VECTOR FIELDS

Definition

- A <u>three-dimensional vector field</u> f is a mapping $f: A \to \mathbb{R}^3$ with $A \subseteq \mathbb{R}^3$.
- If f is a vector field, we write:
  $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$
- The scalar fields f_1, f_2, f_3 are the <u>components</u> of the vector field f.

Derivatives of a vector field

- Let $f: A \to \mathbb{R}^3$ be a vector field with components f_1, f_2, f_3 that are assumed to be partially differentiable. Then we define:

a) The divergence of f

$$\boxed{\nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}}$$

b) The curl of f

$$\boxed{\nabla \times f = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}}$$

- Note that

  $\nabla \cdot f$ is a scalar field

  $\nabla \times f$ is a vector field.

- Let $\varphi : A \to \mathbb{R}$ be a scalar field with $A \subseteq \mathbb{R}^3$.

  Then we define:

a) The gradient of φ

$$\boxed{\nabla \varphi = (\partial \varphi / \partial x, \partial \varphi / \partial y, \partial \varphi / \partial z)}$$

b) The Laplacian of φ

$$\boxed{\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}}$$

Tensor notation

Tensor notation makes it easier to work with the derivatives of vector fields. We follow the following guidelines:

a) We write the vector field "f" as "f_a", representing the "a" component of f. With tensor notation we always work in terms of the components of the involved vector fields.

b) Repeating indices are automatically summed over all components, when associated with a product

example: $f_a g_a = f_1 g_1 + f_2 g_2 + f_3 g_3 = f \cdot g$
represents the dot product
However, for the vector sum
$(f+g)_a = f_a + g_a$
no summation is implied.

c) We abbreviate the partial derivative $\partial / \partial x_a$ as ∂_a.
We may thus write:
$$\nabla \cdot f = \partial_a f_a \; ; \; (\nabla \varphi)_a = \partial_a \varphi \; ; \; \nabla^2 \varphi = \partial_a \partial_a \varphi$$

d) To define the curl, we introduce the Levi-Civita tensor ε_{abc} as:

$$\boxed{\varepsilon_{abc} = \frac{(a-b)(b-c)(c-a)}{2} = \begin{cases} +1, & \text{if } (a,b,c) \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1, & \text{if } (a,b,c) \in \{(3,2,1),(1,3,2),(2,1,3)\} \\ 0, & \text{if } a=b \vee b=c \vee c=a \end{cases}}$$

Then, the curl reads:
$$(\nabla \times f)_a = \varepsilon_{abc} \partial_b f_c$$
Likewise, for two vector fields f, g, the cross-product reads:
$$(f \times g)_a = \varepsilon_{abc} f_b g_c$$

• To summarize: Given the vector fields f, g and the scalar field φ:

$(f+g)_a = f_a + g_a$ $f \cdot g = f_a g_a$ $f \times g = \varepsilon_{abc} f_b g_c$	$\nabla\varphi = \partial_a \varphi$ $\nabla^2 \varphi = \partial_a \partial_a \varphi$	$\nabla \cdot f = \partial_a f_a$ $\nabla \times f = \varepsilon_{abc} \partial_b f_c$

▸ Kronecker delta

We define:

$$\delta_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that $\delta_{ab} f_b = f_a$

(because in the implied summation $b = 1, 2, 3$ the only non-zero contribution occurs when $a = b$).

Similar contractions are possible. For example:

$\delta_{ab}\, \varepsilon_{acd} = \varepsilon_{bcd}$

(because in the implied summation $a = 1, 2, 3$ the only non-zero contribution occurs when $a = b$).

▸ Properties of Levi-Civita tensor

We can show that ε_{abc} satisfy the following properties:

1) $$\boxed{\begin{aligned} \varepsilon_{abc} f_b f_c &= 0_a \\ \varepsilon_{abc} \partial_b \partial_c &= 0_a \end{aligned}}$$ $\leftarrow$ 0_a is the "a" component of the field $0 = (0,0,0)$.

2) $$\boxed{\begin{aligned} \varepsilon_{abc} &= \varepsilon_{bca} = \varepsilon_{cab} \\ \varepsilon_{abc} &= -\varepsilon_{cba} \end{aligned}}$$

3) Relation to the kronecker delta

$$\boxed{\varepsilon_{abc}\,\varepsilon_{pqr} = \begin{vmatrix} \delta_{ap} & \delta_{aq} & \delta_{ar} \\ \delta_{bp} & \delta_{bq} & \delta_{br} \\ \delta_{cp} & \delta_{cq} & \delta_{cr} \end{vmatrix}}$$

4) Contracted epsilon identities

$$\boxed{\begin{aligned} \varepsilon_{abc}\,\varepsilon_{apq} &= \delta_{bp}\delta_{cq} - \delta_{bq}\delta_{cp} \\ \varepsilon_{abp}\,\varepsilon_{abq} &= 2\delta_{pq} \\ \varepsilon_{abc}\,\varepsilon_{abc} &= 6 \end{aligned}}$$

Note that (4) is a consequence of (3). All properties of the curl and cross-product are encapsulated in properties (1), (2), (3).

Second derivatives

Tensor notation can be used to establish the following properties about 2nd derivatives:

1) $\boxed{\nabla \times (\nabla \varphi) = \mathbf{0}}$

Proof

$$[\nabla \times (\nabla \varphi)]_a = \varepsilon_{abc} \partial_b (\nabla \varphi)_c = \varepsilon_{abc} \partial_b (\partial_c \varphi) =$$
$$= (\varepsilon_{abc} \partial_b \partial_c) \varphi = \mathbf{0}_a \varphi = \mathbf{0}_a \Rightarrow$$
$$\Rightarrow \nabla \times \nabla \varphi = \mathbf{0}. \quad \square$$

2) $\boxed{\nabla \cdot (\nabla \times f) = 0}$

Proof

$$\nabla \cdot (\nabla \times f) = \partial_a (\nabla \times f)_a = \partial_a (\varepsilon_{abc} \partial_b f_c) =$$
$$= \varepsilon_{abc} \partial_a \partial_b f_c = (\varepsilon_{cab} \partial_a \partial_b) f_c$$
$$= \mathbf{0}_c f_c = 0$$

3) $\boxed{\nabla \times (\nabla \times f) = \nabla (\nabla \cdot f) - \nabla^2 f}$

Proof

$$[\nabla\times(\nabla\times f)]_a = \varepsilon_{abc}\partial_b(\nabla\times f)_c = \varepsilon_{abc}\partial_b(\varepsilon_{cpq}\partial_p f_q)$$
$$= \varepsilon_{abc}\varepsilon_{cpq}\partial_b\partial_p f_q = \varepsilon_{cab}\varepsilon_{cpq}\partial_b\partial_p f_q =$$
$$= (\delta_{ap}\delta_{bq} - \delta_{aq}\delta_{bp})\partial_b\partial_p f_q =$$
$$= \delta_{ap}\delta_{bq}\partial_b\partial_p f_q - \delta_{aq}\delta_{bp}\partial_b\partial_p f_q =$$
$$= \delta_{ap}\partial_q\partial_p f_q - \delta_{aq}\partial_p\partial_p f_q =$$
$$= \partial_q\partial_a f_q - \partial_p\partial_p f_a = \partial_a(\partial_q f_q) - \partial_p\partial_p f_a =$$
$$= \partial_a(\nabla\cdot f) - \nabla^2 f_a = [\nabla(\nabla\cdot f)]_a - \nabla^2 f_a \Rightarrow$$
$$\Rightarrow \nabla\times(\nabla\times f) = \nabla(\nabla\cdot f) - \nabla^2 f. \quad \square$$

Note that δ_{ab} and ε_{abc} are constants so they freely commute with the operator ∂_a. Be careful, however. Sometimes it is necessary to employ the product rule. For example:

EXAMPLE

Show that: $\nabla\cdot(\varphi f) = f\cdot\nabla\varphi + \varphi\nabla\cdot f$
with φ scalar field and f vector field.

Solution

$$\nabla\cdot(\varphi f) = \partial_a(\varphi f)_a = \partial_a(\varphi f_a) = f_a\partial_a\varphi + \varphi\partial_a f_a$$
$$= (f\cdot\nabla)\varphi + \varphi(\nabla\cdot f) =$$
$$= f\cdot\nabla\varphi + \varphi(\nabla\cdot f).$$

EXERCISES

① Evaluate $\nabla \cdot F$ and $\nabla \times F$ for the following vector fields:

a) $F(x,y,z) = (x^3, 2xy, yz^3)$

b) $F(x,y,z) = (x^a, y^a, z^a)$ with $a \in \mathbb{R}$

c) $F(x,y,z) = (yz, zx, xy)$

d) $F(x,y,z) = (\cos(ax), \sin(by), c)$ with $a, b, c \in \mathbb{R}$

e) $F(x,y,z) = (e^x \cos(ay), e^x \sin(by), z)$

f) $F(x,y,z) = (x+y, y+z, z+x)$

② Evaluate $\nabla^2 f$ for the following scalar fields

a) $f(x,y,z) = x^3+y^3+z^3-3xyz$

b) $f(x,y,z) = \cos(xyz)$

c) $f(x,y,z) = \ln|xyz|$

d) $f(x,y,z) = x^a + y^a + z^a$

e) $f(x,y,z) = xe^y \sin(z)$

f) $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

③ Use tensor notation to show the following identities. Here F, G are vector fields and φ, ψ are scalar fields

a) $\nabla \cdot (F+G) = \nabla \cdot F + \nabla \cdot G$

b) $\nabla \times (F+G) = \nabla \times F + \nabla \times G$

c) $\nabla \times (\nabla \varphi) = \mathbf{0}$

d) $\nabla \cdot (\nabla \times F) = 0$

e) $\nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - \nabla^2 F$

f) $\nabla \cdot (\varphi F) = \varphi(\nabla \cdot F) + F \cdot (\nabla \varphi)$

g) $\nabla \|F\|^2 = 2[(F \cdot \nabla)F + F \times (\nabla \times F)]$

h) $\nabla(F \cdot G) = (F \cdot \nabla)G + (G \cdot \nabla)F + F \times (\nabla \times G) + G \times (\nabla \times F)$

i) $\nabla \cdot (F \times G) = (\nabla \times F) \cdot G - F \cdot (\nabla \times G)$

j) $\nabla \times (F \times G) = F(\nabla \cdot G) - G(\nabla \cdot F) - (F \cdot \nabla)G + (G \cdot \nabla)F$

k) $\nabla \cdot (\varphi \nabla \psi) = \varphi \nabla^2 \psi + (\nabla \varphi) \cdot (\nabla \psi)$

l) $\psi \nabla^2 \varphi - \varphi \nabla^2 \psi = \nabla \cdot (\psi \nabla \varphi - \varphi \nabla \psi)$

m) $\nabla^2(\varphi \psi) = \varphi \nabla^2 \psi + 2(\nabla \varphi) \cdot (\nabla \psi) + \psi \nabla^2 \varphi$

n) $\nabla^2(\varphi F) = F \nabla^2 \varphi + 2(\nabla \varphi \cdot \nabla)F + \varphi \nabla^2 F$

o) $\nabla^2(F \cdot G) = F \cdot (\nabla^2 G) - G \cdot (\nabla^2 F) + 2\nabla \cdot [(G \cdot \nabla)F + G \times (\nabla \times F)]$

p) $\nabla^2(\nabla \varphi) = \nabla(\nabla^2 \varphi)$

r) $\nabla^2(\nabla \times F) = \nabla \times (\nabla^2 F)$.

(4) Let $x = (x_1, x_2, x_3)$ be a vector.

(a) Explain why $\partial_a x_b = \delta_{ab}$ and $\partial_a \|x\| = x_a / \|x\|$.

(b) If f is differentiable scalar function, show that
$$\partial_a f(\|x\|) = f'(\|x\|)\, x_a / \|x\|$$

(c) Use the previous results to show that
$$\partial_a \left[\frac{x_a}{\|x\|} f(\|x\|) \right] = \frac{2 f(\|x\|)}{\|x\|} + f'(\|x\|)$$

(5) Consider the vector field $F(x) = x/\|x\|^m$ with $x = (x_1, x_2, x_3)$ a vector. Show that for all $x \in \mathbb{R}^3 - \{0\}$ we have:

a) $m = 3 \Rightarrow \nabla \cdot F = 0 \wedge \nabla \times F = 0$

b) $m \neq 3 \Rightarrow \nabla \cdot F \neq 0 \wedge \nabla \times F = 0$

(6) The Navier-Stokes equations govern the velocity field $u(x_1, x_2, x_3, t)$ of an incompressible fluid, and written using tensor notation, they read

$$\begin{cases} \partial u_a/\partial t + u_b \partial_b u_a = -\partial_a p + \nu \nabla^2 u_a \\ \partial_a u_a = 0 \leftarrow \text{(incompressibility condition)} \end{cases}$$

with $\nu > 0$ the fluid viscosity and p the pressure field. Show, using tensor notation, that

a) $u_b \partial_b u_a = \partial_b (u_a u_b)$

b) $\nabla^2 p = -\partial_a \partial_b (u_a u_b)$

c) The vorticity field defined as $w = \nabla \times u$ satisfies the equation

$$\frac{\partial w_a}{\partial t} + u_b \partial_b w_a = w_b \partial_b u_a + \nu \nabla^2 w_a$$

▼ Line Integrals

- Line integrals are integrals of vector fields defined over a path.

↳ Definitions concerning paths

- Let $a: I \to \mathbb{R}^n$ be a vector-valued function. We say that

$$a \text{ is a } \underline{\text{path}} \Leftrightarrow \begin{cases} \exists t_1, t_2 \in \mathbb{R} : I = [t_1, t_2] \\ a \text{ continuous at } [t_1, t_2]. \end{cases}$$

Furthermore, if a is a path, we say that

a is a <u>closed path</u> $\Leftrightarrow a(t_1) = a(t_2)$

a is an <u>open path</u> $\Leftrightarrow a(t_1) \neq a(t_2)$.

We say that: $a(t_1) =$ <u>initial point</u>

$a(t_2) =$ <u>final point</u>

- Consider a path $a: [t_1, t_2] \longrightarrow \mathbb{R}^n$. We say that

$$a \text{ } \underline{\text{smooth}} \text{ path} \Leftrightarrow \begin{cases} a \text{ continuous at } [t_1, t_2] \\ a \text{ differentiable at } (t_1, t_2) \\ \dot{a} \text{ continuous at } (t_1, t_2). \end{cases}$$

- Let $a: [t_1, t_2] \longrightarrow \mathbb{R}^n$ be a path, and let $B \subseteq [t_1, t_2]$. The <u>restriction</u> of "a" to B is denoted as:

$$b = a \restriction B \Leftrightarrow \begin{cases} b : B \longrightarrow \mathbb{R}^n \\ \forall t \in B : b(t) = a(t) \end{cases}$$

- Let $a : [t_1, t_2] \to \mathbb{R}^n$ be a path. We say that

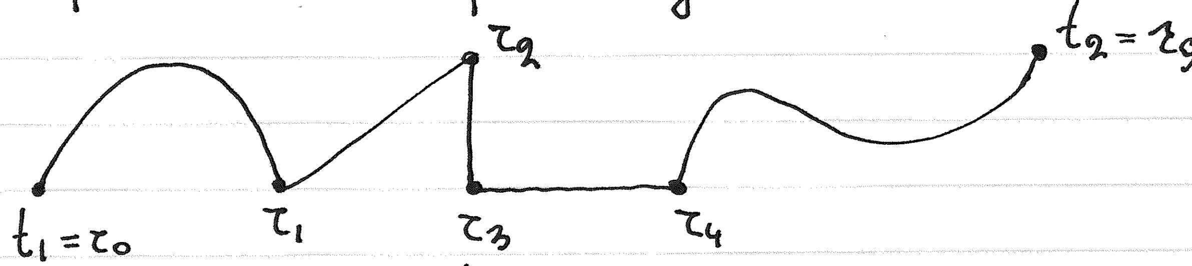
a piecewise smooth path $\iff$

$\iff \exists \tau_0, \tau_1, \ldots, \tau_n \in [t_1, t_2] :$

$$: \begin{cases} t_1 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n = t_2 \\ \forall k \in [n] : a \restriction [\tau_{k-1}, \tau_k] \text{ smooth path} \end{cases}$$

EXAMPLE

A piecewise smooth path may look like this:



Connected sets

- Let $x, y \in \mathbb{R}^n$ be given points. We define:

$\mathcal{P}_A(x, y) =$ the set of all piecewise smooth paths in A with initial point x and final point y.

- Let $A \subseteq \mathbb{R}^n$ be a given region. We say that

$$\boxed{\begin{array}{l} A \text{ path-connected} \iff \forall x, y \in A : \mathcal{P}_A(x, y) \neq \emptyset \\ A \text{ path-disconnected} \iff \exists x, y \in A : \mathcal{P}_A(x, y) = \emptyset \end{array}}$$

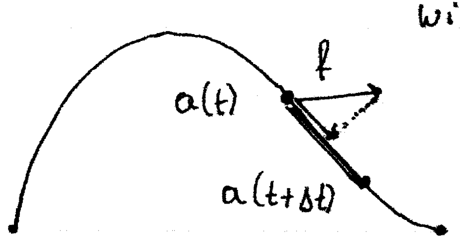
▸ interpretation: In a path-connected set A, every two points $x, y \in A$ are connected by at least one piecewise smooth path.

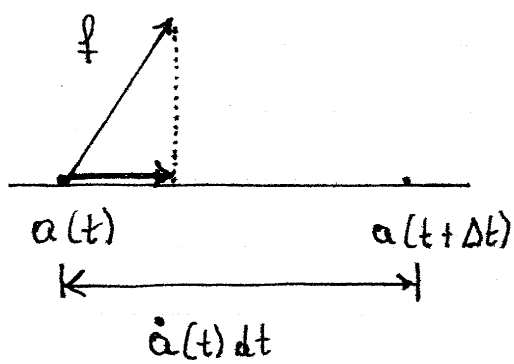
Definition of line integral

- Let $a : [t_1, t_2] \to \mathbb{R}^n$ be a piecewise smooth path and let $f : A \to \mathbb{R}^n$ be a vector field with $A \subseteq \mathbb{R}^n$ such that $a([t_1, t_2]) \subseteq A$. We define the line integral

$$\boxed{\int f \cdot da = \int_{t_1}^{t_2} f(a(t)) \cdot \dot{a}(t)\, dt}$$

▶ Interpretation : Consider two points $a(t)$ and $a(t+dt)$ with $dt \to 0$. Then $\dot{a}(t)\,dt$





represents the distance between the points $a(t)$ and $a(t+\Delta t)$. The dot product $f(a(t)) \cdot \dot{a}(t)\,dt$ gives the product of this point distance with the directed magnitude of the projection of the vector f along the direction defined by the points $a(t)$ and $a(t+dt)$. Consequently, only the tangential component of f contributes to the integral. The normal component of f does not contribute at all.

