VECTORS IN $\mathbb{R}^3$

Cartesian product

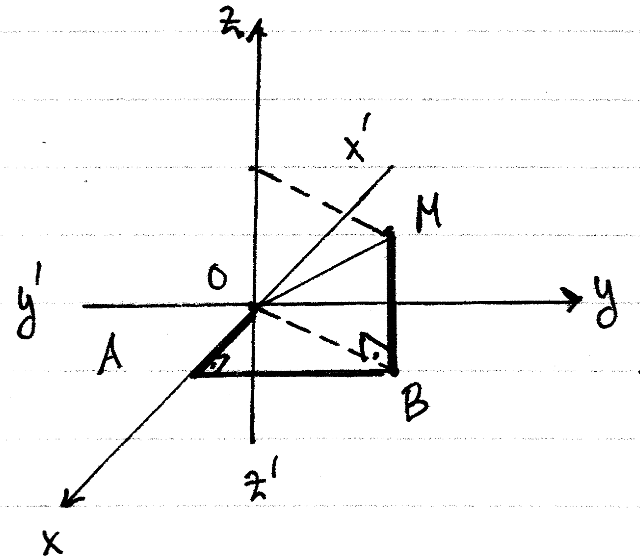
Let A, B, C be three sets. We define the Cartesian products

$$A \times B = \{(a,b) \mid a \in A \land b \in B\}$$
$$A \times B \times C = \{(a,b,c) \mid a \in A \land b \in B \land c \in C\}$$

where (a,b) is an ordered pair and (a,b,c) is an ordered triplet

- The behavior of ordered pairs and ordered triplets is covered by the following axioms:
$$(a_1,a_2) = (b_1,b_2) \Leftrightarrow a_1 = b_1 \land a_2 = b_2$$
$$(a_1,a_2,a_3) = (b_1,b_2,b_3) \Leftrightarrow a_1 = b_1 \land a_2 = b_2 \land a_3 = b_3$$
- Geometrical three-dimensional space can be represented as
$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) \mid x \in \mathbb{R} \land y \in \mathbb{R} \land z \in \mathbb{R}\}$$
with $\mathbb{R}$, the set of all real numbers.
An element $(x,y,z) \in \mathbb{R}^3$ represents a point in space with cartesian coordinates x, y, z, as defined below.
- Likewise, geometrical two-dimensional space can be represented as
$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R} \land y \in \mathbb{R}\}$$
An element $(x,y) \in \mathbb{R}^2$ represents a point on a plane with coordinates (x,y).

Cartesian coordinates



The cartesian coordinate system consists of three lines $(x'x)$, $(y'y)$, $(z'z)$ such that

$$\begin{cases} (x'x) \perp (y'y) \perp (z'z) \perp (x'x) \\ (x'x) \cap (y'y) \cap (z'z) = \{O\} \end{cases}$$

where O is the origin of the coordinate system.

Let $M(x, y, z)$ be a point with coordinates (x, y, z).
We define x, y, z as follows:
Let B be the projection of M to the xy plane.
Let A be the projection of B to the $x'x$ axis.
Then, we define:

$$\begin{cases} x = \overline{OA} \\ y = \overline{AB} \\ z = \overline{BM} \end{cases}$$

The bar indicates using directional distance. For example, $\overline{OA}$ is positive or negative depending on whether A is on the Ox or the Ox' ray.

<u>Terminology</u>: x-axis is the line $x'Ox$
y-axis is the line $y'Oy$
z-axis is the line $z'Oz$

Likewise:

xy plane: plane defined by $(x'x)$ and $(y'y)$
yz plane: plane defined by $(y'y)$ and $(z'z)$
zx plane: plane defined by $(z'z)$ and $(x'x)$

Distance Formula

① $\boxed{M(x,y,z) \Rightarrow OM = \sqrt{x^2+y^2+z^2}}$

Proof

$OA \perp AB \Rightarrow O\hat{A}B$ right triangle with $\hat{A}=90°$

$\Rightarrow OB^2 = OA^2 + AB^2 = x^2+y^2$

$OB \perp BM \Rightarrow O\hat{B}M$ right triangle with $\hat{B}=90°$

$\Rightarrow OM^2 = OB^2 + BM^2 = OB^2 + z^2 =$

$= (x^2+y^2) + z^2$

$\Rightarrow OM = \sqrt{x^2+y^2+z^2}$ □

② $\boxed{\left.\begin{array}{l} A(x_1,y_1,z_1) \\ B(x_2,y_2,z_2) \end{array}\right\} \Rightarrow AB = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}$

Proof

Using A as the origin of a coordinates system where the axises are parallel and similarly oriented with the coordinate system around O, the coordinates of B are $(x_2-x_1, y_2-y_1, z_2-z_1)$. Using the previous result we immediately calculate AB □.

↳ The <u>sphere (A,r)</u> with <u>center A</u> and <u>radius r</u> is defined as:

$(A,r) = \{M \in \mathbb{R}^3 \mid AM = r\}$

and therefore:

$(A,r) : (x-x_A)^2 + (y-y_A)^2 + (z-z_A)^2 = r^2$

with $A(x_A, y_A, z_A)$.

EXAMPLES

Find all $\lambda \in \mathbb{R}$ such that $AB = a$ with $A(1, \lambda+1, \lambda-1)$ and $B(\lambda, \lambda+1, -1)$

Solution

Since,

$$AB^2 = (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 =$$
$$= (1-\lambda)^2 + [(\lambda+1) - (\lambda+1)]^2 + [(\lambda-1) - (-1)]^2 =$$
$$= (1-\lambda)^2 + 0^2 + (\lambda - 1 + 1)^2 = (1-\lambda)^2 + \lambda^2 =$$
$$= 1 - 2\lambda + \lambda^2 + \lambda^2 = 2\lambda^2 - 2\lambda + 1 \Rightarrow$$
$$\Rightarrow AB = \sqrt{2\lambda^2 - 2\lambda + 1}$$

it follows that

$$AB = a \Leftrightarrow \sqrt{2\lambda^2 - 2\lambda + 1} = a \qquad [\text{Require } a \geq 0]$$
$$\Leftrightarrow 2\lambda^2 - 2\lambda + 1 = a^2 \Leftrightarrow 2\lambda^2 - 2\lambda + 1 - a^2 = 0 \qquad (1)$$

The corresponding discriminant is:

$$\Delta = (-2)^2 - 4 \cdot 2 \cdot (1 - a^2) = 4 - 8(1 - a^2) = 4 - 8 + 8a^2$$
$$= 8a^2 - 4 = 4(2a^2 - 1) = 4(\sqrt{2}a - 1)(\sqrt{2}a + 1)$$

and note the sign of Δ; where we require $a \geq 0$

a		$-1/\sqrt{2}$		0		$1/\sqrt{2}$	
$\sqrt{2}a - 1$	$-$		$-$		$-$	0	$+$
$\sqrt{2}a + 1$	$-$	0	$+$		$+$		$+$
Δ		0			$-$	0	$+$

We distinguish between the following cases

Case 1 : Assume that $a \in (0, 1/\sqrt{2})$. Then $\Delta < 0$, and therefore Eq. (1) has no real solutions.

Case 2: Assume that $a = 1/\sqrt{2}$. Then $\Delta = 0$, and therefore Eq. (1) has one real solution:

$\lambda = -(-2)/(2 \cdot 2) = 1/2$

Case 3 : Assume that $a \in (1/\sqrt{2}, +\infty)$. Then $\Delta > 0$, and therefore Eq. (1) has two real solutions

$$\lambda_{1,2} = \frac{-(-2) \pm \sqrt{4(2a^2-1)}}{2 \cdot 2} = \frac{2 \pm 2\sqrt{2a^2-1}}{2 \cdot 2}$$

$$= \frac{1 \pm \sqrt{2a^2-1}}{2}$$

We conclude that

$AB = a \Leftrightarrow \lambda \in S$

with

$$S = \begin{cases} \{[1+\sqrt{2a^2-1}\,]/2, [1-\sqrt{2a^2-1}\,]/2\} & \text{, if } a \in (1/\sqrt{2}, +\infty) \\ \{1/2\} & \text{, if } a = 1/\sqrt{2} \\ \emptyset & \text{, if } a \in (0, 1/\sqrt{2}) \end{cases}$$

8) Find the center and radius of the sphere

$(c): x^2+y^2+z^2-4x+2y+6z=11.$

<u>Solution</u>

Since

$$(c): x^2+y^2+z^2-4x+2y+6z=11 \Leftrightarrow$$

$$\Leftrightarrow (x^2-4x+4)+(y^2+2y+1)+(z^2+6z+9)=11+4+1+9$$

$$\Leftrightarrow (x-2)^2+(y+1)^2+(z+3)^2=25$$

$$\Leftrightarrow (x-2)^2+(y-(-1))^2+(z-(-3))^2=5^2$$

it follows that (c) has center $A(2,-1,-3)$ and radius $r=5$.

EXERCISES

① Find the distance AB between the points A and B with coordinates:

a) $A(\sqrt{2}+\sqrt{3}, 2, \sqrt{2})$ and $B(\sqrt{2}-\sqrt{3}, 3, 1)$

b) $A(0, \sqrt{3}, \sqrt{5})$ and $B(2, 1+\sqrt{3}, \sqrt{2})$

c) $A(a, b, a)$ and $B(-b, a, b)$

d) $A(ab, b^2, bc)$ and $B(ac, bc, c^2)$

e) $A(at, bt, ct)$ and $B(bt, ct, at)$

② Find all $\lambda \in \mathbb{R}$ such that for $A(\lambda, \lambda+1, \lambda+2)$ and $B(\lambda-1, 2\lambda, 2)$ satisfy $AB = 1$.

③ Let $\hat{ABC}$ be a triangle with $A(a,b,c)$, $B(b,c,a)$, and $C(c,a,b)$. Show that $\hat{ABC}$ is an equilateral triangle.

④ Write the equation of a sphere with center C and radius r given by:

a) $C(1,2,3)$ and $r=5$

b) $C(-2,-3,-6)$ and $r=\sqrt{5}$

c) $C(\pi, e, \sqrt{2})$ and $r=\sqrt{\pi}$

⑤ Find the radius and center for the following spheres:

a) $(C): x^2+y^2+z^2-12x+14y-8z+1=0$

b) $(C): x^2+y^2+z^2+2x-6y-10z+34=0$

c) $(C): 4x^2+4y^2+4z^2-4x+8y+16z-13=0$

d) $(C): x^2+y^2+z^2+8x-4y-22z+77=0$

6. Find the equation of the sphere with center C and passing through the point A with

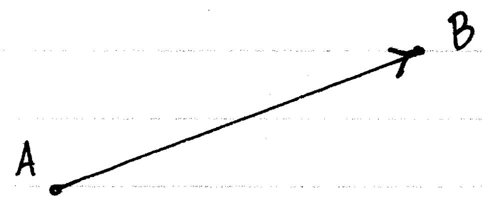
a) $C(1,2,1)$ and $A(3,2,-1)$

b) $C(2,0,3)$ and $A(-1,-1,-2)$

c) $C(0,1,-1)$ and $A(2,2,3)$

Geometric vectors

- Let A, B be two points. The vector $\vec{AB}$ is a segment AB in which we define a direction from A to B.
  We say that: A is the initial point of $\vec{AB}$
  B is the terminal point of $\vec{AB}$



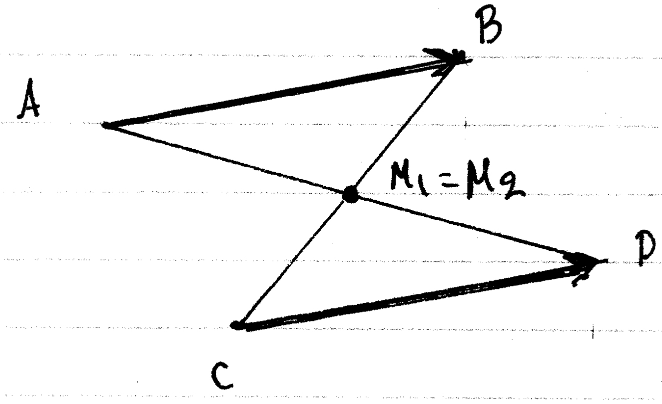
Geometric vector equivalence

Given the vectors $\vec{AB}$ and $\vec{CD}$, we say that $\vec{AB} = \vec{CD}$ if and only if the vectors $\vec{AB}$ and $\vec{CD}$ have the same length, are parallel to each other and have the same direction.

