MULTIPLE INTEGRALS

Definition of Multiple integrals

We are mainly interested in defining the double and triple integral. However, in order to eliminate repetitiveness, we will just give one definition for the general n-dimensional integral. The definition is made in two steps:

a) We define the multiple integral over a box region

b) We then extend the definition over a general bounded region.

Finally we introduce the Fubini theorems that reduce the multiple integral to simpler integrals.

① → Multiple integral over a box region

Let us consider a scalar field $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ and also consider a box region

$$S = \prod_{k=1}^{n} [a_k, b_k] =$$

$$= \{(x_1, x_2, \dots, x_n) \mid \forall k \in [n] : x_k \in [a_k, b_k]\}$$

with $\forall k \in [n] : (a_k, b_k \in \mathbb{R} \land a_k < b_k)$.

We define the integral

$$I = \int_{\$} dx\, f(x)$$

using Riemann sums as follows:
1) We divide each interval $[a_K, b_K]$ with $K \in [n]$ involved in forming the box $\$$ onto N subintervals $[x_{Km}, x_{K,m+1}]$ with
$\forall m \in \{0\} \cup [N] : x_{Km} = a_K + (K/N)(b_K - a_K)$
This results in dividing the box $\$$ into N^n smaller boxes that we can index via a set $Map([n],[N])$ that contains all possible mappings $\sigma : [n] \to [N]$. Then, for each choice $\sigma \in Map([n],[N])$ the corresponding small box is given by

$$\forall \sigma \in Map([n],[N]) : \$(\sigma) = \prod_{K=1}^{n} [x_{K,\sigma(K)-1}, x_{K,\sigma(K)}]$$

2) For each small box, we define the minimum and maximum value of the scalar field f as

$$\forall \sigma \in Map([n],[N]) : m_\sigma(f \mid \$, N) = \min_{x \in \$(\sigma)} f(x)$$

$$\forall \sigma \in Map([n],[N]) : M_\sigma(f \mid \$, N) = \max_{x \in \$(\sigma)} f(x)$$

3) We form the Riemann sums $L_N(f|\$)$ and $U_N(f|\$)$ given by:

$$L_N(f|\$) = \sum_{\sigma \in Map([n],[N])} m_\sigma(f|\$,N) \left[\prod_{k=1}^{n} (x_{k,\sigma(k)} - x_{k,\sigma(k)-1}) \right]$$

$$U_N(f|\$) = \sum_{\sigma \in Map([n],[N])} M_\sigma(f|\$,N) \left[\prod_{k=1}^{n} (x_{k,\sigma(k)} - x_{k,\sigma(k)-1}) \right]$$

Based on the above, we define integrability and the n-dimensional integral as follows:

Def: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ and let $\$ \subseteq A$ be a box domain. Then, we say that

f integrable on $\$ \Leftrightarrow \exists l \in \mathbb{R}: \lim_{m \in \mathbb{N}^*} L_m(f|\$) = \lim_{m \in \mathbb{N}^*} U_m(f|\$) = l$

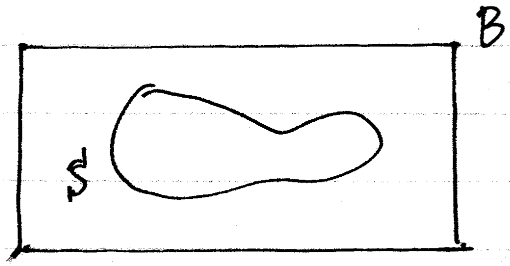
If f is indeed integrable on $\$$, then we write

$$l = \int_{\$} dx\, f(x)$$

and say that l is the integral of f over $\$$.

② → Multiple integral over a closed bounded set

- Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $S \subseteq A$ be a closed and bounded set. It follows that we can define a box domain $B \subseteq \mathbb{R}^n$ such that $S \subseteq B$.



Given such a box domain, we define a new scalar field $F: B \to \mathbb{R}$ such that

$$F(x) = \begin{cases} f(x), & \text{if } x \in S \\ 0, & \text{if } x \in B - S \end{cases}$$

and we use F to define the integral of f over S as:

$$\int_S dx\, f(x) = \int_B dx\, F(x)$$

- We also extend the definition further to the cases $S = \emptyset$ and $S = \mathbb{R}^n$ such that

$$\int_\emptyset dx\, f(x) = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} dx\, f(x) = \lim_{a \to +\infty} \int_{B(0,a)} dx\, f(x)$$

notation : In the previous notation $x \in \mathbb{R}^n$ is a vector. For the cases $n=2$ and $n=3$ we use the following notation and terminology:

a) For $n=2$, we write:

$$I = \iint_S dx dy\, f(x,y) = \iint_S f(x,y)\, dx dy$$

with $x,y \in \mathbb{R}$ and $S \subseteq \mathbb{R}^2$ and designate I as a double integral.

b) For $n=3$, we write

$$I = \iiint_S dx dy dz\, f(x,y,z) = \iiint_S f(x,y,z)\, dx dy dz$$

with $x,y,z \in \mathbb{R}$ and $S \subseteq \mathbb{R}^3$ and designate I as a triple integral.

Properties of multiple integrals

Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be scalar fields. Let $S \subseteq A$ be a closed and bounded domain.

1) Linearity

$$\forall \lambda_1, \lambda_2 \in \mathbb{R}: \int_S dx\, [\lambda_1 f(x) + \lambda_2 g(x)] =$$

$$= \lambda_1 \int_S f(x) dx + \lambda_2 \int_S g(x) dx$$

2) Comparison theorem

$$\left[\forall x \in \$: f(x) \leq g(x)\right] \Rightarrow \int_{\$} dx\, f(x) \leq \int_{\$} dx\, g(x)$$

3) Inclusion - Exclusion principle

Let $\$_1 \subseteq A$ and $\$_2 \subseteq A$ be closed and bounded domains. Then:

$$\int_{\$_1 \cup \$_2} f(x)\,dx = \int_{\$_1} f(x)\,dx + \int_{\$_2} f(x)\,dx - \int_{\$_1 \cap \$_2} f(x)\,dx$$

▶ Special case:

$$\$_1 \cap \$_2 = \emptyset \Rightarrow \int_{\$_1 \cup \$_2} f(x)dx = \int_{\$_1} f(x)dx + \int_{\$_2} f(x)dx$$

Evaluate double integrals in boxed domains

Double integrals in box domains can be evaluated using Fubini's theorem. Below, we give a simplified corollary of Fubini's theorem that is specific to double integrals

Thm: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ and let $S = [a_1, a_2] \times [b_1, b_2] \subseteq A$ be a boxed domain. Then:

f continuous on $S \Rightarrow$

$$\Rightarrow \iint_S f(x,y)\,dx\,dy = \int_{a_1}^{a_2} dx \int_{b_1}^{b_2} dy\, f(x,y) = \int_{b_1}^{b_2} dy \int_{a_1}^{a_2} dx\, f(x,y)$$

notation: The notation

$$\int_{a_1}^{a_2} dx \int_{b_1}^{b_2} dy\, f(x,y)$$

is an iterated integral, and it is equivalent to

$$\int_{a_1}^{a_2} dx \left[\int_{b_1}^{b_2} dy\, f(x,y) \right]$$

where we first do the y integral and then do the x integral. There is an equivalent notation that reads

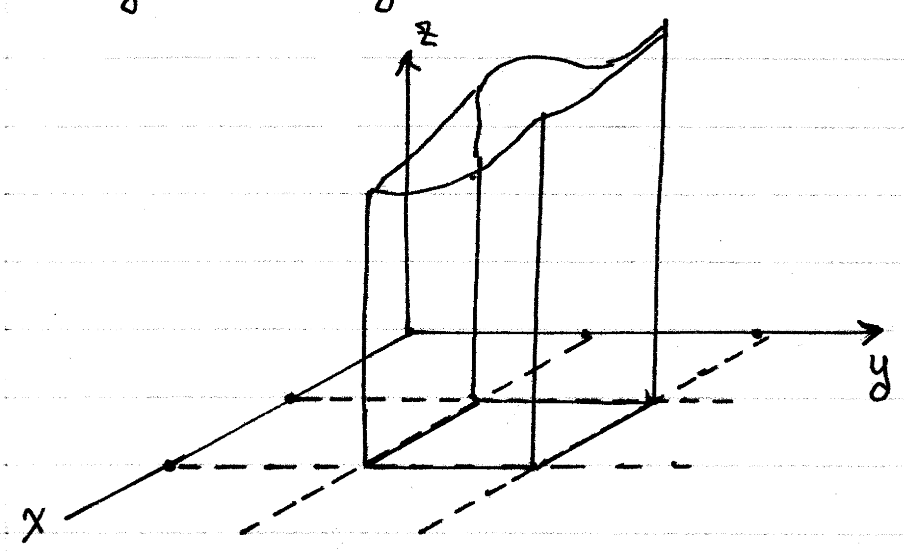
$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(x,y)\,dy\,dx$$

in common use, but it should be avoided because it is confusing, stupid, idiotic, pathetic, annoying, and absolutely horrible.