▸ <u>Notation</u>

a) When we write a line integral in the <u>indefinite form</u> as

$$I = \int f \cdot da$$

then the differential da points to the path $a : [t_1, t_2] \longrightarrow \mathbb{R}^n$ over which f is integrated.

b) Alternatively, if we define the path as:

$$C : a(t) = (a_1(t), \ldots, a_n(t)), \quad t \in [t_1, t_2]$$

with C representing the path, we may write the line integral in <u>definite form</u> as:

$$I = \int f \cdot da = \int_C f \cdot dl$$

Here, the l in dl is just a dummy variable that doesn't really mean anything. It is standard to use the letter "l" as the dummy variable in line integrals when written in the definite form.

c) Consider a vector field f with components:

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

and the path $a : [t_1, t_2] \longrightarrow \mathbb{R}^n$ with components:

$\mathbf{a}(t) = (a_1(t), a_2(t), \ldots, a_n(t))$, $\forall t \in [t_1, t_2]$

The line integral can be written in indefinite component form as:

$$I = \int f \cdot da = \int f_1\, da_1 + f_2\, da_2 + \cdots + f_n\, da_n$$

and in definite component form as:

$$I = \int_C f \cdot dl = \int_C f_1\, dx_1 + f_2\, dx_2 + \cdots + f_n\, dx_n$$

d) For a two-dimensional vector field

$f(x,y) = (f_1(x,y), f_2(x,y))$

the component forms of a line integral of f can be written as:

$$I = \int f \cdot da = \int f_1(x,y)\, da_1 + f_2(x,y)\, da_2 =$$

$$= \int_C f_1(x,y)\, dx + f_2(x,y)\, dy$$

e) Similarly, for a three-dimensional vector field

$f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$

the component forms of a line integral of f can be written as:

$$I = \int f \cdot da =$$

$$= \int f_1(x,y,z)\, da_1 + f_2(x,y,z)\, da_2 + f_3(x,y,z)\, da_3 =$$

$$= \int_C f_1(x,y,z)\, dx + f_2(x,y,z)\, dy + f_3(x,y,z)\, dz.$$

EXAMPLE

Evaluate the integral $I = \int_C (x^2-y^2)dx + 2xy\,dy$ over the curve:

$C : a(t) = (t^2, t^3), \ \forall t \in [0,1].$

Solution

We note that $\dot{a}(t) = (2t, 3t^2), \ \forall t \in [0,1]$.
It follows that

$$I = \int_C (x^2-y^2)dx + 2xy\,dy =$$

$$= \int_0^1 \left((t^2)^2 - (t^3)^2,\ 2(t^2)(t^3)\right) \cdot (2t, 3t^2)\, dt =$$

$$= \int_0^1 (t^4 - t^6, 2t^5) \cdot (2t, 3t^2)\, dt =$$

$$= \int_0^1 \left[2t(t^4 - t^6) + (2t^5)(3t^2)\right] dt =$$

$$= \int_0^1 (2t^5 - 2t^7 + 6t^7)\,dt = \int_0^1 (2t^5 + 4t^7)\,dt =$$

$$= \left[\frac{2t^6}{6} + \frac{4t^8}{8} \right]_0^1 = \left[\frac{t^6}{3} + \frac{t^8}{2} \right]_0^1 =$$

$$= \frac{1^6 - 0^6}{3} + \frac{1^8 - 0^8}{2} = \frac{1}{3} + \frac{1}{2} =$$

$$= \frac{2+3}{6} = \frac{5}{6}$$

Basic Properties of line integrals

Linearity

Let f, g be two vector fields, $\lambda_1, \lambda_2 \in \mathbb{R}$, and let C be a path. Then:

$$\boxed{\int_C (\lambda_1 f + \lambda_2 g) \cdot dl = \lambda_1 \int_C f \cdot dl + \lambda_2 \int_C g \cdot dl}$$

Path Equivalence

Consider two paths $a : [a_1, a_2] \longrightarrow \mathbb{R}^n$ and $b : [b_1, b_2] \longrightarrow \mathbb{R}^n$. It is possible for a and b to sketch the same curve but with different velocities. In that case we want to be able to say that a and b are <u>equivalent</u> (notation: $a \equiv b$). Furthermore, we would like equivalent paths over the same function to give equal line integrals. We give a formal definition as follows:

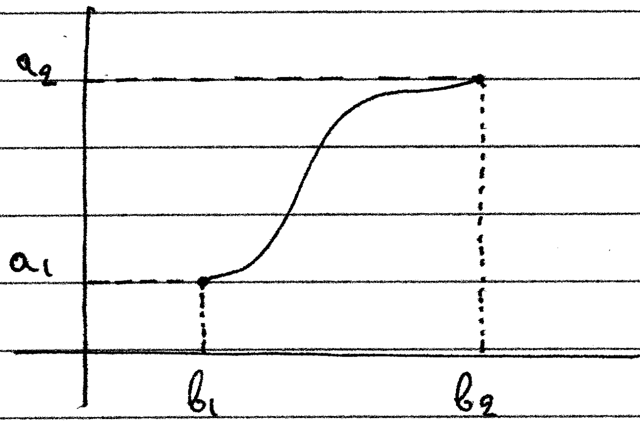
<u>Def</u>: We say that $a \equiv b$ (a is equivalent to b) if and only if there is a function $u : [b_1, b_2] \to [a_1, a_2]$ such that

a) $u([b_1, b_2]) = [a_1, a_2]$

b) u differentiable on $[b_1, b_2]$

c) $\forall t \in [b_1, b_2] : u'(t) > 0$

d) $\forall t \in [b_1, b_2] : b(t) = a(u(t))$



▶ Note that an immediate consequence of the above conditions is that

$$\begin{cases} u(b_1) = a_1 \\ u(b_2) = a_2 \end{cases}$$

<u>Thm</u> : Let $a : [a_1, a_2] \longrightarrow \mathbb{R}^n$ and $b : [b_1, b_2] \longrightarrow \mathbb{R}^n$ be two paths. Then:

$$\boxed{a \equiv b \Rightarrow \int f \cdot da = \int f \cdot db}$$

<u>Proof</u>

Since $a \equiv b$, there is a function $u : [b_1, b_2] \to [a_1, a_2]$ such that $u(b_1) = a_1 \wedge u(b_2) = a_2$ and

$\forall t \in [b_1, b_2] : b(t) = a(u(t)) \Rightarrow$

$\Rightarrow \forall t \in [b_1, b_2] : \dot{b}(t) = \dot{a}(u(t))\, u'(t)$

It follows that:

$$\int f \cdot db = \int_{b_1}^{b_2} f(b(t)) \cdot \dot{b}(t)\, dt =$$

$$= \int_{b_1}^{b_2} f(a(u(t))) \cdot \dot{a}(u(t))\, u'(t)\, dt.$$

Let $\tau = u(t) \Rightarrow \begin{cases} d\tau = u'(t)\, dt \\ u(b_1) = a_1 \\ u(b_2) = a_2 \end{cases} \Rightarrow$

$$\Rightarrow \int f \cdot db = \int_{a_1}^{a_2} f(a(\tau)) \cdot \dot{a}(\tau)\, d\tau = \int f \cdot da \qquad \square$$

Path Merging

- Consider 3 path defined as:

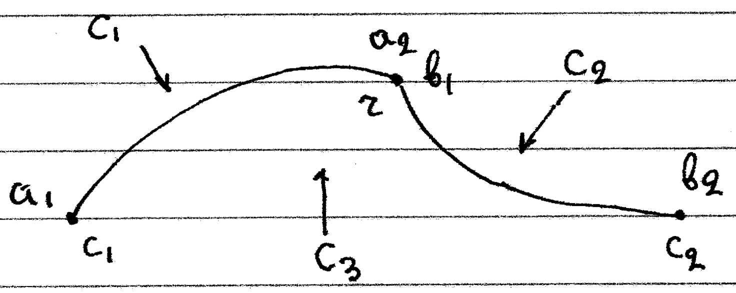
$C_1 : a(t) \;,\; \forall t \in [a_1, a_2]$

$C_2 : b(t) \;,\; \forall t \in [b_1, b_2]$

$C_3 : c(t) \;,\; \forall t \in [c_1, c_2]$

We say that

$$\boxed{C_3 = C_1 \cup C_2 \iff \exists\, \tau \in [c_1, c_2] : \begin{cases} a \equiv c \restriction [c_1, \tau] \\ b \equiv c \restriction [\tau, c_2] \end{cases}}$$



- Let f be a vector field. Then it can be shown that:

$$\boxed{\int_{C_1 \cup C_2} f \cdot dl = \int_{C_1} f \cdot dl + \int_{C_2} f \cdot dl.}$$

Path Reversal

- Let $C : a(t)$, $\forall t \in [t_1, t_2]$ be a path. We define the reverse path $-C$ as follows:

$$-C : b(t) = a(t_1 + t_2 - t), \ \forall t \in [t_1, t_2]$$

- The reverse path $-C$ traverses the same points in space as C, but in the reverse direction.
- Given a vector field f, we can show that:

$$\boxed{\int_{-C} f \cdot dl = -\int_{C} f \cdot dl}$$

EXAMPLES

⑦ Evaluate the following line integrals over the given paths

a) $I = \int_{C_1} e^x dx + e^{-y} dy$

with $C_1 : (x,y) = (\ln t, \ln(bt))$, $\forall t \in [1, a+1]$
and $a, b \in (0, +\infty)$.

b) $I = \int_{C_1} (x+y)\,dx + (x-y)\,dy$

with $C_1 : (x,y) = (a\cos t, b\sin t)$, $\forall t \in [0, \pi/2]$
and $a, b \in (0, +\infty)$.

c) $I = \int_{C_1} (x+3y)\,dx + (2y-z)\,dy + (z+x)\,dz$

with $C_1 : (x,y,z) = (0,0,0) + t(a,a,b)$, $\forall t \in [0,1]$
and $a, b \in (0, +\infty)$.

d) $I = \int_{C_1} z\,dx + x^2 dy + y\,dz$

with $C_1 : (x,y,z) = (\cos t, \tan t, t)$, $\forall t \in [0, \pi/4]$

e) $I = \int_{C} \frac{-y\,dx + x\,dy}{x^2+y^2}$

with $C: (x,y) = (1,0) + t(-1,1)$, $\forall t \in [0,1]$

f) $I = \int_{C} y^2\,dx + z^2\,dy + (1-x^2)\,dz$

with $C: (x,y,z) = (a\cos t, 1, a\sin t)$, $\forall t \in [0, \pi/4]$
and $a \in (0,+\infty)$

g) $I = \int_{C} \frac{-y\,dx + x\,dy}{(x^2+y^2)^2}$

with $C: (x,y) = (a\cos t, a\sin t)$, $\forall t \in [0, \pi/6]$
and $a \in (0,+\infty)$

h) $I = \int_{C} \frac{(x+y)\,dx - (x-y)\,dy}{x^2+y^2}$

with $C: (x,y) = (a\cos t, a\sin t)$, $\forall t \in [0, 2\pi]$
and $a \in (0,+\infty)$

i) $I = \int_{C} \frac{dx + dy}{|x|+|y|}$

with C the square with vertices $(1,0), (0,1), (-1,0), (0,-1)$
traversed once in the counterclockwise direction.

j) $I = \int_{C} (x^2+y^2)\,dx + (x^2-y^2)\,dy$

with $C: (x,y) = (t, 1-|1-t|), \ \forall t \in [0,2]$

▼ Conservative fields and potential functions

Thm: Let $\varphi : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field. Let $x, y \in A$ be given. Assume that

a) A is open and path-connected.

b) φ differentiable in A

c) $\nabla\varphi$ continuous in A

Then:

$$\boxed{\forall C \in \mathcal{P}_A(x,y) : \int_C \nabla\varphi \cdot d\ell = \varphi(y) - \varphi(x)}$$

Proof

Let $C \in \mathcal{P}_A(x,y)$ with $C : a(t), \forall t \in [a_1, a_2]$ be given such that $a(a_1) = x$ and $a(a_2) = y$. Define:

$$\forall t \in [a_1, a_2] : g(t) = \varphi(a(t))$$

Then: $\forall t \in [a_1, a_2] : g'(t) = \nabla\varphi(a(t)) \cdot \dot{a}(t)$

Since $a(t)$ is piecewise smooth, let $a_1 = t_0 < t_1 < t_2 < \cdots < t_n = a_2$ be a partition of the interval $[a_1, a_2]$ such that

$\forall k \in [n] : a \restriction [t_{k-1}, t_k]$ is a smooth path

It follows that:

$$\int_C \nabla\varphi \cdot d\ell = \int_{a_1}^{a_2} \nabla\varphi(a(t)) \cdot \dot{a}(t)\,dt = \int_{a_1}^{a_2} g'(t)\,dt =$$

$$= \sum_{k\in[n]} \int_{t_{k-1}}^{t_k} g'(t)\,dt = \sum_{k\in[n]} [g(t_k) - g(t_{k-1})] =$$

$$= g(t_n) - g(t_0) = g(a_2) - g(a_1) =$$
$$= \varphi(a(a_2)) - \varphi(a(a_1)) = \varphi(y) - \varphi(x) \qquad \square$$

- We see that the line integral depends only on the initial and final points, and is independent of the path connecting the two points.

Potential Functions

<u>Def</u>: Let $f: A \to \mathbb{R}^n$ be a vector field with $A \subseteq \mathbb{R}^n$. We say that

$$\boxed{f \text{ conservative} \iff \exists \varphi : \begin{cases} \varphi : A \to \mathbb{R} \\ \forall x \in A: f(x) = \nabla\varphi(x) \end{cases}}$$

- If f is a conservative vector field and $f = \nabla\varphi$, then φ = potential function of f. and furthermore:

$$\boxed{C \in \mathcal{P}_A(a,b) \Rightarrow \int_C f \cdot dl = \varphi(b) - \varphi(a)}$$

If a vector field is conservative, the fastest way to find its potential function is as follows:

▶ Method: How to find the potential.

Consider, for example, a three-dimensional ~~function~~ vector field $f = (f_1, f_2, f_3)$. If $f = \nabla\varphi$, then:
$$\frac{\partial\varphi}{\partial x} = f_1 \quad \text{and} \quad \frac{\partial\varphi}{\partial y} = f_2 \quad \text{and} \quad \frac{\partial\varphi}{\partial z} = f_3$$

Integrating with respect to x, y, z gives:

$$\varphi(x,y,z) = \int f_1(x,y,z)\,dx + A(y,z) \qquad (1)$$

$$\varphi(x,y,z) = \int f_2(x,y,z)\,dy + B(z,x) \qquad (2)$$

$$\varphi(x,y,z) = \int f_3(x,y,z)\,dz + C(x,y) \qquad (3)$$

Here $A(y,z), B(z,x), C(x,y)$ are integration constants. To find φ, it is sufficient to define A, B, C such that equations (1), (2), (3) agree with each other. Then any one of the equations (1), (2), (3) yields the potential function $\varphi(x,y,z)$.

EXAMPLE

Show that the function
$$f(x,y,z) = (2xyz + z^2 - 2y^2 + 1,\ x^2z - 4xy,\ x^2y + 2xz - 2)$$
Find a potential function φ such that $f = \nabla\varphi$.

Solution

Given $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$

with $f_1(x,y,z) = 2xyz + z^2 - 2y^2 + 1$

$f_2(x,y,z) = x^2z - 4xy$

$f_3(x,y,z) = x^2y + 2xz - 2$

We note that

$$\varphi(x,y,z) = \int f_1(x,y,z)\,dx + A(y,z) =$$

$$= \int (2xyz + z^2 - 2y^2 + 1)\,dx + A(y,z) =$$

$$= x^2yz + xz^2 - 2xy^2 + x + A(y,z).$$

$$\varphi(x,y,z) = \int f_2(x,y,z)\,dy + B(z,x) =$$

$$= \int (x^2z - 4xy)\,dy + B(z,x) =$$

$$= x^2yz - 2xy^2 + B(z,x)$$

$$\varphi(x,y,z) = \int f_3(x,y,z)\,dz + C(x,y) =$$

$$= \int (x^2y + 2xz - 2)\,dz + C(x,y) =$$

$$= x^2yz + xz^2 - 2z + C(x,y).$$

•1 We guess the potential function by merging all the terms from the above 3 equations, noting that they share some but not all terms.

$$\varphi(x,y,z) = x^2yz + xz^2 - 2xy^2 + x - 2z$$

•2 Now we define the functions $A(y,z)$, $B(z,x)$, $C(x,y)$ by identifying the missing terms:

$$A(y,z) = -2z$$
$$B(z,x) = xz^2 + x - 2z$$
$$C(x,y) = -2xy + x$$

•3 In order for the proposed potential function φ to be a legitimate potential, it is necessary that in the proposed definitions of A, B, C that A be independent of x, B be independent of y, C be independent of z. If this is not possible, then the original vector field is probably not conservative.

Potential as a line integral

<u>Def</u> : Let $f: A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^n$ be a vector field. We say that:

$$\boxed{\begin{array}{l} f \text{ path-independent in } A \Leftrightarrow \\ \Leftrightarrow \forall x, y \in A : \exists I \in \mathbb{R} : \forall C \in \mathcal{P}_A(x,y) : \displaystyle\int_C f \cdot dl = I \end{array}}$$

<u>Notation</u> : If f is a path-independent vector field in A, then for each given $x, y \in A$, the corresponding I is denoted as

$$I = \int_x^y f \cdot dl$$

<u>Thm</u> : Let $f: A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^n$ be a vector field and let $a \in A$ be given. Assume that:

a) f is path-independent in A

b) A is open and path-connected

c) $\forall x \in A : \varphi(x) = \displaystyle\int_a^x f \cdot dl$

Then : $\forall x \in A : \nabla\varphi(x) = f(x)$

- This theorem is the generalization of the first fundamental theorem of calculus to line integrals.

