SCALAR FIELDS

▼ Definitions

- We define the n-dimensional space $\mathbb{R}^n$ as follows:
  $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid \forall k \in [n] : x_k \in \mathbb{R}\}$
  with $[n]$ defined as
  $[n] = \{1, 2, 3, \dots, n\}$
- The elements $x \in \mathbb{R}^n$ are <u>n-dimensional vectors</u> with $x = (x_1, x_2, \dots, x_n)$. The numbers $x_1, x_2, \dots, x_n \in \mathbb{R}$ are the <u>components</u> of x.

▶ Algebra on $\mathbb{R}^n$

Let $x, y, z \in \mathbb{R}^n$ with $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$, and $z = (z_1, z_2, \dots, z_n)$. We define:

$$x = y \Leftrightarrow \forall k \in [n] : x_k = y_k$$
$$z = x + y \Leftrightarrow \forall k \in [n] : z_k = x_k + y_k$$
$$z = ax \Leftrightarrow \forall k \in [n] : z_k = a x_k \quad \text{(with } a \in \mathbb{R}\text{)}$$
$$x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{k=1}^{n} x_k y_k \quad \text{(inner product)}$$
$$\|x\| = (x \cdot x)^{1/2} = \sqrt{\sum_{k=1}^{n} x_k^2} \quad \text{(norm).}$$

▶ Balls on $\mathbb{R}^n$

- Let $x, y \in \mathbb{R}^n$ be two vectors. Assume that x, y represent two points in an n-dimensional space.. Then $\|x-y\|$ is the distance between x and y.
- We therefore define:

  $$B(x,\rho) = \{ u \in \mathbb{R}^n \mid \|x-u\| < \rho \}$$

  with $x \in \mathbb{R}^n$ and $\rho \in (0, +\infty)$
- The set $B(x,\rho)$ contains all the points in $\mathbb{R}^n$ whose distance from x is less than ρ.

▶ Open and closed sets

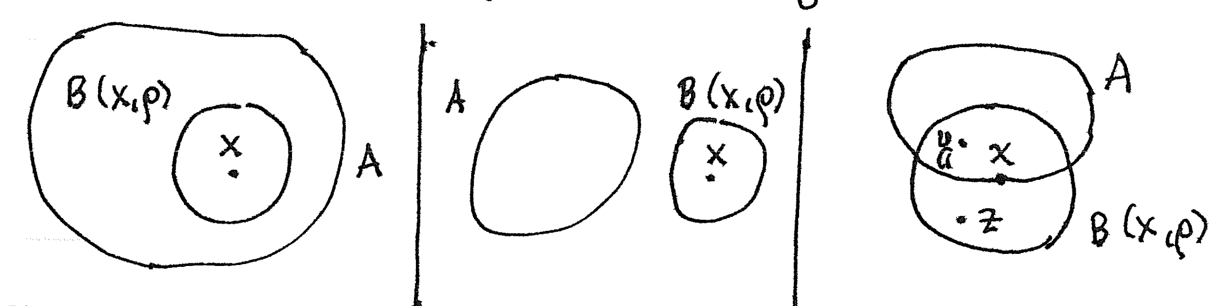
- Let $A \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. We say that

  x interior to $A \Leftrightarrow \exists \rho \in (0,+\infty): B(x,\rho) \subseteq A$

  x exterior to $A \Leftrightarrow \exists \rho \in (0,+\infty): B(x,\rho) \cap A = \emptyset$

  x boundary point to $A \Leftrightarrow$

  $\Leftrightarrow \forall \rho \in (0,+\infty): \exists y, z \in B(x,\rho): (y \in A \wedge z \notin A)$.



x interior to A	x exterior to A	x boundary point to A.

- We may therefore define:
  $\text{int}(A) = \{x \in \mathbb{R}^n \mid x \text{ interior to } A\}$
  $\text{ext}(A) = \{x \in \mathbb{R}^n \mid x \text{ exterior to } A\}$
  $\partial A = \{x \in \mathbb{R}^n \mid x \text{ boundary point to } A\}$.
- We say that:
  A is an <u>open</u> set $\Leftrightarrow A \cap \partial A = \emptyset \Leftrightarrow A = \text{int}(A)$
  A is a <u>closed</u> set $\Leftrightarrow \partial A \subseteq A$
- It follows that
  a) An open set does not contain any of the points in its boundary.
  b) A closed set includes all of the points in its boundary.

Scalar fields

- A <u>scalar field</u> is a mapping $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$.
- A is the <u>domain</u> of f, and we write $\text{dom}(f) = A$.
- The range $f(A)$ of A is defined as:
  $f(A) = \{f(x) \mid x \in A\}$.

▼ Limits of scalar fields

- Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $x_0 \in \text{int}(A)$. We define

$$\boxed{\lim_{x \to x_0} f(x) = \ell \Leftrightarrow \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (x \in B(x_0, \delta) - \{x_0\} \Rightarrow |f(x) - \ell| < \varepsilon)}$$

- This definition is similar to the Weierstrass definition of $\lim_{x \to a} f(x) = \ell$ for functions of one variable.

↳ Properties of limits

Thm: Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$, and let $a \in \text{int}(A)$. Assume that

$$\lim_{x \to a} f(x) = \ell_1 \wedge \lim_{x \to a} g(x) = \ell_2. \quad \text{Then:}$$

a) $\lim_{x \to a} [f(x) + g(x)] = \ell_1 + \ell_2$

b) $\lim_{x \to a} [f(x) g(x)] = \ell_1 \ell_2$

c) $\forall \lambda \in \mathbb{R} : \lim_{x \to a} [\lambda f(x)] = \lambda \lim_{x \to a} f(x)$

d) $\ell_2 \neq 0 \Rightarrow \lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\ell_1}{\ell_2}$

Limit of polynomials

Def : A polynomial $f \in \mathbb{R}^n[x]$ is a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ with

$$\boxed{f(x) = \sum_{(k_1,\dots,k_n)\in([m]\cup\{0\})^n} A_{k_1 k_2 \cdots k_n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}}$$

with $x = (x_1, x_2, \dots, x_n)$ and $[m] = \{1, 2, 3, \dots, m\}$.

► $\mathbb{R}^n[x]$ = the set of all polynomials on $\mathbb{R}^n$.

Thm : $$\boxed{f \in \mathbb{R}[x] \Rightarrow \lim_{x \to x_0} f(x) = f(x_0)\ ,\ \forall x_0 \in \mathbb{R}^n}$$

EXAMPLE

Evaluate: $\lim\limits_{(x,y)\to(1,3)} [3 + x^2 + y^2 + 3xy]$

Solution

$$\lim_{(x,y)\to(1,3)} [3+x^2+y^2+3xy] = 3 + 1^2 + 3^2 + 3\cdot1\cdot3 =$$
$$= 3 + 1 + 9 + 9 =$$
$$= 22.$$

Zero-bounded theorem

Def: Let $f: A \to \mathbb{R}$ be a scalar field with $A \subseteq \mathbb{R}^n$. Let $B \subseteq A$. We say that

f bounded in $B \iff \exists a \in (0, +\infty): \forall x \in B: |f(x)| \leq a$

Let $N(x_0, \delta)$ be an open ball with center x_0 and radius δ defined as:

$$N(x_0, \delta) = \{ x \in \mathbb{R}^n \mid 0 < \|x - x_0\| < \delta \}$$

The zero-bounded theorem reads:

Thm: Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ be scalar fields with $A \subseteq \mathbb{R}^n$ and let $x_0 \in \operatorname{int}(A)$. Then

$$\begin{cases} \exists \delta \in (0, +\infty): f \text{ bounded at } N(x_0, \delta) \\ \lim_{x \to x_0} g(x) = 0 \end{cases} \Rightarrow \lim_{x \to x_0} [f(x) g(x)] = 0$$

EXAMPLE

Evaluate the limit $\displaystyle\lim_{(x,y) \to (0,0)} \frac{3x^2(x^2 - y^2)}{x^2 + y^2}$

Solution

Define $b(x,y) = \dfrac{3x^2}{x^2 + y^2}$, $\forall (x,y) \in \mathbb{R}^2 - \{(0,0)\}$

and $g(x,y) = x^2 - y^2$, $\forall (x,y) \in \mathbb{R}^2$.

