exercise : Write out ② and ③ in words.

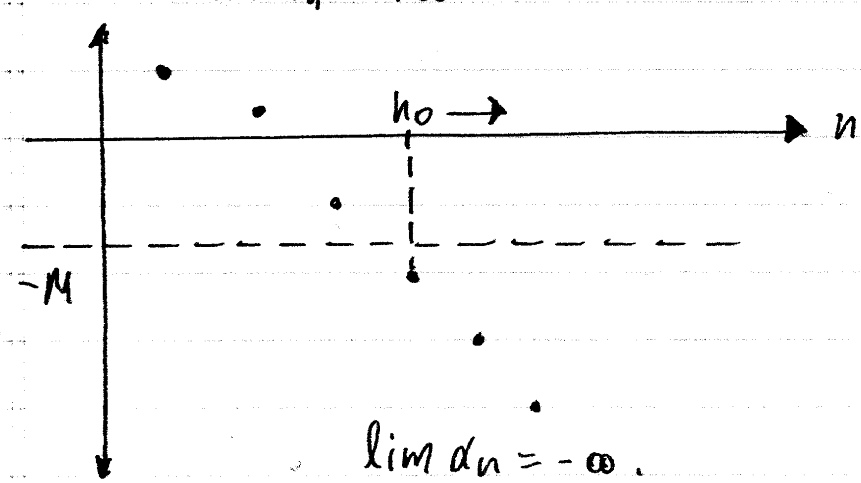
Limits on sequences
Geometric interpretations



$\lim a_n = l$



$\lim a_n = +\infty$



$\lim a_n = -\infty.$

Limits on Functions

Definition : Let $\sigma \in \mathbb{R} \cup \{+\infty, -\infty\}$
and $L \in \mathbb{R} \cup \{+\infty, -\infty\}$

We say that

$$\lim_{x \to \sigma} f(x) = L$$

if and only if for all sequences $a_n \in A_f$

$$\boxed{\lim a_n = \sigma \Rightarrow \lim f(a_n) = L}$$

↳ The limit $\lim_{x \to \sigma} f(x)$ does not exist if it is impossible to find an L that satisfies the definition

example : $\lim_{x \to +\infty} \sin x$ does not exist

Choose $a_n = 2n\pi \Rightarrow \sin a_n = 0 \Rightarrow \lim \sin a_n = 0$
$b_n = 2n\pi + \pi/4 \Rightarrow \sin b_n = \sqrt{2}/2$
$\Rightarrow \lim \sin b_n = \sqrt{2}/2.$

so agreement on a common L by all sequences is impossible

Limits and Domain

A necessary condition for the existence of the limit

$$\lim_{x \to \sigma} f(x)$$

is:

1) If $\sigma \in \mathbb{R}$, then $(a,\sigma) \cup (\sigma, b) \subseteq A_f$, $\exists\, a, b \in \mathbb{R}$
2) If $\sigma = +\infty$, then $(a, +\infty) \subseteq A_f$, $\exists\, a \in \mathbb{R}$
3) If $\sigma = -\infty$, then $(-\infty, a) \subseteq A_f$, $\exists\, a \in \mathbb{R}$

<u>example:</u> $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \sqrt{1-x}$ does not exist

Domain: $1 - x \geq 0 \Leftrightarrow x \leq 1$

so $A_f = (-\infty, 1]$

Thus A_f does not contain an interval $(a, +\infty)$.

Limits and operations.

(1) Assume that $\lim_{\sigma} f = l_1 \in \mathbb{R}$ and

$\lim_{\sigma} g = l_2 \in \mathbb{R}$.

a) $\lim_{\sigma} (f+g) = \lim_{\sigma} f + \lim g$.

b) $\lim_{\sigma} (fg) = (\lim_{\sigma} f)(\lim_{\sigma} g)$

c) $\lim_{\sigma} (\lambda f) = \lambda \lim_{\sigma} f$, $\lambda \in \mathbb{R}$.

d) $\lim_{\sigma} g \neq 0 \Rightarrow \lim_{\sigma} \left(\frac{f}{g}\right) = \dfrac{\lim_{\sigma} f}{\lim_{\sigma} g}$

(2) Assume that $\lim_{\sigma} f = +\infty$, $\lim_{\sigma} g$ exists

a) $\lim_{\sigma} g \in \mathbb{R} \cup \{+\infty\} \Rightarrow \lim_{\sigma} (f+g) = +\infty$

b) $\lim_{\sigma} g \in (0, +\infty) \cup \{+\infty\} \Rightarrow \lim_{\sigma} (fg) = +\infty$

c) $\lim_{\sigma} g \in (-\infty, 0) \cup \{-\infty\} \Rightarrow \lim_{\sigma} (fg) = -\infty$

