VECTOR-VALUED FUNCTIONS

Def: A vector-valued function is a mapping $f: A \to \mathbb{R}^3$ with $A \subseteq \mathbb{R}$

As such, vector-valued functions are mapping a NUMBER to a VECTOR

Limit of a vector-valued function

Def: Let $f: A \to \mathbb{R}^3$ be a vector-valued function, let $a \in A$ be a limit point of A, and let $u \in \mathbb{R}^3$. Then

$$\lim_{t \to a} f(t) = u \iff \lim_{t \to a} \|f(t) - u\| = 0$$

- This definition piggybacks on the Calculus 1 definition of the limit for a function $g: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$:

$$\lim_{t \to a} g(t) = l \iff \forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall t \in A : (0 < |t - a| < \delta \Rightarrow |g(t) - l| < \varepsilon)$$

The negation of this definition is given by

$$\lim_{t \to a} g(t) \neq l \iff \exists \varepsilon \in (0, +\infty) : \forall \delta \in (0, +\infty) : \exists t \in A : (0 < |t - a| < \delta \land |g(t) - l| \geq \varepsilon)$$

and we use it to prove the following theorem:

Thm: Let $f: A \to \mathbb{R}^3$ be a vector-valued function with $A \subseteq \mathbb{R}$ and let $a \in A$ be a limit point of A and assume that
$$\forall t \in A : f(t) = (x(t), y(t), z(t))$$
Let $u = (u_1, u_2, u_3) \in \mathbb{R}^3$. Then:
$$\lim_{t \to a} f(t) = u \Leftrightarrow$$
$$\Leftrightarrow \left(\lim_{t \to a} x(t) = u_1 \wedge \lim_{t \to a} y(t) = u_2 \wedge \lim_{t \to a} z(t) = u_3 \right)$$

▶ It follows that when the limits converge, we have:
$$\lim_{t \to a} f(t) = \left(\lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right)$$

<u>Proof</u>

<u>($\Rightarrow$)</u>: Assume that $\lim_{t \to a} f(t) = u$. To show that $\lim_{t \to a} x(t) = u_1$, assume that $\lim_{t \to a} x(t) \neq u_1$, in order to derive a contradiction. Then,
$$\lim_{t \to a} x(t) \neq u_1 \Rightarrow \exists \varepsilon \in (0, +\infty) : \forall \delta \in (0, +\infty) : \exists t \in A :$$
$$: (0 < |t-a| < \delta \wedge |x(t) - u_1| \geq \varepsilon)$$
Choose some $\varepsilon \in (0, +\infty)$ such that
$$\forall \delta \in (0, +\infty) : \exists t \in A : (0 < |t-a| < \delta \wedge |x(t) - u_1| \geq \varepsilon)$$
Let $\delta \in (0, +\infty)$ be given and choose some $t \in A$

such that

$$\begin{cases} 0 < |t-a| < \delta \\ |x(t) - u_1| \geq \varepsilon \end{cases}$$

Then, it follows that

$$\|f(t) - u\| = \sqrt{[x(t)-u_1]^2 + [y(t)-u_2]^2 + [z(t)-u_3]^2}$$
$$\geq \sqrt{[x(t)-u_1]^2} = |x(t) - u_1| \geq \varepsilon \Rightarrow$$
$$\Rightarrow \|f(t) - u\| \geq \varepsilon$$

We have thus shown that

$$\exists \varepsilon \in (0, +\infty) : \forall \delta \in (0, +\infty) : \exists t \in A : \begin{cases} 0 < |t-a| < \delta \\ \|f(t) - u\| \geq \varepsilon \end{cases}$$

$$\Rightarrow \lim_{t \to a} \|f(t) - u\| \neq 0 \Rightarrow \lim_{t \to a} f(t) \neq u$$

which is a contradiction, since by hypothesis we have: $\lim_{t \to a} f(t) = u$. It follows that $\lim_{t \to a} x(t) = u_1$

Similarly, we can show that $\lim_{t \to a} y(t) = u_2$ and

$\lim_{t \to a} z(t) = u_3$.

($\Leftarrow$): Assume that $\lim_{t \to a} x(t) = u_1 \wedge \lim_{t \to a} y(t) = u_2 \wedge \lim_{t \to a} z(t) = u_3$.

