Sets and Mappings

- A **set** is a collection of **elements**, (usually numbers or points).

notation

1) $x \in A$: x belongs to set A
2) $x \notin A$: x does not belong to set A.
3) $A = \{1, 2, 3, 4\}$: the set with elements 1,2,3,4.
4) $A = \{x \in B \mid p(x)\}$: all the elements of B that also satisfy the statement $p(x)$.

number sets

1) Natural numbers : $\mathbb{N} = \{0, 1, 2, 3, \dots\}$
2) Integers : $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$
   (German: Zahl)
3) Rational numbers : $\mathbb{Q}$.
   A number $x \in \mathbb{Q}$ if and only if there are two integers $a, b \in \mathbb{Z}$ such that $x = a/b$.
   Symbolically:

$$\boxed{x \in \mathbb{Q} \Leftrightarrow \exists\, a, b \in \mathbb{Z} : x = a/b}$$

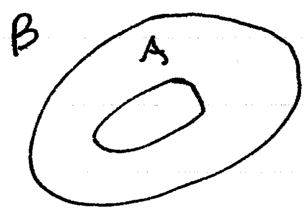
$\exists x : p(x) \longrightarrow$ there is a x such that $p(x)$ true

4) Real Numbers : $\mathbb{R}$.

- Any number that can be appoximated as a sequence of rational numbers.

Interrals

A set A is a subset of B (notation: $A \subseteq B$) if all the elements of A belong to B



"$\wedge$" = and

$x \in [a,b] \iff x \in \mathbb{R} \wedge a \leq x \leq b$

$x \in (a,b) \iff x \in \mathbb{R} \wedge a < x < b$

$x \in (a,b] \iff x \in \mathbb{R} \wedge a < x \leq b$

$x \in [a,b) \iff x \in \mathbb{R} \wedge a \leq x < b$

$x \in [a,+\infty) \iff x \in \mathbb{R} \wedge a \leq x$

$x \in (a,+\infty) \iff x \in \mathbb{R} \wedge a < x$

$x \in (-\infty,a) \iff x \in \mathbb{R} \wedge x < a$

$x \in (-\infty,a] \iff x \in \mathbb{R} \wedge x \leq a$

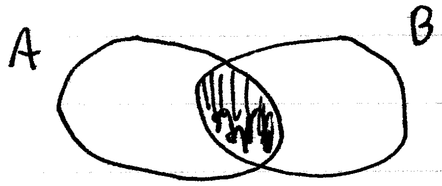
Set operations

Let A, B be two sets

$\wedge$ = and
$\vee$ = or

1) Intersection

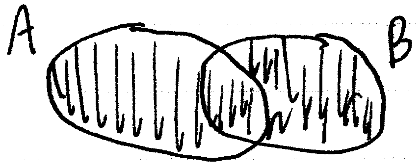
$x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$



Elements A, B have in common

2) Union

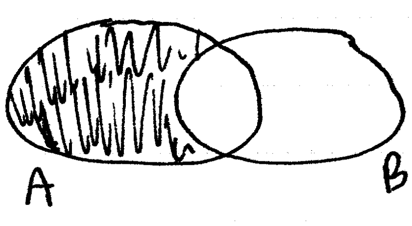
$x \in A \cup B \Leftrightarrow x \in A \vee x \in B$



Elements that belong to A or B (or both)

3) Difference

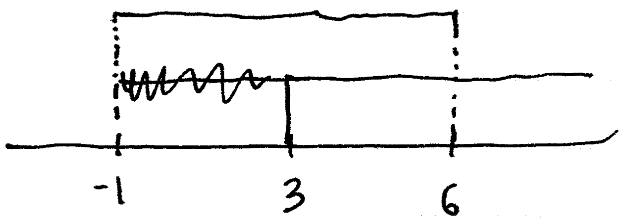
$x \in A - B \Leftrightarrow x \in A \wedge x \notin B$



Elements of A that do not belong to B.