- **Geometric interpretations of the double integral.**

(1) → Volume of solid under a surface

Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ be a scalar field such that $\forall (x,y) \in A : f(x,y) \geq 0$



Now consider a solid V under the surface $(S) : z = f(x,y)$, shown above, defined by

$$V = \{ (x,y,z) \mid (x,y) \in A \wedge 0 \leq z \leq f(x,y) \}$$

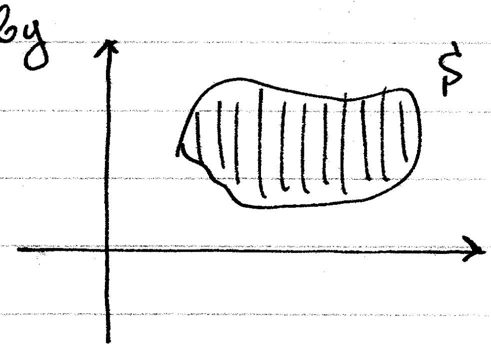
Then, the volume of V is given by

$$\boxed{vol(V) = \iint_A f(x,y)\,dx\,dy}$$

② → <u>Area of a surface S</u>

Let $S \subseteq \mathbb{R}^2$ be a closed and bounded set. Then, the area of S is given by

$$\boxed{area(S) = \iint_S dx\,dy}$$



③ → <u>Center of mass</u>

If the closed and bounded set $S \subseteq \mathbb{R}^2$ represents a two-dimensional solid object with density $\rho(x,y)$ then it has <u>mass</u>

$$\boxed{m = \iint_S dx\,dy\; \rho(x,y)}$$

and <u>center of mass</u> $M(\bar{x}, \bar{y})$ with $\bar{x}, \bar{y}$ given by

$$\boxed{\begin{aligned} \bar{x} &= \frac{1}{m} \iint_S x\rho(x,y)\,dx\,dy \\ \bar{y} &= \frac{1}{m} \iint_S y\rho(x,y)\,dx\,dy \end{aligned}}$$

For the special case $\forall (x,y) \in S : \rho(x,y) = 1$ the calculation above gives the geometric center of mass, also known as <u>Barycenter.</u>

EXAMPLES

a) Evaluate the integral:

$$I = \iint_A \frac{1+x^2}{1+y^2}\,dx\,dy \quad \text{for } A = [0,1]\times[0,1]$$

Solution

$$I = \iint_A \frac{1+x^2}{1+y^2}\,dx\,dy = \int_0^1 dx \int_0^1 dy\, \frac{1+x^2}{1+y^2} =$$

$$= \int_0^1 dx\,(1+x^2)\int_0^1 \frac{dy}{1+y^2} = \int_0^1 dx\,(1+x^2)\left[\operatorname{Arctan} y\right]_0^1 =$$

$$= \int_0^1 dx\,(1+x^2)(\operatorname{Arctan} 1 - \operatorname{Arctan} 0) =$$

$$= \left(\frac{\pi}{4} - 0\right)\int_0^1 dx\,(1+x^2) = \frac{\pi}{4}\left[x + \frac{x^3}{3}\right]_0^1 =$$

$$= \frac{\pi}{4}\left[(1-0) + \frac{1^3 - 0^3}{3}\right] = \frac{\pi}{4}\,\frac{4}{3} =$$

$$= \frac{\pi}{3}$$

6) Find the volume of the solid under the surface $(s): z = x\sqrt{x^2+y}$ and above the rectangle $A = [0,1]\times[0,1]$ on the xy-plane.

Solution

$$V = \iint_A dxdy\ x\sqrt{x^2+y} = \int_0^1 dx \int_0^1 dy\ x\sqrt{x^2+y} =$$

$$= \int_0^1 dx\ x\left[\int_0^1 dy\ \sqrt{x^2+y}\right]$$

Let $t = g(y) = \sqrt{x^2+y} \Leftrightarrow t^2 = x^2+y \Leftrightarrow y = t^2-x^2$.
Thus: $dy = 2t\,dt$ and furthermore:
for $y=0$: $g(0) = \sqrt{x^2+0} = \sqrt{x^2}$
for $y=1$: $g(1) = \sqrt{x^2+1}$ -
It follows that:

$$V = \int_0^1 dx\ x\left[\int_{\sqrt{x^2}}^{\sqrt{1+x^2}} dt\ (2t)\,t\right] = 2\int_0^1 dx\ x\left[\int_x^{\sqrt{1+x^2}} t^2\,dt\right] =$$

$$= 2\int_0^1 dx\ x\left[\frac{t^3}{3}\right]_x^{\sqrt{1+x^2}} = 2\int_0^1 dx\ x\left[\frac{(\sqrt{1+x^2})^3 - x^3}{3}\right] =$$

$$= \frac{2}{3}\int_0^1 dx\ x\left[(1+x^2)\sqrt{1+x^2} - x^3\right] =$$

$$= \frac{2}{3}\int_0^1 dx\, x(1+x^2)\sqrt{1+x^2} - \frac{2}{3}\int_0^1 dx\, x^4 =$$

$$= \frac{2}{3}(I_1 - I_2) \quad \text{with}$$

$$I_1 = \int_0^1 dx\, x(1+x^2)\sqrt{1+x^2} \quad \text{and} \quad I_2 = \int_0^1 x^4 dx$$

- For I_2:

$$I_2 = \int_0^1 x^4 dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1^5 - 0^5}{5} = \frac{1}{5}$$

- For I_1:

Let $x = \tan\vartheta \Rightarrow \begin{cases} dx = d\vartheta/\cos^2\vartheta \\ \text{For } x=0 : \vartheta = 0 \\ \text{For } x=1 : \vartheta = \pi/4 \end{cases} \Rightarrow$

$$\Rightarrow I_1 = \int_0^{\pi/4} \frac{d\vartheta}{\cos^2\vartheta} \tan\vartheta\,(1+\tan^2\vartheta) \cdot \sqrt{1+\tan^2\vartheta} =$$

$$= \int_0^{\pi/4} \frac{d\vartheta}{\cos^2\vartheta} \tan\vartheta \frac{1}{\cos^2\vartheta}\sqrt{\frac{1}{\cos^2\vartheta}} =$$

$$= \int_0^{\pi/4} d\vartheta \frac{\tan\vartheta}{\cos^4\vartheta\,|\cos\vartheta|} = \int_0^{\pi/4} d\vartheta \frac{\tan\vartheta}{\cos^5\vartheta} =$$

$$= \int_0^{\pi/4} d\vartheta \frac{\sin\vartheta}{\cos^5\vartheta\cos\vartheta} = \int_0^{\pi/4} d\vartheta \frac{\sin\vartheta}{\cos^6\vartheta}$$

Let $s = \cos\vartheta = g(\vartheta) \Rightarrow \begin{cases} ds = -\sin\vartheta\, d\vartheta \Rightarrow \sin\vartheta\, d\vartheta = -ds \\ g(0) = \cos 0 = 1 \\ g(\pi/4) = \cos(\pi/4) = \sqrt{2}/2 \end{cases} \Rightarrow$

$$\Rightarrow I_1 = \int_1^{\sqrt{2}/2} d\varsigma \cdot (-1) \frac{1}{\varsigma^6} = \int_1^{\sqrt{2}/2} d\varsigma \, (-\varsigma^{-6}) =$$

$$= \left[\frac{-\varsigma^{-5}}{-5} \right]_1^{\sqrt{2}/2} = \left[\frac{1}{5\varsigma^5} \right]_1^{\sqrt{2}/2} =$$

$$= \frac{1}{5} \left[\frac{1}{(1/\sqrt{2})^5} - \frac{1}{1^5} \right] =$$

$$= \frac{1}{5} \left[(\sqrt{2})^5 - 1 \right] = \frac{4\sqrt{2}-1}{5}$$

It follows that

$$V = \frac{2}{3}(I_1 - I_2) = \frac{2}{3}\left[\frac{4\sqrt{2}-1}{5} - \frac{1}{5} \right] =$$

$$= \frac{2}{3} \, \frac{4\sqrt{2}-1-1}{5} = \frac{2(4\sqrt{2}-2)}{15} = \frac{4(2\sqrt{2}-1)}{15}$$

EXERCISES

① Evaluate the following double integrals

a) $I = \iint_A xy(x+y)^2 \, dx\, dy$ with $A = [1,2] \times [1,3]$

b) $I = \iint_A (x^3 + y^3 - 3xy) \, dx\, dy$ with $A = [0,a] \times [0,a]$

c) $I = \iint_A x^2 \sin y \, dx\, dy$ with $A = [1,b] \times [0, \pi/4]$

d) $I = \iint_A \sqrt{x+y} \, dx\, dy$ with $A = [0,a] \times [0,b]$

e) $I = \iint_A \frac{dx\, dy}{x+y}$ with $A = [1,a] \times [0,b]$

f) $I = \iint_A \frac{dx\, dy}{\sqrt{x+y}}$ with $A = [0,a] \times [0,b]$

g) $I = \iint_A \frac{\ln(xy)}{y} \, dx\, dy$ with $A = [1,a] \times [1,b]$

h) $I = \iint_A \frac{y}{1+xy} \, dx\, dy$ with $A = [0,1] \times [0,1]$

i) $I = \iint_A \exp(ax + by) \, dx\, dy$ with $A = [0,1] \times [0,2]$

(9) Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ be a continuous scalar field. Show that if there is a field $F: A \to \mathbb{R}$ such that

$$\forall (x,y) \in A, \quad f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

Then, it follows that over the region

$$A = [a_1, a_2] \times [b_1, b_2]$$

we have

$$\iint_A f(x,y)\,dx\,dy = F(a_2,b_2) - F(a_1,b_2) - F(a_2,b_1) + F(a_1,b_1)$$

Double integrals over simple regions

① x-simple regions

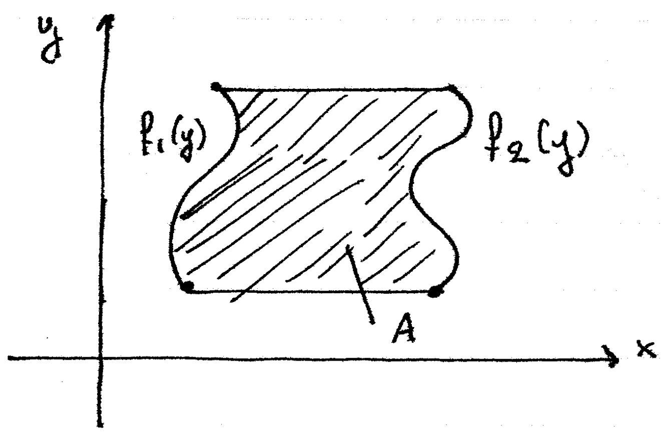
- Let $A \subseteq \mathbb{R}^2$. We say that A is <u>x-simple</u> if and only if it can be written us:

$$A = \{(x,y) \in \mathbb{R}^2 \mid f_1(y) \leq x \leq f_2(y) \wedge y \in [a,b]\}$$

with f_1, f_2 functions with $f_1 : [a,b] \longrightarrow \mathbb{R}$ and $f_2 : [a,b] \longrightarrow \mathbb{R}$.

- If A is x-simple then

$$\boxed{\iint_A f(x,y)\,dx\,dy = \int_a^b dy \int_{f_1(y)}^{f_2(y)} dx\; f(x,y)}$$

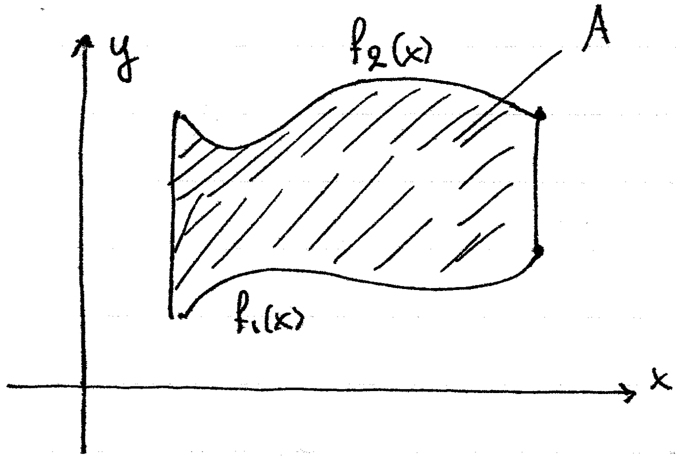


② y-simple regions

- Let $A \subseteq \mathbb{R}^2$. We say that A is y-simple if and only if it can be written as:

$$A = \{(x,y) \in \mathbb{R}^2 \mid x \in [a,b] \wedge f_1(x) \leq y \leq f_2(x)\}$$

with f_1, f_2 functions with $f_1 : [a,b] \to \mathbb{R}$ and $f_2 : [a,b] \to \mathbb{R}$.