Then:

$$\lim_{t \to a} \|f(t) - u\| = \lim_{t \to a} \sqrt{[x(t)-u_1]^2 + [y(t)-u_2]^2 + [z(t)-u_3]^2}$$
$$= \sqrt{(u_1-u_1)^2 + (u_2-u_2)^2 + (u_3-u_3)^2}$$
$$= 0 \Rightarrow \lim_{t \to a} f(t) = u.$$

Continuity of vector-valued functions

Def: Let $f: A \to \mathbb{R}^3$ be a vector-valued function, and let $J \subseteq A$, and $a \in A$.

a) f continuous at $t=a \Leftrightarrow \lim_{t \to a} f(t) = f(a)$

b) f continuous at $J \Leftrightarrow \forall a \in J$: f continuous at $t=a$.

Derivatives of vector-valued functions

Def: Let $f: A \to \mathbb{R}^3$ be a vector-valued function. We say that:

a) f differentiable at $t \in A \Leftrightarrow$

$$\Leftrightarrow \exists v \in \mathbb{R}^3: \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = v$$

b) f differentiable at $J \subseteq A \Leftrightarrow$

$\Leftrightarrow \forall a \in J$: f differentiable at $t=a$.

Def: Let $f: A \to \mathbb{R}^3$ be a vector-valued function that is differentiable in $J \subseteq A$. We define the derivative function $\dot{f}: J \to \mathbb{R}^3$ as:

$$\boxed{\forall t \in J: \dot{f}(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}}$$

▸ <u>notation</u> : $\dot{f}(t) = \dfrac{df(t)}{dt}$

For higher-order derivatives:

$$\ddot{f}(t) = \frac{d\dot{f}(t)}{dt} = \frac{d^2 f(t)}{dt^2}$$

$$\dddot{f}(t) = \frac{d\ddot{f}(t)}{dt} = \frac{d^3 f(t)}{dt^3}$$

<u>Thm</u>: Let $f : A \to \mathbb{R}^3$ with $f(t) = (x(t), y(t), z(t))$, $\forall t \in A$ and assume that f differentiable on B with $B \subseteq A$. Then, $\forall t \in B : \dot{f}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$

▸ <u>interpretation</u>

If a vector-valued function $r : [0, +\infty) \to \mathbb{R}^3$ represents the position vector of an object in motion, then:

$u(t) = \dot{r}(t)$ = the velocity of the object
$a(t) = \dot{u}(t) = \ddot{r}(t)$ = the acceleration of the object
$\|u(t)\| = \|\dot{r}(t)\|$ = the speed of the object

▸ Note the careful distinction between the terms <u>speed</u> and <u>velocity</u>.

EXAMPLE

For circular motion given by
$r(t) = (\rho\cos(\omega t), \rho\sin(\omega t), 0)$, $\forall t \in \mathbb{R}$
show that

a) $u(t) \perp r(t)$, $\forall t \in \mathbb{R}$

b) $a(t) = -\omega^2 r(t)$, $\forall t \in \mathbb{R}$

c) $\|a(t)\| = \|u(t)\|^2/\rho$, $\forall t \in \mathbb{R}$

Solution

a) $u(t) = \dot{r}(t) = (d/dt)(\rho\cos(\omega t), \rho\sin(\omega t), 0) =$
$= (-\omega\rho\sin(\omega t), \omega\rho\cos(\omega t), 0) =$
$= \omega\rho(-\sin(\omega t), \cos(\omega t), 0)$, $\forall t \in \mathbb{R} \Rightarrow$
$\Rightarrow u(t) \cdot r(t) = [\omega\rho(-\sin(\omega t), \cos(\omega t), 0)] \cdot (\rho\cos(\omega t), \rho\sin(\omega t), 0)$
$= \omega\rho^2[(-\sin(\omega t))\cos(\omega t) + \cos(\omega t)\sin(\omega t) + 0 \cdot 0] =$
$= \omega\rho^2 \cdot 0 = 0$, $\forall t \in \mathbb{R} \Rightarrow$
$\Rightarrow u(t) \perp r(t)$, $\forall t \in \mathbb{R}$.

b) $a(t) = \dot{u}(t) = (d/dt)[\omega\rho(-\sin(\omega t), \cos(\omega t), 0)] =$
$= \omega\rho(-\omega\cos(\omega t), -\omega\sin(\omega t), 0) =$
$= -\omega^2(\rho\cos(\omega t), \rho\sin(\omega t), 0)$
$= -\omega^2 r(t)$, $\forall t \in \mathbb{R}$.