③ Assume that $\lim_{\sigma} f = -\infty$ and $\lim g$ exists

a) $\lim_{\sigma} g \in \mathbb{R} \cup \{-\infty\} \Rightarrow \lim_{\sigma} (f+g) = -\infty$

b) $\lim_{\sigma} g \in (0, +\infty) \cup \{+\infty\} \Rightarrow \lim_{\sigma} (fg) = -\infty$

c) $\lim_{\sigma} g \in (-\infty, 0) \cup \{-\infty\} \Rightarrow \lim_{\sigma} (fg) = +\infty$.

↳ Rules 2+3 can be summarized as follows:

Let $a \in \mathbb{R}$, $p \in (0, +\infty)$, $n \in (-\infty, 0)$

$$\boxed{\begin{aligned}
a + (+\infty) &= +\infty \\
a + (-\infty) &= -\infty \\
(+\infty) + (+\infty) &= +\infty \\
(-\infty) + (-\infty) &= -\infty \\
p(+\infty) &= +\infty \\
p(-\infty) &= -\infty \\
n(+\infty) &= \cancel{+\infty} -\infty \\
n(-\infty) &= +\infty \\
(+\infty)(+\infty) &= +\infty \\
(+\infty)(-\infty) &= -\infty \\
(-\infty)(-\infty) &= +\infty
\end{aligned}}$$

Indeterminate Forms

$(+\infty) - (+\infty)$

$0 \cdot (+\infty)$

$0 \cdot (-\infty)$

$\dfrac{\infty}{\infty}$

↳ The limit can exist but it can be ANYTHING. No fixed rule.

Methods for limits $x \to \pm\infty$

We use the following applications

1) If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ then

$$\boxed{\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} [a_n x^n]}$$

2) If
$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
$$Q(x) = b_m x^n + b_{m-1} x^{n-1} + \cdots + b_1 x + b_0$$
then

$$\boxed{\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)} = \lim_{x \to \pm\infty} \left[\frac{a_n x^n}{b_m x^n}\right]}$$

3) $k \in (0, +\infty) \Longrightarrow \lim_{x \to +\infty} x^k = +\infty$

$$k \in \mathbb{N} - \{0\} \Longrightarrow \begin{cases} \lim_{x \to -\infty} x^{2k} = +\infty \\ \lim_{x \to -\infty} x^{2k+1} = -\infty \end{cases}$$

$k \in (0, +\infty) \Rightarrow \lim_{x \to +\infty} (1/x^k) = 0$

$k \in \mathbb{N} - \{0\} \Rightarrow \lim_{x \to -\infty} (1/x^k) = 0$

examples (monomials)

1) $\lim\limits_{x \to +\infty} (3x^5)$

2) $\lim\limits_{x \to -\infty} (2x^4)$

3) $\lim\limits_{x \to -\infty} (-7x^3)$

4) $\lim\limits_{x \to +\infty} \dfrac{3}{2x^2}$

5) $\lim\limits_{x \to +\infty} \dfrac{5}{7x\sqrt{x}}$

examples

1) $\lim_{x \to +\infty} (2x^4 + 3x + 1)$

2) $\lim_{x \to -\infty} \dfrac{x + x^3 + 1}{2x - x^2}$

3) $\lim_{x \to +\infty} \dfrac{x^2 + 3x + 1}{x^2 - 2}$

4) $\lim_{x \to -\infty} \dfrac{x^2 + 1}{x^4 - x}$

Indeterminate forms

1) Form ∞/∞

$$f(x) = \frac{\sqrt{g(x)}}{h(x)}, \quad f(x) = \frac{g(x)}{\sqrt{h(x)}}, \quad f(x) = \sqrt{\frac{g(x)}{h(x)}}$$

Factor highest order term and simplify.

CAUTION: $\boxed{\begin{array}{l} \sqrt{x^2} = |x| \\ (\sqrt{x})^2 = x \end{array}}$

example: $\lim_{x \to -\infty} \dfrac{\sqrt{9x^2 - 2x + 5}}{x + 4}$

2) Form $\infty - \infty$

a) $f(x) = Q_1(x) - Q_2(x)$
with Q_1, Q_2 rational functions
▸ **Method**: Combine to one fraction

example $f(x) = \dfrac{x^3}{x^2+1} - \dfrac{2x^2}{x-1} \longrightarrow \lim\limits_{x \to \pm\infty} f(x)$