More rigorously, we give the following definition:

Def: Let $\vec{AB}$ and $\vec{CD}$ be two vectors and let
M_1 = midpoint of AD
M_2 = midpoint of BC
Then:
$$\vec{AB} = \vec{CD} \iff M_1 = M_2$$

which is illustrated in the following figure:



Recall the definition of the midpoint:

$$\boxed{M = \text{midpoint of } AB \Leftrightarrow \begin{cases} x_M = (1/2)(x_A + x_B) \\ y_M = (1/2)(y_A + y_B) \\ z_M = (1/2)(z_A + z_B) \end{cases}}$$

▸ Coordinate representation of geometric vectors.

Choose a coordinate system and assume that $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ with respect to the chosen coordinate system. We define the coordinate representation of the vector $\vec{AB}$ as

$$\vec{AB} = (x_B - x_A, y_B - y_A, z_B - z_A)$$

Note that this representation is dependent on our choice of coordinate system.

It follows that

$$\vec{AA} = (0, 0, 0) = \mathbf{0}$$

with $\mathbf{0}$ the zero vector.

We will now show that equivalent geometric vectors have the same coordinate representation.

Prop: Let $\vec{AB}$ and $\vec{CD}$ be two vectors with
$\vec{AB} = (a_1, a_2, a_3)$ and $\vec{CD} = (b_1, b_2, b_3)$

Then:

$$\vec{AB} = \vec{CD} \Leftrightarrow a_1 = b_1 \land a_2 = b_2 \land a_3 = b_3$$
$$\Leftrightarrow (a_1, a_2, a_3) = (b_1, b_2, b_3)$$

Proof

We note that

$$\vec{AB} = (a_1, a_2, a_3) \Rightarrow \begin{cases} a_1 = x_B - x_A \\ a_2 = y_B - y_A \\ a_3 = z_B - z_A \end{cases}$$

$$\vec{CD} = (b_1, b_2, b_3) \Rightarrow \begin{cases} b_1 = x_D - x_C \\ b_2 = y_D - y_C \\ b_3 = z_D - z_C \end{cases}$$

Let M be the midpoint of AD and let N be the midpoint of BC. Then:

$$\begin{cases} x_M = (1/2)(x_A + x_D) \\ y_M = (1/2)(y_A + y_D) \\ z_M = (1/2)(z_A + z_D) \end{cases} \land \begin{cases} x_N = (1/2)(x_B + x_C) \\ y_N = (1/2)(y_B + y_C) \\ z_N = (1/2)(z_B + z_C) \end{cases}$$

It follows that

$$a_1 = b_1 \Leftrightarrow x_B - x_A = x_D - x_C \Leftrightarrow x_A + x_D = x_B + x_C \Leftrightarrow$$
$$\Leftrightarrow (1/2)(x_A + x_D) = (1/2)(x_B + x_C) \Leftrightarrow$$
$$\Leftrightarrow x_M = x_N$$

and similarly, we have:

$a_2 = b_2 \Leftrightarrow y_M = y_N$

$a_3 = b_3 \Leftrightarrow z_M = z_N$

It follows that

$$\vec{AB} = \vec{CD} \Leftrightarrow M = N \Leftrightarrow$$
$$\Leftrightarrow x_M = x_N \wedge y_M = y_N \wedge z_M = z_N$$
$$\Leftrightarrow a_1 = b_1 \wedge a_2 = b_2 \wedge a_3 = b_3$$
$$\Leftrightarrow (a_1, a_2, a_3) = (b_1, b_2, b_3) \qquad \square$$

Vector operations

Vector operations are defined in terms of a particular coordinate system but result in a vector or number that is independent of our choice of coordinate system.

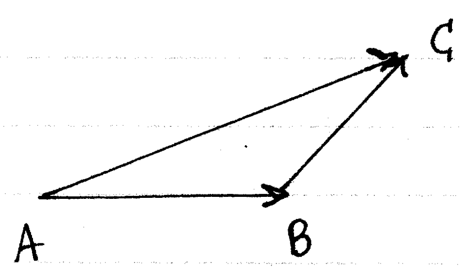
① Vector addition/subtraction

Def: Let $u, v \in \mathbb{R}^3$ be two vectors with $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in some coordinate system.
We define the vectors $u+v$ and $u-v$ such that

$$u+v = (u_1+v_1, u_2+v_2, u_3+v_3)$$
$$u-v = (u_1-v_1, u_2-v_2, u_3-v_3)$$

The following statement shows that the result of adding two vectors is independent of the choice of coordinate system.

Prop: Given three points A, B, C:
$$\vec{AB} + \vec{BC} = \vec{AC}$$



Proof

$$\vec{AB}+\vec{BC} = (x_B-x_A, y_B-y_A, z_B-z_A)+(x_C-x_B, y_C-y_B, z_C-z_B)$$
$$= (x_B-x_A+x_C-x_B, y_B-y_A+y_C-y_B, z_B-z_A+z_C-z_B) =$$
$$= (x_C-x_A, y_C-y_A, z_C-z_A) =$$
$$= \vec{AC} \qquad \square$$

▸ Properties of vector addition

$\forall u,v \in \mathbb{R}^3 : u+v = v+u$	(Commutative)
$\forall u,v,w \in \mathbb{R}^3 : u+(v+w) = (u+v)+w$	(Associative)
$\forall u \in \mathbb{R}^3 : u+\mathbf{0} = u$	(Neutral element)

2 → Scalar multiplication

Def : Let $\lambda \in \mathbb{R}$ and let $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ be a vector represented in some coordinate system. We define the scalar $\lambda u \in \mathbb{R}^3$ such that

$$\lambda u = (\lambda u_1, \lambda u_2, \lambda u_3)$$

We also define:

$$-u = (-1)u = (-u_1, -u_2, -u_3)$$

▶ Properties of scalar multiplication

1) $-u$ additive inverse of u
$\forall u \in \mathbb{R}^3 : u + (-u) = \mathbf{0}$
2) Distributive over a vector sum
$\forall \lambda \in \mathbb{R} : \forall u, v \in \mathbb{R}^3 : \lambda(u+v) = \lambda u + \lambda v$
3) Distributive over a scalar sum
$\forall \lambda, \mu \in \mathbb{R} : \forall u \in \mathbb{R}^3 : (\lambda + \mu) u = \lambda u + \mu u$
4) Associative property on a mixed product
$\forall \lambda, \mu \in \mathbb{R} : \forall u \in \mathbb{R}^3 : (\lambda \mu) u = \lambda(\mu u) = \mu(\lambda u)$

EXAMPLES

a) Let $u=(1+\sqrt{2}, 2-\sqrt{2}, 3)$ and $v=(1-\sqrt{2}, 3-2\sqrt{2}, 1)$ and define $p=u+v$ and $q=u-v$. Evaluate the vector $w=2p-3q-u$

Solution

$$
\begin{aligned}
w &= 2p-3q-u = 2(u+v)-3(u-v)-u = \\
&= 2u+2v-3u+3v-u = (2-3-1)u+(2+3)v = \\
&= -2u+5v \\
&= -2(1+\sqrt{2}, 2-\sqrt{2}, 3)+5(1-\sqrt{2}, 3-2\sqrt{2}, 1) = \\
&= (-2-2\sqrt{2}, -4+2\sqrt{2}, -6)+(5-5\sqrt{2}, 15-10\sqrt{2}, 5) \\
&= (-2-2\sqrt{2}+5-5\sqrt{2}, -4+2\sqrt{2}+15-10\sqrt{2}, -6+5) \\
&= (3-7\sqrt{2}, 11-8\sqrt{2}, -1)
\end{aligned}
$$

b) Prove: $\forall \lambda, \mu \in \mathbb{R} : \forall u \in \mathbb{R}^3 : (\lambda+\mu)u = \lambda u + \mu u$

Solution

Let $\lambda, \mu \in \mathbb{R}$ and $u \in \mathbb{R}^3$ be given with $u=(u_1, u_2, u_3)$. Then:

$$
\begin{aligned}
(\lambda+\mu)u &= (\lambda+\mu)(u_1, u_2, u_3) = \\
&= ((\lambda+\mu)u_1, (\lambda+\mu)u_2, (\lambda+\mu)u_3) = \\
&= (\lambda u_1+\mu u_1, \lambda u_2+\mu u_2, \lambda u_3+\mu u_3) \\
&= (\lambda u_1, \lambda u_2, \lambda u_3)+(\mu u_1, \mu u_2, \mu u_3) = \\
&= \lambda(u_1, u_2, u_3)+\mu(u_1, u_2, u_3) = \lambda u+\mu u.
\end{aligned}
$$

EXAMPLE

⑦ Let $u=(1,3,2)$ and $v=(-2,1,5)$ and define $p=u+\sqrt{2}v$ and $q=2u-\sqrt{2}v$. Evaluate the vectors $w=2p-q$ and $x=p+3q$.

⑧ Let $u=(1,-1,3)$ and $v=(2,0,-1)$, and define the vectors $p=u+\sqrt{3}v$ and $q=v-2\sqrt{3}u$ and $r=u+2v$. Evaluate the vectors

a) $w=p+q+r$

b) $x=2p+\sqrt{3}q-r$

c) $y=p-\sqrt{3}q-\sqrt{3}r$

d) $z=\sqrt{3}p+2q-3\sqrt{3}r$

⑨ Prove the following properties of vector addition and scalar multiplication

a) $\forall u,v\in\mathbb{R}^3 : u+v=v+u$

b) $\forall u,v,w\in\mathbb{R}^3 : u+(v+w)=(u+v)+w$

d) $\forall \lambda\in\mathbb{R} : \forall u,v\in\mathbb{R}^3 : \lambda(u+v)=\lambda u+\lambda v$

e) $\forall \lambda,\mu\in\mathbb{R} : \forall u\in\mathbb{R}^3 : (\lambda\mu)u=\lambda(\mu u)$.