- If A is y-simple then:

$$\boxed{\iint_A f(x,y)\,dx\,dy = \int_a^b dx \int_{f_1(x)}^{f_2(x)} dy\; f(x,y)}$$

EXAMPLES

a) Evaluate the integral $I = \iint_A 2ye^x \, dxdy$ with

$A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 \wedge 0 \leq x \leq y^2\}$

Solution

$$I = \iint_A 2y e^x \, dxdy = \int_0^1 dy \int_0^{y^2} dx \; 2ye^x = \int_0^1 dy \, 2y \left[\int_0^{y^2} e^x \, dx \right] =$$

$$= \int_0^1 dy \, 2y \left[e^x \right]_0^{y^2} = \int_0^1 dy \, 2y \left(e^{y^2} - e^0 \right) =$$

$$= \int_0^1 2y e^{y^2} dy - \int_0^1 2y \, dy = I_1 - I_2 \qquad (1)$$

For $I_1 = \int_0^1 2y e^{y^2} dy$

Let $t = e^{y^2} = g(y) \Rightarrow \begin{cases} dt = 2ye^{y^2} dy \\ g(0) = e^{0^2} = 1 \\ g(1) = e^{1^2} = e \end{cases} \Rightarrow$

$$\Rightarrow I_1 = \int_1^e dt = \left[t \right]_1^e = e - 1 .$$

For $I_2 = \int_0^1 2y dy = \left[y^2 \right]_0^1 = 1^2 - 0^2 = 1$

It follows that $I = I_1 - I_2 = (e-1) - 1 = e - 2.$

b) Evaluate the following integral:

$$I = \iint_A (4x+10y)\,dx\,dy$$

over the region

$$A = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1] \wedge -x \le y \le x^2\}$$

<u>Solution</u>

$$I = \iint_A (4x+10y)\,dx\,dy = 4\iint_A x\,dx\,dy + 10\iint_A y\,dx\,dy =$$

$$= 4I_1 + 10I_2 \qquad \text{with}$$

$$I_1 = \iint_A x\,dx\,dy = \int_0^1 dx \int_{-x}^{x^2} dy\; x = \int_0^1 dx\; x\left[\int_{-x}^{x^2} dy\right] =$$

$$= \int_0^1 dx\; x\,[y]_{-x}^{x^2} = \int_0^1 dx\; x\,(x^2-(-x)) =$$

$$= \int_0^1 (x^3+x^2)\,dx = \left[\frac{x^4}{4}+\frac{x^3}{3}\right]_0^1 =$$

$$= \frac{1^4-0^4}{4} + \frac{1^3-0^3}{3} = \frac{1}{4}+\frac{1}{3} = \frac{4+3}{12} = \frac{7}{12}$$

and

$$I_2 = \iint_A y\,dx\,dy = \int_0^1 dx \int_{-x}^{x^2} dy\, y = \int_0^1 dx \left[\frac{y^2}{2}\right]_{-x}^{x^2} =$$

$$= \int_0^1 dx\, \frac{(x^2)^2 - (-x)^2}{2} = \int_0^1 dx\, \frac{x^4 - x^2}{2} =$$

$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2}\left[\frac{1^5 - 0^5}{5} - \frac{1^3 - 0^3}{3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{5} - \frac{1}{3}\right] = \frac{1}{2}\,\frac{3-5}{15} = \frac{1}{2}\,\frac{-2}{15} =$$

$$= \frac{-1}{15}.$$

It follows that

$$I = 4I_1 + 10I_2 = 4\cdot\left(\frac{7}{12}\right) + 10\cdot\left(\frac{-1}{15}\right) = \frac{7}{3} - \frac{2}{3} = \frac{5}{3}.$$

▶ Application to volumes

- Before setting up a volume integral, you should:

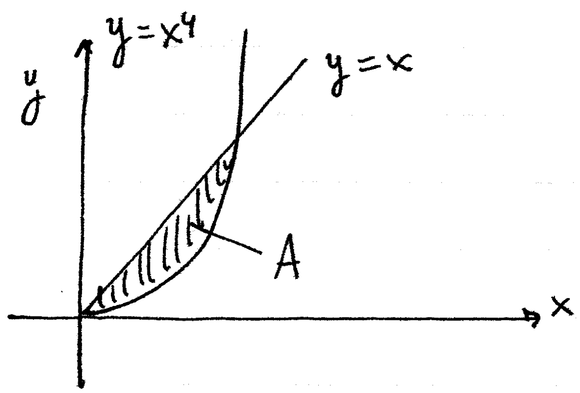
a) Define the region $A \subseteq \mathbb{R}^2$ over which we integrate using set-theoretic notation.

b) Define the solid S in terms of A, again using set theoretic notation.

EXAMPLE

Find the volume of the solid under the plane $(p): x+2y-z=0$ and bounded by the surfaces $(p_1): y=x$ and $(p_2): y=x^4$.

Solution



We note that

$$x=x^4 \Leftrightarrow x^4-x=0 \Leftrightarrow$$
$$\Leftrightarrow x(x^3-1)=0$$
$$\Leftrightarrow x=0 \vee x^3-1=0$$
$$\Leftrightarrow x=0 \vee x^3=1$$
$$\Leftrightarrow x=0 \vee x=1$$

Thus, the region on the xy-plane bounded by (p_1) and (p_2) is given by:

$$A=\{(x,y)\in\mathbb{R}^2 \mid x\in[0,1] \wedge x^4 \leq y \leq x\}$$

It follows that the solid we are interested in is given by:

$$S = \{ (x,y,z) \in \mathbb{R}^2 \mid (x,y) \in A \wedge 0 \leq z \leq x+2y \}$$

It follows that the volume of S is given by:

$$V = \iint_A (x+2y)\,dx\,dy = \iint_A x\,dx\,dy + 2\iint_A y\,dx\,dy = I_1 + 2I_2$$

with

$$I_1 = \iint_A x\,dx\,dy = \int_0^1 dx \int_{x^4}^{x} dy\ x = \int_0^1 dx\ x \left[\int_{x^4}^{x} dy \right] =$$

$$= \int_0^1 dx\ x \left[y \right]_{x^4}^{x} = \int_0^1 dx\ x(x - x^4) = \int_0^1 (x^2 - x^5)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^6}{6} \right]_0^1 = \frac{1^3 - 0^3}{3} - \frac{1^6 - 0^6}{6} = \frac{1}{3} - \frac{1}{6} =$$

$$= \frac{2-1}{6} = \frac{1}{6}$$

and

$$I_2 = \iint_A y\,dx\,dy = \int_0^1 dx \int_{x^4}^{x} dy\ y = \int_0^1 dx \left[\frac{y^2}{2} \right]_{x^4}^{x} =$$

$$= \int_0^1 dx\ \frac{x^2 - (x^4)^2}{2} = \frac{1}{2}\int_0^1 (x^2 - x^8)\,dx =$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^9}{9} \right]_0^1 = \frac{1}{2}\left[\frac{1^3 - 0^3}{3} - \frac{1^9 - 0^9}{9} \right] =$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{9}\right) = \frac{1}{2}\,\frac{3-1}{9} = \frac{1}{2}\,\frac{2}{9} = \frac{1}{9}$$

It follows that:

$$V = I_1 + 2I_2 = \frac{1}{6} + 2 \cdot \frac{1}{9} = \frac{3+2\cdot 2}{18} =$$

$$= \frac{3+4}{18} = \frac{7}{18}.$$

EXERCISES

③ Evaluate the following double integrals

a) $I = \iint_A x^2 y \, dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq a \wedge x \leq y \leq 3x+1\}$

b) $I = \iint_A dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq 4 \wedge 0 \leq y \leq \ln x\}$

c) $I = \iint_A x \, dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq a \wedge 1 \leq y \leq \exp(x2)\}$

d) $I = \iint_A \dfrac{dx dy}{x^2+y^2}$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq a \wedge 0 \leq y \leq x\}$

e) $I = \iint_A e^x \cos y \, dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq \pi/4 \wedge 0 \leq x \leq \sin y\}$

f) $I = \iint_A (x+y)\,dx\,dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq a \wedge 0 \leq y \leq \sqrt{a^2 - x^2}\}$

Changing the order of iterated integrals

In some double integrals, defined over an x-simple or y-simple domain, we may find that in the corresponding iterated integral, the first integral may not have an antiderivative that can be defined using elementary functions, while the second integral evaluates to an outcome that can be expressed with elementary functions. If the domain of integration is both x-simple and y-simple, then we can try reversing the order of the corresponding iterated integral, and see whether this results in iterated integrals that can be evaluated one at a time.

Note that in order to apply this technique, it is important that the solution include a mathematical proof that the x-simple and y-simple representations of the domain of integration are equal.

