SEQUENCES

▼ Preliminaries

- Recall the following set definitions:

  $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ (set of natural numbers)

  $\mathbb{N}^* = \{1, 2, 3, \dots\} = \mathbb{N} - \{0\}$

  $[n] = \{x \in \mathbb{N}^* \mid x \leq n\} = \{1, 2, 3, \dots, n\}$

  The set of all natural numbers $n > n_0$ with $n_0 \in \mathbb{N}^*$ can be represented as:

  $\mathbb{N}^* - [n_0] = \{x \in \mathbb{N} \mid x > n_0\}$.

- For the theory of sequences and series we make heavy use of quantifiers $\forall$ and $\exists$. (see my Calculus 1 lecture notes).

  Let $p(x)$ be a predicate (i.e. a statement about x which is either true or false depending on the value of x) and let A be a set. Then:

  1) $\forall x \in A : p(x)$ (universal quantifier)

     means: "For all $x \in A$, $p(x)$ is true"

  2) $\exists x \in A : p(x)$ (existential quantifier)

     means: "There exists (at least one) $x \in A$ such that $p(x)$ is true".

  We can now construct nested statements:

  e.g. $\forall x \in A : \exists y \in B : p(x, y)$

  "For all $x \in A$, there is at least one $y \in B$, such that $p(x, y)$ is true".

Sequences and convergent sequences

- A sequence a_n is a list of numbers

$$a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$$

written down in a definite order.

- More rigorously, a sequence a_n is a mapping of the form

$$a : n \in \mathbb{N} \longrightarrow a_n \in \mathbb{R}$$

or $$a : n \in \mathbb{N}^* \longrightarrow a_n \in \mathbb{R}$$

- A sequence can be defined:

a) Explicitly: By providing an algebraic expression that yields a_n in terms of n.

e.g. $\forall n \in \mathbb{N} : a_n = n^2 \cdot 2^n$

b) Recursively: By providing an algorithm that gives the next element of a sequence in terms of previous elements

e.g. $$\begin{cases} a_1 = 1 \wedge a_2 = 1 \\ \forall n \in \mathbb{N}^* : a_{n+2} = a_n + a_{n+1} \end{cases}$$

↳ This example is the famous Fibonacci sequence.

Convergent sequences

Let a_n be a sequence and let $l \in \mathbb{R}$. Then we define:

$$\boxed{\begin{array}{l} \lim a_n = l \Leftrightarrow \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - l| < \varepsilon \\ a_n \text{ convergent} \Leftrightarrow \exists l \in \mathbb{R} : \lim a_n = l \end{array}}$$

• These definitions read:

a) "$\lim a_n = l$ if and only if for all real numbers $\varepsilon > 0$ there is at least one $n_0 \in \mathbb{N}^*$ such that for all $n \in \mathbb{N}^* - [n_0]$ (i.e. natural numbers n with $n > n_0$) we have $|a_n - l| < \varepsilon$"

b) "a_n convergent if and only if there is at least one $l \in \mathbb{R}$ such that $\lim a_n = l$."

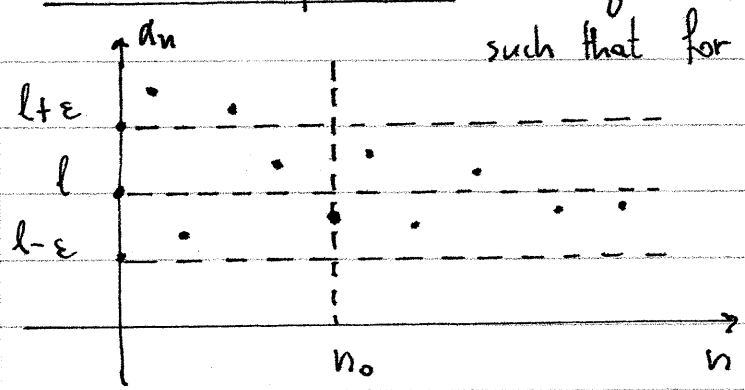
• We note that:

$$|a_n - l| < \varepsilon \Leftrightarrow -\varepsilon < a_n - l < \varepsilon \Leftrightarrow$$
$$\Leftrightarrow l - \varepsilon < a_n < l + \varepsilon$$

and therefore the inequality $|a_n - l| < \varepsilon$ means that the distance between a_n and l is less than ε.