3 Scalar Product

Def: Let $u, v \in \mathbb{R}^3$ be two vectors with

$$u = (u_1, u_2, u_3) \text{ and } v = (v_1, v_2, v_3)$$

in some coordinate system. We define:

a) The dot product $u \cdot v \in \mathbb{R}$

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

b) The vector norm $\|u\| \in \mathbb{R}$

$$\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{u \cdot u}$$

- Immediate properties of the dot product

$$\forall u, v \in \mathbb{R}^3 : u \cdot v = v \cdot u$$
$$\forall u, v, w \in \mathbb{R}^3 : u \cdot (v + w) = u \cdot v + u \cdot w$$
$$\forall \lambda \in \mathbb{R} : \forall u, v \in \mathbb{R}^3 : (\lambda u) \cdot v = u \cdot (\lambda v) = \lambda (u \cdot v)$$
$$\forall u \in \mathbb{R}^3 : \mathbf{0} \cdot u = 0$$

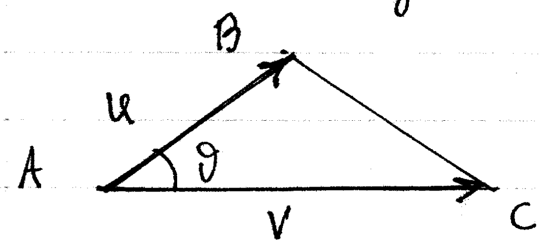
- Immediate properties of the vector norm

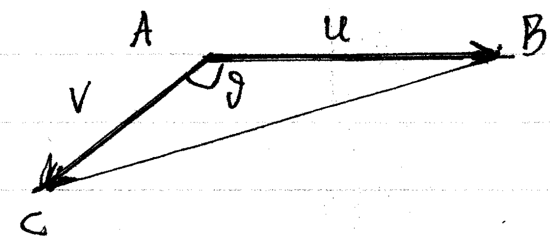
$$\forall u \in \mathbb{R}^3 : \|u\|^2 = u \cdot u$$
$$\forall \lambda \in \mathbb{R} : \forall u \in \mathbb{R}^3 : \|\lambda u\| = |\lambda| \, \|u\|$$

We will now use these properties to show that the dot product $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$ is independent of our choice of coordinate system.

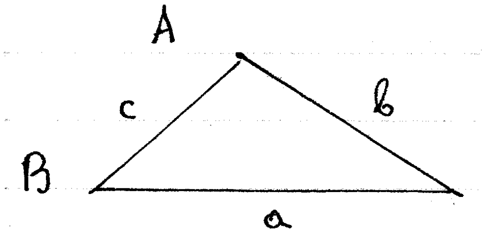
▸ Geometric interpretation of the dot product

Def: Given two vectors $u, v \in \mathbb{R}^3$ with $u = \vec{AB}$ and $v = \vec{AC}$, we define the interior angle $\vartheta(u,v)$ as the angle $\vartheta(u,v) = \hat{A}$ of the triangle $A\overset{\triangle}{B}C$





Our main result is based on the law of cosines from Precalculus:



Define: $a = BC \wedge b = CA \wedge c = AB$

$$\begin{cases} a^2 = b^2 + c^2 - 2bc\cos A \\ b^2 = c^2 + a^2 - 2ca\cos B \\ c^2 = a^2 + b^2 - 2ab\cos C \end{cases}$$

Thm: $\boxed{\forall u, v \in \mathbb{R}^3 : \ u \cdot v = \|u\|\,\|v\| \cos\vartheta(u,v)}$

Proof

Let $u = \vec{AB}$ and $v = \vec{AC}$ for some points A, B, C and let $\vartheta(u,v)$ be the interior angle of the vectors u, v. For the triangle $A\overset{\triangle}{B}C$ we have

$$a = BC = \|\vec{BC}\| = \|(\vec{AB} + \vec{BC}) - \vec{AB}\| = \|\vec{AC} - \vec{AB}\| =$$
$$= \|v - u\|$$

$b = CA = \|\vec{AC}\| = \|v\|$

$c = AB = \|\vec{AB}\| = \|u\|$

and therefore, from the law of cosines,

$$\cos\vartheta(u,v) = \cos\hat{A} = \frac{b^2+c^2-a^2}{2bc} = \frac{\|v\|^2+\|u\|^2-\|v-u\|^2}{2\|v\|\,\|u\|} =$$

$$= \frac{\|u\|^2+\|v\|^2-(v-u)\cdot(v-u)}{2\|u\|\,\|v\|} =$$

$$= \frac{\|u\|^2+\|v\|^2-v\cdot(v-u)+u\cdot(v-u)}{2\|u\|\,\|v\|} =$$

$$= \frac{\|u\|^2+\|v\|^2-v\cdot v+v\cdot u+u\cdot v-u\cdot u}{2\|u\|\,\|v\|}$$

$$= \frac{\|u\|^2+\|v\|^2-\|v\|^2+u\cdot v+u\cdot v-\|u\|^2}{2\|u\|\,\|v\|}$$

$$= \frac{2u\cdot v}{2\|u\|\,\|v\|} = \frac{u\cdot v}{\|u\|\,\|v\|} \Rightarrow$$

$\Rightarrow u\cdot v = \|u\|\,\|v\|\cos\vartheta(u,v)$. $\square$

▶ Triangle inequalities.

1) $\boxed{\forall u, v \in \mathbb{R}^3 : \ |u \cdot v| \leqslant \|u\| \, \|v\|}$

Proof

Let $u, v \in \mathbb{R}^3$ be given. Then

$$|u \cdot v| = |\, \|u\| \, \|v\| \cos\vartheta(u,v)| = |[\|u\| \, \|v\|]| \cdot |\cos\vartheta(u,v)|$$
$$= \|u\| \, \|v\| \, |\cos\vartheta(u,v)| \leqslant \|u\| \, \|v\| . \qquad \square$$

2) $\boxed{\forall u, v \in \mathbb{R}^3 : \ \|u+v\| \leqslant \|u\| + \|v\|}$

Proof

Let $u, v \in \mathbb{R}^3$ be given. Then:

$$\|u+v\|^2 = (u+v) \cdot (u+v) = u \cdot (u+v) + v \cdot (u+v) =$$
$$= u \cdot u + u \cdot v + v \cdot u + v \cdot v =$$
$$= \|u\|^2 + u \cdot v + u \cdot v + \|v\|^2 =$$
$$= \|u\|^2 + 2u \cdot v + \|v\|^2 =$$
$$= \|u\|^2 + 2\|u\| \, \|v\| \cos\vartheta(u,v) + \|v\|^2 \leqslant$$
$$\leqslant \|u\|^2 + 2\|u\| \, \|v\| + \|v\|^2 = (\|u\| + \|v\|)^2$$

$$\Rightarrow 0 \leqslant \|u+v\|^2 \leqslant (\|u\| + \|v\|)^2 \Rightarrow$$
$$\Rightarrow \|u+v\| \leqslant \|u\| + \|v\| \qquad \square$$

EXERCISES

(10) Let $u = (1+\sqrt{2}, 1-\sqrt{2}, \sqrt{2})$ and $v = (\sqrt{2}+2, 2+3\sqrt{2}, 1+\sqrt{2})$ be given.

a) Evaluate $u \cdot v$, $\|u\|$, $\|v\|$

b) Use the previous results to calculate

1) $(2u-v)\cdot(u+3v)$
2) $[v\cdot(u+v)]\,\|u\|$
3) $(u\cdot v + \|u\|)v$
4) $[3\|u\|v]\cdot(u-2v)$

(11) Find the interior angle $\vartheta(u,v)$ between the vectors

a) $u = (1,0,3)$ and $v = (2,1,0)$

b) $u = (\sqrt{2}, 1-\sqrt{2}, 0)$ and $v = (\sqrt{2}, 1+\sqrt{2}, 1)$

c) $u = (1+\sqrt{3}, 1-\sqrt{3}, \sqrt{3})$ and $v = (1-\sqrt{3}, 1+\sqrt{3}, -\sqrt{3})$

(12) Let $A\hat{B}C$ be a triangle with $A(1,2,3)$, $B(-4,5,6)$, $C(1,0,1)$. Evaluate $\cos\hat{A}$, $\cos\hat{B}$, $\cos\hat{C}$.

(13) Show that

a) $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$

b) $4(u\cdot v) = \|u+v\|^2 - \|u-v\|^2$

(14) Prove the following properties

a) $\forall u,v,w \in \mathbb{R}^3 : u\cdot(v+w) = u\cdot v + u\cdot w$

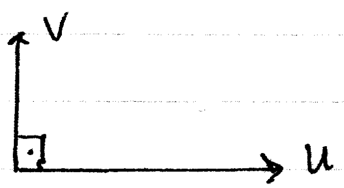
b) $\forall \lambda \in \mathbb{R} : \forall u,v \in \mathbb{R}^3 : (\lambda u)\cdot v = \lambda(u\cdot v)$

c) $\forall \lambda \in \mathbb{R} : \forall u \in \mathbb{R}^3 : \|\lambda u\| = |\lambda|\,\|u\|$

Orthogonality condition

Def: Let $u, v \in \mathbb{R}^3 - \{0\}$ be two vectors. We say that
$u \perp v \Leftrightarrow \vartheta(u,v) = \pi/2$

Terminology:
$u \perp v$ reads: u is orthogonal to v.



Prop: $\forall u, v \in \mathbb{R}^3 - \{0\} : u \perp v \Leftrightarrow u \cdot v = 0$

Proof

Since

$$\cos\vartheta(u,v) = \frac{\|u\| \|v\| \cos\vartheta(u,v)}{\|u\| \|v\|} = \frac{u \cdot v}{\|u\| \|v\|}$$

it follows that

$$u \perp v \Leftrightarrow \vartheta(u,v) = \pi/2 \Leftrightarrow \cos\vartheta(u,v) = 0 \Leftrightarrow$$
$$\Leftrightarrow \frac{u \cdot v}{\|u\| \|v\|} = 0 \Leftrightarrow u \cdot v = 0 \qquad \square$$

EXAMPLE

Find all $x \in \mathbb{R}$ such that $u = (0, x, x+1)$ and $v = (x, x-1, x-2)$ are orthogonal

Solution

Since,

$$
\begin{aligned}
u \cdot v &= u_1 v_1 + u_2 v_2 + u_3 v_3 = \\
&= 0x + x(x-1) + (x+1)(x-2) = \\
&= x^2 - x + x^2 + (1-2)x + 1 \cdot (-2) = \\
&= x^2 - x + x^2 - x - 2 = 2x^2 - 2x - 2
\end{aligned}
$$

it follows that

$$
\begin{aligned}
u \perp v &\Leftrightarrow u \cdot v = 0 \Leftrightarrow 2x^2 - 2x - 2 = 0 \Leftrightarrow \\
&\Leftrightarrow x^2 - x - 1 = 0
\end{aligned}
$$

From the quadratic formula

$$
\begin{aligned}
(a, b, c) = (1, -1, -1) \Rightarrow \Delta &= b^2 - 4ac = (-1)^2 - 4 \cdot 1 \cdot (-1) = \\
&= 1 + 4 = 5 \Rightarrow
\end{aligned}
$$

$$
\Rightarrow x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-1) \pm \sqrt{5}}{2 \cdot 1} = \frac{1 \pm \sqrt{5}}{2}
$$

and therefore

$$
u \perp v \Leftrightarrow x = \frac{1+\sqrt{5}}{2} \vee x = \frac{1-\sqrt{5}}{2}
$$

EXAMPLE

Let $u, v \in \mathbb{R}^3$. Show that

$$u+v \perp u-2v \Rightarrow u \cdot v = \|u\|^2 - 2\|v\|^2$$

Solution

Assume that $u+v \perp u-2v$. Since

$$\begin{aligned}
(u+v)\cdot(u-2v) &= u\cdot(u-2v) + v\cdot(u-2v) = \\
&= u\cdot u + u\cdot(-2v) + v\cdot u + v\cdot(-2v) = \\
&= \|u\|^2 - 2(u\cdot v) + u\cdot v - 2\|v\|^2 = \\
&= \|u\|^2 - u\cdot v - 2\|v\|^2
\end{aligned}$$

it follows that

$$\begin{aligned}
u+v \perp u-2v &\Rightarrow (u+v)\cdot(u-2v) = 0 \Rightarrow \\
&\Rightarrow \|u\|^2 - u\cdot v - 2\|v\|^2 = 0 \Rightarrow \\
&\Rightarrow u\cdot v = \|u\|^2 - 2\|v\|^2
\end{aligned}$$

EXERCISES

(15) Find all $a \in \mathbb{R}$ such that $u \perp v$ when

a) $u = (a+1, a, a-1)$ and $v = (3, 1, 2)$

b) $u = (a^2-1, 3, a+1)$ and $v = (2, a+2, a)$

c) $u = (a, 3a+1, 2a-3)$ and $v = (3, a, a-1)$

(16) Let $u = (a-1, a+1, 2)$ and $v = (2, 0, a)$ be given.
Find all $a \in \mathbb{R}$ such that

a) $u \perp v$

b) $u \perp (2u+3v)$

c) $u-v \perp u+v$

d) $2u+v \perp u+2v$

(Hint: It is useful to precalculate $\|u\|^2$, $\|v\|^2$, $u \cdot v$, before doing any of the subquestions).