$f(x) = \dfrac{x}{x-2} - \dfrac{1}{x+2} \longrightarrow \lim\limits_{x \to +\infty} f(x)$

b) $f(x) = \sqrt{g(x)} - \sqrt{h(x)}$
$f(x) = g(x) - \sqrt{h(x)}$
$f(x) = \sqrt{g(x)} - h(x)$

▸ **Method**: Use identity $\boxed{a - b = \dfrac{a^2 - b^2}{a+b}}$

example: $f(x) = \sqrt{9x^2+x+1} - 3x \longleftarrow \lim\limits_{x \to +\infty} f(x)$

Method for limits $x \to x_0$ with $x_0 \in \mathbb{R}$

1) Polynomials $P(x) = a_n x^n + \cdots + a_1 x + a_0$

P polynomial $\Rightarrow \lim\limits_{x \to x_0} P(x) = P(x_0)$.

example: $\lim\limits_{x \to 3} (x^2 + x + 12)$

2) Rational Functions: $f(x) = P(x)/Q(x)$
with P, Q polynomials.

a) If $Q(x_0) \neq 0 \Rightarrow \lim\limits_{x \to x_0} f(x) = \dfrac{P(x_0)}{Q(x_0)}$

example: $f(x) = \dfrac{x+1}{x^2 - 3x + 2} \longrightarrow \lim\limits_{x \to 3} f(x)$

b) If $P(x_0) = Q(x_0) = 0$ then
a factor "0/0" can be removed by simplification

example: $f(x) = \dfrac{x^2 + 3x + 2}{x^2 + 4x + 4} \longrightarrow \lim\limits_{x \to -2} f(x)$

3) The numerator or denominator have a difference of radicals and the limit is 0/0.

example: $f(x) = \dfrac{\sqrt{x-1} - 2}{x-5} \longrightarrow \lim_{x \to 5} f(x)$

▸ Method: Use the identity $a - b = \dfrac{a^2 - b^2}{a+b}$ AND look for cancellation.

Side limits $x \to x_0^{\pm}$

Let $\sigma \in \mathbb{R}$ and $L \in \mathbb{R} \cup \{+\infty, -\infty\}$.

a) We say that

$$\lim_{x \to \sigma^+} f(x) = L$$

if and only if for all sequences $a_n \in A_f$

$$\boxed{\left.\begin{array}{l} \lim a_n = \sigma \\ a_n > \sigma, \ \forall n \in \mathbb{N} \end{array}\right\} \Rightarrow \lim f(a_n) = L}$$

b) We say that

$$\lim_{x \to \sigma^-} f(x) = L$$

if and only if for all sequences $a_n \in A_f$

$$\boxed{\left.\begin{array}{l} \lim a_n = \sigma \\ a_n < \sigma, \ \forall n \in \mathbb{N} \end{array}\right\} \Rightarrow \lim f(a_n) = L}$$

Properties of side limits

1) $\lim_{x\to\sigma} f(x) = l \Rightarrow \lim_{x\to\sigma^+} f(x) = l$

2) $\lim_{x\to\sigma} f(x) = l \Rightarrow \lim_{x\to\sigma^-} f(x) = l$

3) $\left.\begin{array}{l}\lim_{x\to\sigma^+} f(x) = l \\ \lim_{x\to\sigma^-} f(x) = l\end{array}\right\} \Rightarrow \lim_{x\to\sigma} f(x) = l$

4) $\left.\begin{array}{l}\lim_{x\to\sigma^+} f(x) = l_1 \\ \lim_{x\to\sigma^-} f(x) = l_2 \\ l_1 \neq l_2\end{array}\right\} \Rightarrow \lim_{x\to\sigma} f(x)$ does not exist.

key application

For $a \in \mathbb{R}$:

$$\boxed{\left.\begin{array}{l}\lim_{x\to a^+} \dfrac{1}{x-a} = +\infty \\ \lim_{x\to a^-} \dfrac{1}{x-a} = -\infty\end{array}\right\} \Rightarrow \lim_{x\to a} \dfrac{1}{x-a} \text{ does not exist}}$$

It follows that

$$\boxed{\lim_{x \to a} \frac{1}{(x-a)^{2k}} = +\infty, \ \forall k \in \mathbb{N} - \{0\}}$$

examples

1) $f(x) = \dfrac{3x+5}{1-2x} \rightarrow \lim\limits_{x \to 1/2^+} f(x)$

2) $f(x) = \dfrac{5-2x}{(x-5)^2} \rightarrow \lim\limits_{x \to 5} f(x)$

3) $f(x) = \dfrac{3x}{(3x+2)^3} \rightarrow \lim\limits_{x \to -2/3^+} f(x)$

Examples with Side Limits

1) $f(x) = \dfrac{2x-1}{2-3x} \leftarrow \lim\limits_{x \to 2/3^+} f(x)$.