We note that:

$$|b(x,y)| = \left|\frac{3x^2}{x^2+y^2}\right| = \frac{3|x^2|}{|x^2+y^2|} = \frac{3x^2}{x^2+y^2} \leq$$

$$\leq \frac{3(x^2+y^2)}{x^2+y^2} = 3, \ \forall (x,y) \in \mathbb{R}^2 - \{(0,0)\} \Rightarrow$$

$\Rightarrow b$ bounded at $\mathbb{R}^2 - \{(0,0)\}$. (1)

Also:

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} (x^2-y^2) = 0^2-0^2 = 0 \quad (2)$$

From (1) and (2):

$$\lim_{(x,y)\to(0,0)} \frac{3x^2(x^2-y^2)}{x^2+y^2} = 0 \ .$$

Limit restricted to paths

Let $x_0 \in \mathbb{R}^n$ be given. We define $\mathcal{P}(x_0|A)$ as the set of all continuous mappings $\gamma: (0,a) \to A-\{x_0\}$ with $a \in (0,+\infty)$ and $A \subseteq \mathbb{R}^n$ such that $\lim_{t\to 0^+} \|\gamma(t)-x_0\|=0$. The corresponding belonging condition is:

$$\boxed{\gamma \in \mathcal{P}(x_0|A) \iff \begin{cases} \exists a \in (0,+\infty) : (\gamma : (0,a) \to A-\{x_0\}) \land \\ \qquad \land \gamma \text{ continuous on } (0,a)) \\ \lim_{t\to 0^+} \|\gamma(t) - x_0\| = 0 \end{cases}}$$

We now define path-restricted limits as follows:

Def: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field, let $x_0 \in \text{int}(A)$, and let $\gamma \in \mathcal{P}(x_0|A)$ be a path. Then, we define the limit of f along the path γ as:

$$\lim_{x\in\gamma} f(x) = \lim_{t\to 0^+} f(\gamma(t))$$

These path limits are a generalization of the "side limits" from single-variable calculus. Then, we can show that:

$$\boxed{\begin{array}{l} \lim\limits_{x\to x_0} f(x) = \ell \iff \forall \gamma \in \mathcal{P}(x_0|A) : \lim\limits_{x\in\gamma} f(x) = \ell \\ \exists \gamma_1, \gamma_2 \in \mathcal{P}(x_0|A) : \left(\lim\limits_{x\in\gamma_1} f(x) \neq \lim\limits_{x\in\gamma_2} f(x)\right) \implies \\ \qquad \implies \lim\limits_{x\to x_0} f(x) \text{ does not exist} \end{array}}$$

It follows that $\lim_{x\to x_0} f(x)$ will converge if and only if all paths limits for all paths $\gamma \in \mathcal{P}(x_0|A)$ agree. This includes both linear and nonlinear paths. As we will see from a counterexample below, agreement between just the linear paths is not sufficient to ensure that $\lim_{x\to x_0} f(x)$ converges.

EXAMPLE

Evaluate the following limit or show it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

Solution

Let $\forall (x,y)\in\mathbb{R}^2-\{0\}: f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ and consider the path

$$\gamma(\vartheta): \begin{cases} x = t\cos\vartheta \\ y = t\sin\vartheta \end{cases}, \text{ with } t\to 0^+$$

Then:

$$\lim_{(x,y)\in\gamma(\vartheta)} f(x,y) = \lim_{(x,y)\in\gamma(\vartheta)} \frac{x^2-y^2}{x^2+y^2} = \lim_{t\to 0^+} \frac{(t\cos\vartheta)^2-(t\sin\vartheta)^2}{(t\cos\vartheta)^2+(t\sin\vartheta)^2}$$

$$= \lim_{t\to 0^+} \frac{t^2(\cos^2\vartheta-\sin^2\vartheta)}{t^2(\cos^2\vartheta+\sin^2\vartheta)} = \cos^2\vartheta-\sin^2\vartheta = \cos(2\vartheta)$$

For $\vartheta=0$: $\displaystyle\lim_{(x,y)\in\gamma(0)} f(x,y) = \cos(2\cdot 0) = \cos 0 = 1 \qquad (1)$

For $\vartheta=\pi/4$: $\displaystyle\lim_{(x,y)\in\gamma(\pi/4)} f(x,y) = \cos(2(\pi/4)) = \cos(\pi/2) = 0 \qquad (2)$

From Eq.(1) and Eq.(2):

$$\lim_{(x,y)\in\gamma(0)} f(x,y) \neq \lim_{(x,y)\in\gamma(\pi/4)} f(x,y) \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) \text{ does not exist.}$$

COUNTEREXAMPLE

Consider the function $f(x,y) = \dfrac{xy^2}{x^2+y^4}$, $\forall (x,y) \in \mathbb{R}^2 - \{0\}$

and $\gamma(\vartheta)$ be a linear path towards $(0,0)$ defined as

$$\gamma(\vartheta): \begin{cases} x = t\cos\vartheta \\ y = t\sin\vartheta \end{cases}, \text{ with } t \to 0^+$$

Show that:

a) $\forall \vartheta \in [0, 2\pi)$: $\lim\limits_{(x,y)\in\gamma(\vartheta)} f(x,y) = 0$

b) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Solution

a) Let $\vartheta \in [0,2\pi)$ be given. Then:

$$\lim_{(x,y)\in\gamma(\vartheta)} f(x,y) = \lim_{(x,y)\in\gamma(\vartheta)} \frac{xy^2}{x^2+y^4} = \lim_{t\to 0^+} \frac{(t\cos\vartheta)(t\sin\vartheta)^2}{(t\cos\vartheta)^2+(t\sin\vartheta)^4} =$$

$$= \lim_{t\to 0^+} \frac{t^3\cos\vartheta\sin^2\vartheta}{t^2\cos^2\vartheta + t^4\sin^4\vartheta} = \lim_{t\to 0^+} \frac{t^3\cos\vartheta\sin^2\vartheta}{t^2[\cos^2\vartheta + t^2\sin^4\vartheta]}$$

$$= \lim_{t\to 0^+} \frac{t\cos\vartheta\sin^2\vartheta}{\cos^2\vartheta + t^2\sin^4\vartheta} \qquad (1)$$

We distinguish between the following cases:

Case 1: Assume that $\cos\vartheta \neq 0$. Then:

$$\lim_{(x,y)\in\gamma(\vartheta)} f(x,y) = \frac{0\cos\vartheta\sin^2\vartheta}{\cos^2\vartheta + 0^2\sin^4\vartheta} = \frac{0}{\cos\vartheta} = 0$$

Case 2: Assume that $\cos\vartheta = 0$. Then

$\sin^2\vartheta = 1 - \cos^2\vartheta = 1 - 0 = 1 \Rightarrow$

and it follows that

$$\lim_{(x,y)\in\gamma(\vartheta)} f(x,y) = \lim_{t\to 0^+} \frac{t\cdot 0\cdot 1}{0^2 + t^2\cdot 1^2} = \lim_{t\to 0^+} \frac{0}{t^2} = 0$$

We conclude that $\forall\vartheta\in[0,2\pi): \lim_{(x,y)\in\gamma(\vartheta)} f(x,y) = 0$

b) Consider the path $\gamma_0: \begin{cases} x = t^2 \\ y = t \end{cases}$ with $t\to 0^+$

Then, we have

$$\lim_{(x,y)\in\gamma_0} f(x,y) = \lim_{(x,y)\in\gamma_0} \frac{xy^2}{x^2+y^4} = \lim_{t\to 0^+} \frac{t^2(t)^2}{(t^2)^2+t^4}$$

$$= \lim_{t\to 0^+} \frac{t^4}{t^4+t^4} = \lim_{t\to 0^+} \frac{t^4}{2t^4} = \frac{1}{2} \Rightarrow$$

$$\rightarrow \lim_{(x,y)\in\gamma_0} f(x,y) \neq \lim_{(x,y)\in\gamma(0)} f(x,y) \Rightarrow$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) \text{ does not exist.}$$