▶ <u>2nd method : How to find the potential φ.</u>

- 1, For each $x \in A$, choose a convenient path from some $a \in A$ to x and calculate the line integral:

$$\varphi(x) = \int_a^x f \cdot dl$$

- 2. Check whether $\nabla\varphi = f$. If yes, then f is conservative with potential function φ. If no, then f is not conservative.

EXAMPLE

Show that the function $f(x,y) = (3x^2y, x^2y)$ is not conservative

Solution

Define the path $G(x,y) : a(t) = (xt, yt)$, $\forall t \in [0,1]$ from the point $(0,0)$ to (x,y). Then

$\dot{a}(t) = (x,y)$, $\forall t \in [0,1]$

Define

$$\varphi(x,y) = \int_{G(x,y)} f \cdot dl = \int_{G(x,y)} 3x^2y\,dx + x^2y\,dy =$$

$$= \int_0^1 (3(xt)^2(yt), (xt)^2(yt)) \cdot (x,y)\,dt =$$

$$= \int_0^1 t^3 (3x^2y, x^2y) \cdot (x,y)\,dt =$$

$$= \int_0^1 t^3 (3x^2yx + x^2yxy)\,dt =$$

$$= (3x^3y + x^3y^2) \int_0^1 t^3\,dt =$$

$$= (3x^3y + x^3y^2) \left[\frac{t^4}{4}\right]_0^1 = (3x^3y + x^2y^2) \frac{1^4 - 0^4}{4}$$

$$= (3/4)x^3y + (1/4)x^2y^2$$

We note that

$$\frac{\partial \varphi}{\partial x} = (\partial/\partial x)[(3/4)x^3 y + (1/4)x^2y^2] =$$
$$= (3/4)3x^2y + (1/4)2xy^2 =$$
$$= (9/4)x^2y + (1/2)xy^2 \neq 3x^2y \Rightarrow$$

$$\Rightarrow \nabla\varphi \neq f \Rightarrow f \text{ not conservative.}$$

Loop line integrals of conservative vector fields

Let $A \subseteq \mathbb{R}^n$ be an open path-connected set. Recall that for any given $x, y \in A$, $\mathcal{P}_A(x,y)$ is the set of all piecewise smooth paths from x to y that lie inside the set A. Then the set $\mathrm{Loop}(A)$ of all closed piecewise smooth paths lying inside A can be defined as:

$$\boxed{\mathrm{Loop}(A) = \bigcup_{x \in A} \mathcal{P}_A(x,x)}$$

<u>Notation</u>: Let $C \in \mathrm{Loop}(A)$ be a given closed path.

For closed paths, in general, we use the following notation to denote a line integral, to highlight the fact that the path is closed:

$$I = \oint_C \mathbf{f} \cdot d\mathbf{l}$$

We will now show that:

<u>Thm</u>: Let $\mathbf{f}: A \to \mathbb{R}^n$ be a vector field. Assume that A is <u>open</u> and <u>path-connected</u>. Then, the following statements are equivalent:

a) $\mathbf{f}$ conservative in A

b) $\mathbf{f}$ path independent in A

c) $\forall C \in \mathrm{Loop}(A): \oint_C \mathbf{f} \cdot d\mathbf{l} = 0$

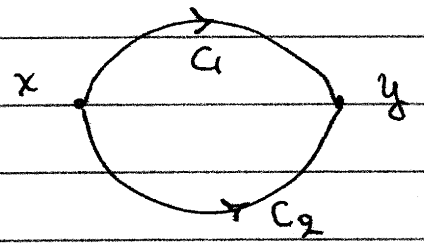
Proof

- $(c) \Rightarrow (b)$: Assume that $\forall C \in Loop(A) : \oint_C f \cdot dl = 0$

We will show that f is path independent.

Let $x, y \in A$ be given. Let $C_1, C_2 \in P_A(x,y)$ be given.

We define the closed path $C = C_1 \cup (-C_2)$.



It follows that

$$\oint_C f \cdot dl = \oint_{C_1 \cup (-C_2)} f \cdot dl = \int_{C_1} f \cdot dl + \int_{-C_2} f \cdot dl =$$

$$= \int_{C_1} f \cdot dl - \int_{C_2} f \cdot dl \quad (1)$$

Since : $\oint_C f \cdot dl = 0 \underset{(1)}{\Rightarrow} \int_{C_1} f \cdot dl - \int_{C_2} f \cdot dl = 0 \Rightarrow$

$$\Rightarrow \int_{C_1} f \cdot dl = \int_{C_2} f \cdot dl$$

Thus:

$\forall x, y \in A : \forall C_1, C_2 \in P_A(x,y) : \int_{C_1} f \cdot dl = \int_{C_2} f \cdot dl \Rightarrow$

$\Rightarrow$ f path independent.

• $(b) \Rightarrow (a)$: Assume that f is path-independent. We will show that f is conservative.

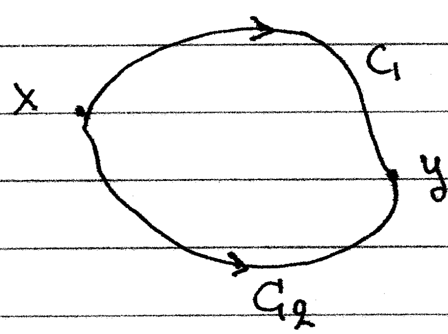
Choose some $a \in A$ and define the scalar field

$$\varphi(x) = \int_a^x f \cdot dl \ , \quad \forall x \in A$$

Since f is path-independent $\Rightarrow \forall x \in A : \nabla \varphi(x) = f(x)$
$\Rightarrow f$ conservative.

• $(a) \Rightarrow (c)$: Assume that f is conservative.

Let $C \in \text{Loop}(A)$ be given. Choose $x, y \in C$ and write $C = C_1 \cup (-C_2)$ with $C_1, C_2 \in \mathcal{P}_A(x, y)$.



Since f is conservative, there is a scalar field $\varphi : A \to \mathbb{R}$ such that $\forall x \in A : \nabla \varphi(x) = f(x)$.
It follows that:

$$\int_{C_1} f \cdot dl = \int_{C_1} \nabla \varphi \cdot dl = \varphi(y) - \varphi(x) \qquad (1)$$

$$\int_{C_2} f \cdot dl = \int_{C_2} \nabla \varphi \cdot dl = \varphi(y) - \varphi(x). \qquad (2)$$

and therefore:

$$\oint_C f \cdot dl = \int_{C_1} f \cdot dl - \int_{C_2} f \cdot dl = (\varphi(y) - \varphi(x)) - (\varphi(y) - \varphi(x))$$

$$= 0 \ , \quad \forall C \in \text{Loop}(A). \qquad \square$$

EXERCISES

(8) Use indefinite integrals to find a potential function for the following conservative vector fields

a) $F(x,y) = (x^2+2xy,\ y^2+2xy)$
b) $F(x,y) = (e^x \cos y,\ e^x \sin y)$
c) $F(x,y) = \left(\dfrac{x^3}{(x^4+y^4)^2},\ \dfrac{y^3}{(x^4+y^4)^2} \right)$
d) $F(x,y) = (y - 1/x^2,\ x - 1/y^2)$
e) $F(x,y,z) = (6xy^3+2z^2,\ 9x^2y^2,\ 4zx+1)$
f) $F(x,y,z) = (yz+1,\ zx+1,\ xy+1)$
g) $F(x,y,z) = (y+z,\ z+x,\ x+y)$
h) $F(x,y,z) = (\cos x + 2yz,\ \sin y + 2zx,\ z+2xy)$

(9) Use line integrals to determine and show whether or not the following vector fields are conservative

a) $F(x,y) = (2xe^y+y,\ x^2e^y+x-2y)$
b) $F(x,y) = (\sin y - y\sin x + x,\ \cos x + x\cos y + y)$
c) $F(x,y) = (\sin(xy) + xy\cos(xy),\ x^2\cos(xy))$
d) $F(x,y,z) = (2xy^3,\ x^2z^3,\ 3x^2yz^2)$
e) $F(x,y,z) = (3y^4z^2,\ 4x^3z^2,\ -3x^2y^2)$
f) $F(x,y,z) = (2x^2+8xy^2,\ 3x^3y-3xy,\ -4y^2z^2-2x^3z)$
g) $F(x,y,z) = (y^2\cos x + z^3,\ 2y\sin x - 4,\ 3xz^2+2)$
h) $F(x,y,z) = (4xy-3x^2z^2+1,\ 2x^2+2,\ -2x^3z-3z^2)$

Green's theorem

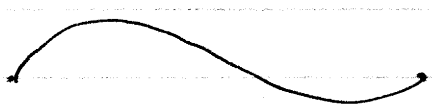
For the following discussion, we restrict ourselves to $\mathbb{R}^2$ (i.e. to two dimension).

Jordan curves

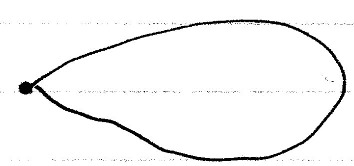
Def : Let $C : a(t)$, $\forall t \in [a_1, a_2]$ be a path. We say that

$$\boxed{C \text{ simple} \iff \forall t_1, t_2 \in [a_1, a_2) : (t_1 \neq t_2 \Rightarrow a(t_1) \neq a(t_2))}$$

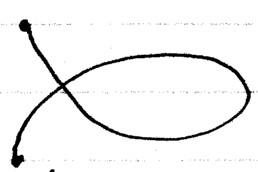
▸ interpretation : A simple path is one that does not intersect with itself short of closing upon itself to possibly form a loop. This is why in the definition above we use the set $[a_1, a_2)$ instead of $[a_1, a_2]$. For example:



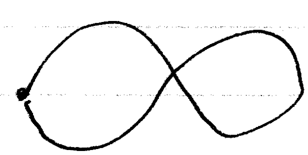
simple



simple



not simple

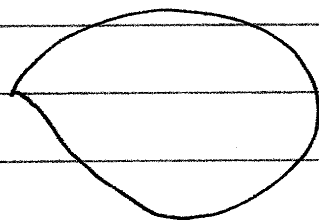


not simple

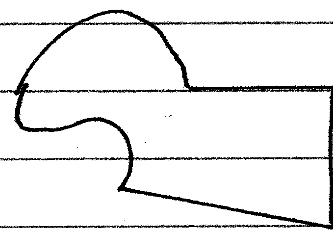
Def : Let G be a path. We say that

$$\boxed{G \text{ Jordan curve} \iff \begin{cases} G \in Loop(\mathbb{R}^2) \\ G \text{ simple} \end{cases}}$$

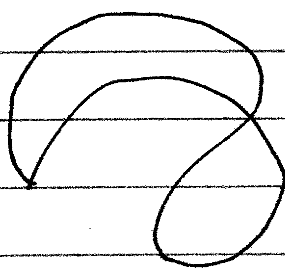
▸ interpretation : A Jordan curve is a simple, closed, and piecewise smooth path.



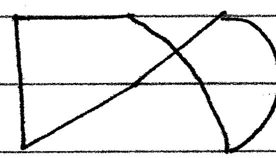
Jordan curve



Jordan curve



NOT a Jordan curve



NOT a Jordan curve.

▸ Notation : The set of all Jordan curves that lie in A is denoted as $Jord(A)$ and the definition of $Jord(A)$ can be written as:

$$\boxed{Jord(A) = \{ G \in Loop(A) \mid G \text{ simple} \}}$$

Jordan's theorem

Thm: For every Jordan curve C, there are two sets $I \subseteq \mathbb{R}^2$ and $E \subseteq \mathbb{R}^2$ such that all of the following statements are true:

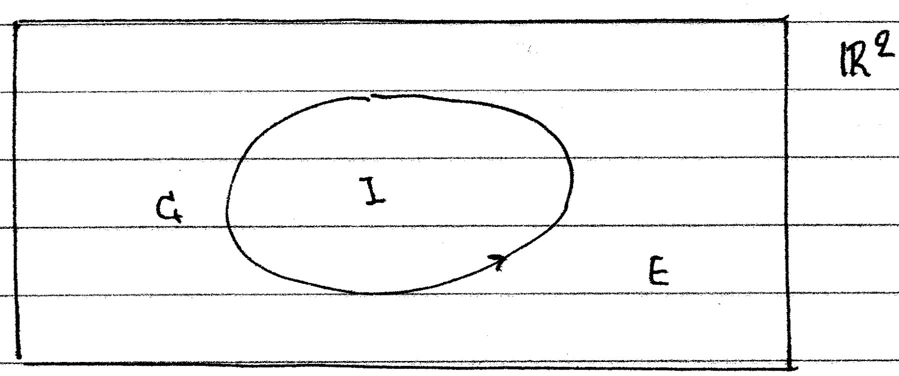
a) $I \cup C \cup E = \mathbb{R}^2$

b) I, E are open sets

c) I bounded $\wedge$ E NOT bounded

d) $\partial I = \partial E = C$ (i.e. C is the boundary set for both I and E).

▸ interpretation: As shown in the figure below, a Jordan curve C divides $\mathbb{R}^2$ into an interior set I which is open and bounded with $\partial I = C$ and an exterior set E which is open but unbounded with $\partial E = C$.



▸ notation: We denote the interior set of C as $I = \text{int}(C)$ and the exterior set of C as $E = \text{ext}(C)$.

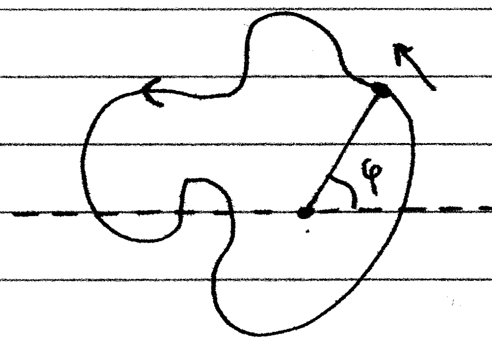
Orientation of Jordan curves

<u>Def</u>: Let $C \in Jord(\mathbb{R}^2)$ be a Jordan curve and let $(x_0, y_0) \in Int(C)$ be a point interior to the curve C. Consider the following polar coordinates representation of C:

$$C : (x,y) = (x_0, y_0) + a(t)(\cos(\varphi(t)), \sin(\varphi(t))), \forall t \in [t_1, t_2]$$

We define the winding number $w(C)$ of C as:

$$\boxed{w(C) = \frac{1}{2\pi}\int_{t_1}^{t_2} \varphi'(t)\, dt}$$



$w(C) = +1$
positive oriented
counterclockwise direction

- We can show that
$\forall C \in Jord(\mathbb{R}^2) : w(C) = +1 \vee w(C) = -1$
and may therefore give the following definition:

<u>Def</u>: Let $C \in Jord(\mathbb{R}^2)$ be a Jordan curve.
We say that:

a) C is positive oriented $\Leftrightarrow w(C) = +1$

b) C is negative oriented $\Leftrightarrow w(C) = -1$.

- Let $C \in \mathrm{Jord}(\mathbb{R}^2)$ be a Jordan curve and let $a = (x_0, y_0) \in \mathrm{int}(C)$ be an interior point.
  Consider the following line integral:

$$W(C|a) = \oint_C \frac{-(y-y_0)\,dx + (x-x_0)\,dy}{(x-x_0)^2 + (y-y_0)^2}$$

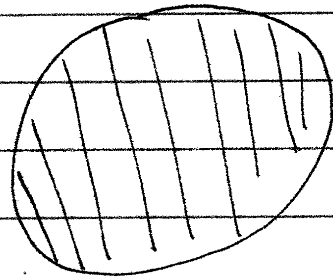
  It can be shown that
  $\forall a \in \mathrm{Int}(C) : W(C|a) = \omega(C)$.
  This line integral is a fairly convenient method for calculating the winding number.

Simply connected sets.

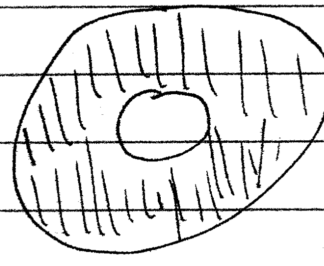
<u>Def</u>: Let $A \subseteq \mathbb{R}^2$. We say that

$$\boxed{A \text{ simply connected} \Leftrightarrow \begin{cases} A \text{ path-connected} \\ \forall C \in \mathrm{Jord}(A) : \mathrm{int}(C) \subseteq A \end{cases}}$$

▶ <u>interpretation</u>: A path-connected set can have "holes" because it is always possible to connect any two points by going around the holes.
A simply connected set is not allowed to have any holes. If there is a hole, then a Jordan curve around the hole will violate the definition above.



path-connected
simply connected



path-connected but
NOT simply connected

Green's theorem

Thm : Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ be two scalar fields. Let $C \in Jord(A)$ be a Jordan curve on A. Assume that:

a) A open $\wedge$ A simply connected
b) f, g differentiable in $C \cup int(C)$
c) $\nabla f, \nabla g$ continuous in $C \cup int(C)$
d) C positive oriented (i.e. $w(C) = +1$).