• Intuitively, this definition means that the sequence a_n approaches l for large n. More precisely: for any maximum allowed distance $\varepsilon > 0$ (which we can make as small as we want) there is some integer $n_0 \in \mathbb{N}^*$ such that all a_n <u>after</u> a_{n_0} (with $n > n_0$) have distance from l that is less than ε. The key is that some $n_0 \in \mathbb{N}$ can be found for any choice made for $\varepsilon > 0$ regardless of how small ε is chosen.

• <u>Geometric interpretation:</u> For any $\varepsilon > 0$, there is some $n_0 \in \mathbb{N}^*$ such that for $n > n_0$, the sequence is thereafter restricted in the strip between $l - \varepsilon$ and $l + \varepsilon$.



Properties of convergent sequences

1) **Uniqueness:** Let a_n be a sequence and $l_1, l_2 \in \mathbb{R}$.

Then:

$$\boxed{\begin{cases} \lim a_n = l_1 \\ \lim a_n = l_2 \end{cases} \Rightarrow l_1 = l_2}$$

2) **Convergence and operations**

Let a_n, b_n be convergent sequences and let $\lambda \in \mathbb{R}$, $k \in \mathbb{N}^*$

Then:

$$\boxed{\begin{array}{l} \lim (a_n + b_n) = \lim a_n + \lim b_n \\ \lim (a_n b_n) = (\lim a_n)(\lim b_n) \\ \lim (\lambda a_n) = \lambda \lim a_n \\ \lim \left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim a_n}{\lim b_n} \\ (\forall n \in \mathbb{N}^* : a_n \geq 0) \Rightarrow \lim \sqrt[k]{a_n} = \sqrt[k]{\lim a_n} \end{array}}$$

3) **Sequence and function limit**

Let a_n be a sequence, $f : [0, +\infty) \to \mathbb{R}$ be a function and let $l \in \mathbb{R}$. Then

$$\boxed{\begin{cases} \forall n \in \mathbb{N}^* : a_n = f(n) \\ \lim\limits_{x \to +\infty} f(x) = l \end{cases} \Rightarrow \lim a_n = l}$$

Useful applications

The following statements are good examples and can also be used to do other exercises.

1) $\boxed{\forall a \in (0,+\infty): \lim \sqrt[n]{a} = 1}$

Proof

We note that

$$a_n \equiv \sqrt[n]{a} = a^{1/n} = \exp((1/n)\ln a) = \exp\left(\frac{\ln a}{n}\right), \forall n \in \mathbb{N}^*.$$

and since

$$\lim_{x\to+\infty} \frac{\ln a}{x} = \ln a \cdot \lim_{x\to+\infty} \frac{1}{x} = 0 \cdot \ln a = 0 \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} \exp\left(\frac{\ln a}{x}\right) = \exp(0) = 1 \Rightarrow \lim \sqrt[n]{a} = 1.$$

2) $\boxed{\lim \sqrt[n]{n} = 1}$

Proof

We note that:

$$\sqrt[n]{n} = n^{1/n} = \exp((1/n)\ln(n)) = \exp\left(\frac{\ln(n)}{n}\right), \forall n \in \mathbb{N}^*$$

and thus, since

$$\lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{(\ln x)'}{(x)'} = \lim_{x\to+\infty} \frac{1/x}{1} = \lim_{x\to+\infty} \frac{1}{x} = 0$$

$$\Rightarrow \lim_{x\to+\infty} \exp\left(\frac{\ln x}{x}\right) = \exp(0) = 1 \Rightarrow \lim \sqrt[n]{n} = 1.$$