(17) Let $v = (1, -2, -3)$ and $w = (-3, 2, 0)$. Find all vectors $u \in \mathbb{R}^3$ such that $u \perp v$ and $u \perp w$.
(Hint: You will find that

$$u \perp v \wedge u \perp w \iff u \in \{tp \mid t \in \mathbb{R}\}$$

for some $p \in \mathbb{R}^3$)

(18) Let $u = (x, 0, 1)$, $v = (0, 2, y)$, and $w = (1, z, 1)$.
Find all $x, y, z \in \mathbb{R}$ such that $u \perp v \wedge v \perp w \wedge w \perp u$.

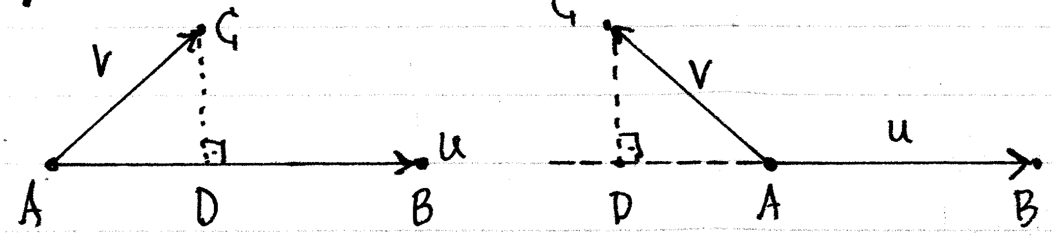
(19) Show that

a) $u \perp v \wedge u \perp w \Rightarrow \forall a, b \in \mathbb{R}: u \perp (av + bw)$

b) $u+v \perp u-v \Rightarrow \|u\| = \|v\|$

c) $\left.\begin{array}{l} u+v \perp u-w \\ u \perp v \wedge u \perp w \end{array}\right\} \Rightarrow \|u\|^2 = v \cdot w$

Projections



Def: Let $u, v \in \mathbb{R}^3 - \{0\}$ be two vectors and write $u = \vec{AB}$ and $v = \vec{AC}$. Let (ℓ) be the line defined by A, B and choose the unique $D \in (\ell)$ such that $CD \perp AB$. Then, we define

a) The *projection* of v onto u:
$$\mathrm{proj}_u(v) = \vec{AD}$$

b) The *component* $\mathrm{comp}_u(v)$ of v onto u:
$$\mathrm{proj}_u(v) = [\mathrm{comp}_u(v)] \frac{u}{\|u\|}$$

The dot product can be used to calculate both the projection and the component of v onto u:

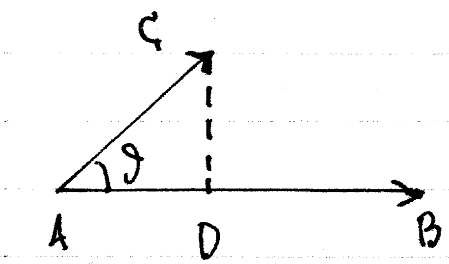
Prop: $\forall u, v \in \mathbb{R}^3 - \{0\}: \mathrm{proj}_u(v) = \frac{u \cdot v}{\|u\|^2} u$

$\forall u, v \in \mathbb{R}^3 - \{0\}: \mathrm{comp}_u(v) = \frac{u \cdot v}{\|u\|}$

Proof

Let $u, v \in \mathbb{R}^3 - \{0\}$ be given and let $\vartheta = \vartheta(u, v)$.
We distinguish between the following cases:

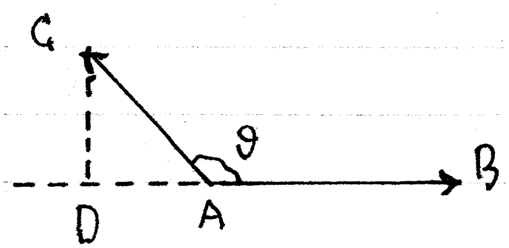
Case 1: Assume $0 \leq \vartheta \leq \pi/2$. Then



$$\mathrm{proj}_u(v) = \vec{AD} = AD\left[\frac{1}{AB}\vec{AB}\right] =$$

$$= \frac{AC\cos\vartheta}{AB}\vec{AB} =$$

Case 2: Assume that $\pi/2 < \vartheta \leq \pi$. Then



$$\mathrm{proj}_u(v) = \vec{AD} = AD\left[\frac{-1}{AB}\vec{AB}\right]$$

$$= \frac{-AC\cos(\pi - \vartheta)}{AB}\vec{AB} =$$

$$= \frac{-AC[-\cos(-\vartheta)]}{AB}\vec{AB} =$$

$$= \frac{AC\cos(-\vartheta)}{AB}\vec{AB} =$$

$$= \frac{AC\cos\vartheta}{AB}\vec{AB}$$

In both cases, we have:

$$\mathrm{proj}_u(v) = \frac{AC\cos\vartheta}{AB}\vec{AB} = \frac{\|\vec{AC}\|\cos\vartheta}{\|\vec{AB}\|}\vec{AB} =$$

$$= \frac{\|v\|\cos\vartheta}{\|u\|}u = \frac{\|u\|\|v\|\cos\vartheta}{\|u\|^2}u =$$

$$= \frac{u \cdot v}{\|u\|^2}u$$

Furthermore,

$$\text{proj}_u(v) = \frac{u \cdot v}{\|u\|^2}\, u = \frac{u \cdot v}{\|u\|}\,\frac{u}{\|u\|} =$$

$$= \text{comp}_u(v)\,\frac{u}{\|u\|} \Rightarrow$$

$$\Rightarrow \text{comp}_u(v) = \frac{u \cdot v}{\|u\|}$$

EXAMPLES

a) Let $u = (x-1, 2, 0)$ and $v = (1, 2x+1, 3)$.
Evaluate $\text{proj}_u(v)$ and $\text{comp}_u(v)$.

Solution

Since

$$u \cdot v = (x-1, 2, 0) \cdot (1, 2x+1, 3) =$$
$$= (x-1)1 + 2(2x+1) + 0 \cdot 3 =$$
$$= x - 1 + 4x + 2 = 5x + 1$$

and

$$\|u\|^2 = u \cdot u = (x-1, 2, 0) \cdot (x-1, 2, 0) =$$
$$= (x-1)(x-1) + 2 \cdot 2 + 0 \cdot 0 = (x-1)^2 + 4$$

then

$$\text{proj}_u(v) = \frac{u \cdot v}{\|u\|^2} u = \frac{5x+1}{(x-1)^2+4}(x-1, 2, 0) =$$
$$= \left(\frac{(5x+1)(x-1)}{(x-1)^2+4}, \frac{2(5x+1)}{(x-1)^2+4}, 0 \right)$$

and

$$\text{comp}_u(v) = \frac{u \cdot v}{\|u\|} = \frac{5x+1}{\sqrt{(x-1)^2+4}}$$

b) Let $u, v \in \mathbb{R}^3 - \{0\}$. Show that $\text{proj}_{(au)}(v) = \text{proj}_u(v)$.

Solution

$$\text{proj}_{(au)}(v) = \frac{(au)\cdot v}{\|au\|^2}(au) = \frac{a(u\cdot v)}{[|a|\,\|u\|]^2}(au) =$$

$$= \frac{a^2(u\cdot v)}{|a|^2\|u\|^2}u = \frac{a^2(u\cdot v)}{a^2\|u\|^2}u =$$

$$= \frac{u\cdot v}{\|u\|^2}u = \text{proj}_u(v).$$

EXERCISES

(20) Let $u, v, w \in \mathbb{R}^3 - \{0\}$. Show that

a) $\text{proj}_u(v+w) = \text{proj}_u(v) + \text{proj}_u(w)$

b) $\forall a \in \mathbb{R}$: $\text{proj}_u(av) = a\,\text{proj}_u(v)$

c) $\forall a \in \mathbb{R} - \{0\}$: $\text{proj}_{(au)}(v) = \text{proj}_u(v)$

d) $u \perp v \Rightarrow \text{proj}_u(v) = 0$

e) $\forall a \in \mathbb{R} - \{0\}$: $\text{comp}_{(au)}(v) = (a/|a|)\,\text{comp}_u(v)$

f) $\text{comp}_u(u) = \|u\|$

g) $\text{proj}_u(u) = u$

(4) Cross-product

The cross-product can only be defined between 3D vectors, as follows:

Def: Let $u, v \in \mathbb{R}^3$ with $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$. We define the cross-product $u \times v$ such that

$$(u \times v)_1 = u_2 v_3 - u_3 v_2$$
$$(u \times v)_2 = u_3 v_1 - u_1 v_3$$
$$(u \times v)_3 = u_1 v_2 - u_2 v_1$$

An alternate practical definition of the cross-product is using 3×3 determinants. Recall that a 3×3 determinant can be evaluated via the Sarrus rule as follows:

$$\left| \begin{array}{ccc|cc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array} \right. =$$

$$= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3$$

We define the unit vectors

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1)$$

noting that in general any vector can be written as:

$$u = (u_1, u_2, u_3) = u_1 e_1 + u_2 e_2 + u_3 e_3$$

It can be shown that

$$\boxed{\begin{aligned} u \times v &= (u_1, u_2, u_3) \times (v_1, v_2, v_3) = \\ &= \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{aligned}}$$

In terms of unit vectors, we can show that

$e_1 \times e_2 = -e_2 \times e_1 = e_3$	$e_1 \times e_1 = 0$
$e_2 \times e_3 = -e_3 \times e_2 = e_1$	$e_2 \times e_2 = 0$
$e_3 \times e_1 = -e_1 \times e_3 = e_2$	$e_3 \times e_3 = 0$

EXAMPLE

Evaluate $u \times v$ with $u = (1,3,4)$ and $v = (2,1,1)$.