Then:

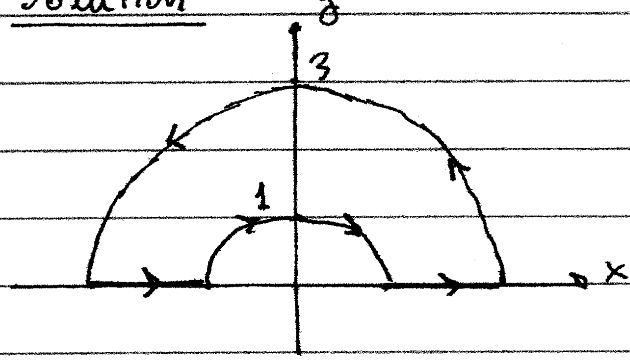
$$\boxed{\oint_C f\,dx + g\,dy = \iint_{int(C)} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\,dy}$$

EXAMPLE

Let $A = \{(r\cos\vartheta, r\sin\vartheta) \mid r \in [1,3] \wedge \vartheta \in [0,\pi]\}$ be a region in $\mathbb{R}^2$ and let $C = \partial A$ be a Jordan curve delineating the boundary of A with C positive oriented. Evaluate the integral

$$I = \oint_C y^2 dx + x^2 dy$$

Solution



$$I = \oint_C y^2 dx + x^2 dy = \iint_{int(C)} \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2) \right] dx dy$$

$$= \iint_A (2x - 2y)\, dx\, dy = 2 \iint_A (x-y)\, dx dy$$

Let $B = \{(r,\vartheta) \mid r \in [1,3] \wedge \vartheta \in [0,\pi]\}$. We change variables:

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases} \Rightarrow dx dy = r dr d\vartheta$$

It follows that:

$$I = 2\iint_B (r\cos\vartheta - r\sin\vartheta)\, r\,dr\,d\vartheta = 2\int_1^3 dr \int_0^{\pi} d\vartheta\, r^2(\cos\vartheta - \sin\vartheta)$$

$$= 2\left[\int_1^3 r^2 dr\right]\left[\int_0^{\pi} (\cos\vartheta - \sin\vartheta)\, d\theta\right] =$$

$$= 2\left[\frac{r^3}{3}\right]_1^3 \left[\sin\vartheta + \cos\vartheta\right]_0^{\pi} =$$

$$= 2\,\frac{3^3 - 1^3}{3}\left[(\sin\pi + \cos\pi) - (\sin 0 + \cos 0)\right] =$$

$$= \frac{2(27-1)}{3}\left[(0-1)-(0+1)\right] = \frac{2\cdot 26\cdot(-2)}{3}$$

$$= \frac{-104}{3}.$$

EXERCISES

(10) Use Green's theorem to evaluate the following line integrals by converting them to double integrals.

a) $I = \oint_{C} y^2 dx + x^2 dy$

with C the boundary $C = \partial S$ of the rectangle $S = [0, a] \times [0, b]$ traversed counterclockwise with $a, b \in (0, +\infty)$.

b) $I = \oint_{C} e^{x+y} dx + e^{x-y} dy$

with C the triangle with vertices $A(0,0)$, $B(a,0)$, $C(a,b)$ traversed counterclockwise with $a, b \in (0, +\infty)$.

c) $I = \oint_{C} x^2 y \, dx + x y^2 \, dy$

with C a circle with center $O(0,0)$ and radius $R > 0$, traversed counterclockwise.

d) $I = \oint_{C} (x+y) \, dx + (x^2 - y) \, dy$

with C the boundary of the region enclosed by

$(c_1): y = x^2$ and $(c_2): y = \sqrt{x}$, traversed counterclockwise.

e) $I = \oint_{C} (\ln x + y)\,dx - x^3\,dy$

with C the rectangle with vertices $A_1(1,1)$, $A_2(1,b)$, $A_3(a,1)$, $A_4(a,b)$, traversed counterclockwise with $a, b \in (1, +\infty)$.

f) $I = \oint_{C} xy\,dx + (x^2+x)\,dy$

with C the triangle with vertices $A_1(-a,0)$, $A_2(a,0)$, $A_3(0,a)$ traversed counterclockwise, with $a \in (0, +\infty)$.

g) $I = \oint_{C} xe^y\,dx + (x + x^2 e^y)\,dy$

with C the boundary of a half-disk given by

$S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq a^2 \wedge y \geq 0\}$

traversed counterclockwise with $a \in (0, +\infty)$.

h) $I = \oint_{C} (\sin x + y)\,dx + (3x + y)\,dy$

with C a polygon with vertices $A_1(0,0)$, $A_2(a,a)$, $A_3(a, a+b)$, $A_4(0, 2a+b)$ traversed counterclockwise with $a, b \in (0, +\infty)$.

i) $I = \oint_{C_1} (x^2+y^2)\,dx - 2xy\,dy$

with C_1 a circle with center $K(0,0)$ and radius $a \in (0,+\infty)$, traversed counterclockwise

V Applications of Green's theorem

① Area calculation

Prop: Let $C \in \mathrm{Jord}(\mathbb{R}^2)$ be a positively-oriented Jordan's curve. The area $A(C)$ of the interior $\mathrm{int}(C)$ of C is given by

$$\boxed{A(C) = \frac{1}{2}\oint_C (x\,dy - y\,dx)}$$

Proof

Note that the area $A(C)$ is given by $A(C) = \iint_{\mathrm{int}(C)} dx\,dy$
The line integral reads:

$$\oint_C (x\,dy - y\,dx) = \oint_C (-y\,dx + x\,dy) =$$

$$= \iint_{\mathrm{int}(C)} \left[\frac{\partial}{\partial x} x - \frac{\partial}{\partial y}(-y)\right] dx\,dy$$

$$= \iint_{\mathrm{int}(C)} (1-(-1))\,dx\,dy = \iint_{\mathrm{int}(C)} 2\,dx\,dy =$$

$$= 2\iint_{\mathrm{int}(C)} dx\,dy = 2A(C) \Rightarrow$$

$$\Rightarrow A(C) = \frac{1}{2}\oint_C (x\,dy - y\,dx) \qquad \square$$

EXAMPLE

Find the area of the ellipse

$$S = \{(x,y) \in \mathbb{R}^2 \mid x^2/a^2 + y^2/b^2 \leq 1\}$$

Solution

The boundary of the ellipse can be written as:

$$\partial S : (x,y) = (a\cos\vartheta, b\sin\vartheta), \ \forall \vartheta \in [0, 2\pi]$$

Differentiating with respect to ϑ gives:

$$(\dot{x}, \dot{y}) = (-a\sin\vartheta, b\cos\vartheta), \ \forall \vartheta \in [0, 2\pi]$$

It follows that the area is given by:

$$A = \iint_S dx\,dy = \frac{1}{2} \oint_{\partial S} (-y\,dx + x\,dy) =$$

$$= \frac{1}{2} \int_0^{2\pi} (-b\sin\vartheta, a\cos\vartheta) \cdot (-a\sin\vartheta, b\cos\vartheta)\, d\vartheta$$

$$= \frac{1}{2} \int_0^{2\pi} [(-b\sin\vartheta)(-a\sin\vartheta) + (a\cos\vartheta)(b\cos\vartheta)]\, d\vartheta$$

$$= \frac{1}{2} \int_0^{2\pi} ab(\sin^2\vartheta + \cos^2\vartheta)\, d\vartheta = \frac{1}{2} \int_0^{2\pi} ab\, d\vartheta =$$

$$= \frac{ab}{2} \int_0^{2\pi} d\vartheta = \frac{ab}{2} \cdot (2\pi) = \pi ab$$

Thus $\underline{A = \pi ab}$.

② Conservative vector fields in $\mathbb{R}^2$

- Green's theorem can be used to derive a simple test for determing whether a two-dimensional vector field is conservative.

Thm : Let $f(x,y) = (f_1(x,y), f_2(x,y))$, $\forall (x,y) \in A$ be a vector field. We assume that:

a) A open set $\land$ A simply connected

b) f_1, f_2 differentiable in A

c) $\nabla f_1, \nabla f_2$ continuous in A

Then:

$$\boxed{f \text{ conservative} \Leftrightarrow \forall (x,y) \in A: \frac{\partial f_2(x,y)}{\partial x} = \frac{\partial f_1(x,y)}{\partial y}}$$

Proof

($\Rightarrow$): Assume that f is conservative.
Let $\varphi: A \to \mathbb{R}$ be the corresponding potential function such that

$$\forall (x,y) \in A : f(x,y) = \nabla\varphi(x,y) \Rightarrow$$
$$\Rightarrow \forall (x,y) \in A: f_1(x,y) = \frac{\partial \varphi(x,y)}{\partial x} \land f_2(x,y) = \frac{\partial \varphi(x,y)}{\partial y}$$

It follows that:

$$\frac{\partial f_2(x,y)}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\partial \varphi(x,y)}{\partial y}\right] = \frac{\partial^2 \varphi(x,y)}{\partial x \partial y} =$$

$$= \frac{\partial^2 \varphi(x,y)}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial \varphi(x,y)}{\partial x}\right] =$$

$$= \frac{\partial f_1(x,y)}{\partial y}, \quad \forall (x,y) \in A$$

($\Leftarrow$): Assume that $\dfrac{\partial f_2(x,y)}{\partial x} = \dfrac{\partial f_1(x,y)}{\partial y}$

▸ We will show that $\forall C \in \text{Loop}(A): \oint_C \mathbf{f} \cdot d\mathbf{l} = 0$

Let $C \in \text{Loop}(A)$ be given.

▸ We note that C can be rewritten as the union of Jordan curves, as explained after the proof:

$\exists C_1, C_2, \ldots, C_n \in \text{Jord}(A): C = C_1 \cup C_2 \cup \cdots \cup C_n$

It follows that:

$$\oint_C \mathbf{f} \cdot d\mathbf{l} = \sum_{a=1}^{n} \oint_{C_a} \mathbf{f} \cdot d\mathbf{l} = \sum_{a=1}^{n} \oint_{C_a} f_1 dx + f_2 dy =$$

$$= \sum_{a=1}^{n} w(C_a) \iint_{\text{int}(C_a)} \left[\frac{\partial f_2(x,y)}{\partial x} - \frac{\partial f_1(x,y)}{\partial y}\right] dx dy$$

$$= \sum_{a=1}^{n} w(C_a) \iint_{\text{int}(C_a)} 0 \, dx dy = 0, \ \forall C \in \text{Loop}(A)$$

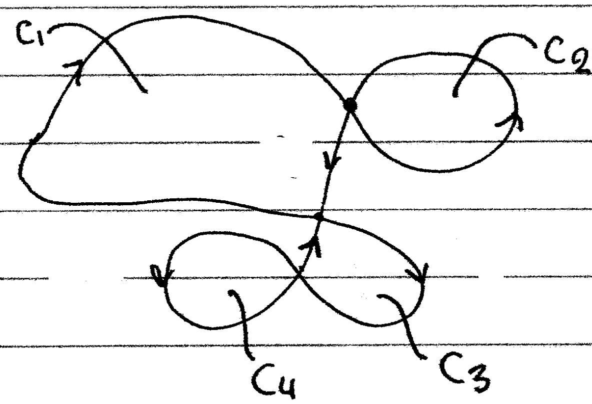
$\Rightarrow \mathbf{f}$ conservative □

↳ In the proof above we claimed that any loop $C \in Loop(A)$ can be written as a union of Jordan curves:

$C = C_1 \cup C_2 \cup \cdots \cup C_n$

with $C_1, C_2, \ldots, C_n \in Jord(A)$.

This is not easy to prove but we can illustrate it with a graphical example:



In this figure we decompose $C = C_1 \cup C_2 \cup C_3 \cup C_4$ with the Jordan curves C_1, C_2, C_3, C_4 identified by pointing towards their interior sets: $int(C_1)$, $int(C_2)$, $int(C_3)$, and $int(C_4)$.

↳ The contrapositive statement of the theorem reads:

$$\boxed{f \text{ NOT conservative} \iff \exists (x,y) \in A : \frac{\partial f_2(x,y)}{\partial x} \neq \frac{\partial f_1(x,y)}{\partial y}}$$

EXAMPLES

a) Examine whether the field $f(x,y)=(x-y,x-2), \forall (x,y)\in\mathbb{R}^2$ is conservative

Solution

Define $\begin{cases} f_1(x,y) = x-y, \forall (x,y)\in\mathbb{R}^2 \\ f_2(x,y) = x-2, \forall (x,y)\in\mathbb{R}^2 \end{cases}$

Then:

$$\frac{\partial f_2(x,y)}{\partial x} = \frac{\partial}{\partial x}(x-2) = 1, \forall (x,y)\in\mathbb{R}^2$$

$$\frac{\partial f_1(x,y)}{\partial y} = \frac{\partial}{\partial y}(x-y) = -1, \forall (x,y)\in\mathbb{R}^2$$

It follows that:

$$\exists (x,y)\in\mathbb{R}^2: \frac{\partial f_2(x,y)}{\partial x} \neq \frac{\partial f_1(x,y)}{\partial y} \Rightarrow$$

$\Rightarrow f$ NOT conservative.

b) Examine whether the field $f(x,y) = (3+2xy, x^2-3y^2), \forall (x,y)\in\mathbb{R}^2$ is conservative.

Solution

Define $\begin{cases} f_1(x,y) = 3+2xy, \forall (x,y)\in\mathbb{R}^2 \\ f_2(x,y) = x^2-3y^2, \forall (x,y)\in\mathbb{R}^2. \end{cases}$

Then:

$$\frac{\partial f_2(x,y)}{\partial x} = \frac{\partial}{\partial x}(x^2-3y^2) = 2x \quad , \quad \forall (x,y) \in \mathbb{R}^2$$

$$\frac{\partial f_1(x,y)}{\partial y} = \frac{\partial}{\partial y}(3+2xy) = 2x \quad , \quad \forall (x,y) \in \mathbb{R}^2$$

It follows that

$$\forall (x,y) \in \mathbb{R}^2 : \frac{\partial f_2(x,y)}{\partial x} = \frac{\partial f_1(x,y)}{\partial y} \Rightarrow f \text{ conservative.}$$

EXERCISES

(11) Use line integrals to find the area of the "asteroid" S given by
$$S=\{(x,y)\in\mathbb{R}^2 \mid x^{2/3}+y^{2/3}\leq a^{2/3}\}$$
with $a\in(0,+\infty)$.
(Hint: The boundary of S can be parameterized as:
$$(\partial S): (x,y)=(a\cos^3 t, a\sin^3 t), \forall t\in[0,2\pi]$$)

(12) Similarly to the previous problem, find the area of the "elliptical asteroid" S given by
$$S=\{(x,y)\in\mathbb{R}^2 \mid (x/a)^{2/3}+(y/b)^{2/3}\leq 1\}$$
with $a,b\in(0,+\infty)$

(13) Use line integrals to find the area between the cycloid (c) given by
$$(c): \begin{cases} x=t-\sin t \\ y=1-\cos t \end{cases}, \forall t\in[0,2\pi]$$
and the x-axis

↳ Note that a different technique for calculating the area under a cycloid is given in my Calculus 2 Online Lecture Notes.

(14) An epicycloid is traced by rolling a circle with radius 1 around and outside another circle with radius $a-1$. The epicycloid (c) is given by

$$(c): \begin{cases} x = a\cos t - \cos(at) \\ y = a\sin t - \sin(at) \end{cases}, \quad \forall t \in [0, 2\pi]$$

Use line integrals to find the area enclosed by the epicycloid, with $a \in \mathbb{N} - \{0,1\}$.

(15) <u>Area of a polygon</u>

a) Consider the line segment (ℓ) from the point $A_1(x_1, y_1)$ to the point $A_2(x_2, y_2)$, given by

$$(\ell): \begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \end{cases}, \quad \forall t \in [0,1]$$

Show that

$$\int_{(\ell)} x\,dy - y\,dx = x_1 y_2 - x_2 y_1$$

b) Now consider a polygon traced by the vertices $A_1(x_1, y_1), A_2(x_2, y_2), \ldots, A_n(x_n, y_n), A_1(x_1, y_1)$. Show, using part (a), that the area of the polygon is given by:

$$A = \left| \sum_{a=1}^{n-1} (x_a y_{a+1} - x_{a+1} y_a) + (x_n y_1 - x_1 y_n) \right|$$

▼ Parametric surfaces

<u>Def</u>: Let $x: A \to \mathbb{R}$, $y: A \to \mathbb{R}$, $z: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ be three scalar fields and consider the set $S = \{(x(t,s), y(t,s), z(t,s)) \mid (t,s) \in A\}$.

We say that

$$\boxed{S \text{ surface} \iff \begin{cases} A \text{ simply connected} \land A \text{ closed} \\ x, y, z \text{ continuous in } A \end{cases}}$$

$$\boxed{S \text{ differentiable surface} \iff \begin{cases} S \text{ surface} \\ x, y, z \text{ differentiable in } A \end{cases}}$$

<u>notation</u>: Alternatively, we write the definition of S as:

$$(S): a(t,s) = (x(t,s), y(t,s), z(t,s)), \forall (t,s) \in A$$

with $a: A \to \mathbb{R}^3$.

<u>remark</u>: The parameters (t,s) act as a local curvilinear coordinate system on the surface. Since a surface should be a two-dimensional object, we require 2 coordinates. However, it is possible under the definitions above for a surface to collapse into a one-dimensional structure. For example:

$$(S): a(t,s) = (t+s, t+s, t+s), \forall (t,s) \in A$$

is in fact defining a line. To rule out this possibility we propose the following stronger definition.

Def: Let $(S): a(t,s) = (x(t,s), y(t,s), z(t,s))$, $\forall (t,s) \in A$ be a differentiable surface.

a) We define the fundamental product $\mathcal{R}(t,s|a)$ of the surface (S) as:

$$\boxed{\mathcal{R}(t,s|a) = \frac{\partial a}{\partial t} \times \frac{\partial a}{\partial s}, \quad \forall (t,s) \in A}$$

b) We say that:

$$\boxed{S \text{ smooth surface} \iff \begin{cases} S \text{ differentiable surface} \\ \mathcal{R}(t,s|a) \text{ continuous in } A \\ \forall (t,s) \in \mathrm{int}(A): \mathcal{R}(t,s|a) \neq \mathbf{0} \end{cases}}$$

▸ interpretation: Given a point (t,s) of the surface, a small change in the parameter t should move us in a different direction than a small change in the parameter s. To prevent a surface from being degenerate, it is therefore essential that $\partial a/\partial t$ should NOT be parallel to $\partial a/\partial s$. This requirement is equivalent to the condition

$$\mathcal{R}(t,s|a) \neq \mathbf{0}, \ \forall (t,s) \in \mathrm{int}(A)$$

in the above definition. We allow the condition to be violated at points on the boundary ∂A, and if such points exist, we call them singular points

Remark: The definition of the fundamental product can be rewritten as follows:

$$\boxed{\mathfrak{a}(t,s\,|\,a) = \left(\frac{\partial(y,z)}{\partial(t,s)}, \frac{\partial(z,x)}{\partial(t,s)}, \frac{\partial(x,y)}{\partial(t,s)} \right)}$$

with:

$$\frac{\partial(y,z)}{\partial(t,s)} = \frac{\partial y}{\partial t}\frac{\partial z}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial z}{\partial t}$$

$$\frac{\partial(z,x)}{\partial(t,s)} = \frac{\partial z}{\partial t}\frac{\partial x}{\partial s} - \frac{\partial z}{\partial s}\frac{\partial x}{\partial t}$$

$$\frac{\partial(x,y)}{\partial(t,s)} = \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial t}$$

Simple surfaces

Consider a surface

$$S = \{ a(t,s) \mid (t,s) \in A \}$$

with $A \subseteq \mathbb{R}^2$.