c) $\|a(t)\| = \|-\omega^2 r(t)\| = |-\omega^2|\,\|r(t)\| = \omega^2\|r(t)\| =$
$= \omega^2\|(\rho\cos(\omega t), \rho\sin(\omega t), 0)\| =$
$= \omega^2\|\rho(\cos(\omega t), \sin(\omega t, 0)\| =$
$= \omega^2|\rho|\,\|(\cos(\omega t), \sin(\omega t), 0)\| =$
$= \omega^2\rho\sqrt{\cos^2(\omega t) + \sin^2(\omega t) + 0^2} = \omega^2\rho$

and

$$
\begin{aligned}
\|u(t)\| &= \| \omega\rho\,(-\sin(\omega t),\ \cos(\omega t),\ 0)\| = \\
&= |\omega\rho|\ \|(-\sin(\omega t),\ \cos(\omega t),\ 0)\| = \\
&= \omega\rho \sqrt{(-\sin(\omega t))^2 + \cos^2(\omega t) + 0^2} \\
&= \omega\rho \sqrt{\sin^2(\omega t) + \cos^2(\omega t)} = \omega\rho\ ,\ \forall t\in\mathbb{R}
\end{aligned}
$$

it follows that

$$
\|a(t)\| = \omega^2\rho = \frac{\omega^2\rho^2}{\rho} = \frac{(\omega\rho)^2}{\rho} = \frac{\|u(t)\|^2}{\rho}\ ,\ \forall t\in\mathbb{R}.
$$

EXERCISES

① Find the default domain for the following vector-valued functions:

a) $r(t) = \left(\dfrac{2}{t-1}, \sqrt{3-t}, \ln(t+3) \right)$

b) $r(t) = (2t^2+1, \text{Arcsin}(2t-3), \text{Arctan}(2t+3))$

c) $r(t) = (\ln[(t-2)^2(3t-2)^3], t^2+3, t/(t-3))$

d) $r(t) = (\text{Arcsin}(3t+1), \text{Arcsin}(2t-1), 1/t)$

e) $r(t) = (\log_x(2x+1), 1/(x+1), 0)$

↳ The default domain is defined as the widest possible subset of $\mathbb{R}$ for which the function definition can be evaluated into real-valued vectors.

② Evaluate the following limits:

a) $\lim\limits_{t \to 3} (2(t-3)^2, -7t^3, 0)$

b) $\lim\limits_{t \to 2} \left(\dfrac{t-2}{t^2-4}, \dfrac{t^2+3t-10}{t-2}, 0 \right)$

c) $\lim\limits_{t \to 0} \left(\dfrac{\sin(2t)\cos(3t)}{5t}, \dfrac{e^t-1}{3t}, \dfrac{2t}{t+2} \right)$

d) $\lim_{t\to 0}\left(\dfrac{t^2\cos(1/t)}{e^t-1}, \dfrac{\ln(t)}{t^2}, 0\right)$

e) $\lim_{t\to +\infty}\left(\log_{2x}(3x+1), \log_{x-2}(2x+5), 0\right)$

f) $\lim_{t\to 0}\left(t^t, t^{2t}, t^{3t}\right)$

g) $\lim_{t\to 0^+}\left(t^2\ln t, \ln(t^3+1), 2t-1\right)$

(3) Find the derivatives $\dot{r}(t)$ and $\ddot{r}(t)$ for the following vector-valued functions.

a) $r(t) = \left(2t^2, (t-3)^3, (2t+1)^2\right)$

b) $r(t) = \left(t^2e^{-t}, te^{-t^2}, t^3e^{-t}\right)$

c) $r(t) = \left(\ln(t^2+1), \ln(t^3+2t^2-t), \ln(3t)\right)$

d) $r(t) = \left(t^t, t^{2t}, t^{3t}\right)$

Properties of differentiation

Thm: Let $u : A \to \mathbb{R}^3$ and $v : A \to \mathbb{R}^3$ be two vector-valued functions. Then:

a) $(d/dt)[u(t) + v(t)] = \dot{u}(t) + \dot{v}(t)$

b) $(d/dt)[\lambda u(t)] = \lambda \dot{u}(t)$

c) $(d/dt)[f(t) u(t)] = f'(t) u(t) + f(t) \dot{u}(t)$

d) $(d/dt)[u(t) \cdot v(t)] = \dot{u}(t) \cdot v(t) + u(t) \cdot \dot{v}(t)$

e) $(d/dt)[u(t) \times v(t)] = \dot{u}(t) \times v(t) + u(t) \times \dot{v}(t)$

We give selected proofs for properties (d), (e) using tensor notation

Proof of (d)

$$(d/dt)(u(t) \cdot v(t)) = (d/dt)[u_a(t) v_a(t)] =$$
$$= \dot{u}_a(t) v_a(t) + u_a(t) \dot{v}_a(t) =$$
$$= \dot{u}(t) \cdot v(t) + u(t) \cdot \dot{v}(t).$$