example : Convert intersection of intervals to union of intervals

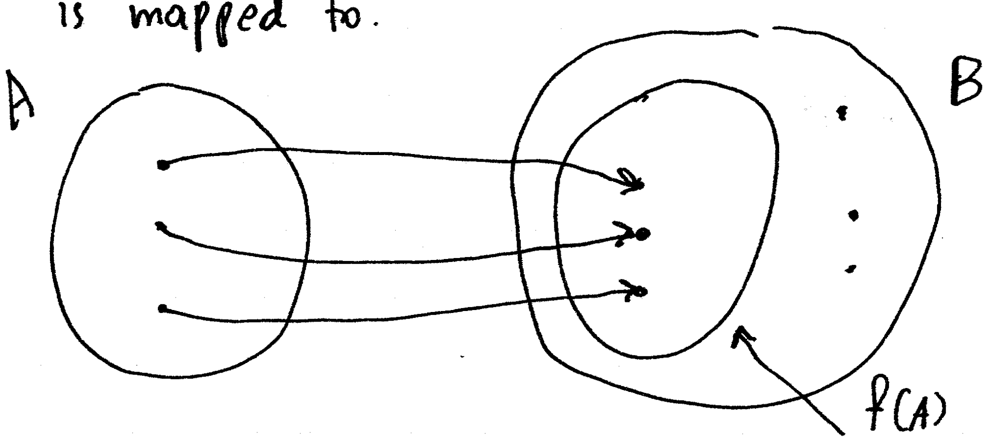
$[3,+\infty) \cap (-1,6) = [3,6)$



$[2,4) \cap (0,1) = \emptyset \leftarrow$ empty set $\rightarrow$ it has no elements

Mappings / Functions / Sequences

- A mapping $f : A \rightarrow B$ is a rule that maps every element of A to a unique element of B.
  A = domain of f
  B = destination set of f.
  For $x \in A$, then $f(x) \in B$ is the element that x is mapped to.



- $f(A)$ is the set of all elements of B for which some $x \in A$ maps to these elements:

$$\boxed{y \in f(A) \Leftrightarrow \exists\, x \in A : f(x) = y}$$

$f(A)$ = range of f.

- A <u>function</u> ~~is~~ f is a mapping

$$f : A \to \mathbb{R} \text{ with } A \subseteq \mathbb{R}$$

- A <u>sequence</u> is a mapping $f : \mathbb{N} \longrightarrow \mathbb{R}$.

Defining a function

To define a function f you must give
1) The rule for $f(x)$
2) The domain A_f of f.

<u>example</u> : Let f be the function $f(x) = 3x+2$ with domain $A = [5, +\infty)$.

- <u>Default Domain</u> : If the domain is not given then we assume the default domain to be the <u>largest possible subset of $\mathbb{R}$.</u>

Cases:

1) Polynomial function

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x$

Default domain: $A = \mathbb{R}$.

2) Rational function

$f(x) = \dfrac{P(x)}{Q(x)}$ with P, Q polynomial functions

Default domain: All of $\mathbb{R}$ except for the numbers that zero the denominator.