Recall that in order to show that the two sets A, B satisfy $A = B$, we have to show that

$$\begin{cases} x \in A \Rightarrow x \in B \\ x \in B \Rightarrow x \in A \end{cases}$$

EXAMPLE

Evaluate the integral

$$I = \iint_A \exp(y^2)\, dx\, dy$$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 4 \wedge x/2 \leq y \leq 2\}$

Solution

1st method: (without exchanging the order of integration)

$$I = \iint_A \exp(y^2)\, dx\, dy = \int_0^4 dx \int_{x/2}^2 dy \, \exp(y^2) =$$

$$= \int_0^4 dx \left[[(d/dx)x] \int_{x/2}^2 dy \, \exp(y^2) \right] =$$

$$= \left[x \int_{x/2}^2 dy \, \exp(y^2) \right]_{x=0}^{x=4} - \int_0^4 dx \left[x \frac{d}{dx} \int_{x/2}^2 dy \, \exp(y^2) \right] =$$

$$= \left[4 \int_2^2 dy \, \exp(y^2) - 0 \int_0^2 dy \, \exp(y^2) \right]$$

$$- \int_0^4 dx \; x \left[-(x/2)' \exp((x/2)^2) \right] =$$

$$= (4 \cdot 0 - 0) + \int_0^4 (1/2)\, x \exp((x/2)^2)\, dx$$

Let $t = \exp((x/2)^2)$. Then:

$$dt = [\exp((x/2)^2)]' dx = [(x/2)^2]' \exp((x/2)^2)\, dx$$

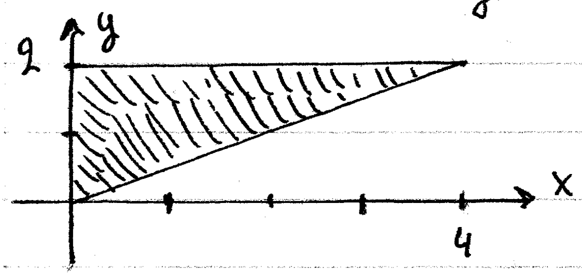
$$= 2(x/2)(x/2)' \exp((x/2)^2)\, dx =$$

$$= (1/2)\, x \exp((x/2)^2)\, dx$$

For $x = 0 \Rightarrow t = \exp(0) = 1$
$x = 4 \Rightarrow t = \exp((4/2)^2) = e^4$
It follows that

$$I = \int_1^{e^4} dt = e^4 - 1$$

2nd method: (with exchanging order of integration).
Recall that $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 4 \wedge x/2 \leq y \leq 2\}$



To exchange the integral order, we define
$B = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 2 \wedge 0 \leq x \leq 2y\}$
and claim that $A = B$.
Proof of claim:
($\Rightarrow$): Assume that $(x,y) \in A$. Then:

$(x,y) \in A \Rightarrow 0 \leq x \leq 4 \wedge x/2 \leq y \leq 2$
and it follows that
$y \geq x/2 \geq 0/2 = 0 \Rightarrow y \geq 0$
$y \leq 2$, by hypothesis
$0 \leq x$, by hypothesis
$x = 2(x/2) \leq 2y \Rightarrow x \leq 2y$

From the above, it follows that

$0 \leq y \leq 2 \wedge 0 \leq x \leq 2y \Rightarrow (x,y) \in B$

($\Leftarrow$): Assume that $(x,y) \in B$. Then

$(x,y) \in B \Rightarrow 0 \leq y \leq 2 \wedge 0 \leq x \leq 2y$

It follows that

$0 \leq x$, by hypothesis

$x \leq 2y \leq 2 \cdot 2 = 4 \Rightarrow x \leq 4$

$x/2 \leq (2y)/2 = y \Rightarrow x/2 \leq y$

$y \leq 2$, by hypothesis

From the above, it follows that

$0 \leq x \leq 4 \wedge x/2 \leq y \leq 2 \Rightarrow (x,y) \in A$

This proves the claim: $A = B$.

It follows that

$$I = \iint_A \exp(y^2)\,dxdy = \iint_B \exp(y^2)\,dxdy =$$

$$= \int_0^2 dy \int_0^{2y} dx \exp(y^2) = \int_0^2 dy \left[\exp(y^2) \int_0^{2y} dx\right] =$$

$$= \int_0^2 dy \exp(y^2)(2y) = \int_0^2 dy \exp(y^2)\,(y^2)' =$$

$$= \int_0^2 dy \left[\exp(y^2)\right]' = \left[\exp(y^2)\right]_0^2 =$$

$$= \exp(2^2) - \exp(0^2) = e^4 - 1.$$

EXERCISES

(4) Evaluate the following double integrals by changing the order of integration

a) $I = \iint_A \sqrt{x^3+1}\, dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid \sqrt{y} \leq x \leq 2 \wedge 0 \leq y \leq 4\}$

b) $I = \iint_A x \exp(y^3)\, dx dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \wedge x \leq y \leq 1\}$

c) $I = \iint_A \frac{x\, dx dy}{\sqrt{3x^2+y}}$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq \sqrt{y} \wedge 0 \leq y \leq 9\}$

d) $I = \iint_A \sqrt{ax^2+by}\, dx dy$

with $a, b \in (0, +\infty)$ and

$A = \{(x,y) \in \mathbb{R}^2 \mid \sqrt{y} \leq x \leq 2 \wedge 0 \leq y \leq 4\}$

⑤ Let $f: \mathbb{R} \to \mathbb{R}$ be a function, continuous on $\mathbb{R}$, and define

$$\forall t \in \mathbb{R}: \; g(t) = \iint_{A(t)} f(y)\, dx\, dy$$

with

$$\forall t \in (0,+\infty): \; A(t) = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq t \wedge 0 \leq y \leq x\}$$

a) Use the fundamental theorem of calculus to show that

$$\forall t \in (0,+\infty): \; g''(t) = f(t)$$

b) Exchange the order of the integrals to show that

$$\forall t \in (0,+\infty): \; g(t) = \int_0^t (t-y) f(y)\, dy.$$

Change of variables in multiple integrals

Consider a change of variables defined by the equations

$$\begin{cases} x_1 = g_1(u_1, u_2, \dots, u_n) \\ x_2 = g_2(u_1, u_2, \dots, u_n) \quad , \forall (u_1, u_2, \dots, u_n) \in B \\ \vdots \\ x_n = g_n(u_1, u_2, \dots, u_n) \end{cases}$$

and define $g: B \to \mathbb{R}^n$ such that

$\forall u \in B: g(u) = (g_1(u), g_2(u), \dots, g_n(u))$

with $u = (u_1, u_2, \dots, u_n)$.

We begin with the following definitions

Smooth change of variables

1) We define the derivative matrix of g as:

$$Dg(u) = \begin{bmatrix} \partial g_1/\partial u_1 & \partial g_1/\partial u_2 & \cdots & \partial g_1/\partial u_n \\ \partial g_2/\partial u_1 & \partial g_2/\partial u_2 & \cdots & \partial g_2/\partial u_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial g_n/\partial u_1 & \partial g_n/\partial u_2 & \cdots & \partial g_n/\partial u_n \end{bmatrix}, \forall u \in B$$

and define the Jacobian of g as the determinant of the derivative matrix $Dg(u)$:

$$\frac{\partial(g_1, g_2, \dots, g_n)}{\partial(u_1, u_2, \dots, u_n)} = \det(Dg(u)), \quad \forall u \in B$$

2) We say that, given $S \subseteq B$,

g one-to-one on $S \Leftrightarrow \forall u, v \in S : (g(u) = g(v) \Rightarrow u = v)$

<u>Def</u>: A transformation $g : B \to \mathbb{R}^n$ with $B \subseteq \mathbb{R}^n$ and components:

$\forall u \in B : g(u) = (g_1(u), g_2(u), \dots, g_n(u))$

is a <u>smooth change of variables</u> if and only if

$$\begin{cases} g_1, g_2, \dots, g_n \text{ differentiable on } B \\ Dg \text{ continuous on } B \\ g \text{ one-to-one on } B - \partial B \\ \dfrac{\partial(g_1, g_2, \dots, g_n)}{\partial(u_1, u_2, \dots, u_n)} \neq 0, \ \forall (u_1, \dots, u_n) \in B - \partial B \end{cases}$$

Note that in the above definition, the last two requirements can be violated on the boundary ∂B of the domain B. The corresponding points $u \in \partial B$ are then considered <u>singular points</u> of the otherwise smooth change of variables.

→ <u>Changing variables on multiple integrals</u>

Let us consider the integral of a scalar field $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ over a region $S \subseteq A$ defined as

$$S = \{ g(u) \mid u \in S_0 \} = g(S_0)$$

with $x = g(u)$ representing a smooth change of variables. Then, it follows that

$$\boxed{\int_{S} f(x)\,dx = \int_{S_0} f(g(u)) \left| \frac{\partial g}{\partial u} \right| du}$$

with $dx = dx_1 dx_2 \cdots dx_n$
$du = du_1 du_2 \cdots du_n$
and $\dfrac{\partial g}{\partial u} = \dfrac{\partial(g_1, g_2, \dots, g_n)}{\partial(u_1, u_2, \dots, u_n)}$

Formally, the corresponding transformation of the integral differentials reads:

$$\boxed{dx_1 dx_2 \cdots dx_n = \frac{\partial(g_1, g_2, \dots, g_n)}{\partial(u_1, u_2, \dots, u_n)}\, du_1 du_2 \cdots du_n}$$

- This change of variables can be beneficial when one of the following happens:
  a) The domain S_0 is substantially simpler than S.
  b) The resulting integrand is simpler.
- Note that a change of variables from x to u requires a mapping g from u back to x (i.e. in the reverse direction). This is similar to the backsubstitution method in Calculus 1.

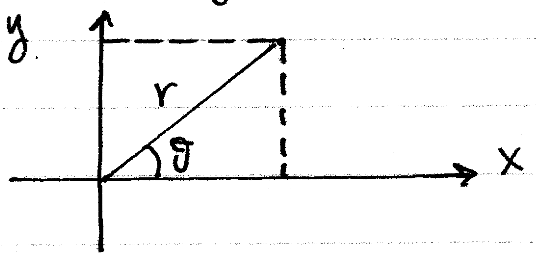
Change of variables to polar coordinates

Consider the change of variables

$$\begin{cases} x = g_1(r,\vartheta) = r\cos\vartheta, & \forall r \in [0,+\infty), \forall \vartheta \in [0,2\pi] \\ y = g_2(r,\vartheta) = r\sin\vartheta \end{cases}$$

or equivalently

$$(x,y) = g(r,\vartheta) = (r\cos\vartheta, r\sin\vartheta), \quad \forall (r,\vartheta) \in [0,+\infty) \times [0,2\pi]$$



The corresponding Jacobian is given by

$$\frac{\partial(g_1,g_2)}{\partial(r,\vartheta)} = \begin{vmatrix} \partial g_1/\partial r & \partial g_1/\partial\vartheta \\ \partial g_2/\partial r & \partial g_2/\partial\vartheta \end{vmatrix} = \begin{vmatrix} \cos\vartheta & -r\sin\vartheta \\ \sin\vartheta & r\cos\vartheta \end{vmatrix}$$

$$= (\cos\vartheta)(r\cos\vartheta) - (\sin\vartheta)(-r\sin\vartheta) =$$

$$= r\cos^2\vartheta + r\sin^2\vartheta = r(\cos^2\vartheta + \sin^2\vartheta) = r$$

and therefore

$$dx\,dy = \left| \frac{\partial(g_1,g_2)}{\partial(r,\vartheta)} \right| dr\,d\vartheta = |r|\,dr\,d\vartheta = r\,dr\,d\vartheta$$