Solution:

$$f(x) = \frac{2x-1}{2-3x} = (2x-1)\frac{1}{-3} \cdot \frac{1}{x-2/3} =$$

$$= -\frac{2x-1}{3} \frac{1}{x-2/3}.$$

$$\lim_{x \to 2/3^+} \left[-\frac{2x-1}{3} \right] = -\frac{2\cdot(2/3)-1}{3} = -\frac{1/3}{3} < 0 \quad (1)$$

$$\lim_{x \to 2/3^+} \frac{1}{x-2/3} = +\infty \quad (2)$$

Multiply (1) and (2): $\lim\limits_{x \to 2/3^+} f(x) = -\infty$.

2) $f(x) = \dfrac{5-2x}{(x-5)^2} \leftarrow \lim\limits_{x \to 5} f(x)$.

Solution:

$$f(x) = (5-2x) \cdot \frac{1}{(x-5)^2}.$$

$$\left.\begin{array}{l} \lim\limits_{x \to 5} (5-2x) = 5-10 < 0 \\ \lim\limits_{x \to 5} \dfrac{1}{(x-5)^2} = +\infty \end{array}\right\} \Rightarrow \lim_{x \to 5} f(x) = -\infty .$$

3) $f(x) = \dfrac{x+1}{(x-1)^3} \leftarrow \lim\limits_{x \to 1} f(x)$

Solution

$$f(x) = (x+1)\frac{1}{(x-1)^3}$$

$$\lim_{x \to 1} (x+1) = 1+1 = 2 \qquad (1)$$

$$\lim_{x \to 1^-} \frac{1}{(x-1)^3} = (-\infty)(-\infty)(-\infty) = -\infty \qquad (2)$$

$$\lim_{x \to 1^+} \frac{1}{(x-1)^3} = (+\infty)^3 = +\infty \qquad (3)$$

From (1) and (2): $\lim\limits_{x \to 1^-} f(x) = -\infty \qquad (4)$

From (1) and (3): $\lim\limits_{x \to 1^+} f(x) = +\infty \qquad (5)$

From (4) and (5): $\lim\limits_{x \to 1} f(x)$ does not exist.

Convergence Theorems

Def: For a given $\sigma \in \mathbb{R} \cup \{+\infty, -\infty\}$ a set $N(\sigma) \subseteq \mathbb{R}$ is a neighbohrhood of σ if it has the form:

σ	$N(\sigma)$
$\sigma = x_0 \in \mathbb{R}$	$(x_0-\delta, x_0) \cup (x_0, x_0+\delta)$
$\sigma = x_0^+$	$(x_0, x_0+\delta)$
$\sigma = x_0^-$	$(x_0-\delta, x_0)$
$\sigma = +\infty$	$(\alpha, +\infty)$
$\sigma = -\infty$	$(-\infty, \alpha)$

1) $$\boxed{\left.\begin{array}{l} \lim\limits_{x \to \sigma} f(x) = 0 \\ f(x) > 0, \forall x \in N(\sigma) \end{array}\right\} \Rightarrow \lim\limits_{x \to \sigma} \frac{1}{f(x)} = +\infty}$$

2) $$\boxed{\left.\begin{array}{l} \lim\limits_{x \to \sigma} f(x) = 0 \\ f(x) < 0, \forall x \in N(\sigma) \end{array}\right\} \Rightarrow \lim\limits_{x \to \sigma} \frac{1}{f(x)} = -\infty}$$

$\hookrightarrow$ Schematically $\frac{1}{0^+} = +\infty$

$$\frac{1}{0^-} = -\infty$$

③ If $f(x) \leq g(x)$, $\forall x \in N(\sigma)$ and

a) $\lim\limits_{\sigma} f, \lim\limits_{\sigma} g \in \mathbb{R} \Rightarrow \lim\limits_{\sigma} f \leq \lim\limits_{\sigma} g$

b) $\lim\limits_{\sigma} f = +\infty \Rightarrow \lim\limits_{\sigma} g = +\infty$

c) $\lim\limits_{\sigma} g = -\infty \Rightarrow \lim\limits_{\sigma} f = -\infty$

④ Squeeze Theorem

$$\boxed{\left.\begin{array}{l} g_1(x) \leq f(x) \leq g_2(x), \ \forall x \in N(\sigma) \\ \lim\limits_{\sigma} g_1 = \lim\limits_{\sigma} g_2 = L \in \mathbb{R} \end{array}\right\} \Rightarrow \lim\limits_{\sigma} f = L}$$