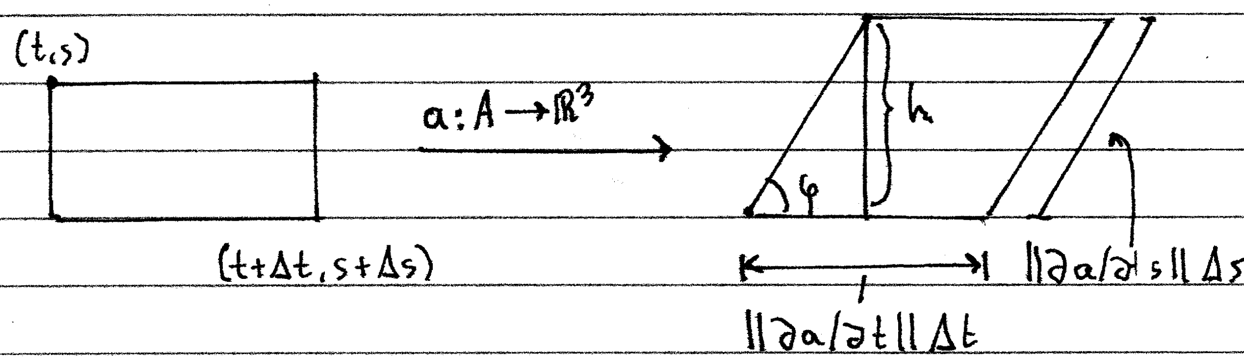
Def: S is simple $\Leftrightarrow$ $a : A \to \mathbb{R}^3$ one-to-one $\Leftrightarrow \forall (t_1,s_1),(t_2,s_2) \in A : ((t_1,s_1) \neq (t_2,s_2) \Rightarrow a(t_1,s_1) \neq a(t_2,s_2))$

An immediate consequence of the definition is that every loop $C \in \mathrm{Loop}(A)$ in the (t,s) plane is mapped by "a" to a loop in the surface $\mathcal{S}$:

Prop : $\boxed{\mathcal{S} \text{ simple} \Rightarrow \forall\, \mathcal{C} \in \mathrm{Loop}(A) : a(\mathcal{C}) \in \mathrm{Loop}(\mathcal{S})}$

Surface Area

Consider a smooth surface $\mathcal{S} = \{ a(t) \mid ($
$\mathcal{S} = \{ a(t,s) \mid (t,s) \in A \}$
with $a(t,s) = (x(t,s), y(t,s), z(t,s))$, $\forall (t,s) \in A$.
Consider a "patch" on the surface $\mathcal{S}$ defined by a rectangle in the (t,s)-plane which is in turn defined by the vertices (t,s) and $(t+\Delta t, s+\Delta s)$ as shown in the following figure:



We see that:
Bottom side of patch length $\ell \approx \|\partial a/\partial t\| \Delta t$
Height $h \simeq \|\partial a/\partial s\| \Delta s \cdot \sin\varphi$
and therefore, the area of the patch is:
$$\Delta A = \ell h \cong (\|\partial a/\partial t\| \Delta t)(\|\partial a/\partial s\| \Delta s \cdot \sin\varphi) =$$
$$= (\|\partial a/\partial t\| \cdot \|\partial a/\partial s\| \cdot \sin\varphi)\, \Delta t\, \Delta s =$$

$$= \| (\partial a/\partial t) \times (\partial a/\partial s) \| \Delta t \Delta s =$$
$$= \| R(t,s|a) \| \Delta t \Delta s$$

with $\Delta t \to 0$ and $\Delta s \to 0$. It follows that the total area of the surface is given by:

$$\boxed{\text{Area}(S) = \iint_A \| R(t,s|a) \| \, dt\, ds}$$

Expanding the norm of the fundamental product, the above equation can be rewritten as follows:

$$\boxed{\text{Area}(S) = \iint_A \sqrt{\left(\frac{\partial(x,y)}{\partial(t,s)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(t,s)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(t,s)}\right)^2} \, dt\, ds}$$

Surface integrals

This result motivates the following definition of the surface integral:

Def : Let $S = \{a(t,s) \mid (t,s) \in A\}$ with $a : A \to \mathbb{R}^3$ be a smooth surface. Let $f : B \to \mathbb{R}$ with $B \subseteq \mathbb{R}^3$ be a scalar field such that f is continuous at B and $a(A) \subseteq B$. We define the surface integral as:

$$\boxed{\iint_S f \, dS = \iint_A f(a(t,s)) \, \| R(t,s|a) \| \, dt\, ds}$$

- Note that in the above definition, both f and $d\mathcal{S}$ are scalars. The condition $a(A) \subseteq B$ ensures that the surface $\mathcal{S}$ lies in the domain of the scalar field f.
- A similar definition is possible for vector fields:

Def: Let $\mathcal{S} = \{a(t,s) \mid (t,s) \in A\}$ with $a: A \to \mathbb{R}^3$ be a smooth surface. Let $f: B \to \mathbb{R}^3$ with $B \subseteq \mathbb{R}^3$ be a vector field such that f is continuous at B and $a(A) \subseteq B$. We define the following surface integrals:

$$\boxed{\begin{aligned} \iint_{\mathcal{S}} f \cdot d\mathcal{S} &= \iint_A \left[f(a(t,s)) \cdot \mathcal{R}(t,s \mid a) \right] dt\, ds \\ \iint_{\mathcal{S}} f \times d\mathcal{S} &= \iint_A \left[f(a(t,s)) \times \mathcal{R}(t,s \mid a) \right] dt\, ds \end{aligned}}$$

→ Normal vector and surface integrals

Let $\mathcal{S} = \{a(t,s) \mid (t,s) \in A\}$ be a smooth curve. The vectors $\partial a / \partial t$ and $\partial a / \partial s$ define a plane tangent to the surface $\mathcal{S}$ at the given point (t,s). Since the fundamental product reads:

$$\mathcal{R}(t,s \mid a) = \frac{\partial a(t,s)}{\partial t} \times \frac{\partial a(t,s)}{\partial s}$$

it follows that:

$$\forall (t,s) \in A : \begin{cases} \mathcal{R}(t,s\,|\,a) \perp \partial a(t,s)/\partial t \\ \mathcal{R}(t,s\,|\,a) \perp \partial a(t,s)/\partial s \end{cases}$$

Consequently, the fundamental product $\mathcal{R}(t,s|a)$ essential is normal to the surface $\mathcal{S}$, and we may formally define a unit-normal vector as follows:

Def: The unit-normal vector $n(t,s|a)$ of the smooth surface $\mathcal{S}$ given by
$\mathcal{S} = \{a(t,s) \mid (t,s) \in A\}$
is defined as:

$$\boxed{n(t,s|a) = \frac{\mathcal{R}(t,s|a)}{\|\mathcal{R}(t,s|a)\|}, \quad \forall (t,s) \in A}$$

It follows that:

$$\boxed{\iint_{\mathcal{S}} f \cdot d\mathcal{S} = \iint_{\mathcal{S}} (f \cdot n)\, d\mathcal{S}}$$

$$\boxed{\iint_{\mathcal{S}} f \times d\mathcal{S} = \iint_{\mathcal{S}} (f \times n)\, d\mathcal{S}}$$

- Note that the integrals on the right-hand side of the above equations are scalar surface integrals.

• Using the unit normal vector n, we can define the normal derivative of a scalar field on a surface as the directional derivative of f in the direction defined by the unit vector n

<u>Def</u> : Let $f: B \to \mathbb{R}$ be a scalar field with $a(A) \subseteq B$. Assume that f has partial derivatives in B. We define the <u>normal derivative</u>

$$\boxed{\frac{\partial f(t,s)}{\partial n} = \nabla f(x(t,s), y(t,s), z(t,s)) \bullet n(t,s|a), \ \forall (t,s) \in A}$$

EXERCISES

(16) Evaluate and simplify the fundamental product $\mathcal{R}(a|t,s)$ and the norm of the fundamental product $\|\mathcal{R}(a|t,s)\|$ for the following surfaces:

a) Elliptic paraboloid

$$a(t,s) = (as\cos t,\ bs\sin t,\ s^2),\ \forall (s,t) \in [0,1]\times[0,2\pi]$$

b) Ellipsoid

$$a(\varphi,\vartheta) = (a\sin\varphi\cos\vartheta,\ b\sin\varphi\sin\vartheta,\ c\cos\varphi),\ \forall (\varphi,\vartheta)\in[0,\pi]\times[0,2\pi]$$

c) Cylinder

$$a(x,\vartheta) = (x,\ r\cos\vartheta,\ r\sin\vartheta),\ \forall (x,\vartheta)\in[0,\ b]\times[0,2\pi]$$

d) Torus

$$a(\vartheta,\varphi) = ((a+b\cos\vartheta)\sin\varphi,\ (a+b\cos\vartheta)\cos\varphi,\ b\sin\varphi),$$
$$\forall (\vartheta,\varphi)\in[0,2\pi]\times[0,2\pi]$$

with $a,b\in(0,+\infty)$ and $0<b<a$.

(17) Evaluate the area of the surfaces defined in the previous exercise by evaluating the corresponding surface integral

$$\text{area}(S) = \iint_S dS$$

(18) Use a surface integral to evaluate the surface area of a sphere

$$S = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 = R^2\}$$

with radius R, using spherical coordinates.

Fundamental product for special surfaces

We now derive the fundamental product for the special cases

① → Surface defined as $(S): z = f(x,y)$

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ and consider the surface given by

$$S = \{(x,y,z) \mid z = f(x,y) \land (x,y) \in A\}$$
$$= \{(x,y,f(x,y)) \mid (x,y) \in A\}$$

Then:

$$\boxed{\begin{aligned} N(x,y|a) &= \left(\frac{-\partial f}{\partial x}, \frac{-\partial f}{\partial y}, 1 \right) \\ \|N(x,y|a)\| &= \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \end{aligned}}$$

Proof

Define $a(x,y) = (x,y,f(x,y))$, $\forall (x,y) \in A$.

It follows that

$$\begin{cases} \partial a/\partial x = (1,0,\partial f/\partial x) \\ \partial a/\partial y = (0,1,\partial f/\partial y) \end{cases} \Rightarrow$$

$$\Rightarrow N(x,y|a) = (\partial a/\partial x) \times (\partial a/\partial y) =$$
$$= (1,0,\partial f/\partial x) \times (0,1,\partial f/\partial y) =$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & \partial f/\partial x \\ 0 & 1 & \partial f/\partial y \end{vmatrix} \begin{matrix} e_1 & e_2 \\ 1 & 0 \\ 0 & 1 \end{matrix} =$$

$$= 0e_1 + 0e_2 + e_3 - 0e_3 - (\partial f/\partial x)e_1 - (\partial f/\partial y)e_2 =$$
$$= -(\partial f/\partial x)e_1 - (\partial f/\partial y)e_2 + e_3$$
$$= (-\partial f/\partial x, -\partial f/\partial y, 1)$$

and therefore

$$\|R(x,y|a)\| = \|(-\partial f/\partial x, -\partial f/\partial y, 1)\| =$$

$$= \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

2 → Surface of revolution of $(c): y = f(x)$ around x-axis

Let $f: A \to \mathbb{R}$ with $A = [a,b]$ be a function with $\forall x \in [a,b]: f(x) > 0$. Define a surface S by rotating the curve $(c): y = f(x)$ around the x-axis. It follows that

$$S = \{(x, f(x)\cos\theta, f(x)\sin\theta) \mid x \in [a,b] \land \theta \in [0, 2\pi]\}$$

Then

$$\boxed{\begin{aligned} &R(x,\theta|a) = (f(x)f'(x), -f(x)\cos\theta, -f(x)\sin\theta) \\ &\|R(x,\theta|a)\| = f(x)\sqrt{1 + [f'(x)]^2} \end{aligned}}$$

Proof

Define $a(x,\vartheta) = (x, f(x)\cos\vartheta, f(x)\sin\vartheta)$, $\forall (x,\vartheta) \in [a,b] \times [0, 2\pi]$

Then

$$\left.\begin{aligned} \partial a/\partial x &= (1, f'(x)\cos\vartheta, f'(x)\sin\vartheta) \\ \partial a/\partial\vartheta &= (0, -f(x)\sin\vartheta, f(x)\cos\vartheta) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow R(x,\vartheta|a) = (\partial a/\partial x) \times (\partial a/\partial\vartheta) =$$

$$= (1, f'(x)\cos\vartheta, f'(x)\sin\vartheta) \times (0, -f(x)\sin\vartheta, f(x)\cos\vartheta)$$

$$= \left|\begin{matrix} e_1 & e_2 & e_3 \\ 1 & f'(x)\cos\vartheta & f'(x)\sin\vartheta \\ 0 & -f(x)\sin\vartheta & f(x)\cos\vartheta \end{matrix}\right| \begin{matrix} e_1 & e_2 \\ 1 & f'(x)\cos\vartheta \\ 0 & -f(x)\sin\vartheta \end{matrix} =$$

$$= e_1 (f'(x)\cos\vartheta)(f(x)\cos\vartheta) + 0e_2 + e_3(-f(x)\sin\vartheta)$$

$$- 0e_3 - e_1 (f'(x)\sin\vartheta)(-f(x)\sin\vartheta) - e_2 (f(x)\cos\vartheta)$$

$$= e_1 f(x) f'(x)(\cos^2\vartheta + \sin^2\vartheta) - (f(x)\cos\vartheta)e_2 - (f(x)\sin\vartheta)e_3$$

$$= f(x)f'(x)e_1 - (f(x)\cos\vartheta)e_2 - (f(x)\sin\vartheta)e_3 =$$

$$= (f(x)f'(x), -f(x)\cos\vartheta, -f(x)\sin\vartheta)$$

$$\Rightarrow \|R(x,\vartheta|a)\|^2 = \|(f(x)f'(x), -f(x)\cos\vartheta, -f(x)\sin\vartheta)\|^2 =$$

$$= [f(x)f'(x)]^2 + [-f(x)\cos\vartheta]^2 + [-f(x)\sin\vartheta]^2$$

$$= [f(x)f'(x)]^2 + [f(x)]^2(\cos^2\vartheta + \sin^2\vartheta) =$$

$$= [f(x)f'(x)]^2 + [f(x)]^2$$

$$= [f(x)]^2(1 + [f'(x)]^2) \Rightarrow$$

$$\Rightarrow \|R(x,\vartheta|a)\| = \sqrt{[f(x)]^2(1+[f'(x)]^2)}$$

$$= |f(x)|\sqrt{1+[f'(x)]^2}$$

$$= f(x)\sqrt{1+[f'(x)]^2}$$

③ → Sphere $(S): x^2+y^2+z^2=\rho^2$

The surface of a sphere $(S): x^2+y^2+z^2=\rho^2$ with radius ρ is given by

$$S=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=\rho^2\}$$
$$=\{(\rho\sin\varphi\cos\vartheta, \rho\sin\varphi\sin\vartheta, \rho\cos\varphi) \mid \varphi\in[0,\pi]\wedge\vartheta\in[0,2\pi]\}$$

Then:

$$\boxed{\begin{array}{l} \mathcal{R}(\varphi,\vartheta\,|\,a)=\rho^2(\sin^2\varphi\cos\vartheta,\ \sin^2\varphi\sin\vartheta,\ \cos\varphi\sin\varphi) \\ \|\mathcal{R}(\varphi,\vartheta\,|\,a)\|=\rho^2\sin\varphi \end{array}}$$

Proof

Let $a(\varphi,\vartheta)=(\rho\sin\varphi\cos\vartheta, \rho\sin\varphi\sin\vartheta, \rho\cos\varphi), \forall(\varphi,\vartheta)\in[0,\pi]\times[0,2\pi]$

It follows that

$$\left.\begin{array}{l} \partial a/\partial\varphi=(\rho\cos\varphi\cos\vartheta, \rho\cos\varphi\sin\vartheta, -\rho\sin\varphi) \\ \partial a/\partial\vartheta=(-\rho\sin\varphi\sin\vartheta, \rho\sin\varphi\cos\vartheta, 0 \end{array}\right\}\Rightarrow$$

$$\Rightarrow \mathcal{R}(\varphi,\vartheta\,|\,a)=(\partial a/\partial\varphi)\times(\partial a/\partial\vartheta)=$$
$$=(\rho\cos\varphi\cos\vartheta, \rho\cos\varphi\sin\vartheta, -\rho\sin\varphi)\times(-\rho\sin\varphi\sin\vartheta, \rho\sin\varphi\cos\vartheta, 0)$$
$$=\begin{vmatrix} e_1 & e_2 & e_3 \\ \rho\cos\varphi\cos\vartheta & \rho\cos\varphi\sin\vartheta & -\rho\sin\varphi \\ -\rho\sin\varphi\sin\vartheta & \rho\sin\varphi\cos\vartheta & 0 \end{vmatrix}$$

$$=0e_1+e_2(-\rho\sin\varphi)(-\rho\sin\varphi\sin\vartheta)+e_3(\rho\cos\varphi\cos\vartheta)(\rho\sin\varphi\cos\vartheta)$$
$$-e_3(\rho\cos\varphi\sin\vartheta)(-\rho\sin\varphi\sin\vartheta)-e_1(-\rho\sin\varphi)(\rho\sin\varphi\cos\vartheta)-0e_2$$

$$= e_1 \rho^2 \sin^2\varphi \cos\vartheta + e_2 \rho^2 \sin^2\varphi \sin\vartheta +$$
$$+ e_3 \rho^2 \cos\varphi \sin\varphi (\cos^2\vartheta + \sin^2\vartheta)$$
$$= e_1 \rho^2 \sin^2\varphi \cos\vartheta + e_2 \rho^2 \sin^2\varphi \sin\vartheta + e_3 \rho^2 \cos\varphi \sin\varphi$$
$$= \rho^2 (\sin^2\varphi \cos\vartheta,\ \sin^2\varphi \sin\vartheta,\ \cos\varphi \sin\varphi)$$