Proof of (e)

$$(d/dt)[u(t) \times v(t)]_a = (d/dt)[\varepsilon_{abc} u_b(t) u_c(t)] =$$
$$= \varepsilon_{abc} (d/dt)[u_b(t) u_c(t)] = \varepsilon_{abc}[\dot{u}_b(t) v_c(t) + u_b(t) \dot{v}_c(t)]$$
$$= \varepsilon_{abc} \dot{u}_b(t) v_c(t) + \varepsilon_{abc} u_b(t) \dot{v}_c(t) =$$
$$= [\dot{u}(t) \times v(t)]_a + [u(t) \times \dot{v}(t)]_a =$$
$$= [\dot{u}(t) \times v(t) + u(t) \times \dot{v}(t)]_a \Rightarrow$$
$$\Rightarrow (d/dt)[u(t) \times v(t)] = \dot{u}(t) \times v(t) + u(t) \times \dot{v}(t)$$

APPLICATION

Show that

$\|u(t)\|$ constant on $\mathbb{R} \Leftrightarrow \forall t \in \mathbb{R}: \dot{u}(t) \perp u(t)$.

Solution

We note that

$$(d/dt)\|u(t)\|^2 = (d/dt)[u(t) \bullet u(t)] =$$
$$= \dot{u}(t) \bullet u(t) + u(t) \bullet \dot{u}(t) = \dot{u}(t) \bullet u(t) + \dot{u}(t) \bullet u(t)$$
$$= 2\dot{u}(t) \bullet u(t)$$

$(\Rightarrow)$: Assume that $\|u(t)\|$ constant on $\mathbb{R}$. Then:

$\|u(t)\|$ constant on $\mathbb{R} \Rightarrow \|u(t)\|^2$ constant on $\mathbb{R} \Rightarrow$

$\Rightarrow (d/dt)\|u(t)\|^2 = 0 \Rightarrow 2\dot{u}(t) \bullet u(t) = 0, \forall t \in \mathbb{R} \Rightarrow$

$\Rightarrow \dot{u}(t) \bullet u(t) = 0, \forall t \in \mathbb{R} \Rightarrow \dot{u}(t) \perp u(t), \forall t \in \mathbb{R}$.

$(\Leftarrow)$: Assume that $\forall t \in \mathbb{R}: \dot{u}(t) \perp u(t)$. Then:

$\dot{u}(t) \perp u(t), \forall t \in \mathbb{R} \Rightarrow \dot{u}(t) \bullet u(t) = 0, \forall t \in \mathbb{R} \Rightarrow$

$\Rightarrow (d/dt)\|u(t)\|^2 = 2\dot{u}(t) \bullet u(t) = 2 \cdot 0 = 0 \Rightarrow$

$\Rightarrow \|u(t)\|^2$ constant on $\mathbb{R} \Rightarrow$

$\Rightarrow \exists a \in [0, +\infty) : \forall t \in \mathbb{R} : \|u(t)\|^2 = a$.

Choose $a \in (0, +\infty)$ such that $\forall t \in \mathbb{R} : \|u(t)\|^2 = a$.

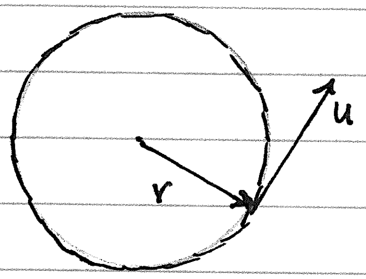
Then:

$\|u(t)\| \geq 0, \forall t \in \mathbb{R} \Rightarrow$

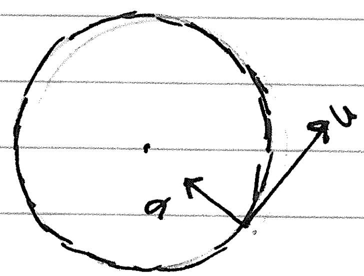
$\Rightarrow \|u(t)\| = |\|u(t)\|| = \sqrt{\|u(t)\|^2} = \sqrt{a}, \forall t \in \mathbb{R} \Rightarrow$

$\Rightarrow \|u(t)\|$ constant on $\mathbb{R}$.