$A = \mathbb{R} - \{ x \in \mathbb{R} \mid Q(x) = 0 \}$

3) Root function

$f(x) = \sqrt{g(x)}$

Default domain: The expression under the root must be positive

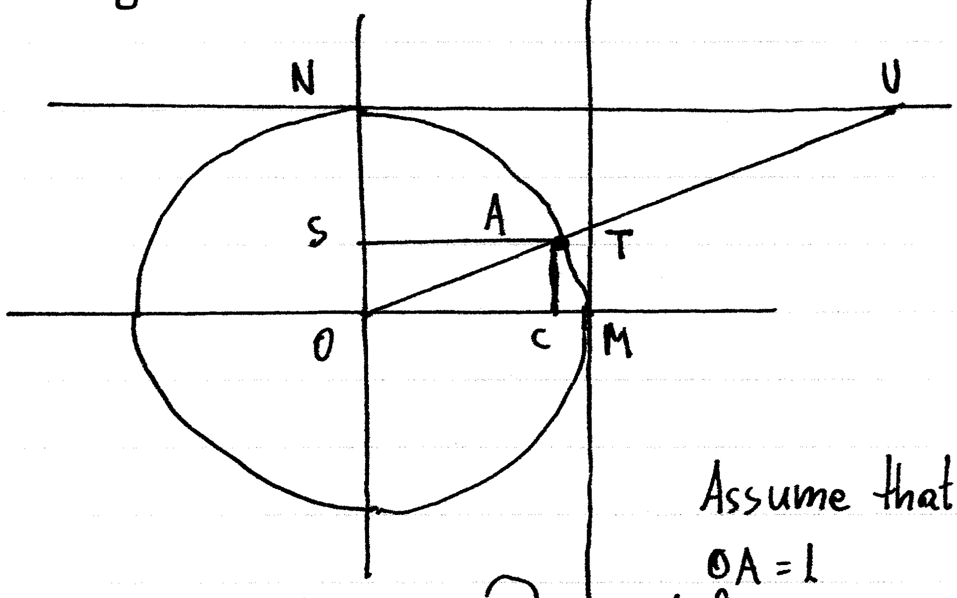
$A = A_g \cap \{ x \in \mathbb{R} \mid g(x) \geq 0 \}$

e.g. $f(x) = 3x + 5$

$f(x) = \dfrac{3x}{(x+1)(x-1)}$

$f(x) = \sqrt{x} + \sqrt{1-x}$

Trigonometric Functions



Assume that the radius $OA = 1$

For the angle $\varphi = \overset{\frown}{MA}$ we define

$\sin\varphi = \overline{OS}$ $\quad$ $\tan\varphi = \overline{MT}$

$\cos\varphi = \overline{OC}$ $\quad$ $\cot\varphi = \overline{NU}$

Default domains

1) $f(x) = \sin[g(x)] \longrightarrow A_f = A_g$

2) $f(x) = \cos[g(x)] \longrightarrow A_f = A_g$

3) $f(x) = \tan[g(x)] \longrightarrow A_f = \{x \in A_g : g(x) \neq k\pi + \pi/2\}$

4) $f(x) = \cot[g(x)] \longrightarrow A_f = \{x \in A_g : g(x) \neq k\pi\}$

example

$f(x) = \tan(2x+1)$

Need $2x+1 \neq k\pi + \pi/2$.

Solve $2x+1 = k\pi + \pi/2 \Leftrightarrow 2x = k\pi + \pi/2 - 1$

$$\Leftrightarrow x = \frac{k\pi}{2} + \frac{\pi}{4} - \frac{1}{2}$$

Thus $A_f = \mathbb{R} - \{k\pi/2 + \pi/4 - 1/2 \mid k \in \mathbb{Z}\}$.

CAUTION: Simplifying or manipulating your function can change the domain!

examples

1) $f(x) = \sqrt{x(x+1)} \longleftarrow A_f = (-\infty, -1] \cup [0, +\infty)$

$f(x) = \sqrt{x}\sqrt{x+1} \longleftarrow A_f = [0, +\infty)$

2) $f(x) = \dfrac{x^2+2x+1}{x+1} \longleftarrow A_f = \mathbb{R} - \{-1\}$

$f(x) = x+1 \longleftarrow A_f = \mathbb{R}$.

Function operations

Let $f: A_f \to \mathbb{R}$ and $g: A_g \longrightarrow \mathbb{R}$ be two functions.

a) $h = f+g \Leftrightarrow \begin{cases} A_h = A_f \cap A_g \\ \forall x \in A_h : h(x) = f(x) + g(x) \end{cases}$

b) $h = fg \Leftrightarrow \begin{cases} A_h = A_f \cap A_g \\ \forall x \in A_h : h(x) = f(x)g(x) \end{cases}$

c) $h = f/g \Leftrightarrow \begin{cases} A_h = (A_f \cap A_g) - \{x \in A_g : g(x) = 0\} \\ \forall x \in A_h : h(x) = f(x)/g(x). \end{cases}$