It follows that the integral

$$I = \iint_A f(x,y)\,dx\,dy$$

over a set A given by

$$A = \{ (r\cos\vartheta, r\sin\vartheta) \mid (r,\vartheta) \in B \}$$

with $B \subseteq [0,+\infty) \times [0, 2\pi]$ can be evaluated with the change of variables

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}$$

as follows:

$$\boxed{\iint_A f(x,y)\,dx\,dy = \iint_B f(r\cos\vartheta, r\sin\vartheta)\, r\,dr\,d\vartheta}$$

- The resulting integral in terms of $dr\,d\vartheta$ can be evaluated if B is a boxed domain or an r-simple or ϑ-simple domain.
- It is also worth noting that:

$$\boxed{x^2 + y^2 = r^2.}$$

- The corresponding formal relation between the cartesian and polar differentials is given by:

$$\boxed{dx\,dy = r\,dr\,d\vartheta}$$

EXAMPLES

Evaluate the integral $I = \iint_A \cos(\pi x^2 + \pi y^2)\, dx dy$ with A given by

$A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leqslant x^2 + y^2 \leqslant 4\}$

Solution

We note that

$$A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leqslant x^2+y^2 \leqslant 4\} =$$
$$= \{(r\cos\vartheta, r\sin\vartheta) \mid r \in [1,2] \wedge \vartheta \in [0, 2\pi]\}$$

and it follows that:

$$I = \iint_A \cos(\pi x^2 + \pi y^2)\, dx dy = \int_1^2 dr \int_0^{2\pi} d\vartheta \; r\cos(\pi r^2) =$$

$$= \int_1^2 dr \; r\cos(\pi r^2) \left[\int_0^{2\pi} d\vartheta\right] = \int_1^2 dr \; r\cos(\pi r^2)\, 2\pi =$$

$$= 2\pi \int_1^2 r\cos(\pi r^2)\, dr$$

Let $t = \pi r^2 = g(r) \Rightarrow \begin{cases} dt = 2\pi r\, dr \\ g(1) = \pi \cdot 1^2 = \pi \\ g(2) = \pi \cdot 2^2 = 4\pi \end{cases} \Rightarrow$

$$\Rightarrow I = \int_{\pi}^{4\pi} \cos(t)\,dt = [\sin(t)]_{\pi}^{4\pi} = \sin(4\pi) - \sin(\pi)$$

$$= \sin 0 - \sin \pi = 0 - 0 = 0.$$

▶ Application: Gaussian integral

$$\boxed{\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}.}$$

Proof

Define $I = \int_{-\infty}^{+\infty} e^{-x^2}\,dx$ and note that:

$$J = \iint_{\mathbb{R}^2} \exp(-x^2-y^2)\,dx\,dy = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, \exp(-x^2-y^2) =$$

$$= \int_{-\infty}^{+\infty} dx \, e^{-x^2} \left[\int_{-\infty}^{+\infty} dy \, e^{-y^2} \right] = \int_{-\infty}^{+\infty} dx \, e^{-x^2} I =$$

$$= I \int_{-\infty}^{+\infty} dx \, e^{-x^2} = I \cdot I = I^2.$$

We now calculate J:

$$J = \iint_{\mathbb{R}^2} \exp(-x^2-y^2)\,dx\,dy = \int_0^{+\infty} dr \int_0^{2\pi} d\vartheta \; r e^{-r^2} =$$

$$= \int_0^{+\infty} dr\; r e^{-r^2}\left[\int_0^{2\pi} d\vartheta\right] = \int_0^{+\infty} 2\pi r e^{-r^2}\, dr =$$

$$= 2\pi \int_0^{+\infty} r e^{-r^2}\, dr .$$

Let $\underset{= g(r)}{\rho = -r^2} \Rightarrow \begin{cases} d\rho = -2r\,dr \Rightarrow r\,dr = (-1/2)\,d\rho \\ g(0) = 0 \\ g(+\infty) = \lim\limits_{r \to +\infty} (-r^2) = -\infty \end{cases} \Rightarrow$

$$\Rightarrow J = 2\pi \int_0^{-\infty} e^{\rho} (-1/2)\, d\rho = -\pi \int_0^{-\infty} e^{\rho}\, d\rho = -\pi \left[e^{\rho} \right]_0^{-\infty} =$$

$$= -\pi \left[\lim_{\rho \to -\infty} e^{\rho} - e^0 \right] = -\pi [0 - 1] = \pi \Rightarrow$$

$$\Rightarrow I^2 = \pi \Rightarrow I = \sqrt{\pi} \lor I = -\sqrt{\pi}.$$

Since $\forall x \in \mathbb{R}: e^{-x^2} > 0 \Rightarrow \int_{-\infty}^{+\infty} e^{-x^2}\, dx > 0 \Rightarrow I > 0$

and therefore $I = \sqrt{\pi}$
(the solution $I = -\sqrt{\pi}$ is rejected).

EXERCISES

⑥ Evaluate the following integrals using change of variables to polar coordinates

a) $I = \iint_A \sqrt{x^2+y^2}\, dxdy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x^2+y^2 \leqslant a^2\}$

b) $I = \iint_A xy\, dxdy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leqslant 4 \wedge x \geqslant 0 \wedge y \geqslant 0\}$

c) $I = \iint_A y(x^2+y^2)^3\, dxdy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid y \geqslant 0 \wedge x^2+y^2 \leqslant a^2\}$

d) $I = \iint_A \dfrac{y\, dxdy}{x^2+y^2}$

with $A = \{(x,y) \in \mathbb{R}^2 \mid y \geqslant 1/2 \wedge x^2+y^2 \leqslant 1\}$

e) $I = \iint_A \operatorname{Arctan}\left(\dfrac{y}{x}\right) dxdy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant a \wedge 0 \leqslant y \leqslant \sqrt{a^2-x^2}\}$

f) $I = \iint_A x\,dx\,dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1/2 \wedge x\sqrt{3} \leq y \leq \sqrt{1-x^2}\}$

g) $I = \iint_A |xy|\,dx\,dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq a^2\}$

h) $I = \iint_A \frac{dx\,dy}{(x^2+y^2)\sqrt{x^2+y^2}}$

with $A = \{(x,y) \in \mathbb{R}^2 \mid x+y \geq 1 \wedge x^2+y^2 \leq 1\}$

i) $I = \iint_A (x-y)\,dx\,dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1 \wedge x+y \geq 1\}$

j) $I = \iint_A y\,dx\,dy$

with $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1 \wedge (x-1)^2+y^2 \leq 1\}$.

Evaluation of triple integrals over boxed domains

Let $A \subseteq \mathbb{R}^3$ be a boxed domain given by

$$A = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid x \in [a_1, a_2] \land y \in [b_1, b_2] \land z \in [c_1, c_2]\}$$

and consider a scalar field $f : A \to \mathbb{R}$ such that f is continuous on A. The triple integral of f over A can be evaluated, according to Fubini's theorem by:

$$\boxed{\iiint_A f(x,y,z)\, dx\, dy\, dz = \int_{a_1}^{a_2} dx \int_{b_1}^{b_2} dy \int_{c_1}^{c_2} dz\, f(x,y,z)}$$

It can also be shown that permuting the order of the iterated integrals gives the same result.

Separable scalar fields

We say that the scalar field $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^3$ is <u>separable</u> if and only if it satisfies $\forall (x,y,z) \in A : f(x,y,z) = f_1(x) f_2(y) f_3(z)$

Then, we can show that:

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \left[\int_{a_1}^{a_2} f_1(x)\,dx\right]\left[\int_{b_1}^{b_2} f_2(y)\,dy\right]\left[\int_{c_1}^{c_2} f_3(z)\,dz\right]$$

Proof

$$I = \iiint_A f(x,y,z)\,dx\,dy\,dz = \iiint_A f_1(x) f_2(y) f_3(z)\,dx\,dy\,dz$$

$$= \int_{a_1}^{a_2} dx \int_{b_1}^{b_2} dy \int_{c_1}^{c_2} dz\; f_1(x) f_2(y) f_3(z) =$$

$$= \int_{a_1}^{a_2} dx\, f_1(x) \left[\int_{b_1}^{b_2} dy \int_{c_1}^{c_2} dz\; f_2(y) f_3(z)\right] =$$

$$= \left[\int_{b_1}^{b_2} dy \int_{c_1}^{c_2} dz\; f_2(y) f_3(z)\right]\left[\int_{a_1}^{a_2} f_1(x)\,dx\right] =$$

$$= \left[\int_{b_1}^{b_2} dy\, f_2(y) \left(\int_{c_1}^{c_2} dz\, f_3(z)\right)\right]\left[\int_{a_1}^{a_2} f_1(x)\,dx\right] =$$

$$= \left[\int_{a_1}^{a_2} f_1(x)\,dx\right]\left[\int_{b_1}^{b_2} f_2(y)\,dy\right]\left[\int_{c_1}^{c_2} f_3(z)\,dz\right]$$

EXAMPLE

Evaluate $I = \iiint_A x^2yz \, dx\,dy\,dz$

with $A = [1,2] \times [0,1] \times [0,2]$

Solution

$$I = \iiint_A x^2yz \, dx\,dy\,dz = \int_1^2 dx \int_0^1 dy \int_0^2 dz \; x^2yz =$$

$$= \left[\int_1^2 x^2\,dx \right] \left[\int_0^1 y\,dy \right] \left[\int_0^2 z\,dz \right] = I_1 I_2 I_3$$

with

$$I_1 = \int_1^2 x^2\,dx = \left[\frac{x^3}{3} \right]_1^2 = \frac{2^3 - 1^3}{3} = \frac{8-1}{3} = \frac{7}{3}$$

$$I_2 = \int_0^1 y\,dy = \left[\frac{y^2}{2} \right]_0^1 = \frac{1^2 - 0^2}{2} = \frac{1}{2}$$

$$I_3 = \int_0^2 z\,dz = \left[\frac{z^2}{2} \right]_0^2 = \frac{2^2 - 0^2}{2} = \frac{4}{2} = 2$$

and therefore

$$I = I_1 I_2 I_3 = \frac{7}{3} \cdot \frac{1}{2} \cdot 2 = \frac{7}{3}$$

EXERCISES

⑦ Evaluate the following triple integrals

a) $I = \iiint_A z^3 \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [1,3] \land y \in [-1,2] \land z \in [1,2]\}$

b) $I = \iiint_A x \exp(ay + bz) \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,1] \land y \in [0,2] \land z \in [0,2]\}$

c) $I = \iiint_A \frac{x \, dx\,dy\,dz}{(y+z)^2}$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,1] \land y \in [1,a] \land z \in [1,a]\}$
and $a \in (1, +\infty)$.