It follows that

$$\|Q(\varphi,\vartheta|a)\|^2 = \|\rho^2 (\sin^2\varphi \cos\vartheta,\ \sin^2\varphi \sin\vartheta,\ \cos\varphi \sin\varphi)\|^2 =$$
$$= \rho^4 [(\sin^2\varphi \cos\vartheta)^2 + (\sin^2\varphi \sin\vartheta)^2 + (\cos\varphi \sin\varphi)^2] =$$
$$= \rho^4 [\sin^4\varphi \cos^2\vartheta + \sin^4\varphi \sin^2\vartheta + \cos^2\varphi \sin^2\varphi] =$$
$$= \rho^4 [\sin^4\varphi (\cos^2\vartheta + \sin^2\vartheta) + \cos^2\varphi \sin^2\varphi] =$$
$$= \rho^4 [\sin^4\varphi + \cos^2\varphi \sin^2\varphi] =$$
$$= \rho^4 \sin^2\varphi [\sin^2\varphi + \cos^2\varphi] = \rho^4 \sin^2\varphi \Rightarrow$$
$$\Rightarrow \|Q(\varphi,\vartheta|a)\| = \sqrt{\rho^4 \sin^2\varphi} = \rho^2 \sqrt{\sin^2\varphi} =$$
$$= \rho^2 |\sin\varphi| = \rho^2 \sin\varphi$$

noting that $\varphi \in [0,\pi] \Rightarrow \sin\varphi \geq 0 \Rightarrow |\sin\varphi| = \sin\varphi$. $\square$

EXAMPLES

a) Evaluate $I = \iint_S x^2 z \, dS$ with S given by

$(S): (x,y,z) = (\rho\cos\vartheta, \rho\sin\vartheta, \rho), \ \forall(\rho,\vartheta) \in [0,a] \times [0,2\pi]$

Solution

Define

$S(\rho,\vartheta) = (\rho\cos\vartheta, \rho\sin\vartheta, \rho), \ \forall(\rho,\vartheta) \in [0,a] \times [0,2\pi]$

It follows that

$\partial S / \partial \rho = (\cos\vartheta, \sin\vartheta, 1)$

$\partial S / \partial \vartheta = (-\rho\sin\vartheta, \rho\cos\vartheta, 0)$

and therefore the fundamental product is given by

$$R(\rho,\vartheta | S) = (\partial S/\partial\rho) \times (\partial S/\partial\vartheta) =$$

$$= (\cos\vartheta, \sin\vartheta, 1) \times (-\rho\sin\vartheta, \rho\cos\vartheta, 0) =$$

$$= \left| \begin{array}{ccc} e_1 & e_2 & e_3 \\ \cos\vartheta & \sin\vartheta & 1 \\ -\rho\sin\vartheta & \rho\cos\vartheta & 0 \end{array} \right| \begin{array}{cc} e_1 & e_2 \\ \cos\vartheta & \sin\vartheta \\ -\rho\sin\vartheta & \rho\cos\vartheta \end{array} =$$

$$= 0e_1 + 1(-\rho\sin\vartheta)e_2 + (\rho\cos\vartheta)(\cos\vartheta)e_3 - (-\rho\sin\vartheta)(\sin\vartheta)e_3$$
$$- (\rho\cos\vartheta)1e_1 - 0e_2 =$$

$$= (-\rho\cos\vartheta)e_1 + (-\rho\sin\vartheta)e_2 + \rho(\cos^2\vartheta + \sin^2\vartheta)e_3 =$$

$$= -\rho\cos\vartheta \, e_1 - \rho\sin\vartheta \, e_2 + \rho e_3 \Rightarrow$$

$$\Rightarrow \|R(\rho,\vartheta|S)\|^2 = \|(-\rho\cos\vartheta, -\rho\sin\vartheta, \rho)\|^2 =$$

$$= (-\rho\cos\vartheta)^2 + (-\rho\sin\vartheta)^2 + \rho^2 =$$

$$= \rho^2\cos^2\vartheta + \rho^2\sin^2\vartheta + \rho^2 =$$

$$= \rho^2(\cos^2\vartheta + \sin^2\vartheta) + \rho^2 = \rho^2 + \rho^2 = 2\rho^2 \Rightarrow$$

$$\Rightarrow \|\mathcal{R}(\rho,\vartheta|\mathcal{S})\| = \sqrt{2\rho^2} = |\sqrt{2}\rho| = \rho\sqrt{2}$$

Define $A = \{(\rho,\vartheta) \mid \rho \in [0,a] \land \vartheta \in [0,2\pi]\}$.

We have

$$I = \iint_{\mathcal{S}} x^2 z \, d\mathcal{S} = \iint_A d\rho \, d\vartheta \, (\rho\cos\vartheta)^2 \rho \, \|\mathcal{R}(\rho,\vartheta|\mathcal{S})\| =$$

$$= \iint_A d\rho \, d\vartheta \, (\rho\cos\vartheta)^2 \rho \, (\rho\sqrt{2}) = \iint_A d\rho \, d\vartheta \, \rho^4 \sqrt{2} \cos^2\vartheta =$$

$$= \sqrt{2} \int_0^a d\rho \int_0^{2\pi} d\vartheta \, \rho^4 \cos^2\vartheta =$$

$$= \sqrt{2} \left[\int_0^a d\rho \, \rho^4 \right] \left[\int_0^{2\pi} d\vartheta \cos^2\vartheta \right] = \sqrt{2} I_1 I_2$$

with

$$I_1 = \int_0^a d\rho \, \rho^4 = \left[\frac{\rho^5}{5} \right]_0^a = \frac{a^5 - 0^5}{5} = \frac{a^5}{5}$$

$$I_2 = \int_0^{2\pi} \cos^2\vartheta \, d\vartheta = \int_0^{2\pi} \frac{1 + \cos(2\vartheta)}{2} \, d\vartheta =$$

$$= \left[\frac{\vartheta}{2} + \frac{\sin(2\vartheta)}{4} \right]_0^{2\pi} =$$

$$= \left[\frac{2\pi}{2} + \frac{\sin(4\pi)}{4} \right] - \left[0 + \frac{\sin 0}{4} \right] =$$

$$= \pi + (1/4)\sin 0 = \pi$$

and therefore

$$I = \sqrt{2} I_1 I_2 = \sqrt{2} (a^5/5) \pi = \frac{\pi a^5 \sqrt{2}}{5}$$

6) Evaluate the surface integral $I = \iint_S (z-x)\,dS$ with
$S = \{(x,y,z)\in\mathbb{R}^3 \mid z = x+y^2 \wedge x\in[0,y] \wedge y\in[0,1]\}$

Solution

Define $g(x,y) = x+y^2$
and $A = \{(x,y)\in\mathbb{R}^2 \mid x\in[0,y] \wedge y\in[0,1]\}$
and note that

$$\partial g/\partial x = (\partial/\partial x)(x+y^2) = 1$$
$$\partial g/\partial y = (\partial/\partial y)(x+y^2) = 2y$$

It follows that

$$dS = \|R(x,y|a)\|\,dx\,dy = \sqrt{1+(\partial g/\partial x)^2+(\partial g/\partial y)^2}\,dx\,dy = \sqrt{1+1^2+(2y)^2}\,dx\,dy = \sqrt{2+4y^2}\,dx\,dy$$

and therefore

$$I = \iint_S (z-x)\,dS = \iint_A dx\,dy\,[(x+y^2)-x]\,\|R(x,y|a)\| =$$
$$= \iint_A dx\,dy\; y^2\sqrt{2+4y^2} = \int_0^1 dy\int_0^y dx\; y^2\sqrt{2+4y^2} =$$
$$= \int_0^1 dy\left[y^2\sqrt{2+4y^2}\int_0^y dx\right] = \int_0^1 dy\; y^2\left(\sqrt{2+4y^2}\right)y =$$
$$= \int_0^1 dy\; y^3\sqrt{2+4y^2}$$

Let $p = 2+4y^2 \Leftrightarrow 4y^2 = p-2 \Leftrightarrow y^2 = \frac{p-2}{4}$

Then

$dp = (2+4y^2)' dy = 8y\,dy \Rightarrow y\,dy = (1/8)\,dp$

For $y=0 \Rightarrow p = 2+4\cdot 0^2 = 2$

For $y=1 \Rightarrow p = 2+4\cdot 1^2 = 2+4 = 6$

It follows that

$$I = \int_2^6 dp\, \frac{p-2}{4} \sqrt{p}\, (1/8) = \frac{1}{32}\int_2^6 dp\, \sqrt{p}\,(p-2)$$

$$= \frac{1}{32}\int_2^6 dp\,(p^{3/2} - 2p^{1/2}) = \frac{1}{32}\left[\frac{p^{5/2}}{5/2} - \frac{2p^{3/2}}{3/2}\right]_2^6$$

$$= \frac{1}{16}\left[\frac{p^2\sqrt{p}}{5} - \frac{2p\sqrt{p}}{3}\right]_2^6 =$$

$$= \frac{1}{16}\left[\left(\frac{6^2\sqrt{6}}{5} - \frac{2\cdot 6\sqrt{6}}{3}\right) - \left(\frac{2^2\sqrt{2}}{5} - \frac{2\cdot 2\sqrt{2}}{3}\right)\right]$$

$$= \frac{1}{16}\left[\left(\frac{36}{5} - 4\right)\sqrt{6} - \left(\frac{4}{5} - \frac{4}{3}\right)\sqrt{2}\right]$$

$$= \frac{1}{16}\left[\frac{36-5\cdot 4}{5}\sqrt{6} - \frac{4\cdot 3 - 4\cdot 5}{5\cdot 3}\sqrt{2}\right] =$$

$$= \frac{1}{16}\left[\frac{36-20}{5}\sqrt{6} - \frac{-8}{15}\sqrt{2}\right]$$

$$= \frac{1}{16}\left(\frac{16}{5}\sqrt{6} + \frac{8}{15}\sqrt{2}\right) = \frac{\sqrt{6}}{5} + \frac{\sqrt{2}}{30}$$

$$= \frac{6\sqrt{6}+\sqrt{2}}{30}$$

c) Evaluate $I = \iint_S (z, x, 1) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \wedge z \geq 0\}$ oriented away from the origin.

<u>Solution</u>

Define $F(x,y,z) = (z, x, 1)$, $\forall (x,y,z) \in \mathbb{R}^3$.

Since

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1 \wedge z \geq 0\} =$$
$$= \{(\sin\varphi\cos\vartheta, \sin\varphi\sin\vartheta, \cos\varphi) \mid \varphi \in [0, \pi/2] \wedge \vartheta \in [0, 2\pi]\}$$

we define

$$A = \{(\varphi, \vartheta) \mid \varphi \in [0, \pi/2] \wedge \vartheta \in [0, 2\pi]\}.$$

and note that the fundamental product is given by:

$$R(\varphi, \vartheta) = \rho^2(\sin^2\varphi\cos\vartheta, \sin^2\varphi\sin\vartheta, \sin\varphi\cos\varphi)$$

It follows that

$$I = \iint_S (z, x, 1) \cdot dS =$$
$$= \iint_A d\varphi\, d\vartheta\, F(\sin\varphi\cos\vartheta, \sin\varphi\sin\vartheta, \cos\varphi) \cdot R(\varphi, \vartheta \mid S) =$$
$$= \iint_A d\varphi\, d\vartheta\, (\cos\varphi, \sin\varphi\cos\vartheta, 1) \cdot (\sin^2\varphi\cos\vartheta, \sin^2\varphi\sin\vartheta, \sin\varphi\cos\varphi)$$
$$= \iint_A d\varphi\, d\vartheta \left[\cos\varphi\sin^2\varphi\cos\vartheta + \sin\varphi\cos\vartheta\sin^2\varphi\sin\vartheta + \sin\varphi\cos\varphi\right]$$
$$= \iint_A d\varphi\, d\vartheta \left[(\cos\varphi\sin^2\varphi)\cos\vartheta + (1/2)\sin^3\varphi\sin(2\vartheta) + \sin\varphi\cos\varphi\right]$$

$$= I_1 + (1/2) I_2 + I_3$$

with

$$I_1 = \iint_A d\varphi\, d\vartheta\, (\cos\varphi \sin^2\varphi) \cos\vartheta =$$

$$= \int_0^{\pi/2} d\varphi \int_0^{2\pi} d\vartheta\, (\cos\varphi \sin^2\varphi) \cos\vartheta =$$

$$= \left[\int_0^{2\pi} d\vartheta \cos\vartheta\right]\left[\int_0^{\pi/2} d\varphi \cos\varphi \sin^2\varphi\right] =$$

$$= \left[\sin\vartheta\right]_0^{2\pi} \left[\int_0^{\pi/2} d\varphi \cos\varphi \sin^2\varphi\right] =$$

$$= (\sin(2\pi) - \sin 0) \left[\int_0^{\pi/2} d\varphi \cos\varphi \sin^2\varphi\right] =$$

$$= 0 \int_0^{\pi/2} d\varphi \cos\varphi \sin^2\varphi = 0$$

and

$$I_2 = \iint_A d\varphi\, d\vartheta\, \sin^3\varphi \sin(2\vartheta) = \int_0^{\pi/2} d\varphi \int_0^{2\pi} d\vartheta\, \sin^3\varphi \sin(2\vartheta)$$

$$= \left[\int_0^{2\pi} d\vartheta \sin(2\vartheta)\right]\left[\int_0^{\pi/2} d\varphi \sin^3\varphi\right] =$$

$$= \left[\frac{-\cos(2\vartheta)}{2}\right]_0^{2\pi} \int_0^{\pi/2} d\varphi \sin^3\varphi =$$

$$= \left[\frac{(-\cos(2\pi)) - (-\cos 0)}{2}\right] \int_0^{\pi/2} d\varphi \sin^3\varphi =$$

$$= 0$$

and

$$I_3 = \iint_A d\varphi d\vartheta \sin\varphi \cdot \cos\varphi = (1/2) \iint_A d\varphi d\vartheta \sin(2\varphi) =$$

$$= \int_0^{\pi/2} d\varphi \int_0^{2\pi} d\vartheta \sin(2\varphi) = \int_0^{\pi/2} d\varphi \left[\sin(2\varphi) \int_0^{2\pi} d\vartheta \right] =$$

$$= \int_0^{\pi/2} d\varphi \, 2\pi \sin(2\varphi) = 2\pi \left[\frac{-\cos(2\varphi)}{2} \right]_0^{\pi/2} =$$

$$= \pi \left[-\cos(2\varphi) \right]_0^{\pi/2} = \pi \left[(-\cos(\pi)) - (-\cos 0) \right] =$$

$$= \pi \left[(-(-1)) - (-1) \right] = \pi \left[1 + 1 \right] = 2\pi$$

and therefore

$$I = I_1 + (1/2) I_2 + I_3 = 0 + (1/2) 0 + 2\pi = 2\pi$$

EXERCISES

(19) Evaluate the following surface integrals using the definition of the surface integral

a) $I = \iint_S z\, dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x,z \in [0,a] \wedge y = a^2 - z^2\}$ and $a \in (0,+\infty)$

b) $I = \iint_S x^2\, dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = R^2 \wedge x,y,z \in [0,+\infty)\}$ and $R \in (0,+\infty)$.

c) $I = \iint_S \frac{x^2\, dS}{a-z}$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid z \in [0,b] \wedge z = a^2 - x^2 - y^2\}$ and $a \in (0,+\infty)$ and $b \in (0,a)$

d) $I = \iint_S e^{-z}\, dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = a \wedge z \in [0,a]\}$ and $a \in [0,+\infty)$

e) $I = \iint_S \frac{z\, dS}{x^2+y^2+z^2}$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid z \in [0,a] \wedge x^2+y^2+z^2 = a^2\}$ and $a \in (0,+\infty)$

f) $I = \iint_S y \, dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = a^2 \wedge z \in [0,b]\}$
and $a \in (0,+\infty)$ and $b \in (0,a)$.

g) $I = \iint_S z \, dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x,y \in [0,a] \wedge z = x^3\}$
and $a \in [0,+\infty)$

h) $I = \iint_S \frac{dS}{z}$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = a^2 \wedge 1 \leq x^2+y^2 \leq b^2 \wedge z \geq 0\}$
and $a \in (0,+\infty)$ and $b \in (0,a)$

(10) Evaluate the following surface integrals using the surface integral definition.

a) $I = \iint_S (e^z, z, x) \cdot dS$

with (S): $(x,y,z) = (st, s+t, s)$, $\forall (s,t) \in [0,a] \times [0,a]$
and $a \in (0,+\infty)$.

b) $I = \iint_S (0,0,z^2) \cdot dS$

with (S): $(x,y,z) = (r\cos\varphi, r\sin\varphi, r)$, $\forall (r,\varphi) \in [0,a] \times [0,2\pi]$
and $a \in (0,+\infty)$

c) $I = \iint_S (y, z, 0) \cdot dS$

with $(S): (x, y, z) = (s^3 - t, s + t, t^2), \forall (s,t) \in [0,a] \times [0,b]$
and $a, b \in (0, +\infty)$

d) $I = \iint_S (z, 0, z^2) \cdot dS$

with $(S): (x, y, z) = (s \cosh(t), s \sinh(t), s)$,
$\forall (s,t) \in [0,a] \times [0,b]$
and $a, b \in (0, +\infty)$

e) $I = \iint_S (0, 3, x) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = a^2 \wedge x, y, z \in [0, +\infty)\}$
and $a \in (0, +\infty)$, oriented away from the origin.

f) $I = \iint_S (x, y, z) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = a^2 \wedge z \in [b, c]\}$
and $a \in (0, +\infty)$ and $b, c \in (-a, a)$, oriented away from the origin.

g) $I = \iint_S (y^2, -2, x) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = a \wedge x, y, z \in [0, +\infty)\}$
and $a \in (0, +\infty)$, oriented away from the origin.