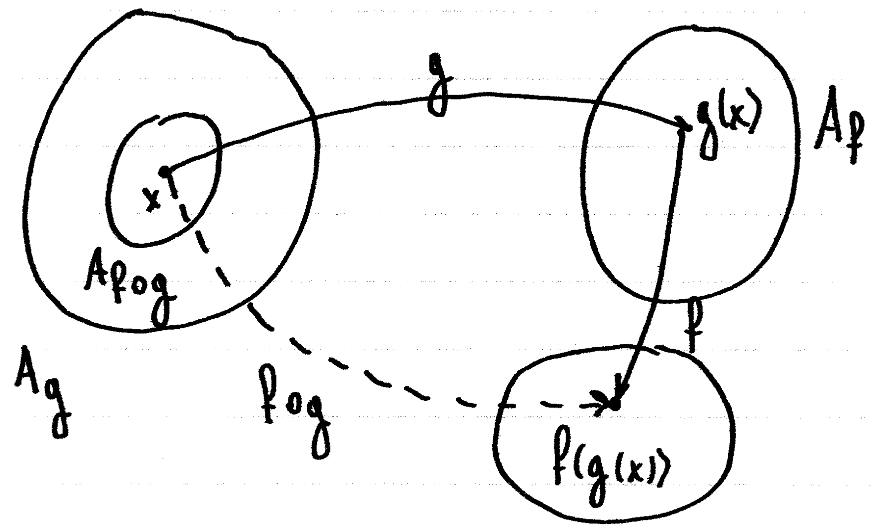
d) $h = \lambda f, \lambda \in \mathbb{R} \Leftrightarrow \begin{cases} A_h = A_f \\ \forall x \in A_h : h(x) = \lambda f(x). \end{cases}$

example

1) $\left.\begin{array}{l} f(x) = \sqrt{x+5} \\ g(x) = 3x+1 \end{array}\right\} \Rightarrow f/g ?$

Function composition

Let $f: A_f \to \mathbb{R}$ and $g: A_g \to \mathbb{R}$.



We define $h = f\circ g$ as follows:

$$h = f\circ g \Leftrightarrow \begin{cases} A_{f\circ g} = \{x \in A_g \mid g(x) \in A_f\} \\ (f\circ g)(x) = f(g(x)),\ \forall x \in A_{f\circ g} \end{cases}$$

- $A_{f\circ g}$ is defined such that $g(A_{f\circ g}) \subseteq A_f$.
- To calculate $f\circ g$:
  - 1 Solve the system $\begin{cases} x \in A_g \\ g(x) \in A_f \end{cases} \Leftrightarrow \dots \Leftrightarrow x \in A_{f\circ g}$.
  - 2 Calculate $(f\circ g)(x) = f(g(x))$.

examples

1) $f(x) = 2x+1$, $g(x) = x^2+2$ $\rightarrow$ $f\circ g$, $g\circ f$.

2) $f(x) = x^2+1$, $g(x) = \sqrt{2x-1}$ $\rightarrow$ $f\circ g$ $\leftarrow$ Be careful with domain!

Absolute Value

Def: $$\boxed{|x| = \begin{cases} x &, x \geq 0 \\ -x &, x < 0 \end{cases}}$$

$|a-b|$ = distance between a, b on real line

CAUTION: $$\boxed{\begin{aligned} (\sqrt{x})^2 &= x \\ \sqrt{x^2} &= |x| \end{aligned}}$$

Properties

1) $|xy| = |x| \cdot |y|$
2) $|x+y| \leq |x| + |y|$
3) $|x|^{2k} = x^{2k} \ , \ \forall k \in \mathbb{N}$
4) $|x|^{2k+1} = |x^{2k+1}| \ , \ \forall k \in \mathbb{N}$
5) $|x| = |-x|$
6) $|x| = a \geq 0 \Leftrightarrow x = \pm a$
7) $|x| < a \Leftrightarrow -a < x < a \Leftrightarrow x \in (-a, a)$
8) $|x| > a \Leftrightarrow x \leq -a \vee x \geq a \Leftrightarrow x \in (-\infty, -a) \cup (a, +\infty)$
9) $|\sin x| \leq 1 \ , \ \forall x \in \mathbb{R}$
   $|\cos x| \leq 1 \ , \ \forall x \in \mathbb{R}$.