↳ This result admits two interpretations for the case of circular motion.



a) Since in circular motion the distance from the origin is constant, then the velocity vector is always tangent to the circle.



b) Since in circular motion the speed is constant, the accelaration is perpendicular to the velocity and therefore parallel to the corresponding radius. Note that this result holds for any motion where the speed is constant.

EXERCISES

(4) Prove the differentiation properties a, b, c. from the lecture notes.

(5) Let $u : A \to \mathbb{R}^3$ be a vector-valued function and $f : A \to \mathbb{R}$ a function with $A \subseteq \mathbb{R}$. If both u and f are differentiable on A, then prove the scalar quotient rule:

$$\frac{d}{dt}\left[\frac{1}{f(t)}\, u(t)\right] = \left[\frac{1}{f(t)}\right]^2 \left(f(t)\,\dot{u}(t) - f'(t)\,u(t)\right)$$

(6) Let $r : \mathbb{R} \to \mathbb{R}^3$ be a differentiable vector-valued function representing the motion of an object through space. We define:

Angular momentum : $L = r(t) \times \dot{r}(t)$

torque : $\tau = r(t) \times \ddot{r}(t)$

a) Show that $dL/dt = \tau$.

b) Let $a, b \in \mathbb{R}^3$ and $r(t) = a\cos(\omega t) + b\sin(\omega t)$ with $\omega \in \mathbb{R}$. Show that $L(t) = \omega(a \times b)$.

(7) Let $u : \mathbb{R} \to \mathbb{R}^3$ be a differentiable vector-valued function and let $v \in \mathbb{R}^3$. Show that:

$(\forall t \in \mathbb{R} : \|u(t) - v\| = c) \Rightarrow \dot{u}(t) \perp u(t) - v$

(8) Let $u : \mathbb{R} \to \mathbb{R}^3$ and $v : \mathbb{R} \to \mathbb{R}^3$ be differentiable vector-valued functions. Show that:

a) $\forall t \in \mathbb{R} : \|u(t) - v(t)\| = \|v(t)\| \Rightarrow$

$\Rightarrow \forall t \in \mathbb{R} : u(t) \cdot \dot{v}(t) = [u(t) - v(t)] \cdot \dot{u}(t)$

b) $\forall t \in \mathbb{R} : \begin{cases} \|u(t)\| = c_1 \\ \|\dot{u}(t)\| = c_2 \end{cases} \Rightarrow \forall t \in \mathbb{R} : u(t) \,//\, \ddot{u}(t)$

(Hint: First show that $(d/dt)(\|u(t)\| \|\dot{u}(t)\|) =$
$= u(t) \times \ddot{u}(t)$, using $\forall t \in \mathbb{R} : \|u(t)\| = c_1$)

c) $\forall t \in \mathbb{R} : |(d/dt)\|u(t)\|| \leq \|(d/dt)\, u(t)\|$

(Hint: Use $|u|^2 = u \cdot u$ and $|\cos\vartheta| \leq 1$)

d) $\forall t \in \mathbb{R} : (d/dt)\|u(t) \times v(t)\| \leq \|\dot{u}(t)\| \|v(t)\| + \|u(t)\| \|\dot{v}(t)\|$

(Hint: Use the previous result and the triangle inequality $\|u + v\| \leq \|u\| + \|v\|$).

(9) Let $r : \mathbb{R} \to \mathbb{R}^3$ be a ~~twice~~ triple-differentiable vector-valued function, and let $u : \mathbb{R} \to \mathbb{R}$ be a function given by $u(t) = r(t) \cdot (\dot{r}(t) \times \ddot{r}(t))$.
Show that

$$du/dt = r(t) \cdot (\dot{r}(t) \times \dddot{r}(t))$$

Arclength

- Let $(c): r(t)$, $\forall t \in [a,b]$ be a finite curve. The length of the curve is given by:

$$\boxed{\ell = \int_a^b \|\dot{r}(t)\| dt}$$

- For the more general case of an infinite curve $(c): r(t)$, $\forall t \in \mathbb{R}$, we define the arclength function $s(t)$ as:

$$\boxed{s(t) = \int_{t_0}^{t} \|\dot{r}(\tau)\| d\tau \quad , \forall t \in \mathbb{R}}$$

Here $t_0 \in \mathbb{R}$ represents an initial time, usually chosen by default as $t_0 = 0$. The arclength function $s(t)$ gives the distance travelled during the interval $[t_0, t]$ for $t > t_0$.