EXERCISES

① Evaluate the following limits or show that they do not exist.

a) $\lim\limits_{(x,y)\to(1,3)} \left[x^2y(x+y)\right]$

b) $\lim\limits_{(x,y,z)\to(2,-1,4)} \left[x^3+y^3+z^3-3xyz\right]$

c) $\lim\limits_{(x,y)\to(1,0)} \dfrac{xy+x^2}{x^2+y^2+3}$

d) $\lim\limits_{(x,y)\to(3,3)} \dfrac{x-3}{\sqrt{y^2-9}}$

e) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{x^2+y^2}$

f) $\lim\limits_{(x,y)\to(0,0)} \dfrac{|x|}{|x|+|y|}$

g) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$

h) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{1+y^2}$

i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2}$

j) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x(x+y)^2}{x^2+y^2}$

k) $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x}{x+y^3}$

l) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x^2-y^2}$

② Show that

a) $\forall x,y\in\mathbb{R}: \begin{cases} |x^3| \leq |x|(x^2+y^2) \\ |y^3| \leq |y|(x^2+y^2) \end{cases}$

b) Consider the function

$$\forall (x,y)\in\mathbb{R}^2-\{\mathbf{0}\}: f(x,y)=\frac{x^3+y^3}{x^2+y^2}$$

Use (a) to show that: $\forall (x,y)\in\mathbb{R}^2-\{\mathbf{0}\}: |f(x,y)|\leq |x|+|y|$.

c) Use (b) to show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

③ Let $f(x,y) = \dfrac{x^a y^b}{x^2+y^2}$, $\forall (x,y) \in \mathbb{R}^2 - \{0\}$

Show that:

a) $a+b > 2 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$

b) $a+b \leq 2 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Continuity of scalar fields

Def: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $x_0 \in \text{int}(A)$. We say that:

$$f \text{ continuous at } x_0 \Longleftrightarrow \lim_{x \to x_0} f(x) = f(x_0)$$

$$f \text{ continuous on } B \Longleftrightarrow \forall x_0 \in B : f \text{ continuous at } x_0$$

Composition theorem

The composition theorem for scalar fields reads:

Thm: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $g: B \to \mathbb{R}$ with $B \subseteq \mathbb{R}$ be a function and let $x_0 \in A$. Define $\forall x \in G: h(x) = f(g(x))$ with $G = \{x \in B \mid g(x) \in A\}$ and assume that $x_0 \in G$. Then

$$\left.\begin{array}{l} f \text{ continuous at } x_0 \\ g \text{ continuous at } f(x_0) \end{array}\right\} \Longrightarrow h \text{ continuous at } x_0$$

- The following are immediate consequences of the composition theorem:

$$\lim_{x \to x_0} f(x) = a \Rightarrow \lim_{x \to x_0} \sin(f(x)) = \sin a$$

$$\lim_{x \to x_0} f(x) = a \Rightarrow \lim_{x \to x_0} \cos(f(x)) = \cos a$$

$$\left.\begin{array}{l} \forall k \in \mathbb{Z}: a \neq k\pi + \pi/2 \\ \lim\limits_{x \to x_0} f(x) = a \end{array}\right\} \Rightarrow \lim_{x \to x_0} \tan(f(x)) = \tan a$$

$$\left.\begin{array}{l} \forall k \in \mathbb{Z}: a \neq k\pi \\ \lim\limits_{x \to x_0} f(x) = a \end{array}\right\} \Rightarrow \lim_{x \to x_0} \cot(f(x)) = \cot a$$

- In certain limit calculations we can take advantage of the composition theorem, but the following composition theorem corollary can result in more economical solutions

Prop: (Composition theorem corollary)

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $g: B \to \mathbb{R}$ with $B \subseteq \mathbb{R}$ be a function. Let $x_0 \in \mathbb{R}^n$ be a limit point of A and let l_0 be a limit point of B. Then:

$$\left.\begin{array}{l} \lim\limits_{x \to x_0} f(x) = l_0 \wedge \lim\limits_{t \to l_0} g(t) = l \\ \exists \delta \in (0, +\infty): \forall x \in N(x_0, \delta): f(x) \neq l_0 \end{array}\right\} \Rightarrow \lim_{x \to x_0} g(f(x)) = l$$

EXAMPLES

a) $f(x,y) = \dfrac{x^4[\sin(xy)+\sin(x+y)]}{(x^2+y^2)^2} \leftarrow \lim_{(x,y)\to(0,0)} f(x,y)$

Solution

Since

$$\lim_{(x,y)\to(0,0)} (xy) = 0\cdot 0 = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \sin(xy) = \sin 0 = 0 \qquad (1)$$

$$\lim_{(x,y)\to(0,0)} (x+y) = 0+0 = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \sin(x+y) = \sin 0 = 0 \qquad (2)$$

then from Eq.(1) and Eq.(2):

$$\lim_{(x,y)\to(0,0)} [\sin(xy)+\sin(x+y)] = 0+0 = 0 \qquad (3)$$

Define $b(x,y) = \dfrac{x^4}{(x^2+y^2)^2}$, $\forall (x,y) \in \mathbb{R}^2 - \{(0,0)\}$

and note that

$$|b(x,y)| = \left|\frac{x^4}{(x^2+y^2)^2}\right| = \frac{x^4}{(x^2+y^2)^2} =$$

$$= \frac{x^4}{x^4+2x^2y^2+y^4} \leq \frac{x^4+2x^2y^2+y^4}{x^4+2x^2y^2+y^4}$$

$$= 1, \ \forall (x,y) \in \mathbb{R}^2 - \{(0,0)\} \Rightarrow$$

$$\Rightarrow b \text{ bounded on } \mathbb{R}^2 - \{(0,0)\} \qquad (4)$$

From Eq.(3) and Eq.(4), via the zero-bounded theorem,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

b) $f(x,y)=\dfrac{\sin(x^2+y^2)}{3(x^2+y^2)} \leftarrow \lim_{(x,y)\to(0,0)} f(x,y)$

<u>Solution</u>

<u>1st method</u>: (via composition theorem - not recommended!)
Define the auxiliary functions

$$g(x)=\begin{cases}\sin(x)/(3x), & \text{if } x\in\mathbb{R}-\{0\}\\ 1/3, & \text{if } x=0\end{cases}$$

$$h(x,y)=x^2+y^2, \quad \forall (x,y)\in\mathbb{R}^2$$

and note that

$$f(x,y)=g(h(x,y)), \quad \forall (x,y)\in\mathbb{R}^2-\{(0,0)\}$$

We show the assumptions needed by the composition theorem:

$$\lim_{(x,y)\to(0,0)} h(x,y)=\lim_{(x,y)\to(0,0)}(x^2+y^2)=0^2+0^2=0=h(0,0)$$

$\Rightarrow$ h continuous at $(x,y)=(0,0)$ (1)

and

$$\lim_{x\to h(0,0)} g(x)=\lim_{x\to 0}\frac{\sin x}{3x}=\frac{1}{3}\lim_{x\to 0}\frac{\sin x}{x}=$$

$$=\frac{1}{3}=g(0)\Rightarrow$$

$\Rightarrow$ g continuous at $x=h(0,0)$. (2)

From Eq.(1) and Eq.(2), via the composition theorem, it follows that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} g(h(x,y)) =$$

$$= g(h(0,0)) = g(0) = 1/3 \qquad \square$$

<u>2nd method</u>: (via the composition corollary).

We note that

$$\lim_{(x,y)\to(0,0)} (x^2+y^2) = 0^2+0^2 = 0$$

and for $t = x^2+y^2$

$$\lim_{t\to 0} \frac{\sin(t)}{3t} = \frac{1}{3} \lim_{t\to 0} \frac{\sin t}{t} = \frac{1}{3}$$

and

$$x^2+y^2 \neq 0 \ , \ \forall (x,y) \in \mathbb{R}^2 - \{(0,0)\}$$

so from the composition theorem it follows that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{3(x^2+y^2)} = \frac{1}{3} \qquad \square$$

→ Using the composition theorem corollary eliminates the need to explicitly define the auxilliary functions $g(x)$ and $h(x,y)$, and substantially simplifies the writing of the solution.