Solution

$$u \times v = (1,3,4) \times (2,1,1) =$$

$$= \left| \begin{array}{ccc|cc} e_1 & e_2 & e_3 & e_1 & e_2 \\ 1 & 3 & 4 & 1 & 3 \\ 2 & 1 & 1 & 2 & 1 \end{array} \right| =$$

$$= e_1 3 \cdot 1 + e_2 4 \cdot 2 + e_3 1 \cdot 1 - e_1 4 \cdot 1 - e_2 1 \cdot 1 - e_3 3 \cdot 2$$

$$= 3e_1 + 8e_2 + e_3 - 4e_1 - e_2 - 6e_3 =$$

$$= (3-4)e_1 + (8-1)e_2 + (1-6)e_3 =$$

$$= -e_1 + 7e_2 - 5e_3 = (-1,7,-5)$$

▸ Norm of the cross-product

> **Thm:** $\forall u, v \in \mathbb{R}^3 : \|u \times v\| = \|u\| \|v\| \sin\vartheta(u,v)$

The proof of this result is based on the Lagrange identity, which reads:

$$\boxed{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}^2}$$

Proof

Let $u, v \in \mathbb{R}^3$ be given. Then; for $\vartheta = \vartheta(u,v)$, we have:

$$\|u \times v\|^2 = \|(u_1,u_2,u_3) \times (v_1,v_2,v_3)\|^2 =$$
$$= \|(u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1)\|^2$$
$$= (u_2v_3-u_3v_2)^2 + (u_3v_1-u_1v_3)^2 + (u_1v_2-u_2v_1)^2 =$$
$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2 =$$
$$= (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2$$
$$= \|u\|^2\|v\|^2 - (u \cdot v)^2 =$$
$$= \|u\|^2\|v\|^2 - [\|u\|\|v\| \cos\vartheta]^2 =$$
$$= \|u\|^2\|v\|^2 - \|u\|^2\|v\|^2 \cos^2\vartheta =$$
$$= \|u\|^2\|v\|^2(1-\cos^2\vartheta) = \|u\|^2\|v\|^2 \sin^2\vartheta$$

and since $\vartheta \in [0,\pi] \Rightarrow \sin\vartheta \geq 0$.

It follows that

$$\|u\times v\| = \sqrt{\|u\|^2\|v\|^2 \sin^2\vartheta} = \sqrt{[\|u\|\,\|v\|\sin\vartheta]^2} =$$
$$= |\,\|u\|\,\|v\|\sin\vartheta\,| = \|u\|\,\|v\|\,|\sin\vartheta| =$$
$$= \|u\|\,\|v\|\sin\vartheta. \qquad \square$$

► <u>Algebraic properties of the cross product</u>

We can show that the cross-product satisfies the following properties:

$$\boxed{\begin{array}{l}
\forall u\in\mathbb{R}^3: \; u\times u = 0 \\
\forall u,v\in\mathbb{R}^3: \; u\times v = -v\times u \\
\forall u,v\in\mathbb{R}^3 : \forall \lambda\in\mathbb{R}: \; (\lambda u)\times v = u\times(\lambda v) = \lambda(u\times v) \\
\forall u,v,w\in\mathbb{R}^3: \; \begin{cases} u\times(v+w) = u\times v + u\times w \\ (v+w)\times u = v\times u + w\times u \end{cases} \\
\forall u,v,w\in\mathbb{R}^3: \; u\cdot(v\times w) = v\cdot(w\times u) = w\cdot(u\times v) \\
\forall u,v,w\in\mathbb{R}^3: \; u\times(v\times w) = (u\cdot w)v - (u\cdot v)w
\end{array}}$$

To prove these properties we use tensor notation, as follows:

a) u_a with $a\in\{1,2,3\}$ will represent the a^{th} component of the vector u

b) Repeating indices are automatically summed over all components when associated with a product

e.g. $u_a v_a = u_1 v_1 + u_2 v_2 + u_3 v_3$ (dot product)

However, for the vector sum $u+v$ we write

$(u+v)_a = u_a + v_a$

and no summation over $a \in \{1,2,3\}$ is implied.

c) We define the Levi-Cirita tensor ε_{abc}:

$$\boxed{\begin{aligned}\varepsilon_{abc} &= (1/2)(a-b)(b-c)(c-a) = \\ &= \begin{cases} +1 , & \text{if } (a,b,c) \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1 , & \text{if } (a,b,c) \in \{(3,2,1),(2,1,3),(1,3,2)\} \\ 0 , & \text{if } a=b \vee b=c \vee c=a \end{cases}\end{aligned}}$$

and note that the cross-product definition can be rewritten as:

$$\boxed{(u \times v)_a = \varepsilon_{abc} u_b v_c}$$

where summation is implied over b, c.

d) We define the Kronecker tensor δ_{ab} as

$$\boxed{\delta_{ab} = \begin{cases} 1 , & \text{if } a=b \\ 0 , & \text{if } a \neq b \end{cases}}$$

and note the following basic properties:

1) $$\boxed{\begin{aligned}\delta_{ab}u_b &= u_a \\ \varepsilon_{abc}u_b u_c &= 0_a\end{aligned}}$$

2) $$\boxed{\begin{aligned}\varepsilon_{abc} &= \varepsilon_{bca} = \varepsilon_{cab} \\ \varepsilon_{abc} &= -\varepsilon_{cba}\end{aligned}}$$

3) Relation to Kronecker delta

$$\boxed{\varepsilon_{abc}\,\varepsilon_{pqr} = \begin{vmatrix} \delta_{ap} & \delta_{aq} & \delta_{ar} \\ \delta_{bp} & \delta_{bq} & \delta_{br} \\ \delta_{cp} & \delta_{cq} & \delta_{cr} \end{vmatrix}}$$

4) Contracted epsilon identities

$$\boxed{\begin{aligned}\varepsilon_{abc}\,\varepsilon_{apq} &= \delta_{bp}\delta_{cq} - \delta_{bq}\delta_{cp} \\ \varepsilon_{abp}\,\varepsilon_{abq} &= 2\delta_{pq} \\ \varepsilon_{abc}\,\varepsilon_{abc} &= 6\end{aligned}}$$

Note that (4) is consequence of (3). All properties of the cross-product are consequences of (1), (2), (3), (4).

EXAMPLE

a) Prove $u \times (v+w) = u \times v + u \times w$

Proof

$$
\begin{aligned}
[u \times (v+w)]_a &= \varepsilon_{abc} u_b (v+w)_c \\
&= \varepsilon_{abc} u_b (v_c + w_c) = \\
&= \varepsilon_{abc} u_b v_c + \varepsilon_{abc} u_b w_c = \\
&= (u \times v)_a + (u \times w)_a = \\
&= (u \times v + u \times w)_a \Rightarrow
\end{aligned}
$$

$$\Rightarrow u \times (v+w) = u \times v + u \times w .$$

b) Prove $u \cdot (v \times w) = v \cdot (w \times u)$

Proof

$$
\begin{aligned}
u \cdot (v \times w) = u_a (v \times w)_a &= u_a \varepsilon_{abc} v_b w_c = \\
&= v_b \varepsilon_{abc} w_c u_a = v_b \varepsilon_{bca} w_c u_a = \\
&= v_b (w \times u)_b = v \cdot (w \times u)
\end{aligned}
$$

c) Prove $u \times (v \times w) = (u \cdot w) v - (u \cdot v) w$

Proof

$$
\begin{aligned}
[u \times (v \times w)]_a &= \varepsilon_{abc} u_b (v \times w)_c = \\
&= \varepsilon_{abc} u_b \varepsilon_{cpq} v_p w_q = \\
&= \varepsilon_{cab} \varepsilon_{cpq} u_b v_p w_q =
\end{aligned}
$$

$$= (\delta_{ap}\delta_{bq} - \delta_{aq}\delta_{bp}) u_b v_p w_q =$$
$$= (\delta_{ap} v_p)(\delta_{bq} w_q) u_b - (\delta_{aq} w_q)(\delta_{bp} v_p) u_b$$
$$= u_b v_a w_b - u_b v_b w_a =$$
$$= (u \cdot w) v_a - (u \cdot v) w_a =$$
$$= [(u \cdot w) v]_a - [(u \cdot v) w]_a =$$
$$= [(u \cdot w) v - (u \cdot v) w]_a$$

d) Let $u, v, w \in \mathbb{R}^3$ with $\|u\| = 4$, $\|v\| = \sqrt{5}$, $\|w\| = 1+\sqrt{2}$, $\vartheta(u,w) = \pi/6$, and $\vartheta(v, w \times u) = \pi/4$.
Evaluate $u \cdot (v \times w)$.

<u>Solution</u>

$$u \cdot (v \times w) = v \cdot (w \times u) = \|v\| \, \|w \times u\| \cos\vartheta(v, w \times u) =$$
$$= \|v\| \left[\|w\| \|u\| \sin\vartheta(u,w) \right] \cos\vartheta(v, w \times u) =$$
$$= \sqrt{5} \left[(1+\sqrt{2}) 4 \sin(\pi/6) \right] \cos(\pi/4) =$$
$$= \sqrt{5} \left[(1+\sqrt{2}) 4 (1/2) \right] (\sqrt{2}/2) =$$
$$= \sqrt{2}\sqrt{5} (1+\sqrt{2}) = \sqrt{5} (\sqrt{2}+2).$$

EXERCISES

(21) Let $u=(\sqrt{2},\sqrt{2}+\sqrt{3},1)$, and $v=(2,\sqrt{3},\sqrt{2})$, and $w=(1,\sqrt{2},\sqrt{3})$ be given. Evaluate the following:

a) $u\times v$ c) $u\cdot(v\times w)$

b) $u\times(v+w)$ d) $u\times(v\times w)$

(22) Let $u=(x,2y-1,1)$, $v=(2,1,z)$, and $w=(1,1,2)$.
Find all $x,y,z\in\mathbb{R}$ such that $u\times(v\times w)=0$
(Hint: Use the identity for $u\times(v\times w)$ to expedite your calculations)

(23) Let $u,v\in\mathbb{R}^3$ with $\|u\|=\|v\|=1$.
Show that: $(u\cdot v)^2+\|u\times v\|^2=1$.

(24) Let $u,v,w\in\mathbb{R}^3$ with $\|u\|=2$, $\|v\|=3$, and $\|w\|=1$.
If the angle φ from u to v is $\pi/4$, and the angle ϑ from w to $u\times v$ is $\pi/3$, then show that:

a) $u\cdot v=3\sqrt{2}$ c) $\|u\times v\|=3\sqrt{2}$

b) $\|u+v\|=\sqrt{13+6\sqrt{2}}$ d) $u\cdot(v\times w)=3\sqrt{6}/2$

- **Parallel vectors**

Def : Let $u, v \in \mathbb{R}^3 - \{\mathbf{0}\}$ be two vectors. We say that $u \parallel v \Leftrightarrow \exists a \in \mathbb{R} : u = av$

Thm: $\forall u, v \in \mathbb{R}^3 - \{\mathbf{0}\} : (u \parallel v \Leftrightarrow u \times v = \mathbf{0})$

Proof

Let $u, v \in \mathbb{R}^3 - \{\mathbf{0}\}$ be given.