↳ The squeeze theorem can be easily proved from ③. From the squeeze theorem we can show that

$$\boxed{\left.\begin{array}{l} |f(x)| \leq g(x), \ \forall x \in N(\sigma) \\ \lim\limits_{\sigma} g = 0 \end{array}\right\} \Rightarrow \lim\limits_{\sigma} f = 0}$$

⑤ Composition theorem

$$\boxed{\left.\begin{array}{l} \lim\limits_{x \to \sigma} g(x) = a \in \mathbb{R} \\ \lim\limits_{x \to a} f(x) = f(a) \end{array}\right\} \Rightarrow \lim\limits_{x \to \sigma} f(g(x)) = f\left(\lim\limits_{x \to \sigma} g(x)\right)}$$

Trigonometric Limits

To establish the theory we use the squeeze theorem in conjuction with the following inequalities from trigonometry:

$$\boxed{\begin{array}{l} |\sin x| \leqslant |x| \ , \ \forall x \in \mathbb{R} \\ |x| \leqslant |\tan x| \ , \ \forall x \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\} \end{array}}$$

Fundamental trigonometric limits

1) $\boxed{\lim\limits_{x \to x_0} \sin x = \sin x_0 \ , \ \forall x_0 \in \mathbb{R}}$

Proof

$$|\sin x - \sin x_0| = \left|2 \sin\left(\frac{x - x_0}{2}\right) \cos\left(\frac{x + x_0}{2}\right)\right| =$$

$$= 2 \left|\sin \frac{x - x_0}{2}\right| \left|\cos \frac{x + x_0}{2}\right| \leqslant$$

$$\leqslant 2 \left|\sin \frac{x - x_0}{2}\right| \leqslant 2 \, \frac{|x - x_0|}{2} = |x - x_0|$$

thus $\lim\limits_{x \to x_0} |x - x_0| = 0 \Rightarrow \lim\limits_{x \to x_0} (\sin x - \sin x_0) = 0$

$\Rightarrow \lim\limits_{x \to x_0} \sin x = \sin x_0$ $\square$

2) $\boxed{\lim_{x\to x_0} \cos x = \cos x_0 \quad , \forall x_0 \in \mathbb{R}}$

Proof

$$\lim_{x\to x_0} \cos x = \lim_{x\to x_0} \sin\left(\frac{\pi}{2} - x\right) =$$

$$= \sin\left(\frac{\pi}{2} - x_0\right) = \cos x_0$$

3) $\lim_{x\to x_0} \tan x = \tan x_0 \quad , \forall x_0 \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}$

4) $\lim_{x\to x_0} \cot x = \cot x_0 \quad , \forall x_0 \in \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}$

→ Method: The above can be combined with the composition theorem to get the limits of functions such as $f(x) = \sin(g(x))$.

examples

1) $f(x) = \cos(3x + \pi) \longrightarrow \lim_{x\to 0} f(x)$

In general,

$$\lim_{x\to a} f(x) = b \Rightarrow \begin{cases} \lim_{x\to a} \sin(f(x)) = \sin b \\ \lim_{x\to a} \cos(f(x)) = \cos b \\ \lim_{x\to a} \tan(f(x)) = \tan b \quad \text{if } b \neq k\pi + \pi/2,\ k\in\mathbb{Z}. \end{cases}$$

Review Squeeze theorem.

Trig limits with $x \to \pm\infty$ will either

a) Not exist.
(use method of sequences)

b) Go to 0
(use squeeze theorem and properties of absolute values)

$$|x-y| \leq |x+y| \leq |x|+|y|$$
$$|xy| = |x|\cdot|y|, \quad |x/y| = |x|/|y|$$

$$|\sin x| \leq 1 \text{ and } |\cos x| \leq 1.$$

examples

1) $\lim_{x\to+\infty} \sin x \cos x$

Choose

$$a_n = 2n\pi \Rightarrow f(a_n) = \sin(2n\pi)\cos(2n\pi)$$
$$= 0 \cdot 1 = 0$$

$$b_n = 2n\pi + \pi/4 \Rightarrow f(b_n) = \sin(2n\pi + \pi/4)\cos(2n\pi + \pi/4)$$
$$= \sin(\pi/4)\cos(\pi/4)$$
$$= \frac{\sqrt{2}}{2}\,\frac{\sqrt{2}}{2} = \frac{1}{2}$$

So

$$\left.\begin{array}{l} \lim a_n = \lim b_n = +\infty \\ \lim f(a_n) = 0 \\ \lim f(b_n) = 1/2 \end{array}\right\} \Rightarrow \lim_{x\to+\infty} f(x) \text{ does not exist.}$$