d) $I = \iiint_A (x-y)(y-z)(z-x) \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,a] \land y \in [0,b] \land z \in [0,c]\}$
and $a, b, c \in (0, +\infty)$.

e) $I = \iiint_A (x+y)^3 \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,a] \land y \in [0,a] \land z \in [0,a]\}$
and $a \in (0, +\infty)$

f) $I = \iiint_A (x+y+z)^2 \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [1,a] \wedge y \in [1,a] \wedge z \in [1,a]\}$

with $a \in (1, +\infty)$

Evaluation of triple integrals on simple domains

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^3$ be a scalar field continuous at A. We distinguish between the following cases:

(1) z-simple regions

For $A = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in S_{xy} \wedge g_1(x,y) \leq z \leq g_2(x,y)\} \Rightarrow$

$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \iint_{S_{xy}} dx\,dy \int_{g_1(x,y)}^{g_2(x,y)} dz\ f(x,y,z)$$

(2) x-simple regions

For $A = \{(x,y,z) \in \mathbb{R}^3 \mid (y,z) \in S_{yz} \wedge g_1(y,z) \leq x \leq g_2(y,z)\} \Rightarrow$

$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \iint_{S_{xy}} dy\,dz \int_{g_1(y,z)}^{g_2(y,z)} dx\ f(x,y,z)$$

(3) y-simple regions

For $A = \{(x,y,z) \in \mathbb{R}^3 \mid (x,z) \in S_{xz} \wedge g_1(x,z) \leq y \leq g_2(x,z)\}$

$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \iint_{S_{xz}} dx\,dz \int_{g_1(x,z)}^{g_2(x,z)} dy\ f(x,y,z).$$

④ xy-simple regions

For $A = \{(x,y,z) \in \mathbb{R}^3 \mid z \in [a,b] \wedge (x,y) \in S_{xy}(z)\} \Rightarrow$

$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \int_a^b dz \iint_{S_{xy}(z)} dx\,dy\; f(x,y,z)$$

⑤ yz-simple regions

For $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [a,b] \wedge (y,z) \in S_{yz}(x)\} \Rightarrow$

$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \int_a^b dx \iint_{S_{yz}(x)} dy\,dz\; f(x,y,z)$$

⑥ xz-simple regions

For $A = \{(x,y,z) \in \mathbb{R}^3 \mid y \in [a,b] \wedge (x,z) \in S_{xz}(y)\} \Rightarrow$

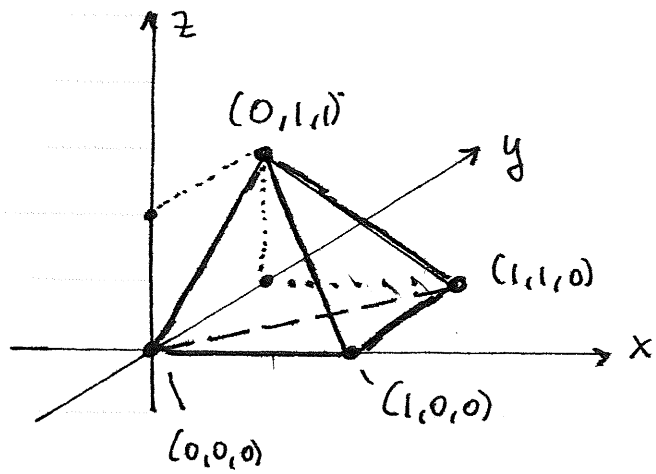
$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \int_a^b dy \iint_{S_{xz}(y)} dx\,dz\; f(x,y,z).$$

EXAMPLES

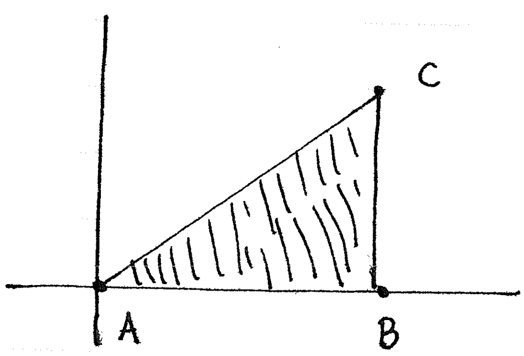
Evaluate the integral $I = \iiint_T xy \, dx\,dy\,dz$

if T is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(0,1,1)$.

Solution



Consider the xy-cross-section of the trapezoid at $z=0$:



A $(0,0,0)$
B $(1,0,0)$
C $(1,1,0)$

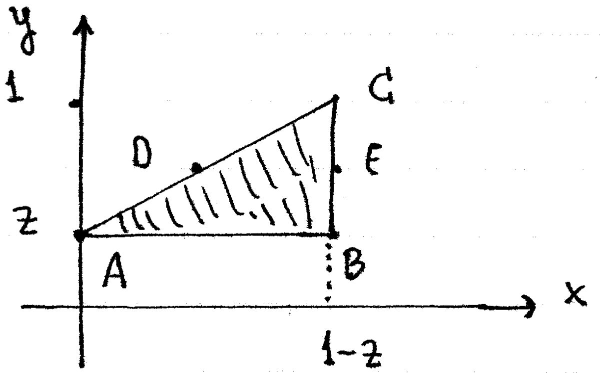
Note that for $0<z<1$, the point A, B, C come together linearly at the point $(0,1,1)$ at the top of the tetrahedron. It follows that for a given z, the coordinates of the vertices A, B, C are:

For $(0,0,0) \longrightarrow (0,1,1)$: $A(0, z, z)$

For $(1,0,0) \longrightarrow (0,1,1)$: $B(1-z, z, z)$

For $(1,1,0) \longrightarrow (0,1,1)$: $C(1-z, 1, z)$

with $0 \leq z \leq 1$. We sketch A, B, C:



It follows that the cross-section of the tetrahedron at z is given by:

$$S_{xy}(z) = \{(x,y) \in \mathbb{R}^2 \mid z \leq y \leq 1 \wedge f_1(y,z) \leq x \leq f_2(y,z)\}$$

To determine $f_1(y,z)$ and $f_2(y,z)$, let $D \in AC$ and $E \in BC$ such that $y_D = y_E = y$. Then we see that $f_1(y,z) = x_D$ and $f_2(y,z) = x_E$. We note that:

$$D \in AC \Leftrightarrow \frac{y_D - y_A}{x_D - x_A} = \frac{y_C - y_A}{x_C - x_A} \Leftrightarrow$$

$$\Leftrightarrow \frac{y-z}{f_1(y,z) - 0} = \frac{1-z}{(1-z)-0} \Leftrightarrow$$

$$\Leftrightarrow \frac{y-z}{f_1(y,z)} = 1 \Leftrightarrow f_1(y,z) = y - z.$$

Since BC is vertical (i.e. $x_B = x_C = 1-z$) it follows that

$$f_2(y,z) = x_E = x_B = 1-z$$

and therefore

$$S_{xy}(z) = \{(x,y) \in \mathbb{R}^2 \mid z \leq y \leq 1 \wedge y-z \leq x \leq 1-z\}$$

The trapezoid itself is represented by:

$$T = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in S_{xy}(z) \wedge z \in [0,1]\}$$

We may now evaluate the integral:

$$I = \iiint_T xy \, dx dy dz = \int_0^1 dz \iint_{S_{xy}(z)} dx dy \; xy =$$

$$= \int_0^1 dz \int_z^1 dy \int_{y-z}^{1-z} dx \; xy = \int_0^1 dz \int_z^1 dy \; y \left[\int_{y-z}^{1-z} x \, dx \right] =$$

$$= \int_0^1 dz \int_z^1 dy \; y \left[\frac{x^2}{2} \right]_{y-z}^{1-z} = \int_0^1 dz \int_z^1 dy \; y \left[\frac{(1-z)^2 - (y-z)^2}{2} \right]$$

$$= \frac{1}{2} \int_0^1 dz \int_z^1 dy \; y \left[(1-2z+\cancel{z^2}) - (y^2 - 2yz + \cancel{z^2}) \right] =$$

$$= \frac{1}{2} \int_0^1 dz \int_z^1 dy \; y(1-2z-y^2+2yz) =$$

$$= \frac{1}{2}\int_0^1 dz \int_z^1 dy\, \left(y - 2yz - y^3 + 2y^2 z\right) =$$

$$= \frac{1}{2}\int_0^1 dz \left[\frac{y^2}{2} - y^2 z - \frac{y^4}{4} + \frac{2y^3 z}{3}\right]_z^1 =$$

$$= \frac{1}{2}\,\frac{1}{12}\int_0^1 dz \left[6y^2 - 12y^2 z - 3y^4 + 8y^3 z\right]_z^1 =$$

$$= \frac{1}{24}\int_0^1 dz \left[6(1-z^2) - 12(1-z^2)z - 3(1-z^4) + 8(1-z^3)z\right]$$

$$= \frac{1}{24}\int_0^1 dz \left[6 - 6z^2 - 12z + 12z^3 - 3 + 3z^4 + 8z - 8z^4\right] =$$

$$= \frac{1}{24}\int_0^1 dz \left[(3-8)z^4 + 12z^3 - 6z^2 + (-12+8)z + (6-3)\right] =$$

$$= \frac{1}{24}\int_0^1 dz \left(-5z^4 + 12z^3 - 6z^2 - 4z + 3\right) =$$

$$= \frac{1}{24}\left[-\frac{5z^5}{5} + 12\,\frac{z^4}{4} - 6\,\frac{z^3}{3} - 4\,\frac{z^2}{2} + 3z\right]_0^1 =$$

$$= \frac{1}{24}\left[-z^5 + 3z^4 - 2z^3 - 2z^2 + 3z\right]_0^1 =$$

$$= \frac{1}{24}\left[-1 + 3 - 2 - 2 + 3\right] = \frac{1}{24}$$

EXERCISES

⑧ Evaluate the following triple integrals

a) $I = \iiint_A \exp(x+y+z)\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid y \leq z \leq x \wedge 0 \leq y \leq x \wedge 0 \leq x \leq a\}$
and $a \in (0,+\infty)$

b) $I = \iiint_A xyz\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq z \leq a \wedge 0 \leq y \leq \sqrt{a^2-x^2}$
$\wedge\ 0 \leq x \leq a\}$
and $a \in (0,+\infty)$

c) $I = \iiint_A x\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq z \leq a\}$
and $a \in (0,+\infty)$

d) $I = \iiint_A z^2\,dx\,dy\,dz$

with $A = \{(x,y,z) \in [0,+\infty)^3 \mid x+y+z \leq a\}$
and $a \in (0,+\infty)$.

e) $I = \iiint_A z \, dx\,dy\,dz$

with $A = \{(x,y,z) \in [0,+\infty)^3 \mid x^2 \leq y \leq 2a \wedge 0 \leq x \leq a$
$\wedge\ x-y \leq z \leq x+y\}$
and $a \in (0,+\infty)$

f) $I = \iiint_A \dfrac{y+z}{x} \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid 1 \leq x \leq a \wedge 0 \leq y \leq a-x \wedge 0 \leq z \leq a-x-y\}$
and $a \in (1,+\infty)$

g) $I = \iiint_A \sin\left(\dfrac{x}{y}\right) dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq z \leq a \wedge 0 \leq y \leq \sin(2z) \wedge 0 \leq x \leq 2yz\}$
and $a \in (0, \pi/2)$.

h) $I = \iiint_A xy \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq x \leq a \wedge 0 \leq y \leq \sqrt{a^2-x^2} \wedge$
$\wedge\ 0 \leq z \leq \sqrt{a^2-x^2-y^2}\ \}$
and $a \in (0,+\infty)$

i) $I = \iiint_A e^z \, dx\,dy\,dz$ with $A \subseteq \mathbb{R}^3$ the tetrahedron with vertices $(0,0,0)$, $(a,0,0)$, $(0,a,0)$, $(0,0,b)$ and $a, b \in (0,+\infty)$.