Stokes and Gauss theorems

Jordan-bounded surface

Def: Let $S = \{a(t,s) \mid (t,s) \in A\}$ be a surface with $A \subseteq \mathbb{R}^2$ and $a: A \longrightarrow \mathbb{R}^3$. Let ∂A be the boundary of A.

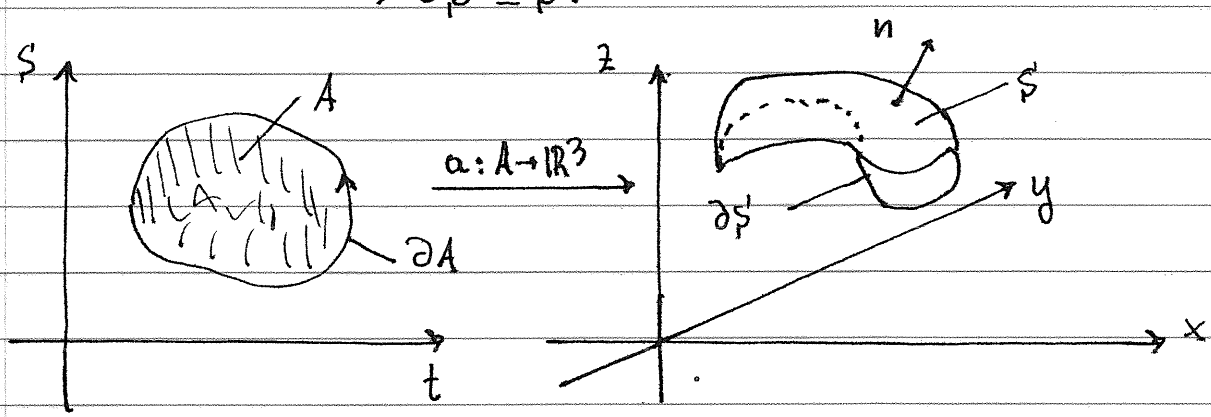
a) We define the boundary ∂S of the surface S as $\partial S = \{a(t,s) \mid (t,s) \in \partial A\}$, if S is a simple surface

b) We say that

$$\boxed{S \text{ Jordan-bounded} \Leftrightarrow \begin{cases} S \text{ simple} \wedge S \text{ smooth} \\ \partial A \in \mathrm{Jord}(\mathbb{R}^2) \end{cases}}$$

▸ Note that by definition:

S surface $\Rightarrow A$ closed set $\Rightarrow \partial A \subseteq A \Rightarrow$
$\Rightarrow \partial S \subseteq S$.



▸ By convention, the direction of ∂A is counterclockwise, so that $\omega(\partial A) = +1$.

→ Stokes theorem

Thm: Let $\mathcal{S} = \{a(t,s) \mid (t,s) \in A\}$ be a surface with $A \subseteq \mathbb{R}^2$, and let $f: B \to \mathbb{R}^3$ be a vector field with $B \subseteq \mathbb{R}^3$ and $\mathcal{S} \subseteq B$. Assume that

a) $\mathcal{S}$ is a Jordan-bounded surface.

b) $a(t,s)$ has continuous 2nd partial derivatives on A

c) f differentiable in B

d) ∇f continuous in B

Then:

$$\boxed{\iint_{\mathcal{S}} (\nabla \times f) \cdot d\mathcal{S} = \oint_{\partial \mathcal{S}} f \cdot dl}$$

- The direction of the path $\partial \mathcal{S}$ is determined by the restriction that $w(\partial A) = +1$ and the mapping $a: A \to \mathbb{R}^3$. Since $\partial A \in \mathrm{Jord}(\mathbb{R}^2)$, it follows that $\partial \mathcal{S} \in \mathrm{Loop}(\mathbb{R}^3)$, which means that the line integral above is circular, hence the $\oint$ notation.

→ To formulate the Gauss divergence theorem we have to define first what we mean by:

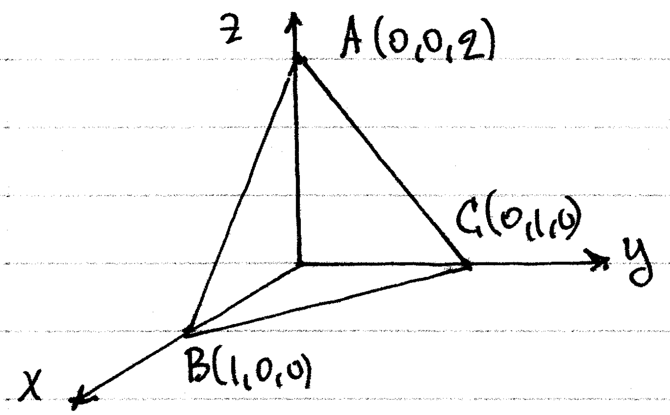
a) Orientable surface

b) Closed surface.

EXAMPLES

a) Evaluate $I = \oint_C F \cdot dl$ with C a triangle with vertices $A(0,0,2)$, $B(1,0,0)$, $C(0,1,0)$ whose projection to the xy-plane is traversed counterclockwise and $f(x,y,z) = (2z, 8x-3y, 3x+y)$, $\forall (x,y,z) \in \mathbb{R}^3$.

Solution



- We need the cartesian equation for the plane (p) defined by the points A, B, C.

$$\begin{cases} A(0,0,2) \\ B(1,0,0) \end{cases} \Rightarrow \vec{AB} = (1-0, 0-0, 0-2) = (1,0,-2)$$

$$\begin{cases} A(0,0,2) \\ C(0,1,0) \end{cases} \Rightarrow \vec{AC} = (0-0, 1-0, 0-2) = (0,1,-2)$$

and therefore

$$\vec{AB} \times \vec{AC} = (1,0,-2) \times (0,1,-2) =$$

$$= \left|\begin{matrix} e_1 & e_2 & e_3 \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{matrix}\right| \begin{matrix} e_1 & e_2 \\ 1 & 0 \\ 0 & 1 \end{matrix} =$$

$= 0e_1 + 0e_2 + (1\cdot 1)e_3 - 0e_3 - 1(-2)e_1 - 1(-2)e_2$

$= e_3 + 2e_1 + 2e_2 = (2,2,1)$

It follows that

$(x,y,z) \in (p) \Leftrightarrow (\vec{AB} \times \vec{AC}) \cdot [(x,y,z) - (0,0,2)] = 0$

$\Leftrightarrow (2,2,1) \cdot (x,y,z-2) = 0 \Leftrightarrow$

$\Leftrightarrow 2x + 2y + (z-2) = 0 \Leftrightarrow z = 2 - 2x - 2y$

and therefore

$(p): z = 2 - 2x - 2y.$

Define $f(x,y) = 2 - 2x - 2y, \ \forall (x,y) \in \mathbb{R}^2$

The corresponding fundamental product is given by

$R(x,y|S) = (-\partial f/\partial x, -\partial f/\partial y, 1) = (2,2,1)$

The projection of the triangle $\overset{\Delta}{ABC}$ onto the xy plane is given by:

$A = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1] \wedge y \in [0, 1-x]\}$

The triangle $\overset{\Delta}{ABC}$ is given by

$S = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in A \wedge z = 2 - 2x - 2y\}.$

To apply the Stokes theorem, we calculate $\nabla \times F$:

$\nabla \times F = \nabla \times (2z, 8x-3y, 3x+y) =$

$$= \left|\begin{matrix} e_1 & e_2 & e_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z & 8x-3y & 3x+y \end{matrix}\right| \begin{matrix} e_1 & e_2 \\ \partial/\partial x & \partial/\partial y \\ 2z & 8x-3y \end{matrix}$$

$$= e_1(\partial/\partial y)(3x+y) + e_2(\partial/\partial z)(2z) + e_3(\partial/\partial x)(8x-3y)$$
$$- e_3(\partial/\partial y)(2z) - e_1(\partial/\partial z)(8x-3y) - e_2(\partial/\partial x)(3x+y) =$$
$$= e_1 + 2e_2 + 8e_3 - 0e_3 - 0e_1 - 3e_2 = e_1 - e_2 + 8e_3$$
$$= (1,-1,8).$$

From the Stokes theorem:

$$I = \oint_{C} F \cdot dl = \iint_{S} (\nabla \times F) \cdot dS = \iint_{A} dx\,dy\, (\nabla \times F) \cdot R(x,y|S)$$
$$= \iint_{A} dx\,dy\, (1,-1,8) \cdot (2,2,1) =$$
$$= [1 \cdot 2 + (-1) \cdot 2 + 8 \cdot 1] \iint_{A} dx\,dy =$$
$$= (2-2+8) \int_0^1 dx \int_0^{1-x} dy = 8 \int_0^1 dx\, (1-x) =$$
$$= 8 \left[x - x^2/2 \right]_0^1 = 8 \left[1 - 1^2/2 \right] = 8 \cdot (1/2) = 4$$

8) Evaluate the integral $I = \oint_{\mathcal{C}} F \cdot d\ell$ with

$$F(x,y,z) = (-y^2, x, z^2), \quad \forall (x,y,z) \in \mathbb{R}^3$$

and $\mathcal{C}$ the curve defined by the intersection of the plane $(p): y+z=2$ with the cylinder $(c): x^2+y^2=1$.

<u>Solution</u>

The curve $\mathcal{C}$ is given by

$$\mathcal{C} = \{(x,y,z) \in \mathbb{R}^3 \mid y+z=2 \land x^2+y^2=1\} =$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid z = 2-y \land x^2+y^2=1\} =$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid z = 2-y \land (x,y) \in \partial A\}$$

with ∂A the projection of $\mathcal{C}$ onto the xy-plane given by

$$\partial A = \{(\cos\vartheta, \sin\vartheta) \mid \vartheta \in [0, 2\pi]\}$$

which is the boundary of the disk A given by

$$A = \{(r\cos\vartheta, r\sin\vartheta) \mid r \in [0,1] \land \vartheta \in [0,2\pi]\}$$

Define the surface $\mathcal{S}$ so that

$$\mathcal{S} = \{(x,y,z) \in \mathbb{R}^3 \mid z = 2-y \land (x,y) \in A\}.$$

and note that $\mathcal{C} = \partial\mathcal{S}$.

To evaluate I via the Stokes theorem, we need $\nabla \times F$ and $\mathcal{R}(x,y|\mathcal{S})$.

- Fundamental product $\mathcal{R}(x,y|\mathcal{S})$

Define $z = f(x,y) = 2-y$ and note that

$$\mathcal{R}(x,y|\mathcal{S}) = (-\partial f/\partial x, -\partial f/\partial y, 1) =$$
$$= (-(\partial/\partial x)(2-y), -(\partial/\partial y)(2-y), 1) =$$
$$= (0, 1, 1)$$

▸ Curl $\nabla \times F$

$$\nabla \times F = \nabla \times (-y^2, x, z^2) =$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix} \begin{matrix} e_1 & e_2 \\ \partial/\partial x & \partial/\partial y \\ -y^2 & x \end{matrix} =$$

$$= e_1(\partial/\partial y)(z^2) + e_2(\partial/\partial z)(-y^2) + e_3(\partial/\partial x)x -$$
$$- e_3(\partial/\partial y)(-y^2) - e_1(\partial/\partial z)x - e_2(\partial/\partial x)z^2 =$$
$$= 0e_1 + 0e_2 + e_3 + 2y e_3 - 0e_1 - 0e_2 =$$
$$= (1+2y)e_3 = (0, 0, 1+2y)$$

▸ The integral

From the Stokes theorem we have:

$$I = \oint_C F \cdot dl = \iint_S (\nabla \times F) \cdot dS = \iint_A dxdy\, (\nabla \times F) \cdot R(x,y|S)$$
$$= \iint_A dxdy\, (0,0,1+2y) \cdot (0,1,1) = \iint_A dxdy\, (1+2y)$$

Change variables to polar coordinates:

$(x,y) = (r\cos\theta, r\sin\theta) \Rightarrow dxdy = r\,dr\,d\theta$

Define $B = \{(r,\theta) \mid r \in [0,1] \wedge \theta \in [0,2\pi]\}$.

It follows that:

$$I = \iint_A dxdy\,(1+2y) = \iint_B dr\,d\theta\, r(1+2r\sin\theta)$$
$$= \iint_B dr\,d\theta\,(r + 2r^2\sin\theta) = \int_0^{2\pi} d\theta \int_0^1 dr\,(r + 2r^2\sin\theta) =$$

$$= \int_0^{2\pi} d\theta \left[\frac{r^2}{2} + \frac{2r^3}{3} \sin\theta \right]_0^1 =$$

$$= \int_0^{2\pi} d\theta \left[\frac{1^2 - 0^2}{2} + \frac{2(1^3 - 0^3)}{3} \sin\theta \right] =$$

$$= \int_0^{2\pi} d\theta \left(\frac{1}{2} + \frac{2}{3} \sin\theta \right) =$$

$$= \left[\frac{\theta}{2} - \frac{2}{3} \cos\theta \right]_0^{2\pi} =$$

$$= \left[\frac{2\pi}{2} - \frac{2}{3} \cos(2\pi) \right] - \left[0 - \frac{2}{3} \cos 0 \right] =$$

$$= \pi - 2/3 + 2/3 = \pi$$

EXERCISES

(21) Evaluate the following line integrals using the Stokes theorem

a) $I = \oint_{C} 2z\,dx + x\,dy + 3y\,dz$

with C the curve that is the intersection of the plane $(p): z = x$ and the cylinder $(S): x^2+y^2 = a^2$, oriented counterclockwise as viewed from above, with $a \in (0, +\infty)$

b) $I = \oint_{C} y\,dx + z\,dy + x\,dz$

with C the triangle with vertices $A_1(0,0,0)$, $A_2(a,0,0)$, $A_3(0,a,a)$, oriented counterclockwise as viewed from above, with $a \in (0,+\infty)$

c) $I = \oint_{C} (y-x)\,dx + (x-z)\,dy + (y-z)\,dz$

with C the boundary of the surface
$S = \{(x,y,z) \in \mathbb{R}^3 \mid x+3y+2z = 1 \wedge x,y,z \in [0,+\infty)\}$
oriented counterclockwise as seen from above.

d) $I = \oint_{C} (z-y)\,dx + y\,dy + x\,dz$

with C the intersection of the cylinder $(S_1): x^2+y^2 = a^2$ with the sphere $(S_2): x^2+y^2+z^2 = b^2$, oriented counterclockwise as seen from above, with $b \in (0,+\infty)$ and $a \in (0, b)$.

e) $I = \oint_{C} (y-z)\,dx + (z-x)\,dy + (x-y)\,dz$

with C the intersection of the plane $(P): x+z=a$ with the cylinder $(S): x^2+y^2=b^2$, oriented counterclockwise as seen from above, with $a,b \in (0,+\infty)$.

f) $I = \oint_{C} x^2\,dx - 2xy\,dy + yz^2\,dz$

with C the boundary of the surface S given by
$S = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,a] \wedge y \in [0,b] \wedge z = xy\}$
oriented counterclockwise as seen from above, with $a,b \in (0,+\infty)$.

g) $I = \oint_{C} 2\,dx + xz\,dy + z^3\,dz$

with C the boundary of the surface S given by
$S = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,a] \wedge y \in [0,b] \wedge z = xy^2\}$
oriented counterclockwise as seen from above, with $a,b \in (0,+\infty)$.

h) $I = \oint_{C} 2\,dx + xz\,dy + z^3\,dz$

with C the boundary of the surface S given by
$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq a^2 \wedge z = x^2y^2\}$
oriented counterclockwise as seen from above, with $a \in (0,+\infty)$.

(22) Let $F(x,y,z) = f(\|(x,y,z)\|)\,(x,y,z)$ with f continuously differentiable on $\mathbb{R}-\{0\}$. Let C_1 be a Jordan curve. Show that $\oint_{C_1} F \cdot dl = 0$

(23) Let S be a Jordan-bounded surface and let f, g be continuously differentiable scalar fields. Show that

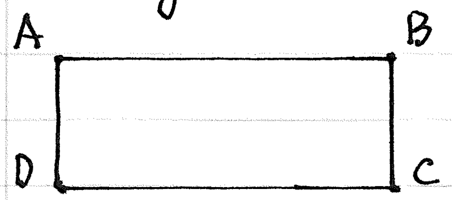
a) $\oint_{\partial S} f(\nabla g) \cdot dl = \iint_S [(\nabla f) \times (\nabla g)] \cdot dS$

b) $\oint_{\partial S} f(\nabla f) \cdot dl = 0$

c) $\oint_{\partial S} (f \nabla g + g \nabla f) \cdot dl = 0$

Orientable surfaces

Intuitively, we say that a surface is orientable if and only if it is possible to consistently define the notion of "above the surface" and "below the surface" for every point on the surface. An example of a non-orientable surface is the <u>Möbius strip</u>. A Möbius strip can be constructed by folding an elongated rectangle ABCD by connecting AD with BC and



and twisting it to connect A with C and B with D. A possible parameteric representation of the Möbius strip is given via the following definition:

Def: We define the Möbius strip M as the surface

$$M = \{ (x(a,\vartheta), y(a,\vartheta), z(a,\vartheta)) \mid a \in [-1,1] \wedge \vartheta \in [0, 2\pi] \}$$

with

$$x(a,\vartheta) = (1 + (a/2)\cos(\vartheta/2))\cos\vartheta$$

$$y(a,\vartheta) = (1 + (a/2)\cos(\vartheta/2))\sin\vartheta$$

$$z(a,\vartheta) = (a/2)\sin(\vartheta/2)$$

We now give the following definitions to capture the concept of an orientable surface:

Def: Let $\mathcal{S} \subseteq \mathbb{R}^3$ be a smooth surface. We define the set $Sub(\mathcal{S})$ of all *subsurfaces* of $\mathcal{S}$ as follows:

$$\boxed{\mathcal{S}_0 \in Sub(\mathcal{S}) \Leftrightarrow \mathcal{S}_0 \subseteq \mathcal{S} \land \mathcal{S}_0 \text{ smooth surface}}$$

Def: Let $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{R}^3$ be two surfaces. We say that *$\mathcal{S}_1$ and $\mathcal{S}_2$ are homeomorphic* (notation: $\mathcal{S}_1 \cong \mathcal{S}_2$) if and only if there is a mapping $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ such that, the following conditions are satisfied:

a) φ one-to-one $\land$ $\varphi(\mathcal{S}_1) = \mathcal{S}_2$

b) φ continuous at $\mathcal{S}_1$

c) φ^{-1} continuous at $\mathcal{S}_2$

▶ *interpretation*: If $\mathcal{S}_1 \cong \mathcal{S}_2$, then $\mathcal{S}_1$ can be deformed into $\mathcal{S}_2$ via "motion" and "stretching" but not "tearing". If that is the case, then the surfaces are considered topologically equivalent.