EXERCISES

(4) Evaluate the following limits or show that they do not exist.

a) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{\tan(x^2+y^2+z^2)}{x^2+y^2+z^2}$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4+2x^2y^2+y^4-4}{x^2+y^2-2}$

d) $\lim\limits_{(x,y)\to(0,0)} \sin(x+y)\exp(-1/(x^2+y^2))$

e) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{\sin x+\sin y+\sin z}{\ln(x^2+y^2+z^2)}$

f) $\lim\limits_{(x,y)\to(0,0)} \left[\dfrac{1}{\sin^2(x^2+y^2)} - \dfrac{1}{x^2+y^2}\right]$

g) $\lim\limits_{(x,y)\to(0,0)} \tan(x^2+y^4)\ln(x^2+y^4)$

h) $\lim\limits_{(x,y,z)\to(0,0,0)} (x^2+y^2+z^2)^{\sin(x^2+y^2+z^2)}$

(5) a) Show that : $\forall (x,y)\in\mathbb{R}^2-\{0\}: x^2+xy+y^2>0$

b) Use the result from (a) to evaluate $\lim\limits_{(x,y)\to(0,0)} \sin(x^2+xy+y^2)\ln(x^2+xy+y^2)$.

Directional and Partial Derivatives

- Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $x \in \operatorname{int} A$ and $y \in \mathbb{R}^n$. The directional derivative $f'(x|y)$ is defined as

$$f'(x|y) = \lim_{h \to 0} \frac{f(x+hy) - f(x)}{h}$$

when the limit exists.

- $f'(x|y)$ give the rate of change of f along the line connecting x and $x+y$ at the point x. However, it is important to realize that $f'(x|y)$ depends on BOTH the direction AND the magnitude of y:

$$\boxed{f'(x|\lambda y) = \lambda f'(x|y) \ , \ \forall \lambda \in \mathbb{R} - \{0\}.}$$

- Method: To calculate the directional derivative $f'(x|y)$ we use the following proposition:

$$g(t) = f(x+ty) \ , \ \forall t \in (-\varepsilon, \varepsilon) \Rightarrow f'(x|y) = g'(0)$$

EXAMPLE

Evaluate $f'(x,y|e)$ for $f(x,y) = xy(x+y)$ and $e = (1,3)$.

Solution

Let $x,y \in \mathbb{R}$ be given. Define:

$$g(t) = f((x,y) + t(1,3)) = f(x+t, y+3t) =$$
$$= (x+t)(y+3t)(x+t+y+3t) =$$
$$= (x+t)(y+3t)(x+y+4t) \Rightarrow$$
$$\Rightarrow g'(t) = (y+3t)(x+y+4t) + (x+t)\cdot 3(x+y+4t) +$$
$$+ (x+t)(y+3t)\cdot 4 \Rightarrow$$
$$\Rightarrow f'(x,y|(1,3)) = g'(0) =$$
$$= y(x+y) + 3x(x+y) + 4xy =$$
$$= xy + y^2 + 3x^2 + 3xy + 4xy =$$
$$= 3x^2 + 8xy + y^2.$$

↳ While calculating $g'(t)$ we treat x,y as given constants.

↳ Mean Value Theorem

Thm: Let $f: A \to \mathbb{R}$ be a scalar field with $A \subseteq \mathbb{R}^n$. Then:

$$\boxed{\begin{aligned} &f'(x+ty|y) \text{ exists}, \ \forall t \in [0,1] \Rightarrow \\ &\Rightarrow \exists \xi \in (0,1): f(x+y) - f(x) = f'(x+\xi y|y) \end{aligned}}$$

EXERCISES

⑥ Use the definition to evaluate the directional derivative $f'(x,y|e)$ for

a) $f(x,y) = x^2y^3$ and $e = (1,-2)$

b) $f(x,y) = x^2(x-y)^3$ and $e = (-2,3)$

c) $f(x,y) = \dfrac{x^2}{x^3+y^3}$ and $e = (2,5)$

d) $f(x,y) = \dfrac{x(x+y)^2}{x-y}$ and $e = (-1,3)$

e) $f(x,y) = \ln(x^2+y^2)$ and $e = (2,1)$

f) $f(x,y) = \sin(xy)$ and $e = (1,1)$

g) $f(x,y) = \mathrm{Arctan}(x^2-y^2)$ and $e = (3,-2)$

Directional derivatives and continuity.

Surprise : The existence of directional derivatives in all directions does not guarantee that your scalar field is continuous!! This is different from single variable calculus.

COUNTEREXAMPLE

Consider the function :

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

- Directional derivatives

For $d = (a,b)$ with $a \neq 0$

$$f'(0,0|d) = \lim_{h\to 0} \frac{f(ha,hb)-f(0,0)}{h} = \lim_{h\to 0} \frac{f(ha,hb)}{h} =$$

$$= \lim_{h\to 0} \left[\frac{1}{h} \frac{(ha)(hb)^2}{(ha)^2+(hb)^4} \right] = \lim_{h\to 0} \left[\frac{h^3ab^2}{h^3(a^2+h^2b^4)} \right]$$

$$= \lim_{h\to 0} \left[\frac{ab^2}{a^2+h^2b^4} \right] = \frac{ab^2}{a^2+0} = \frac{b^2}{a}$$

For $d = (0, b)$:

$$f'(0,0|d) = \lim_{h \to 0} \frac{f(0, hb) - f(0,0)}{h} = \lim_{h}$$

$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

Thus $f'(0,0|d)$ exists for all directions $d \in \mathbb{R}^2 - \{(0,0)\}$.

- <u>Continuity</u>

For $(\gamma_1): \begin{cases} x = 0 \\ y = t \end{cases}$ with $t \to 0^+$, we have:

$$\lim_{\gamma_1} f(x,y) = \lim_{t \to 0^+} f(0,t) = 0 \quad (\text{since } f(0,t) = 0, \forall t \in \mathbb{R}).$$

For $(\gamma_2): \begin{cases} x = t^2 \\ y = t \end{cases}$ with $t \to 0^+$, we have:

$$\lim_{\gamma_2} f(x,y) = \lim_{t \to 0^+} f(t^2, t) = \lim_{t \to 0^+} \frac{t^2 t^2}{(t^2)^2 + t^4} =$$

$$= \lim_{t \to 0^+} \frac{t^4}{t^4 + t^4} = \lim_{t \to 0^+} \frac{t^4}{2t^4} = \frac{1}{2}$$

Since $\lim_{\gamma_1} f(x,y) \neq \lim_{\gamma_2} f(x,y) \Rightarrow$

$\Rightarrow \lim_{(x,y) \to (0,0)} f(x,y)$ does not exist $\Rightarrow$

$\Rightarrow f$ not continuous at $(0,0)$

Partial Derivatives

- Let $f(x) = f(x_1, x_2, \dots, x_n)$, $\forall x \in A$ with $A \subseteq \mathbb{R}^n$ be a scalar field and consider the unit vectors:

$$e_1 = (1, 0, 0, \dots, 0)$$
$$e_2 = (0, 1, 0, \dots, 0)$$
$$e_3 = (0, 0, 1, \dots, 0)$$
$$\vdots$$
$$e_n = (0, 0, 0, \dots, 1)$$

We define the partial derivative of f with respect to x_k as:

$$\frac{\partial f}{\partial x_k} = f'(x \mid e_k)$$

- Method: To calculate $\partial f / \partial x_k$ we differentiate f with respect to x_k treating all other variables as constant.

- Notation: Other notations for partial derivatives, for example for 2 variables:

$$\frac{\partial f(x,y)}{\partial x} = f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = D_1 f(x,y)$$

$$\frac{\partial f(x,y)}{\partial y} = f_y(x,y) = \frac{\partial}{\partial y} f(x,y) \equiv D_2 f(x,y)$$

EXAMPLE

For $f(x,y) = xy^2(x^2+y^2)^3$, evaluate $\partial f/\partial x$ and $\partial f/\partial y$.