2) $$\lim_{x\to+\infty} \frac{\sin x\cos x - 3\sin x}{2x+1}$$

Let $f(x) = \dfrac{\sin x\cos x - 3\sin x}{2x+1}$

▶ Establish an inequality $|f(x)| \leqslant g(x)$ and then show that $\lim\limits_{x\to+\infty} g(x) = 0$. ◀

$$|f(x)| = \left|\frac{\sin x\cos x - 3\sin x}{2x+1}\right| = \frac{|\sin x\cos x - 3\sin x|}{|2x+1|} \leqslant$$
$$\leqslant \frac{|\sin x\cos x| + 3|\sin x|}{|2x+1|} =$$
$$= \frac{|\sin x||\cos x| + 3|\sin x|}{|2x+1|} \leqslant \frac{1\cdot 1 + 3\cdot 1}{|2x+1|}$$

$$= \frac{4}{|2x+1|} = g(x) \quad (1)$$

$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} \frac{4}{|2x+1|} = \lim_{x\to+\infty} \frac{4}{2x+1} =$$

$$= \lim_{x\to+\infty} \frac{4}{2x} = 0 \quad (2)$$

From (1) and (2): $\lim\limits_{x\to+\infty} f(x) = 0$.

→ A similar situation may arise when $x \to a$.

3) $\lim\limits_{x\to 3} (x-3)^2 \cos\left[\frac{1}{(x-3)}\right]$

▶ You can prove that $\lim\limits_{x\to 3} \cos\left[\frac{1}{x-3}\right]$ does not exist.

Define $f(x) = (x-3)^2 \cos\left[\frac{1}{(x-3)}\right]$

$$|f(x)| = (x-3)^2 \left|\cos\left[\frac{1}{(x-3)}\right]\right| \leq$$

$$\left.\begin{aligned} &\leq (x-3)^2 \cdot 1 = g(x) \\ &\lim_{x\to 3} g(x) = (3-3)^2 = 0 \end{aligned}\right\} \Rightarrow \lim_{x\to 3} f(x) = 0.$$

Sometimes you must be careful with the inequalities

4) $f(x) = x[1 - \sin(1/x)] \leftarrow \lim_{x \to 0} f(x)$.

$$|f(x)| = |x[1-\sin(1/x)]| = |x| \cdot |1 - \sin(1/x)|$$

$$\leq |x| \cdot (1 + |\sin(1/x)|) \leq |x| \cdot (1+1)$$

$$= 2|x| = g(x)$$

$$\left.\begin{array}{l} \\ \lim_{x \to 0} g(x) = \lim_{x \to 0} 2|x| = 0 \end{array}\right\} \Rightarrow \lim_{x \to 0} f(x) = 0.$$

A/0 Forms with trig limits

You must use directly the convergence theorems (1) + (2)

$\to$ $1/0^+ = +\infty$, $1/0^- = -\infty$.

Review the $1/0^{\pm}$ theorems.

examples

1) $\lim\limits_{x \to 0} \dfrac{1}{x \sin x}$?

For $f(x) = x\sin x$ we have $\lim\limits_{x\to 0} f(x) = \lim\limits_{x \to 0} (x \sin x) =$

$= 0 \sin 0 = 0$ (1)

and

$\forall x \in \underbrace{(-\pi/6, 0) \cup (0, \pi/6)}_{}$: $x \sin x > 0$ (2)

a neighborhood of $x \to 0$.

From (1) and (2) $\lim\limits_{x\to 0} \dfrac{1}{x \sin x} = +\infty$.

2) $\lim\limits_{x \to \pi/2} \tan x$?

Since $f(x) = \tan x = \dfrac{\sin x}{\cos x}$, let $g(x) = \cos x$.

$\lim\limits_{x \to \pi/2} g(x) = \lim\limits_{x \to \pi/2} \cos x = 0$ (1)

$\cos x > 0 \ , \ \forall x \in (\pi/2 - \pi/10, \pi/2)$ (2)

$\cos x < 0 \ , \ \forall x \in (\pi/2, \pi/2 + \pi/10)$ (3)

$\lim_{x \to \pi/2} \sin x = 1$ (4)

$(1)+(2) \Rightarrow \lim\limits_{x \to \pi/2^-} \dfrac{1}{\cos x} = +\infty \Rightarrow \lim\limits_{x \to \pi/2^-} \tan x = +\infty$ (5)

$(1)+(3) \Rightarrow \lim\limits_{x \to \pi/2^+} \dfrac{1}{\cos x} = -\infty \Rightarrow \lim\limits_{x \to \pi/2^+} \tan x = -\infty$ (6)

(4)

$(5)+(6) \Rightarrow \lim\limits_{x \to \pi/2} \tan x$ does not exist.