- We note, from the fundamental theorem of calculus, that

$$\boxed{\frac{ds(t)}{dt} = \|\dot{r}(t)\|}$$

EXAMPLES

Find the arclength function from $t_0 = 0$ for the curve (c): $r(t) = (e^{2t}\cos 2t,\ 2,\ e^{2t}\sin 2t)$.

Solution

For $x(t) = e^{2t}\cos 2t$, $y(t) = 2$, and $z(t) = e^{2t}\sin 2t$, we find that:

$$\dot{x}(t) = (e^{2t})'\cos 2t + e^{2t}(\cos 2t)' =$$
$$= 2e^{2t}\cos 2t - 2e^{2t}\sin 2t =$$
$$= 2e^{2t}(\cos 2t - \sin 2t),$$
$$\dot{y}(t) = 0,$$
$$\dot{z}(t) = (e^{2t})'\sin 2t + e^{2t}(\sin 2t)' =$$
$$= 2e^{2t}\sin 2t + 2e^{2t}\cos 2t =$$
$$= 2e^{2t}(\sin 2t + \cos 2t).$$

It follows that:

$$|\dot{r}(t)|^2 = [\dot{x}(t)]^2 + [\dot{y}(t)]^2 + [\dot{z}(t)]^2 =$$
$$= [2e^{2t}(\cos 2t - \sin 2t)]^2 + [2e^{2t}(\sin 2t + \cos 2t)]^2 =$$
$$= 4e^{4t}[(\cos 2t - \sin 2t)^2 + (\cos 2t + \sin 2t)^2]$$
$$= 4e^{4t}[\cos^2 2t - 2\cos 2t\sin 2t + \sin^2 2t + \cos^2 2t +$$
$$+ 2\cos 2t\sin 2t + \sin^2 2t]$$
$$= 4e^{4t}[2\cos^2 2t + 2\sin^2 2t] = 4e^{4t}\cdot 2 = 8e^{4t} \Rightarrow$$
$$\Rightarrow |\dot{r}(t)| = 2\sqrt{2}\, e^{2t} \Rightarrow$$
$$\Rightarrow \ell = \int_0^t \|\dot{r}(t)\| dt =$$

$$\Rightarrow s(t) = \int_0^t \|\dot{r}(\tau)\| d\tau = \int_0^t 2\sqrt{2}\, e^{2\tau}\, d\tau =$$

$$= 2\sqrt{2} \int_0^t e^{2\tau}\, d\tau = 2\sqrt{2} \left[\frac{e^{2\tau}}{2} \right]_0^t =$$

$$= 2\sqrt{2}\ \frac{e^{2t} - 1}{2} = \sqrt{2}\,(e^{2t} - 1).$$

EXERCISES

(10) Find the arclength function from $t_0=0$ for the curves defined by

a) $r(t) = (a, bt^2, ct^3)$

b) $r(t) = (at^2, bt^3, ct^3)$

c) $r(t) = (at^3, bt^3, ct^3)$

d) $r(t) = (at, b\sin(\omega t), b\cos(\omega t))$

e) $r(t) = (t\sin(\omega t), t\cos(\omega t), ut)$

(11) Consider the decaying helix curve defined by
$r(t) = (e^{-at}\cos(\omega t), e^{-at}\sin(\omega t), ut)$

a) Find the arclength function from $t_0=0$ for the special case $u=0$

b) Extend the previous result to the case $u \neq 0$.

Curvature of a curve (c)

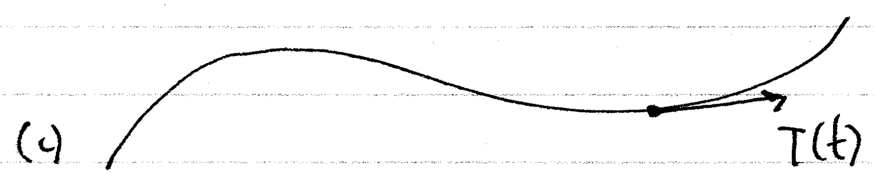
Consider a parameterized curve $(c): (x,y,z) = r(t), \forall t \in A$ representing the motion of an object, and assume that

$$\forall t \in A: \|\dot{r}(t)\| \neq 0$$

meaning that the object is always in motion.