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[xy^2(x^2+y^2)^3\right] =$$

$$= \frac{\partial}{\partial x}\left[xy^2\right] \cdot (x^2+y^2)^3 + xy^2\left[\frac{\partial}{\partial x}(x^2+y^2)^3\right] =$$

$$= y^2(x^2+y^2)^3 + xy^2\, 3(x^2+y^2)^2\left[(\partial/\partial x)(x^2+y^2)\right]$$

$$= y^2(x^2+y^2)^3 + 3xy^2(x^2+y^2)^2 \cdot (2x) =$$

$$= y^2(x^2+y^2)^2\left[(x^2+y^2) + 3x(2x)\right] =$$

$$= y^2(x^2+y^2)^2(x^2+y^2+6x^2) =$$

$$= y^2(x^2+y^2)^2(7x^2+y^2).$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[xy^2(x^2+y^2)^3\right] =$$

$$= \left[\frac{\partial}{\partial y}(xy^2)\right](x^2+y^2)^3 + xy^2\left[\frac{\partial}{\partial y}(x^2+y^2)^3\right] =$$

$$= 2xy(x^2+y^2)^3 + xy^2 \cdot 3(x^2+y^2)^2\left[\frac{\partial}{\partial y}(x^2+y^2)\right] =$$

$$= 2xy(x^2+y^2)^3 + 3xy^2(x^2+y^2)^2\, 2y =$$

$$= 2xy(x^2+y^2)^2\left[(x^2+y^2) + 3y(y)\right] =$$

$$= 2xy(x^2+y^2)^2(x^2+y^2+3y^2) =$$

$$= 2xy(x^2+y^2)^2(x^2+4y^2).$$

EXERCISES

(7) Evaluate the following partial derivatives

a) $f(x,y) = x^3+y^3$; $\partial f/\partial x$, $\partial f/\partial y$

b) $f(x,y) = \sqrt{1-x^2-y^2}$; $\partial f/\partial x$, $\partial f/\partial y$

c) $f(x,y) = \sin(2x)\cos(3y)$; $\partial f/\partial x$, $\partial f/\partial y$

d) $f(x,y) = \tan(x/y)$; $\partial f/\partial x$, $\partial f/\partial y$

e) $f(x,y,z) = \dfrac{x}{(x^2+y^2+z^2)^{3/2}}$; $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$.

f) $f(x,y) = \exp(-y/x^2)$; $\partial f/\partial x$, $\partial f/\partial y$

g) $f(x,y,z) = \ln\left(\sqrt{x^2+y^2+z^2}\right)$; $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$

h) $f(x,y) = \mathrm{Arctan}\left(\sqrt{x^3+y^3}\right)$; $\partial f/\partial x$, $\partial f/\partial y$

i) $f(x,y) = \mathrm{Arcsin}\left(\sqrt{1-x^2y^2}\right)$; $\partial f/\partial x$

Mixed partial derivatives

Mixed partial derivatives are defined by successive partial differentiation, as follows:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = D_1 D_1 f$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy} = D_2 D_2 f$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy} = D_1 D_2 f$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx} = D_2 D_1 f$$

- Clairut's theorem (is $f_{xy} = f_{yx}$?)

Thm: Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ be a scalar field. Let $(a,b) \in A$ be a point and let $S \subseteq A$ be an open set such that $(a,b) \in \operatorname{int}(S)$.
Assume that
a) f_x, f_y, f_{xy} exist in S (i.e. for all points in S)
b) f_{xy} continuous in S
Then: $f_{xy}(a,b) = f_{yx}(a,b)$.

→ In condition (b) we can replace f_{xy} with f_{yx}.

▶ <u>Example with $f_{xy}(a,b) \neq f_{yx}(a,b)$</u>

Consider the function

$$f(x,y)=\begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & , \text{ if } (x,y)\in\mathbb{R}^2-\{(0,0)\} \\ 0 & , \text{ if } (x,y)=(0,0) \end{cases}$$

We will show that $f_{xy}(0,0)=1$ and $f_{yx}(0,0)=-1$.

<u>Solution</u>

• <u>Calculation of $f_{xy}(0,0)$</u>

We begin with evaluating $f_y(x,0)$ for all $x\in\mathbb{R}$.
We note that $\forall x\in\mathbb{R}: f(x,0)=0$. It follows that

$$\forall x\in\mathbb{R}: f_y(x,0)=\lim_{h\to 0}\frac{f((x,0)+h(0,1))-f(x,0)}{h}=$$

$$=\lim_{h\to 0}\frac{f(x,h)-0}{h}\underset{h\neq 0}{=}\lim_{h\to 0}\left[\frac{1}{h}\,\frac{xh(x^2-h^2)}{x^2+h^2}\right]=$$

$$=\lim_{h\to 0}\frac{x(x^2-h^2)}{x^2+h^2} \qquad (1)$$

We distinguish between the following cases

<u>Case 1</u>: Assume that $x\neq 0$. Then, from Eq. (1)

$$f_y(x,0)=\lim_{h\to 0}\frac{x(x^2-h)}{x^2+h^2}=\frac{x(x^2-0)}{x^2+0}=\frac{x^3}{x^2}=x,\ \forall x\in\mathbb{R}-\{0\}$$

<u>Case 2</u>: Assume that $x=0$. Then, from Eq. (1)

$$f_y(0,0) = \lim_{h\to 0} \frac{0(0-h)}{0+h^2} = \lim_{h\to 0} 0 = 0$$

We conclude from both cases that

$$\forall x\in\mathbb{R}: f_y(x,0) = \begin{cases} x & , \text{ if } x\in\mathbb{R}-\{0\} \\ 0 & , \text{ if } x=0 \end{cases} = x$$

$$\Rightarrow \forall x\in\mathbb{R}: f_{xy}(x,0) = (\partial/\partial x)x = 1$$

$$\Rightarrow \underline{f_{xy}(0,0) = 1.}$$

• <u>Calculation of $f_{yx}(0,0)$</u>

We begin by evaluating $f_x(0,y)$ for all $y\in\mathbb{R}$.
We note that $\forall y\in\mathbb{R}: f(0,y)=0$. It follows that

$$\forall y\in\mathbb{R}: f_x(0,y) = \lim_{h\to 0} \frac{f((0,y)+h(1,0))-f(0,y)}{h} =$$

$$= \lim_{h\to 0} \frac{f(h,y)-f(0,y)}{h} = \lim_{h\to 0} \frac{f(h,y)}{h} \underset{h\neq 0}{=}$$

$$= \lim_{h\to 0} \left[\frac{1}{h} \frac{hy(h^2-y^2)}{h^2+y^2} \right] = \lim_{h\to 0} \frac{y(h^2-y^2)}{h^2+y^2} \qquad (2)$$

We distinguish between the following cases.

<u>Case 1</u>: Assume that $y\neq 0$. Then, from Eq. (2)

$$f_x(0,y) = \lim_{h\to 0} \frac{y(h^2-y^2)}{h^2+y^2} = \frac{y(0-y^2)}{0+y^2} =$$

$$= \frac{-y^3}{y^2} = -y \ , \ \forall y\in\mathbb{R}-\{0\}$$

<u>Case 2</u>: Assume that $y=0$. Then, from Eq. (2)

$$f_x(0,0) = \lim_{h\to 0} \frac{0(h^2-0)}{h^2+0} = \lim_{h\to 0} 0 = 0$$

We conclude from both cases that

$$f_x(0,y) = \begin{cases} -y & , \text{ if } y \in \mathbb{R}-\{0\} \\ 0 & , \text{ if } y=0 \end{cases} = -y \ , \ \forall y \in \mathbb{R}$$

$$\Rightarrow f_{yx}(0,y) = (\partial/\partial y)(-y) = -1 \ , \ \forall y \in \mathbb{R}$$

$$\Rightarrow \underline{f_{yx}(0,0) = -1.}$$

EXERCISES

(8) Evaluate the following mixed partial derivatives

a) $f(x,y) = \sin(x^3+2y^2)$; f_{xxy}

b) $f(x,y,z) = x^3y^5z^8$; f_{xyz}

c) $f(x,y) = \dfrac{\exp(-x^2/y)}{\sqrt{y}}$; f_{xx}, f_{xy}

d) $f(x,y,z) = \text{Arctan}(x^2+yz)$; f_{xy}, f_{yz}, f_{zx}

e) $f(x,y,z) = \cos(xy+yz+zx)$; f_{xxy}, f_{yyzz}

(9) Show that the scalar field

$u(x,t) = \sin(nx)\exp(-n^2t)$, $\forall (x,t) \in \mathbb{R}^2$

satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for all $n \in \mathbb{R}$.