Change of variables in $\mathbb{R}^3$

Consider a change of variables

$$\begin{cases} x = g_1(u,v,w) \\ y = g_2(u,v,w) \\ z = g_3(u,v,w) \end{cases}, \quad \forall (u,v,w) \in B$$

that is a smooth change of variables from cartesian coordinates (x,y,z) to a new coordinate system (u,v,w).

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^3$ be a scalar field and let

$$S = \{(g_1(u,v,w), g_2(u,v,w), g_3(u,v,w)) \mid (u,v,w) \in S_0\}$$

be the domain of integration, with $S_0 \subseteq B$. Then, it follows that

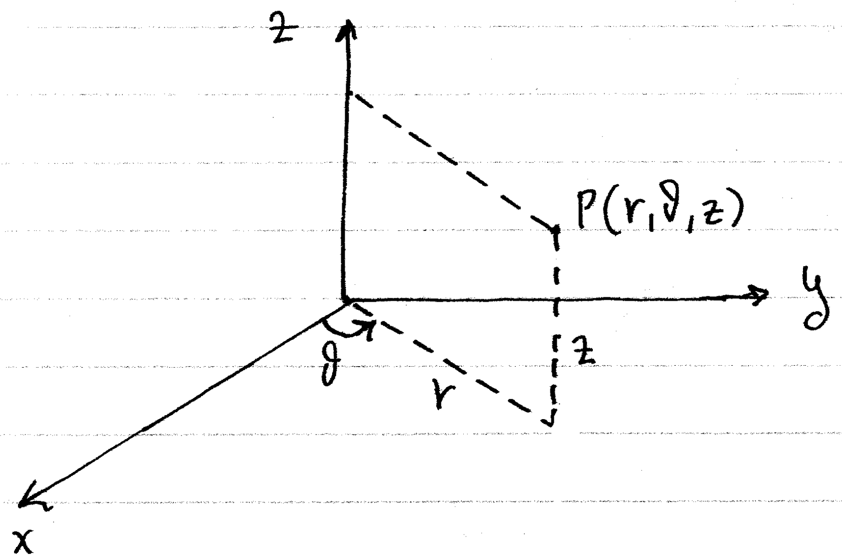
$$\boxed{\begin{aligned} &\iiint_S f(x,y,z)\,dx\,dy\,dz = \\ &= \iiint_{S_0} f(g_1(u,v,w), g_2(u,v,w), g_3(u,v,w)) \left|\frac{\partial(g_1,g_2,g_3)}{\partial(u,v,w)}\right| du\,dv\,dw \end{aligned}}$$

The corresponding transformation of the differential reads:

$$\boxed{dx\,dy\,dz = \left|\frac{\partial(g_1,g_2,g_3)}{\partial(u,v,w)}\right| du\,dv\,dw}$$

1 → Cylindrical coordinates

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \\ z = z \end{cases}, \quad \forall (r,\vartheta,z) \in [0,+\infty) \times [0,2\pi] \times \mathbb{R}$$



$$\boxed{dx\,dy\,dz = r\,dr\,d\vartheta\,dz}$$

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^3$ and consider the domain of integration

$S = \{(r\cos\vartheta, r\sin\vartheta, z) \mid (r,\vartheta,z) \in S_0\}$

Then:

$$I = \iiint_S f(x,y,z)\,dx\,dy\,dz =$$

$$= \iiint_{S_0} f(r\cos\vartheta, r\sin\vartheta, z)\, r\,dr\,d\vartheta\,dz$$

Proof

$x = r\cos\vartheta \wedge y = r\sin\vartheta \wedge z = z \Rightarrow$

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(r,\vartheta,z)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\vartheta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial\vartheta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial\vartheta & \partial z/\partial z \end{vmatrix} =$$

$$= \left|\begin{matrix} \cos\vartheta & -r\sin\vartheta & 0 \\ \sin\vartheta & r\cos\vartheta & 0 \\ 0 & 0 & 1 \end{matrix}\right|\begin{matrix} \cos\vartheta & -r\sin\vartheta \\ \sin\vartheta & r\cos\vartheta \\ 0 & 0 \end{matrix} =$$

$$= (\cos\vartheta)(r\cos\vartheta)1 + 0 + 0 - 0 - 0 - 1(\sin\vartheta)(-r\sin\vartheta)$$

$$= r\cos^2\vartheta + r\sin^2\vartheta = r(\cos^2\vartheta + \sin^2\vartheta) = r \Rightarrow$$

$$\Rightarrow dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(r,\vartheta,z)}\right| dr\,d\vartheta\,dz =$$

$$= |r|\,dr\,d\vartheta\,dz = r\,dr\,d\vartheta\,dz \Rightarrow$$

$$\Rightarrow I = \iiint_S f(x,y,z)\,dx\,dy\,dz =$$

$$= \iiint_{S_0} f(r\cos\vartheta, r\sin\vartheta, z)\,r\,dr\,d\vartheta\,dz.$$

EXAMPLE

Evaluate the integral $I = \iiint_A (x^2+y^2)\,dx\,dy\,dz$

with A given by:

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [-2,2] \wedge y \in [-\sqrt{4-x^2}, \sqrt{4-x^2}] \wedge$$
$$\wedge\, z \in [\sqrt{x^2+y^2}, 2]\}$$

by converting to cylindrical coordinates.

Solution

- First we determine the domain B of the cylindrical integral.

Let $x = r\cos\theta \wedge y = r\sin\theta \wedge z = z$. Then $x^2+y^2 = r^2$.

We note that

$$y \in [-\sqrt{4-x^2}, \sqrt{4-x^2}] \Leftrightarrow -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2} \Leftrightarrow$$
$$\Leftrightarrow y^2 \leq (\sqrt{4-x^2})^2 \Leftrightarrow y^2 \leq 4-x^2 \Leftrightarrow x^2+y^2 \leq 4 \Leftrightarrow 0 \leq r^2 \leq 4$$
$$\Leftrightarrow 0 \leq r \leq 2. \text{ (since } x^2+y^2 \geq 0, \forall x,y \in \mathbb{R}).$$

and

$$z \in [\sqrt{x^2+y^2}, 2] \Leftrightarrow \sqrt{x^2+y^2} \leq z \leq 2$$
$$\Leftrightarrow r \leq z \leq 2$$

The condition $x \in [-2,2]$ is implied by the condition on y via the domain restriction $4-x^2 \geq 0$, so it is redundant and does not introduce further restrictions.

Indeed: $|x| = |r\cos\theta| = |r||\cos\theta| \leq |r| = r \leq 2 \Rightarrow$

$$\Rightarrow |x| \leq 2 \Rightarrow x \in [-2,2].$$

Since there are no restrictions on the angle ϑ, we have $\vartheta \in [0, 2\pi]$.

It follows that

$A = \{(r\cos\vartheta, r\sin\vartheta, z) \mid \vartheta \in [0, 2\pi] \wedge r \in [0, 2] \wedge z \in [r, 2]\}$

Define:

$B = \{(r, \vartheta, z) \mid \vartheta \in [0, 2\pi] \wedge r \in [0, 2] \wedge z \in [r, 2]\}$

• 2 Now we change variables and evaluate the integral.

$$I = \iiint_A (x^2+y^2)\,dx\,dy\,dz = \iiint_B r^2\, r\,dr\,d\vartheta\,dz =$$

$$= \iiint_B r^3\,dr\,d\vartheta\,dz = \int_0^{2\pi} d\vartheta \int_0^2 dr \int_r^2 dz\ r^3 =$$

$$= \int_0^{2\pi} d\vartheta \int_0^2 dr\ r^3 \left[\int_r^2 dz\right] = \int_0^{2\pi} d\vartheta \int_0^2 dr\ r^3(2-r) =$$

$$= \int_0^{2\pi} d\vartheta \int_0^2 dr\ (2r^3 - r^4) = \int_0^{2\pi} d\vartheta \left[\frac{2r^4}{4} - \frac{r^5}{5}\right]_0^2 =$$

$$= \left[\frac{r^4}{2} - \frac{r^5}{5}\right]_0^2 \int_0^{2\pi} d\vartheta = 2\pi\left[\frac{2^4 - 0^4}{2} - \frac{2^5 - 0}{5}\right]$$

$$= 2\pi\left[8 - \frac{32}{5}\right] = 2\pi \cdot \frac{40-32}{5} = 2\pi \cdot \frac{8}{5} = \frac{16\pi}{5}$$

EXERCISES

(9) Evaluate the following integrals using change of variables to cylindrical coordinates

a) $I = \iiint_A (x^2+y^2)\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq a^2 \wedge 0 \leq z \leq b\}$
and $a, b \in (0, +\infty)$.

b) $I = \iiint_A xz\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq a^2 \wedge x \geq 0 \wedge 0 \leq z \leq b\}$
and $a, b \in (0, +\infty)$

c) $I = \iiint_A xy\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid a^2 \leq x^2+y^2 \leq b^2 \wedge x \geq 0 \wedge y \geq 0 \wedge 0 \leq z \leq b\}$
and $a, b \in (0, +\infty)$ with $a \leq b$.

d) $I = \iiint_A z(x+y)^2\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq a^2 \wedge x \geq y \geq 0 \wedge 0 \leq z \leq b\}$
and $a, b \in (0, +\infty)$

e) $I = \iiint_A z\sqrt{x^2+y^2}\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq z \leq a^2-x^2-y^2\}$
and $a \in (0,+\infty)$

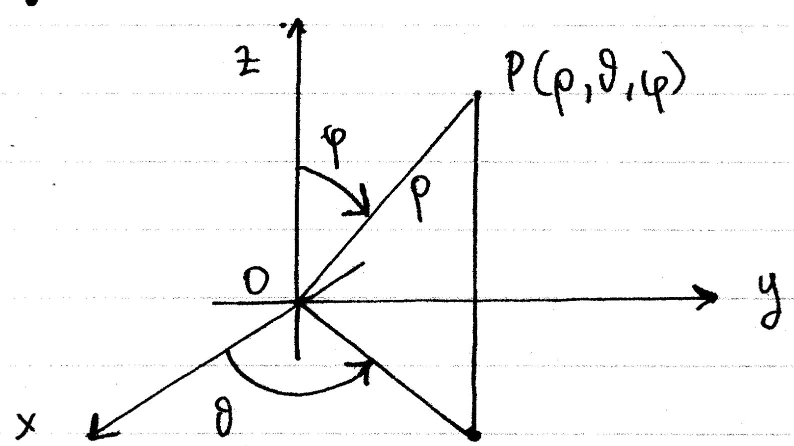
f) $I = \iiint_A z\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq z \leq a\}$
and $a \in (0,+\infty)$.

g) $I = \iiint_A (x-y)^2\,dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq z \leq x^2+y^2 \leq a^2\}$
and $a \in (0,+\infty)$.