Application : $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$

Proof

Calculate the side limit $x \to 0^+$.
For $x > 0$ we have

$$f(x) = \frac{\sin x}{x} = \frac{|\sin x|}{|x|} \leqslant \frac{|x|}{|x|} = 1, \ \forall x \in (0, +\infty)$$

$$f(x) = \frac{\sin x}{x} \geqslant \frac{\sin x}{\tan x} \quad (\text{because } \tan x \geqslant x)$$

$$= \sin x \frac{\cos x}{\sin x} = \cos x.$$

Thus:

$$\left.\begin{array}{l} \forall x \in (0, +\infty): \ \cos x \leqslant f(x) \leqslant 1 \\ \lim_{x\to 0^+} \cos x = 1 \end{array}\right\} \Rightarrow \underline{\lim_{x \to 0^+} f(x) = 1.} \qquad (1)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\sin(-x)}{-x} =$$

$$= \lim_{x\to 0^+} \frac{\sin x}{x} \underset{(1)}{=} 1 \Rightarrow \lim_{x\to 0} f(x) = 1 \quad \square$$

$\mapsto$ Application : $\boxed{\lim_{x\to 0} \frac{\tan x}{x} = 1}$

Proof

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \left[\frac{\sin x}{x} \frac{1}{\cos x} \right]$$
$$= \lim_{x\to 0} \left(\frac{\sin x}{x} \right) \lim_{x\to 0} \frac{1}{\cos x}$$
$$= 1 \cdot \frac{1}{\cos 0} = 1.$$

examples

$$\lim_{x\to 0} \frac{\sin 2x}{\sin 5x}$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \dots = \frac{1}{2} \leftarrow \text{Use } \sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$$

$$\lim_{x\to \pm\infty} \frac{\sin x}{x} = 0$$

$$\lim_{x\to 2} \frac{\sin(x-2)}{x^2-3x+2}$$

Continuity

Def: Let $f: A \to \mathbb{R}$ be a function.

a) f continuous at $x_0 \in A \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)$

b) f continuous at $I \subseteq A \Leftrightarrow \forall x_0 \in I$: f continuous at x_0

↳ The set of all functions continuous at I is denoted as $C_0(I)$.

↦ **Properties/Continuity of basic functions**

1) f polynomial $\Rightarrow$ f continuous at $\mathbb{R}$.
2) If $f(x) = P(x)/Q(x)$ with P, Q polynomials then
   $Q(x_0) \neq 0 \Rightarrow f$ continuous at x_0
3) $\sin, \cos$ continuous at $\mathbb{R}$
   $\tan$ continuous at $\mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}$
   $\cot$ continuous at $\mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}$.

Continuity and operations

Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ with
f continuous at $x_0 \in A$
g continuous at $x_0 \in A$
and let $\lambda \in \mathbb{R}$.

a) $$\left.\begin{aligned} h_1(x) &= f(x) + g(x) \\ h_2(x) &= f(x)\, g(x) \\ h_3(x) &= \lambda f(x) \end{aligned}\right\} \text{ continuous at } x_0$$

b) $g(x_0) \neq 0 \Rightarrow h(x) = f(x)/g(x)$ continuous at x_0.

Remark: There are three possible ways for a function to fail continuity.

a) $f(x_0)$ does not exist

b) $\lim_{x \to x_0} f(x)$ does not exist

c) The numbers
$$a_1 = \lim_{x \to x_0^+} f(x)$$
$$a_2 = \lim_{x \to x_0^-} f(x)$$
$$a_3 = f(x_0)$$

do not agree with each other.

example

For what values of a is

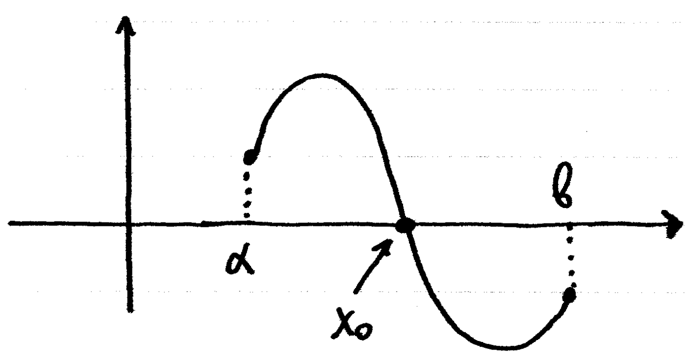
$$f(x)=\begin{cases} 2x+3 & , x\in(-\infty,1] \\ ax^2-a^2x+7 & , x\in(1,+\infty) \end{cases}$$

continuous at $\mathbb{R}$?