(10) A field $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ is <u>harmonic</u> if and only if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Show that the following scalar fields are harmonic:

a) $f(x,y) = x$

b) $f(x,y) = e^x \cos y$

c) $f(x,y) = \text{Arctan}(y/x)$

d) $f(x,y) = \ln(x^2+y^2)$

(11) Find all $a, b, c, d \in \mathbb{R}$ such that
$f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$
is harmonic.

(12) Find all $a, b \in \mathbb{R}$ such that
$f(x,y) = \cos(ax)\exp(by)$
is harmonic.

(13) Use Clairut's theorem to show that there does not exist a scalar field $f: \mathbb{R}^2 \to \mathbb{R}$ with $\partial f/\partial x = y^2$ and $\partial f/\partial y = x$.

Application of partial derivatives to error propagation

Suppose that a variable z is related to a set of variables $x_1, x_2, \dots, x_n$ via the equation
$$z = f(x_1, x_2, \dots, x_n)$$
Let us assume that the value of $x_1, x_2, \dots, x_n$ is not known exactly, but we have error estimates $x_1 \pm \sigma_1$, $x_2 \pm \sigma_2, \dots, x_n \pm \sigma_n$. Then, the error estimate $z \pm \sigma$ is given by the equation

$$\boxed{\sigma = \sqrt{\sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k}\right)^2 \sigma_k^2}}$$

Here, σ is the error in the variables $x_1, x_2, \dots, x_n$ as propagated via f onto the variable z.

- Typically, in experiments, some quantity z is calculated in two different ways from experimental data resulting in $z_1 \pm \sigma_1$ and $z_2 \pm \sigma_2$. We consider these two results to be consistent if
$$[z_1 - \sigma_1, z_1 + \sigma_1] \cap [z_2 - \sigma_2, z_2 + \sigma_2] \neq \emptyset$$

EXAMPLE

The oscillation period of an LC circuit is given by $T = 2\pi\sqrt{LC}$. Write the error of T in terms of the errors in L and C.

Solution

Since,

$$\frac{\partial T}{\partial L} = \frac{\partial}{\partial L}\left[2\pi\sqrt{LC}\right] = 2\pi\sqrt{C}\,(\partial/\partial L)\sqrt{L} =$$

$$= 2\pi\sqrt{C}\,\frac{1}{2\sqrt{L}} = \pi\sqrt{\frac{C}{L}}$$

and

$$\frac{\partial T}{\partial C} = \frac{\partial}{\partial C}\left[2\pi\sqrt{LC}\right] = 2\pi\sqrt{L}\,(\partial/\partial C)\sqrt{C} =$$

$$= 2\pi\sqrt{L}\,\frac{1}{2\sqrt{C}} = \pi\sqrt{\frac{L}{C}}$$

it follows that

$$\sigma_T = \sqrt{(\partial T/\partial L)^2\sigma_L^2 + (\partial T/\partial C)^2\sigma_C^2} =$$

$$= \sqrt{\left[\pi\sqrt{C/L}\right]^2\sigma_L^2 + \left[\pi\sqrt{L/C}\right]^2\sigma_C^2} =$$

$$= \sqrt{\pi^2(C/L)\sigma_L^2 + \pi^2(L/C)\sigma_C^2} =$$

$$= \pi\sqrt{(C/L)\sigma_L^2 + (L/C)\sigma_C^2}$$

EXERCISES

(14) Calculate the error propagation in the following expressions

a) $T = 2\pi\sqrt{l/g}$

b) $F = G\dfrac{m_1 m_2}{r^2}$

c) $E = mc^2$

d) $m = \dfrac{m_0}{\sqrt{1-(u/c)^2}}$

e) $x = x_0\sqrt{1-(u/c)^2}$

f) $y = y_0 \exp(-at)\cos(\omega t)$

g) $p = \exp(-nx^2)$

Differentiable scalar fields

- Directional derivatives account for the rate of change of the scalar field f accross linear directions. A proper definition of differentiability has to account for curved directions as well.
- Def : Let $T:\mathbb{R}^n \longrightarrow \mathbb{R}$ be a scalar field. We say that T linear $\iff \forall \lambda_1, \lambda_2 \in \mathbb{R} : \forall x, y \in \mathbb{R}^n : T(\lambda_1 x + \lambda_2 y) = \lambda_1 T(x) + \lambda_2 T(y)$.

Definition of Total Derivative (Young-Fréchet)

Def : Let $f : S \longrightarrow \mathbb{R}$ with $S \subseteq \mathbb{R}^n$ be a scalar field, and let $a \in \text{int}(S)$.

We say that f is differentiable at a if and only if there exist:

a) A linear scalar field $T_a : \mathbb{R}^n \longrightarrow \mathbb{R}$

b) A scalar field $E_a : \mathbb{R}^n \longrightarrow \mathbb{R}$

c) A number $\rho \in (0, +\infty)$

such that

$$\begin{cases} \forall x \in B(0, \rho) : f(a+x) = f(a) + T_a(x) + \|x\| E_a(x) \\ \lim\limits_{x \to 0} E_a(x) = 0 \end{cases}$$

- $T_a(x)$ is the total derivative of f at a. Note that T_a itself is a scalar field, satisfying:
$\forall \lambda_1, \lambda_2 \in \mathbb{R} : \forall x, y \in \mathbb{R}^n : T_a(\lambda_1 x + \lambda_2 y) = \lambda_1 T_a(x) + \lambda_2 T_a(y)$

↳ Gradient of a scalar field

- Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field. We define the gradient $\nabla f : A \to \mathbb{R}^n$ as

$$\boxed{\nabla f = \langle D_1 f, D_2 f, D_3 f, \dots, D_n f \rangle}$$

provided that the partial derivatives exist.
- Note that ∇f is NOT a scalar field. It is a vector field.

↳ Properties of differentiability.

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field, and let $a \in \operatorname{int}(A)$. It can be shown that:

1) f differentiable at $a \Rightarrow \forall x \in \mathbb{R}^n : T_a(x) = f'(a|x) = \nabla f(a) \cdot x$
2) f differentiable at $a \Rightarrow f$ continuous at a.
3) $\left.\begin{array}{l} \exists \rho \in (0, +\infty) : D_1 f, D_2 f, \dots, D_n f \text{ exist on } B(a, \rho) \\ D_1 f, D_2 f, \dots, D_n f \text{ continuous at } a \end{array}\right\} \Rightarrow$
   $\Rightarrow f$ differentiable at a.

- Properties (1), (2) are consequences of differentiable. Property (3) is a sufficient condition for differentiability.
- Property (1) indicates a method for evaluating directional derivatives via the gradient ∇f.

EXAMPLE

For $f(x,y) = xe^y + x^2y$ evaluate $f'(2,0|(a,b))$ for all $a,b \in \mathbb{R} - \{0\}$.