3) $\boxed{\lim\left(1+\frac{a}{n}\right)^n = e^a \ , \forall a\in\mathbb{R}}$

Proof

We note that

$$\left(1+\frac{a}{n}\right)^n = \exp\left[n\ln\left(1+\frac{a}{n}\right)\right] \ , \ \forall n\in\mathbb{N}^*$$

and since

$$\lim_{x\to+\infty}\left[x\ln\left(1+\frac{a}{x}\right)\right] = \lim_{x\to+\infty}\frac{\ln(1+a/x)}{1/x} \overset{0/0}{=}$$

$$= \lim_{x\to+\infty}\frac{[\ln(1+a/x)]'}{(1/x)'} = \lim_{x\to+\infty}\frac{(1+a/x)^{-1}(1+a/x)'}{-1/x^2}$$

$$= \lim_{x\to+\infty}\frac{(-x^2)(-a/x^2)}{1+a/x} = \lim_{x\to+\infty}\frac{a}{1+a/x} =$$

$$= \frac{a}{1+0} = a \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty}\exp\left(x\ln\left(1+\frac{a}{x}\right)\right) = \exp(a) = e^a$$

$$\Rightarrow \lim(1+a/n)^n = e^a.$$

4) $\boxed{0<a<1 \Rightarrow \lim a^n = 0}$

Proof

Define $\forall x\in\mathbb{R}: f(x) = a^x = \exp(x\ln a)$

Since $0<a<1 \Rightarrow \ln a < 0$, and therefore:

$$\lim_{x\to+\infty}(x\ln a) = -\infty \Rightarrow \lim_{x\to+\infty} a^x = \lim_{x\to+\infty}\exp(x\ln a) = 0 \Rightarrow$$

$$\Rightarrow \lim a^n = 0.$$

EXAMPLES

a) Evaluate the limit of $a_n = \dfrac{3n^2 - 4n + 1}{5 - 2n^2}$, $\forall n \in \mathbb{N}^*$

Solution

Since

$$\lim_{x \to +\infty} \frac{3x^2 - 4x + 1}{5 - 2x^2} = \lim_{x \to +\infty} \frac{3x^2}{-2x^2} = \frac{3}{-2} = \frac{-3}{2} \Rightarrow$$

$$\Rightarrow \lim a_n = \lim_{n \in \mathbb{N}^*} \frac{3n^2 - 4n + 1}{5 - 2n^2} = \frac{-3}{2}$$

b) Evaluate the limit of $a_n = \dfrac{3^n + 4^{n+1}}{3^n - 4^n}$, $\forall n \in \mathbb{N}^*$

Solution

Since:

$$\forall n \in \mathbb{N}^*: a_n = \frac{3^n + 4^{n+1}}{3^n - 4^n} = \frac{3^n + 4 \cdot 4^n}{3^n - 4^n} = \frac{4^n\left[(3/4)^n + 4\right]}{4^n\left[(3/4)^n - 1\right]}$$

$$= \frac{(3/4)^n + 4}{(3/4)^n - 1}$$

$$\Rightarrow \lim a_n = \lim \frac{(3/4)^n + 4}{(3/4)^n - 1} = \frac{4 + \lim (3/4)^n}{-1 + \lim (3/4)^n} = \frac{4+0}{-1+0} = -4.$$

↳ The method above is to factor out the n^{th} power of the largest number from both numerator and denominator, and use the result

$$0 < a < 1 \Rightarrow \lim_{n \in \mathbb{N}^*} a^n = 0.$$

c) Evaluate the limit $a_n = \sqrt{n^2+3n} - \sqrt{n^2-3n}$, $\forall n \in \mathbb{N}^* - \{1,2\}$