- We define the <u>unit tangent vector</u> $T(t)$ such that

$$\boxed{\forall t \in A: \ T(t) = \frac{\dot{r}(t)}{\|\dot{r}(t)\|}}$$



- If $s(t)$ is the arclength function of the curve definition $(x,y,z) = r(t)$, given by:

$$s(t) = \int_0^t \|\dot{r}(\tau)\| \, d\tau \ , \ \forall t \in A$$

we can define a corresponding inverse function

$$\tau(s_0) = t \Leftrightarrow s(t) = s_0$$

and write T as a function of s: $T(\tau(s))$

We can then define the curvature κ as the rate of change in the direction of $T(t)$ with respect to the arclength $s(t)$:

$$\kappa = \left\| \frac{dT}{ds} \right\|$$

→ The reason why we are not using dT/dt to define curvature is because it depends on the speed with which $r(t)$ traces the curve (c) whose curvature we wish to calculate. Multiplying the speed by a factor of 2 would increase dT/dt by the same factor, without an underlying change in the curvature of the curve traced by $r(t)$.

→ <u>Calculation of $\kappa(t)$</u>

<u>Thm</u>: $\boxed{\forall t \in A: \ \kappa(t) = \dfrac{\|\dot{T}(t)\|}{\|\dot{r}(t)\|}}$

<u>Proof</u>

From the chain rule, we have

$$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} = \frac{dT}{ds}\|\dot{r}(t)\| \Rightarrow$$

$$\Rightarrow \frac{dT}{ds} = \frac{1}{\|\dot{r}(t)\|}\frac{dT}{dt} = \frac{\dot{T}(t)}{\|\dot{r}(t)\|} \Rightarrow$$

$$\Rightarrow \kappa(t) = \left\|\frac{dT}{ds}\right\| = \left\|\frac{\dot{T}(t)}{\|\dot{r}(t)\|}\right\| = \left|\frac{1}{\|\dot{r}(t)\|}\right| \|\dot{T}(t)\|$$

$$= \frac{\|\dot{T}(t)\|}{\|\dot{r}(t)\|}$$

Thm : $\boxed{\forall t \in A: \ \kappa(t) = \dfrac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^3}}$

Proof

First, we note that the arclength function $s(t)$ satisfies

$$s'(t) = \frac{d}{dt} \int_{t_0}^{t} \|\dot{r}(\tau)\| \, d\tau = \|\dot{r}(t)\|$$

▶ Begin with calculating $\|\dot{r}(t) \times \ddot{r}(t)\|$ in terms of $T(t)$.

Since $T(t) = \dot{r}(t) / \|\dot{r}(t)\| \Rightarrow$

$\Rightarrow \dot{r}(t) = T(t) \|\dot{r}(t)\| = T(t)\, s'(t) \Rightarrow$

$\Rightarrow \ddot{r}(t) = (d/dt)[T(t)\, s'(t)] =$

$= \dot{T}(t)\, s'(t) + T(t)\, s''(t)$

it follows that

$\dot{r}(t) \times \ddot{r}(t) = [T(t)\, s'(t)] \times [\dot{T}(t)\, s'(t) + T(t)\, s''(t)] =$

$= [s'(t)]^2 [T(t) \times \dot{T}(t)] + s'(t)\, s''(t) [T(t) \times T(t)]$

$= [s'(t)]^2 [T(t) \times \dot{T}(t)] + s'(t)\, s''(t)\, \mathbf{0} =$

$= [s'(t)]^2 [T(t) \times \dot{T}(t)] = \|\dot{r}(t)\|^2 [T(t) \times \dot{T}(t)]$

We also note that

$(\forall t \in A: \|T(t)\| = 1) \Rightarrow (\forall t \in A: T(t) \perp \dot{T}(t))$

$\Rightarrow \vartheta(T(t), \dot{T}(t)) = \pi/2$

and therefore

$\|\dot{r}(t) \times \ddot{r}(t)\| = \| \, \|\dot{r}(t)\|^2 [T(t) \times \dot{T}(t)] \, \| =$

$= \|\dot{r}(t)\|^2 \, \|T(t) \times \dot{T}(t)\| =$

$$= \|\dot{r}(t)\|^2 \, \|T(t)\| \, \|\dot{T}(t)\| \, \sin\theta\,(T(t), \dot{T}(t)) =$$

$$= \|\dot{r}(t)\|^2 \cdot 1 \cdot \|\dot{T}(t)\| \, \sin(\pi/2) =$$

$$= \|\dot{r}(t)\|^2 \, \|\dot{T}(t)\| \Rightarrow$$

$$\Rightarrow \|\dot{T}(t)\| = \frac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^2} \Rightarrow$$

$$\Rightarrow \kappa(t) = \frac{\|\dot{T}(t)\|}{\|\dot{r}(t)\|} = \frac{1}{\|\dot{r}(t)\|} \frac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^2} =$$

$$= \frac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^3}$$

APPLICATION

Show that a circle (c) with radius R has constant curvature $\kappa(t) = 1/R$.