($\Rightarrow$): Assume that $u \parallel v$. Then

$u \parallel v \Rightarrow \exists a \in \mathbb{R} : u = av$

Choose $a \in \mathbb{R}$ such that $u = av$. It follows that

$u \times v = (av) \times v = a(v \times v) = a\mathbf{0} = \mathbf{0}$

($\Leftarrow$): Assume that $u \times v = \mathbf{0}$, and define $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$. Then:

$u \times v = \mathbf{0} \Rightarrow (u_1, u_2, u_3) \times (v_1, v_2, v_3) = (0, 0, 0) \Rightarrow$

$\Rightarrow (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) = (0, 0, 0) \Rightarrow$

$$\Rightarrow \begin{cases} u_1 v_2 - u_2 v_1 = 0 \\ u_2 v_3 - u_3 v_2 = 0 \\ u_3 v_1 - u_1 v_3 = 0 \end{cases}$$

Since $v \in \mathbb{R}^3 - \{\mathbf{0}\} \Rightarrow v \neq \mathbf{0} \Rightarrow v_1 \neq 0 \vee v_2 \neq 0 \vee v_3 \neq 0$

With no loss of generality, let us assume that $v_1 \neq 0$

We choose an $a \in \mathbb{R}$ such that $u_1 = a v_1$. Then, it follows that:

$$u_1v_2 - u_2v_1 = 0 \Rightarrow av_1v_2 - u_2v_1 = 0 \Rightarrow v_1(av_2 - u_2) = 0$$

$$\Rightarrow v_1 = 0 \lor av_2 - u_2 = 0 \Rightarrow \underline{u_2 = av_2}$$

and

$$u_3v_1 - u_1v_3 = 0 \Rightarrow u_3v_1 - (av_1)v_3 = 0 \Rightarrow v_1(u_3 - av_3) = 0$$

$$\Rightarrow v_1 = 0 \lor u_3 - av_3 = 0 \Rightarrow \underline{u_3 = av_3}$$

and therefore

$$u = (u_1, u_2, u_3) = (av_1, av_2, av_3) = a(v_1, v_2, v_3) = av$$

$$\Rightarrow (\exists a \in \mathbb{R} : u = av) \Rightarrow u \parallel v.$$

- <u>Scalar triple product</u>

$$\boxed{\forall u, v, w \in \mathbb{R}^3 : u \cdot (v \times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}}$$

<u>Proof</u>

$$u \cdot (v \times w) = (u_1, u_2, u_3) \cdot [(v_1, v_2, v_3) \times (w_1, w_2, w_3)] =$$

$$= (u_1, u_2, u_3) \cdot (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)$$

$$= u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1) =$$

$$= u_1v_2w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1 - u_2v_1w_3 - u_1v_3w_2 =$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

▸ Vectors perpendicular to the same vector are parallel

Thm: Let $u, v, w \in \mathbb{R}^3 - \{\mathbf{0}\}$ be given. Then
$$(u \perp w \wedge v \perp w) \Leftrightarrow w \parallel u \times v$$

Proof

($\Rightarrow$): Assume that $u \perp w$ and $v \perp w$. Then:

$$\begin{cases} u \perp w \\ v \perp w \end{cases} \Rightarrow \begin{cases} u \cdot w = 0 \\ v \cdot w = 0 \end{cases} \Rightarrow$$

$$\begin{aligned} \Rightarrow w \times (u \times v) &= (w \cdot v)u - (w \cdot u)v = \\ &= (v \cdot w)u - (u \cdot w)v = \\ &= 0u - 0v = \mathbf{0} - \mathbf{0} = \mathbf{0} \Rightarrow \end{aligned}$$

$\Rightarrow w \parallel u \times v$.

($\Leftarrow$): Assume that $w \parallel u \times v$. Then:

$w \parallel u \times v \Rightarrow \exists \lambda \in \mathbb{R}: w = \lambda(u \times v)$

Choose an $\lambda \in \mathbb{R}$ such that $w = \lambda(u \times v)$.

It follows that:

$$\begin{aligned} u \cdot w &= u \cdot [\lambda(u \times v)] = \lambda[u \cdot (u \times v)] = \\ &= \lambda[v \cdot (u \times u)] = \lambda(v \cdot \mathbf{0}) \\ &= \lambda 0 = 0 \Rightarrow u \perp w \end{aligned}$$

and

$$\begin{aligned} v \cdot w &= v \cdot [\lambda(u \times v)] = \lambda[v \cdot (u \times v)] = \lambda[u \cdot (v \times v)] \\ &= \lambda(u \cdot \mathbf{0}) = \lambda 0 = 0 \Rightarrow v \perp w \end{aligned}$$

and therefore: $u \perp w \wedge v \perp w$.

EXERCISES

(25) Use the cross-product to find the set of all vectors u such that $u \perp v$ and $u \perp w$, for $v = (1,2,3)$ and $w = (-2,2,-4)$.

(26) Let $u, v, w \in \mathbb{R}^3$ be given. Show that:

a) $u \parallel v \wedge v \parallel w \Rightarrow u \parallel w$

b) $u \parallel v \wedge u \parallel w \Rightarrow u \parallel (v+w)$

c) $u \cdot (u \times v) = 0$

d) $u \perp v \Rightarrow u \times (u \times v) = -\|u\|^2 v$

e) $(u \times v) \times (v \times w) = [(w \times u) \cdot v] v$

f) $u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = \mathbf{0}$

g) $(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (b \cdot c)(a \cdot d)$

h) $(u - v) \times (u + v) = 2(u \times v)$

Lines in $\mathbb{R}^3$

Let O be the origin of our coordinate system.

- **Def**: The parametric equation for a line (ℓ) going through the points A, B is:

$$\boxed{(\ell): \ \vec{r} = \vec{OA} + t\,\vec{AB}, \ \forall t \in \mathbb{R}}$$

The above statement is equivalent to:

$$\boxed{M \in (\ell) \Leftrightarrow \exists t \in \mathbb{R} : \vec{OM} = \vec{OA} + t\,\vec{AB}}$$

- $\vec{AB}$ = direction vector of (ℓ).

- For $\vec{r} = (x, y, z)$, $\vec{OA} = (x_0, y_0, z_0)$, and $\vec{AB} = (a, b, c)$, the parametric equation is equivalent to:

$$(\ell): \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}, \ \forall t \in \mathbb{R}.$$

- Eliminating t from the above equations gives the <u>symmetric equations</u> representation of the line (ℓ):

$$\boxed{(\ell): \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$

The symemetric equations can be reduced to a system of the form

$$(\ell): \begin{cases} A_1 x + B_1 y + C_1 = 0 \\ A_2 y + B_2 z + C_2 = 0 \end{cases}$$

which essentially defines the line (ℓ) as an intersection of two planes.

→ Relative position of two lines

Consider the lines:

$(\ell_1): \vec{r} = \vec{a}_1 + t\vec{b}_1 \quad , \forall t \in \mathbb{R}$

$(\ell_2): \vec{r} = \vec{a}_2 + t\vec{b}_2 \quad , \forall t \in \mathbb{R}$

Then:

$$(\ell_1) \parallel (\ell_2) \Leftrightarrow \vec{b}_1 \parallel \vec{b}_2 \Leftrightarrow \vec{b}_1 \times \vec{b}_2 = \mathbf{0}$$

EXAMPLES

a) Write the symmetric equations for the line (AB) with $A(1,2,-1)$ and $B(5,4,1)$.

Solution

$$\left.\begin{array}{l} A(1,2,-1) \\ B(5,4,1) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \vec{OA} = (1,2,-1) \\ \vec{AB} = (5-1,\, 4-2,\, 1-(-1)) = (4,2,2) \end{array}\right.$$

therefore:

$$(\ell):\ (x,y,z) = \vec{OA} + t\,\vec{AB} = (1,2,-1) + t\,(4,2,2) =$$
$$= (1+4t,\, 2+2t,\, -1+2t) \Leftrightarrow$$

$$\Leftrightarrow \left\{\begin{array}{l} x = 1+4t \\ y = 2+2t \\ z = -1+2t \end{array}\right. \Leftrightarrow \frac{x-1}{4} = \frac{y-2}{2} = \frac{z+1}{2} \Leftrightarrow$$

$$\Leftrightarrow \left\{\begin{array}{l} 2(x-1) = 4(y-2) \\ 2(y-2) = 2(z+1) \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} 2x-2 = 4y-8 \\ 2y-4 = 2z+2 \end{array}\right. \Leftrightarrow$$

$$\Leftrightarrow \left\{\begin{array}{l} 2x-4y+6=0 \\ 2y-2z-6=0 \end{array}\right. \Leftrightarrow \boxed{\left\{\begin{array}{l} x-2y+3=0 \\ y-z-3=0 \end{array}\right.}$$

6) Write the parametric equation for the line (ℓ) defined by

$$(\ell): \begin{cases} x-3y+2=0 \\ 2y-z+5=0 \end{cases}$$

<u>Solution</u>

Since,

$$(x,y,z) \in (\ell) \Leftrightarrow \begin{cases} x-3y+2=0 \\ 2y-z+5=0 \end{cases} \Leftrightarrow \begin{cases} x=3y-2 \\ z=2y+5 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow (x,y,z) = (3y-2, y, 2y+5) =$$

$$= (3y, y, 2y) + (-2, 0, 5) =$$

$$= y(3, 1, 2) + (-2, 0, 5)$$

$$\Leftrightarrow \exists t \in \mathbb{R}: (x,y,z) = (3,1,2)t + (-2,0,5)$$

it follows that

$$(\ell): (x,y,z) = (-2,0,5) + (3,1,2)t$$

EXERCISES

(27) Write and simplify the symmetric equations for the line (AB) with

a) $A(1,1,3)$ and $B(2,1,1)$

b) $A(5,3,2)$ and $B(7,2,4)$

c) $A(1,9,7)$ and $B(6,2,2)$

d) $A(2,1,6)$ and $B(5,3,2)$

↳ To confirm your results, check whether your symmetric equations are satisfied by the points A and B. Since two points define a <u>unique</u> line, if the answer is yes, then you have in fact <u>proved</u> that your answer is correct.

(28) Write the parametric equations for the following lines, defined by symmetric equations

a) $(\ell): \begin{cases} 2x+y-3=0 \\ y+2z+1=0 \end{cases}$ b) $(\ell): \begin{cases} 3x-2y+2=0 \\ y+2z-5=0 \end{cases}$

c) $(\ell): \begin{cases} 5x+3y-6=0 \\ 2y-6z+1=0 \end{cases}$ d) $(\ell): \begin{cases} 2x+5y+3=0 \\ 4y-3z-2=0 \end{cases}$

↳ To confirm your results, use the parametric equations to obtain two points A, B (e.g. try $t=0$ and $t=2$). Then confirm that the points A, B satisfy the original symmetric equations. If they do, then you have shown that your answer is correct.

Planes in $\mathbb{R}^3$

- Let A, B, C be three non-collinear points (i.e. A, B, C are not on the same line). Then, these three points define a unique plane with equation:

$$\boxed{(p): \vec{r} = \vec{OA} + t\,\vec{AB} + s\,\vec{AC}, \ \forall t, s \in \mathbb{R}}$$

Equivalently, if we let
$\vec{OA} = (x_0, y_0, z_0)$, $\vec{AB} = (a_1, a_2, a_3)$, $\vec{AC} = (b_1, b_2, b_3)$
then:

$$\boxed{(p): \begin{cases} x = x_0 + a_1 t + b_1 s \\ y = y_0 + a_2 t + b_2 s \\ z = z_0 + a_3 t + b_3 s \end{cases}, \ \forall t, s \in \mathbb{R}}$$

Similarly, the belonging condition for (p) is:

$$\boxed{M \in (p) \Leftrightarrow \exists\, t, s \in \mathbb{R} : \vec{OM} = \vec{OA} + t\,\vec{AB} + s\,\vec{AC}}$$

- Eliminating t, s gives an equivalent equation of the form:

$$\boxed{(p): Ax + By + Cz + D = 0}$$

which is called the <u>scalar equation</u> of (p).