Solution

Since,

$f_x(x,y) = (\partial/\partial x)(xe^y + x^2y) = e^y + 2xy \Rightarrow$

$\Rightarrow f_x(2,0) = e^0 + 2\cdot 2\cdot 0 = 1+0 = 1$

and

$f_y(x,y) = (\partial/\partial y)(xe^y + x^2y) = xe^y + x^2 \Rightarrow$

$\Rightarrow f_y(2,0) = 2\cdot e^0 + 2^2 = 2+4 = 6$

it follows that

$$f'(2,0|(a,b)) = \nabla f(2,0)\cdot(a,b) =$$
$$= (f_x(2,0), f_y(2,0))\cdot(a,b) =$$
$$= af_x(2,0) + bf_y(2,0) =$$
$$= a + 6b$$

EXERCISE

(15) Use the gradient to evaluate the directional derivatives $f'(x,y|e)$ for the scalar fields f in the direction $e \in \mathbb{R}^3$.

a) $f(x,y) = (3x^2+y^3)^5(2x^2-y)^3$; $e=(2,4)$

b) $f(x,y) = x^2y^3\sqrt{x^4+y^4}$; $e=(1,-1)$

c) $f(x,y) = \ln(x^3+y^3-3xy)$; $e=(2,-3)$

d) $f(x,y) = xy\exp(-(x+y)^2)$; $e=(1,3)$

e) $f(x,y) = \exp(-2x^2)\sin(3y)$; $e=(-2,3)$

f) $f(x,y) = \mathrm{Arctan}\left(\dfrac{x^2}{x^2+y^2}\right)$; $e=(2,1)$

g) $f(x,y) = \mathrm{Arcsin}\left(\sqrt{x^4+y^4}\right)$; $e=(-1,1)$

▼ The chain rule

Thm : Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $a: I \to \text{int}(A)$ be a vector function with $I \subseteq \mathbb{R}$. Assume that :

a) $a(t)$ differentiable at t

b) f differentiable at $a(t)$.

Then:

$$\boxed{(d/dt)\, f(a(t)) = \nabla f(a(t)) \bullet \dot{a}(t)}$$

▸ Application to scalar fields on $\mathbb{R}^2$

1) For $z = f(x,y)$ with $x = x(t)$ and $y = y(t)$ $\Rightarrow$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

2) For $z = f(x,y)$ with $x = x(t,s)$ and $y = y(t,s)$ $\Rightarrow$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

→ The chain rule can be used to derive rules for transforming partial derivatives to other coordinates systems.

EXAMPLE

Consider the two-dimensional polar coordinates system defined via

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}$$

Given a scalar field $f(x,y)$, write $\partial f/\partial r, \partial f/\partial\vartheta$ in terms of $\partial f/\partial x, \partial f/\partial y$.

Solution

We note that

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\vartheta) = \cos\vartheta \,, \quad \frac{\partial x}{\partial\vartheta} = \frac{\partial}{\partial\vartheta}(r\cos\vartheta) = -r\sin\vartheta$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(r\sin\vartheta) = \sin\vartheta \,, \quad \frac{\partial y}{\partial\vartheta} = \frac{\partial}{\partial\vartheta}(r\sin\vartheta) = r\cos\vartheta$$

and therefore:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} =$$

$$= (\cos\vartheta)\frac{\partial f}{\partial x} + (\sin\vartheta)\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial\vartheta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial\vartheta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\vartheta} =$$

$$= (-r\sin\vartheta)\frac{\partial f}{\partial x} + (r\cos\vartheta)\frac{\partial f}{\partial y}$$

Using matrix notation:

$$\begin{bmatrix} \partial f/\partial r \\ \partial f/\partial \vartheta \end{bmatrix} = \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -r\sin\vartheta & r\cos\vartheta \end{bmatrix} \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix}$$

Let $A = \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -r\sin\vartheta & r\cos\vartheta \end{bmatrix}$. Then

$$\det A = \begin{vmatrix} \cos\vartheta & \sin\vartheta \\ -r\sin\vartheta & r\cos\vartheta \end{vmatrix} = \cos\vartheta \cdot (r\cos\vartheta) - \sin\vartheta\,(-r\sin\vartheta)$$

$$= r(\cos^2\vartheta + \sin^2\vartheta) = r.$$

and therefore:

$$A^{-1} = \frac{1}{r} \begin{bmatrix} r\cos\vartheta & -\sin\vartheta \\ +r\sin\vartheta & \cos\vartheta \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -(\sin\vartheta)/r \\ \sin\vartheta & (\cos\vartheta)/r \end{bmatrix}$$

It follows that

$$\frac{\partial f}{\partial x} = (\cos\vartheta)\,\frac{\partial f}{\partial r} - \frac{\sin\vartheta}{r}\,\frac{\partial f}{\partial \vartheta}$$

$$\frac{\partial f}{\partial y} = (\sin\vartheta)\,\frac{\partial f}{\partial r} + \frac{\cos\vartheta}{r}\,\frac{\partial f}{\partial \vartheta}$$

EXERCISES

(16) Consider the coordinate transformation

$$\begin{cases} x = \xi + t \\ y = \xi - t \end{cases}$$

Show that for a scalar field $f(x,y)$

$$\left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 = \frac{\partial f}{\partial \xi}\frac{\partial f}{\partial t}$$

(17) Consider the function $u(x,y)$ in polar coordinates:

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}$$

a) Show that

$$\|\nabla u\|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \vartheta}\right)^2$$

b) For $u(r,\vartheta) = r^3 \sin^3\vartheta$ in polar coordinates evaluate $\|\nabla u\|^2$.

(18) Consider the spherical coordinate transformation

$$\begin{cases} x = \rho \sin\varphi \cos\vartheta \\ y = \rho \sin\varphi \sin\vartheta \\ z = \rho \cos\varphi \end{cases}$$

Given a function $f(x,y,z)$, write the derivatives $\partial f/\partial \rho$, $\partial f/\partial \vartheta$, $\partial f/\partial \varphi$ in terms of $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$.

(19) Let $r = (x, y, z)$ be a vector and consider a scalar field $f(x,y,z) = F(\|r\|)$, $\forall (x,y,z) \in \mathbb{R}^3 - \{0\}$

a) Show that

$$\nabla f(r) = \frac{F'(\|r\|)}{\|r\|} r \quad , \text{ and}$$

$$\|\nabla f(r)\| = |F'(r)|$$

b) Use part (a) to evaluate ∇f and $\|\nabla f\|$ for the following scalar fields:

1) $f(x,y,z) = \exp(-(x^2+y^2+z^2))$

2) $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

3) $f(x,y,z) = \ln(\sqrt{x^2+y^2+z^2})$

(20) Let $f(x,y,z)$ be a scalar functions such that $f(x,y,z) = u(x-y, y-z, z-x)$, $\forall (x,y,z) \in \mathbb{R}^3$

Show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$$

(21) Let $f(x,y,z)$ be a scalar field such that $\forall t \in (0, +\infty) : \forall (x,y,z) \in \mathbb{R}^3 : f(tx, ty, tz) = t^n f(x,y,z)$

Show that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = nf$$

(22) Consider a scalar field $f(x,y)$ rewritten in terms of polar coordinates

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}$$

Show that:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \vartheta^2}$$

Chain Rule and Implicit Differentiation

Implicit Function Theorem

The Implicit Function Theorem, given below, shows that given a constraint

$$F(x_1, x_2, \ldots, x_n) = 0$$

if the conditions of the theorem are satisfied, then it implicitly defines a new function

$$x_n = f_n(x_1, x_2, \ldots, x_{n-1})$$

such that

$$F(x_1, x_2, \ldots, x_{n-1}, f(x_1, x_2, \ldots, x_{n-1})) = 0.$$

Then, the derivative of x_n with respect to x_a with $a \in [n-1]$ is written:

$$\left(\frac{\partial x_n}{\partial x_a}\right)_{x_1, \ldots, x_{a-1}, x_{a+1}, \ldots, x_n}$$

<u>Thm</u>: Let $F: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field, let $a \in \operatorname{int}(A)$, and let $\rho \in (0, +\infty)$ be given. Assume that $\begin{cases} F(a) = 0 \\ D_1F, D_2F, \ldots, D_nF \text{ continuous on } B(a, \rho) \\ \forall x \in B(a, \rho): D_nF \neq 0 \end{cases}$ Then there is a $f_n: B \to \mathbb{R}$ with $B \subseteq \mathbb{R}^{n-1}$ such that $\forall (x_1, \ldots, x_{n-1}) \in B: F(x_1, \ldots, x_{n-1}, f_n(x_1, \ldots, x_{n-1})) = 0$

• The case $F(x,y)=0$

Consider the function $y=f(x)$ defined implicitly by the equation

$$F(x,y)=0$$

It follows that $F(x,f(x))=0$, and we use the chain rule to differentiate with respect to x:

$$(d/dx)F(x,y)=0 \Rightarrow$$

$$\Rightarrow \frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0 \Rightarrow$$

$$\Rightarrow \frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0 \Rightarrow \frac{\partial F}{\partial y}\frac{dy}{dx}=-\frac{\partial F}{\partial x}$$

$$\Rightarrow \boxed{\frac{dy}{dx}=\frac{-\partial F/\partial x}{\partial F/\partial y}}$$

• The general case

Consider the constraint

$$F(x_1,x_2,\dots,x_n)=0$$

and let it define implicitly a function

$$x_n=f_n(x_1,x_2,\dots,x_{n-1})$$

Then we can show that

$$\boxed{\left(\frac{\partial x_n}{\partial x_a}\right)_{x_1,\dots,x_{a-1},x_{a+1},\dots,x_n}=\frac{\partial f_n}{\partial x_a}=\frac{-\partial F/\partial x_a}{\partial F/\partial x_n}}$$

for all $a\in[n-1]$.