Proof

Consider the circle (c): $r(t) = (R\cos t, R\sin t, 0)$.

It follows that

$$\dot{r}(t) = (-R\sin t, R\cos t, 0)$$
$$\ddot{r}(t) = (-R\cos t, -R\sin t, 0)$$

and therefore

$$\dot{r}(t) \times \ddot{r}(t) = (-R\sin t, R\cos t, 0) \times (-R\cos t, -R\sin t, 0)$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ -R\sin t & R\cos t & 0 \\ -R\cos t & -R\sin t & 0 \end{vmatrix} \begin{matrix} e_1 & e_2 \\ -R\sin t & R\cos t \\ -R\cos t & -R\sin t \end{matrix} =$$

$$= e_3(-R\sin t)(-R\sin t) - e_3(R\cos t)(-R\cos t)$$
$$= e_3\left[R^2\sin^2 t + R^2\cos^2 t\right] = R^2 e_3(\sin^2 t + \cos^2 t)$$
$$= R^2 e_3 \Rightarrow$$

$$\Rightarrow \|\dot{r}(t) \times \ddot{r}(t)\| = \|R^2 e_3\| = R^2\|e_3\| = R^2$$

Also

$$\|\dot{r}(t)\| = \|(-R\sin t, R\cos t, 0)\| = \sqrt{(-R\sin t)^2 + (R\cos t)^2 + 0^2}$$
$$= \sqrt{R^2(\cos^2 t + \sin^2 t)} = \sqrt{R^2} = R$$

and we conclude that

$$\kappa(t) = \frac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^3} = \frac{R^2}{R^3} = \frac{1}{R}$$

APPLICATION

Show that the curvature of a two-dimensional curve $(c): y = f(x)$ at x is given by

$$\boxed{\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}}$$

Proof

We rewrite the definition of the curve parametrically as $(c): r(x) = (x, f(x), 0)$ and note that

$$\begin{cases} \dot{r}(x) = (1, f'(x), 0) \\ \ddot{r}(x) = (0, f''(x), 0) \end{cases}$$

It follows that

$$\dot{r}(x) \times \ddot{r}(x) = (1, f'(x), 0) \times (0, f''(x), 0) =$$

$$= \left| \begin{array}{ccc|cc} e_1 & e_2 & e_3 & e_1 & e_2 \\ 1 & f'(x) & 0 & 1 & f'(x) \\ 0 & f''(x) & 0 & 0 & f''(x) \end{array} \right. =$$

$$= f''(x) e_3 \Rightarrow$$

$$\Rightarrow \|\dot{r}(x) \times \ddot{r}(x)\| = \|f''(x) e_3\| = |f''(x)| \, \|e_3\| = |f''(x)|$$

and

$$\|\dot{r}(x)\| = \|(1, f'(x), 0)\| = \sqrt{1^2 + [f'(x)]^2 + 0^2}$$

$$= \sqrt{1 + [f'(x)]^2}$$

and therefore

$$\kappa(x) = \frac{\|\dot{r}(x) \times \ddot{r}(x)\|}{\|\dot{r}(x)\|^3} = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$$

EXERCISES

(12) Find the curvature $\kappa(t)$ for the curves defined by

a) $r(t) = (at^2, bt, ct^3)$
b) $r(t) = (a\cos(\omega t), a\sin(\omega t), ut)$
c) $r(t) = (\cos^3 t, 0, \sin^3 t)$ at $t = \pi/2$
d) $r(t) = (3\cosh(t/3), t, 0)$ at $t = 1$
e) $r(t) = (e^{at}\cos(\omega t), e^{at}\sin(\omega t), e^{at})$
f) $r(t) = (e^{at}, e^{-at}, ut)$
g) $r(t) = (\ln t, at, bt^2)$ at $t = 1$

(13) For the following curves, find the (x,y) coordinates of the point where the curvature is maximum

a) $(c): y = \ln x$
b) $(c): y = \cosh x$
c) $(c): y = \sin x \wedge x \in [-\pi, \pi]$
d) $(c): y = \sinh x$
e) $(c): y = e^x$
f) $(c): y = \ln(\cos x) \wedge x \in [-\pi/2, \pi/2)$