(2) → Spherical coordinates



ρ = distance from O to P
ϑ = angle from x-axis to projection of OP onto the xy plane
φ = angle from z-axis to OP.

$$\begin{cases} x = \rho \sin\varphi \cos\vartheta \\ y = \rho \sin\varphi \sin\vartheta \\ z = \rho \cos\varphi \end{cases}, \forall (\rho,\vartheta,\varphi) \in [0,+\infty) \times [0,2\pi] \times [0,\pi]$$

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^3$ and let
$S = \{(\rho \sin\varphi \cos\vartheta, \rho \sin\varphi \sin\vartheta, \rho\cos\varphi) \mid (\rho,\vartheta,\varphi) \in S_0\}$
Then:

$$I = \iiint_S f(x,y,z)\,dx\,dy\,dz =$$

$$= \iiint_{S_0} f(\rho\sin\varphi\cos\vartheta, \rho\sin\varphi\sin\vartheta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\vartheta\,d\varphi$$

The formal transformation for the differential is given by:

$$\boxed{dxdydz = \rho^2 \sin\varphi \, d\rho \, d\vartheta \, d\varphi}$$

Proof

$$\begin{cases} x = \rho \sin\varphi \cos\vartheta \\ y = \rho \sin\varphi \sin\vartheta \\ z = \rho \cos\varphi \end{cases} \Rightarrow$$

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(\rho,\vartheta,\varphi)} = \begin{vmatrix} \partial x/\partial\rho & \partial x/\partial\vartheta & \partial x/\partial\varphi \\ \partial y/\partial\rho & \partial y/\partial\vartheta & \partial y/\partial\varphi \\ \partial z/\partial\rho & \partial z/\partial\vartheta & \partial z/\partial\varphi \end{vmatrix} =$$

$$= \begin{vmatrix} \sin\varphi\cos\vartheta & -\rho\sin\varphi\sin\vartheta & \rho\cos\varphi\cos\vartheta \\ \sin\varphi\sin\vartheta & \rho\sin\varphi\cos\vartheta & \rho\cos\varphi\sin\vartheta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix}$$

$$= (\sin\varphi\cos\vartheta)(\rho\sin\varphi\cos\vartheta)(-\rho\sin\varphi) +$$
$$+ (-\rho\sin\varphi\sin\vartheta)(\rho\cos\varphi\sin\vartheta)\cos\varphi + 0$$
$$- (\cos\varphi)(\rho\sin\varphi\cos\vartheta)(\rho\cos\varphi\cos\vartheta) - 0$$

$$-(-\rho\sin\varphi)(\sin\varphi\sin\vartheta)(-\rho\sin\varphi\sin\vartheta)$$

$$= -\rho^2\sin^3\varphi\cos^2\vartheta - \rho^2\sin\varphi\cos^2\varphi\sin^2\vartheta$$
$$-\rho^2\sin\varphi\cos^2\varphi\cos^2\vartheta - \rho^2\sin^3\varphi\sin^2\vartheta =$$

$$= -\rho^2 \sin\varphi \left(\sin^2\varphi \cos^2\vartheta + \cos^2\varphi \sin^2\vartheta + \cos^2\varphi \cos^2\vartheta + \sin^2\varphi \sin^2\vartheta \right) =$$

$$= -\rho^2 \sin\varphi \left[\sin^2\varphi \left(\cos^2\vartheta + \sin^2\vartheta \right) + \cos^2\varphi \left(\cos^2\vartheta + \sin^2\vartheta \right) \right]$$

$$= -\rho^2 \sin\varphi \left[\sin^2\varphi + \cos^2\varphi \right] =$$

$$= -\rho^2 \sin\varphi \Rightarrow$$

$$\Rightarrow dx\,dy\,dz = \left| \frac{\partial(x,y,z)}{\partial(\rho,\vartheta,\varphi)} \right| d\rho\, d\vartheta\, d\varphi =$$

$$= \left| -\rho^2 \sin\varphi \right| d\rho\, d\vartheta\, d\varphi =$$

$$= \rho^2 \left| \sin\varphi \right| d\rho\, d\vartheta\, d\varphi =$$

$$= \rho^2 \sin\varphi\, d\rho\, d\vartheta\, d\varphi. \Rightarrow$$

$$\Rightarrow I = \iiint_S f(x,y,z)\, dx\,dy\,dz =$$

$$= \iiint_{S_0} f(\rho \sin\varphi \cos\vartheta, \rho \sin\varphi \cos\vartheta, \rho \cos\varphi) \rho^2 \sin\varphi\, d\rho\, d\vartheta\, d\varphi \quad \square$$

Note that $\varphi \in [0,\pi] \Rightarrow \sin\varphi \geq 0 \Rightarrow |\sin\varphi| = \sin\varphi.$

EXAMPLE

Use spherical coordinates to evaluate the integral

$$I = \iiint_A \exp\left((x^2+y^2+z^2)^{3/2}\right) dx\,dy\,dz$$

over the region $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\}$

Solution

Let

$$\begin{cases} x = \rho \sin\varphi \cos\vartheta \\ y = \rho \sin\varphi \sin\vartheta \\ z = \rho \cos\varphi \end{cases}$$

with $\rho \in [0, +\infty)$ and $\vartheta \in [0, 2\pi)$ and $\varphi \in [0, \pi]$.

Then $dx\,dy\,dz = \rho^2 \sin\varphi \, d\rho\, d\vartheta\, d\varphi$

and we note that

$x^2+y^2+z^2 \leq 1 \Leftrightarrow \rho^2 \leq 1 \Leftrightarrow 0 \leq \rho \leq 1 \Leftrightarrow \rho \in [0,1]$

There are no constraints on ϑ and φ. It follows that

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\} =$$
$$= \{(\rho \sin\varphi \cos\vartheta, \rho \sin\varphi \sin\vartheta, \rho \cos\varphi) \mid (\rho,\vartheta,\varphi) \in B\}$$

with

$$B = \{(\rho,\vartheta,\varphi) \mid \rho \in [0,1] \land \vartheta \in [0,2\pi) \land \varphi \in [0,\pi]\} =$$
$$= [0,1] \times [0,2\pi) \times [0,\pi]$$

and therefore:

$$I = \iiint_A \exp((x^2+y^2+z^2)^{3/2})\, dx\,dy\,dz =$$

$$= \iiint_B \exp((\rho^2)^{3/2})\, \rho^2 \sin\varphi \, d\rho\, d\vartheta\, d\varphi =$$

$$= \iiint_B \rho^2 e^{\rho^3} \sin\varphi \, d\rho\, d\vartheta\, d\varphi =$$

$$= \int_0^1 d\rho \int_0^{2\pi} d\vartheta \int_0^{\pi} d\varphi \; \rho^2 e^{\rho^3} \sin\varphi =$$

$$= \left[\int_0^1 d\rho\, \rho^2 e^{\rho^3}\right]\left[\int_0^{\pi} d\varphi \sin\varphi\right]\left[\int_0^{2\pi} d\vartheta\right] = 2\pi I_1 I_2$$

with

$$I_1 = \int_0^1 d\rho\, \rho^2 \exp(\rho^3) = \frac{1}{3}\int_0^1 3\rho^2 \exp(\rho^3)\, d\rho$$

$$= \frac{1}{3}\int_0^1 [\exp(\rho^3)]'\, d\rho = \frac{1}{3}\left[\exp(\rho^3)\right]_0^1 =$$

$$= \frac{e^1 - e^0}{3} = \frac{e-1}{3}$$

and

$$I_2 = \int_0^{\pi} \sin\varphi\, d\varphi = \left[-\cos\varphi\right]_0^{\pi} = -\left[\cos\varphi\right]_0^{\pi} =$$

$$= -\left[\cos\pi - \cos 0\right] = -\left[-\cos 0 - \cos 0\right] =$$

$$= -(-1-1) = 2$$

It follows that

$$I = 2\pi I_1 I_2 = 2\pi \frac{e-1}{3} \cdot 2 = \frac{4\pi(e-1)}{3}$$

EXERCISES

(10) Evaluate the following triple integrals using change of variables to spherical coordinates.

a) $I = \iiint_A y \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0 \land y \geq 0 \land z \geq 0 \land x^2+y^2+z^2 \leq a^2\}$ and $a \in (0, +\infty)$

b) $I = \iiint_A \dfrac{dx\,dy\,dz}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid a^2 \leq x^2+y^2+z^2 \leq b^2\}$ and $a, b \in (0, +\infty)$ with $a < b$.

c) $I = \iiint_A (x^2+y^2) \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq a^2\}$ and $a \in (0, +\infty)$.

d) $I = \iiint_A \sqrt{x^2+y^2+z^2} \, dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 2z\}$

e) $I = \iiint_A \frac{z\,dx\,dy\,dz}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 4a^2 \land z \geq a\}$

and $a \in (0, +\infty)$

f) $I = \iiint_A \exp\left(\frac{1}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}\right) dx\,dy\,dz$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid -a \leq x \leq a \land$

$\land -\sqrt{a^2-x^2} \leq y \leq \sqrt{a^2-x^2} \land$

$\land\ 0 \leq z \leq \sqrt{a^2-x^2-y^2}\ \}$

and $a \in (0, +\infty)$.