Proof

We note that $\forall a, b \in [n-1] : \partial x_b / \partial x_a = \delta_{ab}$ with

$$\delta_{ab} = \begin{cases} 1 \ , & \text{if } a = b \\ 0 \ , & \text{if } a \neq b \end{cases}$$

and therefore

$$\begin{aligned} \frac{dF}{dx_a} &= \sum_{b=1}^{n-1} \frac{\partial F}{\partial x_b} \frac{\partial x_b}{\partial x_a} + \frac{\partial F}{\partial x_n} \left(\frac{\partial x_n}{\partial x_a} \right)_{x_1, \dots, x_{a-1}, x_{a+1}, \dots, x_n} \\ &= \sum_{b=1}^{n-1} \frac{\partial F}{\partial x_b} \delta_{ab} + \frac{\partial F}{\partial x_n} \left(\frac{\partial x_n}{\partial x_a} \right)_{x_1, \dots, x_{a-1}, x_{a+1}, \dots, x_n} \\ &= \frac{\partial F}{\partial x_a} + \frac{\partial F}{\partial x_n} \left(\frac{\partial x_n}{\partial x_a} \right)_{x_1, \dots, x_{a-1}, x_{a+1}, \dots, x_n} \end{aligned}$$

It follows that

$$dF / dx_a = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial F}{\partial x_a} + \frac{\partial F}{\partial x_n} \left(\frac{\partial x_n}{\partial x_a} \right)_{x_1, \dots, x_{a-1}, x_{a+1}, \dots, x_n} = 0$$

$$\Rightarrow \left(\frac{\partial x_n}{\partial x_a} \right)_{x_1, \dots, x_{a-1}, x_{a+1}, \dots, x_n} = \frac{-\partial F / \partial x_a}{\partial F / \partial x_n}$$

EXAMPLES

a) If x, y, z are constrained via
$x^3+y^3+z^3+6xyz=1$
evaluate $(\partial z/\partial x)_y$, $(\partial z/\partial y)_x$, $(\partial x/\partial y)_z$
in terms of x, y, z.

Solution

Define $F(x,y,z)=x^3+y^3+z^3+6xyz-1$

We note that

$$\partial F/\partial x = (\partial/\partial x)(x^3+y^3+z^3+6xyz-1) = 3x^2+6yz = 3(x^2+2yz)$$

$$\partial F/\partial y = (\partial/\partial y)(x^3+y^3+z^3+6xyz-1) = 3y^2+6xz = 3(y^2+2xz)$$

$$\partial F/\partial z = (\partial/\partial z)(x^3+y^3+z^3+6xyz-1) = 3z^2+6xy = 3(z^2+2xy)$$

and therefore

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{-\partial F/\partial x}{\partial F/\partial z} = \frac{-3(x^2+2yz)}{3(z^2+2xy)} = \frac{-(x^2+2yz)}{z^2+2xy}$$

$$\left(\frac{\partial z}{\partial y}\right)_x = \frac{-\partial F/\partial y}{\partial F/\partial z} = \frac{-3(y^2+2xz)}{3(z^2+2xy)} = \frac{-(y^2+2xz)}{z^2+2xy}$$

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{-\partial F/\partial y}{\partial F/\partial x} = \frac{-3(y^2+2xz)}{3(x^2+2yz)} = \frac{-(y^2+2xz)}{x^2+2yz}$$

b) Consider the van der Waals equations

$$[P + a(n/V)^2][(V/n) - b] = RT$$

where a, b, R are constants and

P : gas pressure
V : gas volume
T : gas temperature
n : amount of gas molecules.

Use implicit differentiation to evaluate

$(\partial P/\partial n)_{V,T}$, $(\partial T/\partial n)_{P,V}$, $(\partial V/\partial T)_{P,n}$

Solution

Define

$$F(P,V,T,n) = [P+a(n/V)^2][(V/n)-b] - RT$$
$$= P(V/n) - bP + a(V/n)^{-1} - ab(V/n)^{-2} - RT$$

and note that

$$\partial F/\partial P = (V/n) - b$$

$$\partial F/\partial V = (\partial/\partial V)[P(V/n) + a(V/n)^{-1} - ab(V/n)^{-2}] =$$
$$= P/n + (-1)(V/n)^{-2}(1/n) - ab(-2)(V/n)^{-3}(1/n)$$
$$= (1/n)[P - (V/n)^{-2} + 2ab(V/n)^{-3}]$$

$$\partial F/\partial T = -R$$

$$\partial F/\partial n = (\partial/\partial n)[P(n/V)^{-1} + a(n/V) - ab(n/V)^2] =$$
$$= P(-1)(n/V)^{-2}(1/V) + a(1/V) - ab(n/V)2(1/V)$$
$$= (1/V)[-P(n/V)^{-2} + a - 2ab(n/V)]$$

It follows that:

$$\left(\frac{\partial P}{\partial n}\right)_{V,T} = \frac{-\partial F/\partial n}{\partial F/\partial P} =$$

$$= \frac{-(1/V)\left[-P(n/V)^{-2} + a - 2ab(n/V)\right]}{V/n - b}$$

$$\left(\frac{\partial T}{\partial n}\right)_{P,V} = \frac{-\partial F/\partial n}{\partial F/\partial T} =$$

$$= \frac{-(1/V)\left[-P(n/V)^{-2} + a - 2ab(n/V)\right]}{-R}$$

$$= \frac{1}{RV}\left[-P(V/n)^{2} + a - 2ab(n/V)\right]$$

$$\left(\frac{\partial V}{\partial T}\right)_{P,n} = \frac{-\partial F/\partial T}{\partial F/\partial V} =$$

$$= \frac{-(-R)}{(1/n)\left[P - (V/n)^{-2} + 2ab(V/n)^{-3}\right]}$$

$$= \frac{nR}{P - (n/V)^{2} + 2ab(n/V)^{3}}$$

EXERCISES

(23) Evaluate the following partial derivatives using implicit differentiation based on partial derivatives of the implicit definition

a) $x^3y + y^3z + z^3x = (xyz)^2$; $\partial z/\partial x, \partial z/\partial y, \partial x/\partial y$

b) $x^3+y^3+z^3 + (x+y)(y+z)(z+x)$; $\partial y/\partial x, \partial y/\partial z, \partial z/\partial x$

c) $\exp(x^2y) + \cos(y^2z) = xz$; $\partial z/\partial x, \partial x/\partial z, \partial x/\partial y$

d) $\dfrac{1}{x^2+y^2} + \dfrac{1}{y^2+z^2} + \dfrac{1}{z^2+x^2} = 1$; $\partial y/\partial z, \partial x/\partial z, \partial x/\partial y$

e) $\dfrac{xy}{(x+y)^2} + \dfrac{yz}{(y+z)^2} + \dfrac{zx}{(z+x)^2} = xyz$; $\partial y/\partial x, \partial z/\partial y, \partial x/\partial z$

(24) According to the law of cosines, a triangle $A\hat{B}C$ with $a = BC$ and $b = CA$ and $c = AB$ satisfies

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

Use implicit differentiation based on partial derivatives of the implicit definition to evaluate $\partial a/\partial A$, $\partial b/\partial A$, $\partial c/\partial A$.