Scalar equation for plane from 3 points

Let A, B, C be three points with $\vec{AB} \times \vec{AC} \neq \vec{0}$.
The plane (p) defined by A, B, C has scalar equation:

$$\boxed{(p): \ (\vec{AB} \times \vec{AC}) \cdot (\vec{r} - \vec{OA}) = 0}$$

with $r = (x, y, z)$. Equivalently:

$$\boxed{M \in (p) \iff (\vec{AB} \times \vec{AC}) \cdot (\vec{OM} - \vec{OA}) = 0}$$

Here $\vec{n} \equiv \vec{AB} \times \vec{AC}$ = normal vector of (p).
$\vec{n}$ is $\perp$ to every line of (p).

To prove this we use the following lemma:

<u>Lemma:</u> $\boxed{\begin{array}{l} \text{If } \vec{AB} \times \vec{AC} \neq \vec{0}, \text{ then} \\ \forall u \in \mathbb{R}^3 : \exists! x_1, x_2, x_3 \in \mathbb{R} : u = x_1 \vec{AB} + x_2 \vec{AC} + x_3 (\vec{AB} \times \vec{AC}) \end{array}}$

<u>Proof</u>

Define: $(a_1, a_2, a_3) = \vec{AB}$, $(b_1, b_2, b_3) = \vec{AC}$, and
$(c_1, c_2, c_3) = \vec{AB} \times \vec{AC}$.
Let $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ be given. Then:

$$u = x_1 \vec{AB} + x_2 \vec{AC} + x_3 (\vec{AB} \times \vec{AC}) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a_1 x_1 + a_2 x_2 + a_3 x_3 = u_1 \\ b_1 x_1 + b_2 x_2 + b_3 x_3 = u_2 \\ c_1 x_1 + c_2 x_2 + c_3 x_3 = u_3 \end{cases} \quad (1).$$

Since:

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$= [(a_1, a_2, a_3) \times (b_1, b_2, b_3)] \cdot (c_1, c_2, c_3) =$$

$$= (\vec{AB} \times \vec{AC}) \cdot (\vec{AB} \times \vec{AC}) = \| \vec{AB} \times \vec{AC} \|^2 > 0,$$

because $\vec{AB} \times \vec{AC} \neq \vec{0}$. $\Rightarrow$

$\Rightarrow$ equation (1) has a unique solution $(x_1, x_2, x_3) \in \mathbb{R}^3$, which proves the claim. $\square$

$\hookrightarrow$ It follows that the vectors $\vec{AB}$, $\vec{AC}$, and $\vec{AB} \times \vec{AC}$ can be used to define a non-orthogonal coordinate system in which the coordinates of the vector $\vec{u}$ are (x_1, x_2, x_3). We say therefore that $\vec{AB}, \vec{AC}, \vec{AB} \times \vec{AC}$ are linearly independent.

Now we prove the main result:

$$\boxed{M \in (p) \Leftrightarrow (\vec{AB} \times \vec{AC}) \cdot (\vec{OM} - \vec{OA}) = 0}$$

Proof

$(\Rightarrow)$

Assume $M \in (p) \Rightarrow \exists\, t, s \in \mathbb{R} : \vec{OM} = \vec{OA} + t\vec{AB} + s\vec{AC} \Rightarrow$

$$\Rightarrow (\vec{AB} \times \vec{AC}) \cdot (\vec{OM} - \vec{OA}) =$$
$$= (\vec{AB} \times \vec{AC}) \cdot (\vec{OA} + t\vec{AB} + s\vec{AC} - \vec{OA}) =$$
$$= (\vec{AB} \times \vec{AC}) \cdot (t\vec{AB} + s\vec{AC}) =$$
$$= (AB \times \vec{AC}) \cdot (t\vec{AB}) + (\vec{AB} \times \vec{AC}) \cdot (s\vec{AC}) =$$
$$= t\vec{AB} \cdot (\vec{AB} \times \vec{AC}) + s\vec{AC} \cdot (\vec{AB} \times \vec{AC}) =$$
$$= t\vec{AC} \cdot (\vec{AB} \times \vec{AB}) + s\vec{AB} \cdot (\vec{AC} \times \vec{AC}) =$$
$$= t\vec{AC} \cdot \vec{0} + s\vec{AB} \cdot \vec{0} = t0 + s0 = 0$$

$(\Leftarrow)$

Assume that $(\vec{AB} \times \vec{AC}) \cdot (\vec{OM} - \vec{OA}) = 0 \quad (1)$

By the Lemma above, let $x_1, x_2, x_3 \in \mathbb{R}$ such that

$$\vec{AM} = x_1\vec{AB} + x_2\vec{AC} + x_3(\vec{AB} \times \vec{AC}).$$

Also note that from the $(\Rightarrow)$ proof:

$$(\vec{AB} \times \vec{AC}) \cdot (x_1\vec{AB} + x_2\vec{AC}) = 0$$

Since:

$$(\vec{AB} \times \vec{AC}) \cdot (\vec{OM} - \vec{OA}) = (AB \times AC) \cdot (\vec{OA} + \vec{AM} - \vec{OA}) =$$
$$= (\vec{AB} \times \vec{AC}) \cdot \vec{AM} =$$
$$= (\vec{AB} \times \vec{AC}) \cdot (x_1\vec{AB} + x_2\vec{AC} + x_3(\vec{AB} \times \vec{AC})) =$$
$$= (\vec{AB} \times \vec{AC}) \cdot [x_3(\vec{AB} \times \vec{AC})] =$$
$$= x_3(\vec{AB} \times \vec{AC}) \cdot (\vec{AB} \times \vec{AC}) = x_3\|\vec{AB} \times \vec{AC}\|^2.$$

it follows from (1) that

$$\left.\begin{array}{l} x_3\|\vec{AB} \times \vec{AC}\|^2 = 0 \\ \vec{AB} \times \vec{AC} \neq \vec{0} \end{array}\right\} \Rightarrow x_3 = 0 \Rightarrow \vec{AM} = x_1\vec{AB} + x_2\vec{AC}$$

$\Rightarrow \vec{OM} = \vec{OA} + \vec{AM} = \vec{OA} + x_1 \vec{AB} + x_2 \vec{AC} \Rightarrow$

$\Rightarrow M \in (p)$ $\square$

EXAMPLES

Find the equation of the plane (p) containing the points $A(3,1,2)$, $B(4,3,1)$, and $C(2,5,6)$

Solution

$$\left.\begin{array}{l} A(3,1,2) \\ B(4,3,1) \end{array}\right\} \Rightarrow \vec{AB} = (4-3, 3-1, 1-2) = (1,2,-1) \quad (1)$$

$$\left.\begin{array}{l} A(3,1,2) \\ C(2,5,6) \end{array}\right\} \Rightarrow \vec{AC} = (2-3, 5-1, 6-2) = (-1,4,4) \quad (2)$$

From Eq.(1) and Eq.(2):

$$\vec{AB} \times \vec{AC} = (1,2,-1) \times (-1,4,4) =$$

$$= \left|\begin{array}{ccc|cc} e_1 & e_2 & e_3 & e_1 & e_2 \\ 1 & 2 & -1 & 1 & 2 \\ -1 & 4 & 4 & -1 & 4 \end{array}\right| =$$

$$= (2\cdot 4)e_1 + (-1)(-1)e_2 + (1\cdot 4)e_3 - (-1)2e_3 - 4(-1)e_1 - (4\cdot 1)e_2 =$$

$$= 8e_1 + e_2 + 4e_3 + 2e_3 + 4e_1 - 4e_2 =$$

$$= (8+4)e_1 + (1-4)e_2 + (4+2)e_3 = 12e_1 - 3e_2 + 6e_3$$

$$= (12,-3,6) \quad , \quad \text{and therefore}$$

$$(p): (\vec{AB} \times \vec{AC}) \cdot [(x,y,z) - \vec{OA}] = 0 \Leftrightarrow$$

$$\Leftrightarrow (12,-3,6) \cdot (x-3, y-1, z-2) = 0$$

$\Leftrightarrow 12(x-3)-3(y-1)+6(z-2)=0 \Leftrightarrow$
$\Leftrightarrow 4(x-3)-(y-1)+2(z-2)=0 \Leftrightarrow$
$\Leftrightarrow 4x-12-y+1+2z-4=0 \Leftrightarrow$
$\Leftrightarrow 4x-y+2z-15=0 \Leftrightarrow$
$\Leftrightarrow 4x-y+2z=15.$

Thus: $\quad (p): 4x-y+2z=15.$

$\hookrightarrow$ Note that the plane $(p): Ax+By+Cz+D=0$ has normal vector $\vec{n}=(A,B,C)$.

▸ **Plane equation to parametric equation**

The first step is to solve the plane equation for one of the 3 variables and use the result to rewrite (x,y,z) as in the following example.

6) Write the parametric equations for the plane $(p): 2x+y+3z=7$.

Solution

We note that

$2x+y+3z=7 \Leftrightarrow y=7-2x-3z \Leftrightarrow$

$$\begin{aligned}\Leftrightarrow (x,y,z) &= (x, 7-2x-3z, z) = \\ &= (0,7,0)+(x,-2x,0)+(0,-3z,z) \\ &= (0,7,0)+x(1,-2,0)+z(0,-3,1)\end{aligned}$$

$\Leftrightarrow \exists\, t, s \in \mathbb{R}: (x,y,z) = (0,7,0) + t(1,-2,0) + s(0,-3,1)$

and therefore

$(p): (x,y,z) = (0,7,0) + t(1,-2,0) + s(0,-3,1), \forall t, s \in \mathbb{R}.$

▸ Parametric equation to plane equations

We use the parametric equation to obtain ~~3 collinear~~ 3 non-collinear points A, B, C (e.g. use $(t, s) = (0,0), (1,0), (0,1)$). From the 3 points we then derive the plane equation.

c) Write the plane equation for the plane

$$(p): \begin{cases} x = 1 + 2t + s \\ y = 3 - t - 2s \\ z = 2 + 3t + s \end{cases}$$

Solution

We obtain 3 points:

$(t, s) = (0,0) \longrightarrow A(1,3,2)$

$(t, s) = (1,0) \longrightarrow B(3,2,5)$

$(t, s) = (0,1) \longrightarrow C(2,1,3)$

It follows that

$$\left.\begin{array}{l} \vec{AB} = (3-1, 2-3, 5-2) = (2,-1,3) \\ \vec{AC} = (2-1, 1-3, 3-2) = (1,-2,1) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \vec{AB} \times \vec{AC} = (2,-1,3) \times (1,-2,1) =$$

$$= \left| \begin{array}{ccc} e_1 & e_2 & e_3 \\ 2 & -1 & 3 \\ 1 & -2 & 1 \end{array} \right| \begin{array}{cc} e_1 & e_2 \\ 2 & -1 \\ 1 & -2 \end{array} =$$

$$= (-1)1e_1 + 3 \cdot 1 e_2 + 2(-2)e_3 - 1(-1)e_3 - (-2)3e_1 - 1 \cdot 2e_2$$

$$= -e_1 + 3e_2 - 4e_3 + e_3 + 6e_1 - 2e_2 =$$

$= (-1+6)e_1 + (3-2)e_2 + (-4+1)e_3 =$

$= 5e_1 + e_2 - 3e_3 = (5,1,-3)$

and therefore

$(p): (\vec{AB} \times \vec{AC}) \cdot [(x,y,z) - \vec{OA}] = 0 \Leftrightarrow$

$\Leftrightarrow (5,1,-3) \cdot [(x,y,z) - (1,3,2)] = 0 \Leftrightarrow$

$\Leftrightarrow (5,1,-3) \cdot (x-1, y-3, z-2) = 0 \Leftrightarrow$

$\Leftrightarrow 5(x-1) + (y-3) - 3(z-2) = 0 \Leftrightarrow$

$\Leftrightarrow 5x - 5 + y - 3 - 3z + 6 = 0 \Leftrightarrow$

$\Leftrightarrow 5x + y - 3z - 2 = 0 \Leftrightarrow 5x + y - 3z = 2$

and therefore

$\underline{(p): 5x + y - 3z = 2}$

EXERCISES

(29) Write the plane equation $(p): Ax + By + Cz + D = 0$ for the plane (p) defined by three non-collinear points A, B, C with coordinates:

a) $A(1,2,1)$, $B(4,1,0)$, $C(0,3,5)$

b) $A(0,0,0)$, $B(1,1,1)$, $C(1,-1,1)$

c) $A(3,0,0)$, $B(0,1,2)$, $C(1,0,2)$

d) $A(3,2,2)$, $B(3,5,3)$, $C(0,1,2)$

To confirm your answers, it is sufficient to verify that the plane equation is satisfied by the coordinates of the points A, B, C. Note that 3 collinear points define a unique plane.

(30) Write the parametric equations for the planes (p) defined by the following plane equations:

a) $(p): 2x + y + 7z = 3$

b) $(p): x + 3y - 2z = 6$

c) $(p): 3x - 2y - z = 5$

d) $(p): x - y + 3z = -1$

e) $(p): 2x + 5y + 3z = 2$

↳ To confirm your work, use the parametric equations to generate 3 noncollinear points and confirm that these 3 points satisfy the original plane equation.

31 Write the plane equations for the planes (p) defined by the following parametric equations:

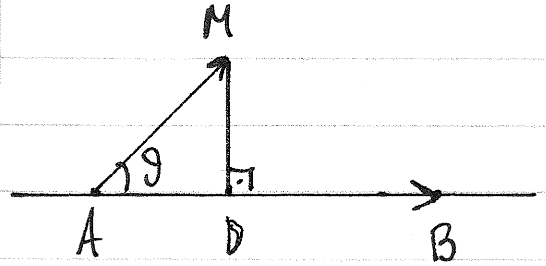
a) $(p): \begin{cases} x = 2 + 3t + 5s \\ y = 1 - t - 2s \\ z = 3 + t + s \end{cases}$ b) $(p): \begin{cases} x = 1 + 2t + s \\ y = 7 + t - 3s \\ z = 2 - t - 9s \end{cases}$

c) $(p): \begin{cases} x = t - s \\ y = t + 2s \\ z = 3t - 2s \end{cases}$

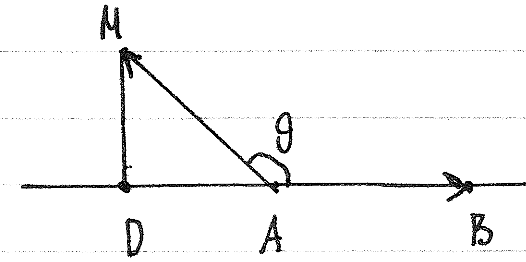
↳ To confirm your work, substitute x, y, z in terms of t, s from the parametric equations into the plane equation, and confirm that the plane equation is satisfied for all values of t and s.

Distances between points, lines, and planes

(1) → Distance of point M from line (AB)



$\vartheta \in [0, \pi/2]$



$\vartheta \in (\pi/2, \pi]$

$$\boxed{d(M, AB) = \frac{\|\vec{AB} \times \vec{AM}\|}{\|\vec{AB}\|}}$$

Proof

Let $D \in (AB)$ such that $MD \perp AB$. Define the interior angle $\vartheta = M\hat{A}B$. We distinguish between the following cases:

Case 1: Assume $\vartheta \in [0, \pi/2]$.

Then $d(M, AB) = MD = AM \sin\vartheta$.

Case 2: Assume $\vartheta \in (\pi/2, \pi]$. Then:

$$d(M, AB) = MD = AM \sin(\pi - \vartheta) = -AM \sin(-\vartheta) =$$
$$= AM \sin\vartheta$$

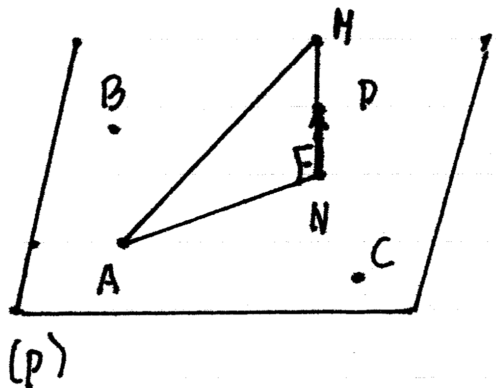
In both cases we find $d(M, AB) = AM \sin\vartheta$.

It follows that:

$$\|\vec{AB} \times \vec{AM}\| = \|\vec{AB}\| \|\vec{AM}\| \sin\vartheta = \|\vec{AB}\| [AM \sin\vartheta] =$$
$$= \|\vec{AB}\| \, d(M, AB) \Rightarrow d(M, AB) = \frac{\|\vec{AB} \times \vec{AM}\|}{\|\vec{AB}\|}$$

② → Distance of point from plane

Case 1 : The distance $d(M,(p))$ between the point M and the plane (p) defined by the points A, B, C is given by



$$\boxed{d(M,(p)) = \frac{|(\vec{AB} \times \vec{AC}) \cdot \vec{MA}|}{\|\vec{AB} \times \vec{AC}\|}}$$

Proof

Let $N \in (p)$ be the projection of M on (p) such that $MN \perp (p)$. Let $D \in (MN)$ such that $\vec{ND} = \vec{AB} \times \vec{AC}$. It follows that:

$$d(M,(p)) = MN = \|\vec{MN}\| = \|\mathrm{proj}_{\vec{ND}}(\vec{AM})\| =$$

$$= |\mathrm{comp}_{\vec{ND}}(\vec{AM})| = \left| \frac{\vec{AM} \cdot \vec{ND}}{\|\vec{ND}\|} \right| =$$

$$= \frac{|\vec{AM} \cdot \vec{ND}|}{\|\vec{ND}\|} = \frac{|\vec{AM} \cdot (\vec{AB} \times \vec{AC})|}{\|\vec{AB} \times \vec{AC}\|} =$$

$$= \frac{|(\vec{AB} \times \vec{AC}) \cdot \vec{MA}|}{\|\vec{AB} \times \vec{AC}\|}$$

<u>Case 2</u>: The distance between $M(x_0, y_0, z_0)$ and the plane $(p): Ax+By+Cz+D=0$ is given by:

$$\boxed{d(M,(p)) = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}}$$

<u>Proof</u>

Let $N, P, Q \in (p)$ be three non-collinear points on the plane (p) with $N(x_1, y_1, z_1)$. Since $N(x_1, y_1, z_1) \in (p) \Rightarrow Ax_1 + By_1 + Cz_1 + D = 0 \Rightarrow$
$\Rightarrow \underline{Ax_1+By_1+Cz_1 = -D}$. (1)

We also note that:

$$\left.\begin{array}{l}(p): Ax+By+Cz+D=0 \Rightarrow \vec{u} = (A, B, C) \perp (p) \\ \qquad\qquad\qquad\qquad\qquad\qquad \vec{NP} \times \vec{NQ} \perp (p)\end{array}\right\} \Rightarrow$$

$\Rightarrow \vec{u} \parallel \vec{NP} \times \vec{NQ} \Rightarrow \exists \lambda \in \mathbb{R}: \underline{\vec{NP} \times \vec{NQ} = \lambda \vec{u}}$ (2)

and

$$\left.\begin{array}{l}M(x_0, y_0, z_0) \\ N(x_1, y_1, z_1)\end{array}\right\} \Rightarrow \underline{\vec{NM} = (x_0-x_1, y_0-y_1, z_0-z_1)} \quad (3)$$

From (1), (2), (3), using the previous result, it follows that

$$d(M,(p)) = \frac{|\vec{NM} \cdot (\vec{NP} \times \vec{NQ})|}{\|\vec{NP} \times \vec{NQ}\|} = \frac{|\vec{NM} \cdot (\lambda \vec{u})|}{\|\lambda \vec{u}\|} =$$

$$= \frac{|\lambda(\vec{NM} \cdot \vec{u})|}{\|\lambda\vec{u}\|} = \frac{|\lambda| \cdot |\vec{NM} \cdot \vec{u}|}{|\lambda| \|\vec{u}\|} = \frac{|\vec{NM} \cdot \vec{u}|}{\|\vec{u}\|} =$$

$$= \frac{|(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (A, B, C)|}{\|(A, B, C)\|} =$$

$$= \frac{|A(x_0 - x_1) + B(y_0 - y_1) + C(z_0 - z_1)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|(Ax_0 + By_0 + Cz_0) - (Ax_1 + By_1 + Cz_1)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|Ax_0 + By_0 + Cz_0 - (-D)|}{\sqrt{A^2 + B^2 + C^2}} =$$

$$= \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} \qquad \square$$

EXERCISES

(32) Find the distance between

a) The point $M(3,1,5)$ and the line (AB) passing through $A(1,2,1)$ and $B(0,0,4)$.

b) The point $M(1,1,-2)$ and the line (AB) given by

$$(AB): \begin{cases} x+y-3=0 \\ y+2z+1=0 \end{cases}$$

c) The point $M(2,2,1)$ and the plane (p) defined by the points $A(1,1,0)$, $B(3,2,4)$, and $C(-1,2,0)$

d) The point $M(1,3,-1)$ and the plane (p) given by

$$(p): \begin{cases} x = 2+t+s \\ y = 1-2t+3s \\ z = 3-t-s \end{cases}$$

e) The point $M(1,2,1)$ and the plane $(p): 2x+3y-z=3$.

(33) Find all $a \in \mathbb{R}$ such that:

a) The distance of $M(1,1,a)$ from the line (AB) passing through $A(0,2,2)$ and $B(3,1,1)$ is equal to 10

b) The distance of $M(a,a,3)$ from the line

$$(AB): \begin{cases} 2x+y-1=0 \\ y+5z-2=0 \end{cases}$$

is minimized

c) The distance between the point $M(1, a, 2a+1)$ and the plane defined by the points $A(2,0,2)$, $B(3,0,0)$, $C(1,-1,3)$ is equal to 6.

d) The distance between the point $M(1,1,2)$ and the plane $(p): x-3y+2z=a$ is equal to a.