Def: Let $\mathcal{S}$ be a smooth surface. We say that:

$$\boxed{\begin{array}{l} \mathcal{S} \text{ orientable} \Leftrightarrow \forall \mathcal{S}_0 \in Sub(\mathcal{S}) : \mathcal{S}_0 \not\cong \mathcal{M} \\ \mathcal{S} \text{ non-orientable} \Leftrightarrow \exists \mathcal{S}_0 \in Sub(\mathcal{S}) : \mathcal{S}_0 \cong \mathcal{M} \end{array}}$$

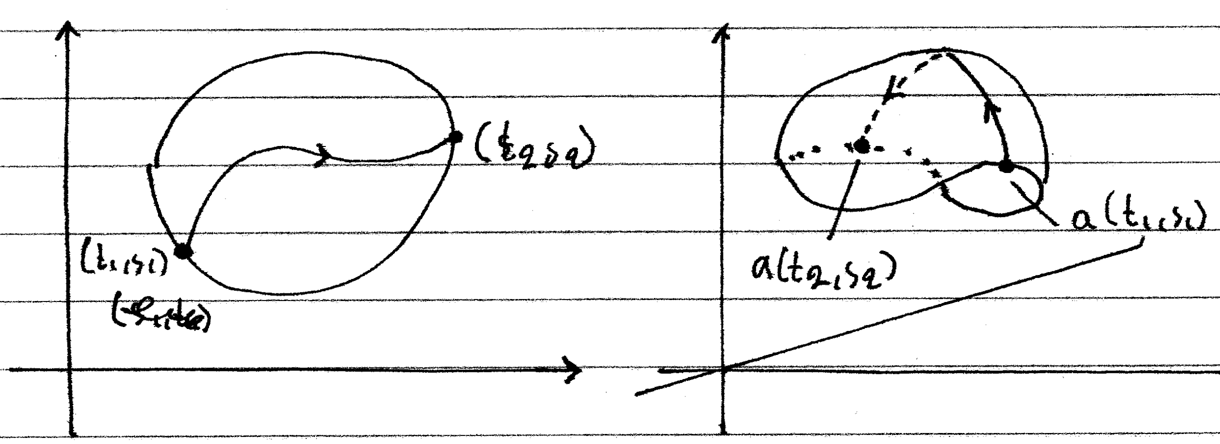
▸ <u>Interpretation</u> : In an orientable surface, none of its subsurfaces is homeomorphic to the Möbius strip M. On the other hand, in a non-orientable surface, there is at least one subsurface that is homeomorphic to the Möbius strip M.

Closed surfaces

▸ <u>Motivation</u> : Consider a smooth surface

$$S = \{ a(t,s) \mid (t,s) \in A \}$$

By definition, A is closed and simply-connected (i.e. no holes), therefore it has a boundary $\partial A \subseteq A$.
For Jordan-bounded surfaces, the image of ∂A, given by $a(\partial A)$, delineates the boundary ∂S of the surface S. To see that, note that any curve C in A from $(t_1,s_1) \in \partial A$ to $(t_2,s_2) \in \partial A$ with $(t_1,s_1) \neq (t_2,s_2)$ is NOT mapped into a closed curve $a(C)$ because



$a(t,s)$ is one-to-one (from the hypothesis that S is simple).

In more general surfaces, a may not necessarily be one-to-one, so it is possible to have $a(t_1,s_1)=a(t_2,s_2)$. Then, the curve C becomes a closed curve, and therefore the point $(x_0,y_0,z_0)=a(t_1,s_1)=a(t_2,s_2)$ should be excluded from the boundary ∂S of the surface S. The remaining points form the boundary ∂S of the surface S. Now we can define a closed surface as one that does not have a boundary.

Def: Let $S=\{a(t,s) \mid (t,s)\in A\}$ be a smooth surface and let the set of excluded points be:

$$P=\{(t,s)\in\partial A \mid \exists (t_0,s_0)\in \partial A-\{(t,s)\} : a(t,s)=a(t_0,s_0)\}$$

a) We define the Boundary ∂S of S as:

$$\partial S = a(\partial A - P) = \{a(t,s) \mid (t,s)\in \partial A - P\}$$

b) We say that:

$$S \text{ is closed} \Longleftrightarrow \partial S = \emptyset$$

▶ Motivation: Closed surfaces have an "inside" and an "outside" (e.g. consider a sphere or a torus) Let $x\in\mathbb{R}^3 - S$ be a point not on the surface. If x is "inside" S, then all rays emanating from x towards any direction will intersect S at an odd number of points. If x is "outside" S, then all such rays will instead intersect S at an even number of points. This observation motivates the following argument:

Def: Let $x \in \mathbb{R}^3$ be given. We define the set $\mathrm{Ray}(x)$ of all rays emanating from x via

$$L \in \mathrm{Ray}(x) \Leftrightarrow \exists a \in \mathbb{R}^3 - \{0\} : L = \{x + ta \mid t \in [0, +\infty)\}$$

Prop: Let $S \subseteq \mathbb{R}^3$ be a smooth surface. Then:

a) $\forall x \in \mathbb{R}^3 - \{0\} : \forall L \in \mathrm{Ray}(x) : L \cap S$ is a finite set.

b) Let $L_0 \in \mathrm{Ray}(x)$ be given. Then

$$|L_0 \cap S| \text{ even} \Rightarrow \forall L \in \mathrm{Ray}(x) : |L \cap S| \text{ even}$$

$$|L_0 \cap S| \text{ odd} \Rightarrow \forall L \in \mathrm{Ray}(x) : |L \cap S| \text{ odd}$$

(Here $|A|$ represents the cardinality of the set A)

Def: Let S be a closed smooth surface. Then:

$$x \in \mathrm{int}(S) \Leftrightarrow x \notin S \wedge (\exists L \in \mathrm{Ray}(x) : |L \cap S| \text{ odd})$$

$$x \in \mathrm{ext}(S) \Leftrightarrow x \notin S \wedge (\exists L \in \mathrm{Ray}(x) : |L \cap S| \text{ even}).$$

▶ Closed surface orientation

By convention, we prefer to define closed surfaces via mappings $a : A \to \mathbb{R}^3$ that yield a normal vector that always points towards the exterior $\mathrm{ext}(S)$ of the surface for every point on the surface. Such surfaces are called positively oriented. We give, therefore the following definition:

<u>Def</u> : Let $S = \{a(t,s) \mid (t,s) \in A\}$ be a <u>closed</u> and <u>smooth</u> surface and let $n(t,s|a)$ be the normal vector corresponding to the given representation $a(t,s)$. We say that:

$a(t,s)$ is <u>positively oriented</u> $\Leftrightarrow$

$$\Leftrightarrow \forall (t,s) \in A : \exists \tau_0 \in (0,+\infty) : \{a(t,s) + \tau n(t,s|a) \mid \tau \in (0,\tau_0)\} \subseteq$$
$$\subseteq \text{ext}(S)$$

$\rightarrow$ <u>Gauss divergence theorem</u>

<u>Thm</u> : Let $S = \{a(t,s) \mid (t,s) \in A\}$ be a surface and let $f : B \to \mathbb{R}^3$ be a vector field with $B \subseteq \mathbb{R}^3$.

We assume that:

a) S smooth $\wedge$ S closed $\wedge$ $a(t,s)$ positively oriented

b) $S \cup \text{int}(S) \subseteq B$

c) f differentiable at $S \cup \text{int}(S)$

d) ∇f continuous at $S \cup \text{int}(S)$

Then:

$$\boxed{\iint_S f \cdot dS = \iiint_{\text{int}(S)} (\nabla \cdot f)\, dx\, dy\, dz}$$

EXAMPLES

a) Evaluate $I = \iint_{S} F \cdot dS$ with

$F(x,y,z) = (x^3+y^3, y^3+z^3, z^3+x^3)$, $\forall (x,y,z) \in \mathbb{R}^3$ and S a hemisphere with center $O(0,0,0)$ and radius 1 above the xy-plane

Solution

Define

$A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1 \wedge z \geq 0\} =$

$= \{(\rho \sin\varphi \cos\vartheta, \rho \sin\varphi \sin\vartheta, \rho \cos\varphi) \mid \rho \in [0,1] \wedge \varphi \in [0,\pi/2] \wedge \vartheta \in [0,2\pi]\}$

and note that $\partial A = S$. Then:

$$I = \iint_{S} F \cdot dS = \iiint_{A} dx\,dy\,dz (\nabla \cdot F) =$$

$$= \iiint_{A} dx\,dy\,dz [\partial F/\partial x + \partial F/\partial y + \partial F/\partial z] =$$

$$= \iiint_{A} dx\,dy\,dz [(\partial/\partial x)(x^3+y^3) + (\partial/\partial y)(y^3+z^3) + (\partial/\partial z)(z^3+x^3)]$$

$$= \iiint_{A} dx\,dy\,dz [3x^2+3y^2+3z^2]$$

$$= \iiint_{A} dx\,dy\,dz\, 3(x^2+y^2+z^2)$$

Define

$B = \{(\rho, \varphi, \vartheta) \mid \rho \in [0,1] \wedge \varphi \in [0,\pi/2] \wedge \vartheta \in [0,2\pi]\}$

and do change of variables:

$(x,y,z) = (\rho\sin\varphi\cos\vartheta, \rho\sin\varphi\sin\vartheta, \rho\cos\varphi) \Rightarrow$

$\Rightarrow dx\,dy\,dz = \rho^2\sin\varphi\, d\rho\, d\varphi\, d\vartheta$

It follows that

$$I = \iiint_A dx\,dy\,dz\; 3(x^2+y^2+z^2) = \iiint_B 3\rho^2 \rho^2 \sin\varphi\; d\rho\, d\varphi\, d\vartheta =$$

$$= \int_0^1 d\rho \int_0^{\pi/2} d\varphi \int_0^{2\pi} d\vartheta\; 3\rho^4 \sin\varphi =$$

$$= \left[\int_0^1 d\rho\, 3\rho^4\right]\left[\int_0^{\pi/2} \sin\varphi\, d\varphi\right]\left[\int_0^{2\pi} d\vartheta\right] =$$

$$= \left[\frac{3\rho^5}{5}\right]_0^1 \left[-\cos\varphi\right]_0^{\pi/2} 2\pi =$$

$$= 2\pi\, \frac{3(1^5-0^5)}{5} \left[(-\cos(\pi/2)) - (-\cos 0)\right] =$$

$$= \frac{6\pi}{5}\,(0-(-1)) = 6\pi/5.$$

6) Evaluate the integral $I=\iint_S F\cdot dS$ where

$S=\partial A$ is the boundary of the solid
$A=\{(x,y,z)\in\mathbb{R}^3 \mid x\in[0,1] \wedge y\in[0,2] \wedge z\in[0,3]\}$
and $F(x,y,z)=(x^2y, xz, yz^3)$.

Solution

$$I=\iint_S F\cdot dS=\iiint_A dxdydz\ \nabla\cdot F=$$

$$=\iiint_A dxdydz\left[\partial F/\partial x+\partial F/\partial y+\partial F/\partial z\right]=$$

$$=\iiint_A dxdydz\left[(\partial/\partial x)(x^2y)+(\partial/\partial y)(xz)+(\partial/\partial z)(yz^3)\right]=$$

$$=\iiint_A dxdydz\left(2xy+0+3yz^2\right)=$$

$$=\int_0^1 dx\int_0^2 dy\int_0^3 dz\ (2xy+3yz^2)=$$

$$=\int_0^1 dx\int_0^2 dy\left[2xyz+yz^3\right]_{z=0}^{z=3}=$$

$$=\int_0^1 dx\int_0^2 dy\ (6xy+27y)=$$

$$=\int_0^1 dx\left[3xy^2+\frac{27y^2}{2}\right]_0^2=\int_0^1 dx\left[3x\cdot 2^2+\frac{27\cdot 2^2}{2}\right]$$

$$=\int_0^1 dx\ (12x+54)=\left[6x^2+54x\right]_0^1=6+54=60.$$

c) Let $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 4 \wedge z \in [0,3]\}$ be a solid cylinder and let

$F(x,y,z) = (x^3 + \tan(yz), y^3 - \exp(xz), 3z + x^3)$

Evaluate the integral

$$I = \iint_{\partial A} F \cdot dS$$

Solution

We note that

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 4 \wedge z \in [0,3]\}$$
$$= \{(r\cos\theta, r\sin\theta, z) \mid r \in [0,2] \wedge \theta \in [0,2\pi] \wedge z \in [0,3]\}$$

and therefore

$$I = \iint_{\partial A} F \cdot dS = \iiint_A dx\,dy\,dz\, \nabla \cdot F =$$
$$= \iiint_A dx\,dy\,dz\, [\partial F/\partial x + \partial F/\partial y + \partial F/\partial z] =$$
$$= \iiint_A dx\,dy\,dz\, \left[(\partial/\partial x)(x^3 + \tan(yz)) + (\partial/\partial y)(y^3 - \exp(xz)) + (\partial/\partial z)(3z + x^3) \right] =$$
$$= \iiint_A dx\,dy\,dz\, (3x^2 + 3y^2 + 3)$$

Change of variables:

$(x,y,z) = (r\cos\theta, r\sin\theta, z) \Rightarrow dx\,dy\,dz = r\,dr\,d\theta\,dz$

Define

$B = \{(r,\theta,z) \mid r \in [0,2] \wedge \theta \in [0,2\pi) \wedge z \in [0,3]\}$

It follows that:

$$I = \iiint_A dx\,dy\,dz\,\left[3(x^2+y^2)+3\right] =$$

$$= \iiint_B \left[3r^2+3\right] r\,dr\,d\vartheta\,dz = \iiint_B (3r^3+3r)\,dr\,d\vartheta\,dz =$$

$$= \int_0^2 dr \int_0^{2\pi} d\vartheta \int_0^3 dz\,(3r^3+3r) =$$

$$= \left[\int_0^2 dr\,(3r^3+3r)\right]\left[\int_0^{2\pi} d\vartheta\right]\left[\int_0^3 dz\right] =$$

$$= \left[\frac{3r^4}{4}+\frac{3r^2}{2}\right]_0^2 2\pi \cdot 3 =$$

$$= 6\pi\left[\frac{3\cdot 2^4}{4}+\frac{3\cdot 2^2}{2}\right] = 6\pi(12+6) =$$

$$= 6\pi \cdot 18 = 108\pi.$$

EXERCISES

(24) Use the Gauss divergence theorem to evaluate the following surface integrals

a) $I = \iint_{\partial S} (z, x, y) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq a^2 \wedge z \geq 0\}$ and $a \in (0,+\infty)$.

b) $I = \iint_{\partial S} (x^3, y^3, z^3) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,a] \wedge y \in [0,b] \wedge z \in [0,c]\}$ and $a,b,c \in (0,+\infty)$.

c) $I = \iint_{\partial S} (\cos(z^2), y, \sin(x^2)) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x,y,z \in [-a,a]\}$ and $a \in (0,+\infty)$.

d) $I = \iint_{\partial S} (x^3, y^3, z^3) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq a^2 \wedge z \geq 0\}$ and $a \in (0,+\infty)$.

e) $I = \iint_{\partial S} (x^3yz, xy^3z, xyz^3) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,a] \wedge y \in [0,b] \wedge z \in [0,c]\}$ and $a,b,c \in (0,+\infty)$

f) $I = \iint_{\partial S} (ax, by, cz) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq \rho^2\}$ and $a,b,c \in \mathbb{R}$ and $\rho \in (0,+\infty)$.

g) $I = \iint_{\partial S} (x^2, y^2, z^2) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq z \leq a^2 - x^2 - y^2\}$

and $a \in (0, +\infty)$

h) $I = \iint_{\partial S} (x^2 + y^2, y^2 + z^2, xy) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq y^2 + z^2 \leq a^2 \wedge 0 \leq x \leq b\}$

and $a, b \in (0, +\infty)$

i) $I = \iint_{\partial S} (x^2, y^2, z^2) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z \leq a \wedge x, y, z \in [0, +\infty)\}$

and $a \in (0, +\infty)$.

j) $I = \iint_{\partial S} (x^3 + y, y^3 + z, z^3 + x) \cdot dS$

with $S = \{(x,y,z) \in \mathbb{R}^3 \mid a^2 \leq x^2 + y^2 + z^2 \leq b^2\}$

and $a, b \in (0, +\infty)$ with $a < b$.

25) Let f, g be scalar fields that are continuously twice differentiable on $\mathbb{R}^3$. Let S be a solid $S \subseteq \mathbb{R}^3$ with boundary ∂S a smooth, closed, and positively oriented surface. Show that:

a) $$\iint_{\partial S} \frac{\partial f}{\partial n}\, dS = \iiint_S (\nabla^2 f)\, dx\,dy\,dz$$

b) $\nabla^2 f(x,y,z) = 0, \forall (x,y,z) \in S \Rightarrow$

$$\Rightarrow \iint_{\partial S} f \frac{\partial f}{\partial n}\, dS = \iiint_S \|\nabla f\|^2\, dx\,dy\,dz$$

c) $$\iint_{\partial \$} f \frac{\partial g}{\partial n} \, d\$ = \iiint_{\$} [f \nabla^2 g + (\nabla f) \cdot (\nabla g)] \, dx\,dy\,dz$$

d) $$\iint_{\partial \$} \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) d\$ = \iiint_{\$} [f \nabla^2 g - g \nabla^2 f] \, dx\,dy\,dz$$

(26) Let F be a vector field that is continuously twice differentiable on $\mathbb{R}^3$ and let $\$$ be a solid $\$ \subseteq \mathbb{R}^3$ with boundary $\partial \$$ a closed, smooth, and positively-oriented surface. Use tensor notation and the Gauss theorem to show that

$$\iint_{\partial \$} (\nabla \times F) \cdot d\$ = 0$$