Consequences of continuity

1) Bolzano-Weierstrass theorem

$$\boxed{\left.\begin{array}{l} f \text{ continuous at } [a,b] \\ f(a)\cdot f(b)<0 \end{array}\right\} \Rightarrow \exists\, x_0\in(a,b):\ f(x_0)=0}$$



2) Intermediate Value Theorem

$$\left.\begin{array}{l} f \text{ continuous at } [a,b] \\ f(a) < f(b) \end{array}\right\} \Rightarrow [f(a), f(b)] \subseteq f([a,b])$$

$$\left.\begin{array}{l} f \text{ continuous at } [a,b] \\ f(a) > f(b) \end{array}\right\} \Rightarrow [f(b), f(a)] \subseteq f([a,b])$$

▸ <u>interpretation:</u> If f is continuous at $[a,b]$ then f takes all the values between $f(a)$ and $f(b)$.

<u>examples</u>

1) Show that $\sin(\cos 3x) = 0$ has a solution in $(0, \pi)$

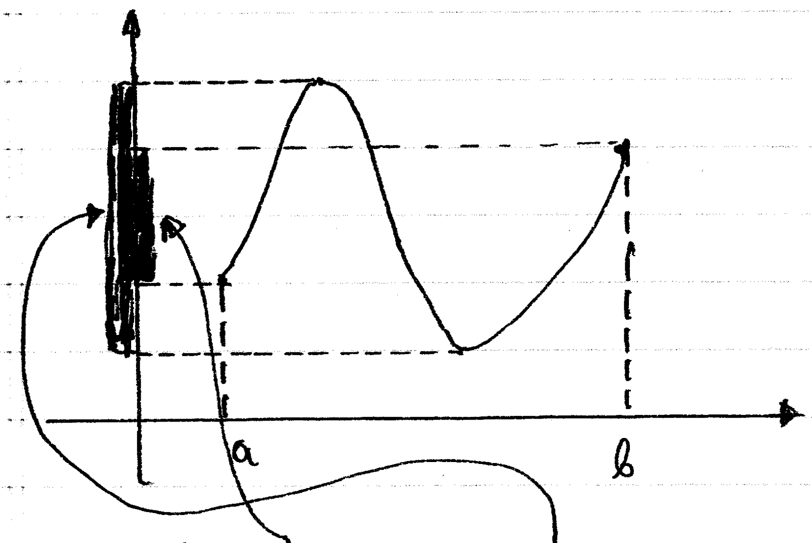
2) Show that
$$\frac{x^2+1}{x-a} + \frac{x^6+1}{x-b} = 0$$
with $a<b$ has a solution in (a,b).

(3) Extremum Value Theorem

$$\left.\begin{array}{l} f \text{ continuous} \\ \text{at } [a,b] \end{array}\right\} \Rightarrow \exists\, \xi_1, \xi_2 \in [a,b] : \forall x \in [a,b] : f(\xi_1) \leq f(x) \leq f(\xi_2)$$

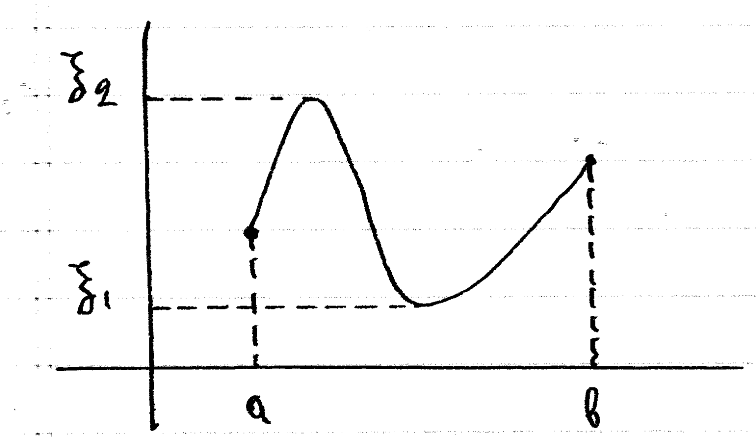
Visualizations (IVP and EVT)

Intermediate Value Theorem



$$\boxed{[f(a), f(b)] \subseteq f([a,b])}$$

Extremum Value Theorem



$$\boxed{\exists\, \xi_1, \xi_2 \in [a,b] : \forall x \in [a,b] : f(\xi_1) \leq f(x) \leq f(\xi_2)}$$