Solution

Since

$$\lim_{x\to+\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-3x}\right) = \lim_{x\to+\infty} \frac{\left(\sqrt{x^2+3x}\right)^2 - \left(\sqrt{x^2-3x}\right)^2}{\sqrt{x^2+3x} + \sqrt{x^2-3x}}$$

$$= \lim_{x\to+\infty} \frac{(x^2+3x)-(x^2-3x)}{\sqrt{x^2+3x} + \sqrt{x^2-3x}} = \lim_{x\to+\infty} \frac{6x}{\sqrt{x^2+3x} + \sqrt{x^2-3x}}$$

$$= \lim_{x\to+\infty} \frac{6x}{|x|\sqrt{1+3/x} + |x|\sqrt{1-3/x}} =$$

$$= \lim_{x\to+\infty} \frac{6}{\sqrt{1+3/x} + \sqrt{1-3/x}} =$$

$$= \frac{6}{\sqrt{1+0} + \sqrt{1-0}} = \frac{6}{1+1} = 3 \Rightarrow$$

$\Rightarrow \lim a_n = 3$.

EXERCISES

① Evaluate the limit of the following sequences

a) $a_n = \dfrac{2n^3+4n^2-2}{6n+3n^3-5}$

b) $a_n = \dfrac{n^2+2n+3}{3n^3+n^2-1}$

c) $a_n = \dfrac{n-3}{n^3+n+4}$

d) $a_n = \dfrac{2n^2-5n+3}{1-5n^2}$

e) $a_n = \dfrac{2^n+5^n}{4^n+7^n}$

f) $a_n = \dfrac{2\cdot 8^n - 5^{2n}}{4+6^{2n+1}}$

g) $a_n = \dfrac{5+3^n+5^{n+1}}{7+2^n+5^{n+4}}$

h) $a_n = \dfrac{7^n-2\cdot 5^n+2}{3^n+7^{n+1}-3}$

i) $a_n = \dfrac{3\cdot 4^n - 5^{2n}}{7+6^{2n+1}}$

j) $a_n = \dfrac{2\cdot 5^n - 3^{2n}}{6+4^{2n+1}}$

k) $a_n = (\sqrt{n+4} - \sqrt{n+3})\sqrt{n+2}$

l) $a_n = \sqrt{n+\sqrt{n}} - \sqrt{n-\sqrt{n}}$

m) $a_n = \sqrt{n^2+5x+1} - \sqrt{n^2+2n+3}$

o) $a_n = \sqrt{4n^2+n} - 2n$

② Show that $\lim\limits_{n\in\mathbb{N}^*} \left[\sqrt{(n+a)(n+b)} - n\right] = \dfrac{a+b}{2}$

Divergent sequences

- Let a_n be a sequence. We say that:

$$\boxed{a_n \text{ divergent} \Leftrightarrow a_n \text{ not convergent}}$$

- **Preliminaries**: If p is a proposition (i.e. a statement that is true or false) then the negation $\bar{p}$ is the statement that "p is false".

e.g.

$$\overline{x=3} \Leftrightarrow x \neq 3$$
$$\overline{x>3} \Leftrightarrow x \leq 3$$
$$\overline{x \leq 3} \Leftrightarrow x > 3$$

To negate quantified statements we use the following negation rules:

$$\boxed{\begin{array}{l} \overline{\forall x \in A : p(x)} \Leftrightarrow \exists x \in A : \overline{p(x)} \\ \overline{\exists x \in A : p(x)} \Leftrightarrow \forall x \in A : \overline{p(x)} \end{array}}$$

- It follows that:

$$\begin{aligned} a_n \text{ divergent} &\Leftrightarrow \overline{a_n \text{ convergent}} \\ &\Leftrightarrow \overline{\exists l \in \mathbb{R} : \lim a_n = l} \\ &\Leftrightarrow \forall l \in \mathbb{R} : \overline{\lim a_n = l} \\ &\Leftrightarrow \forall l \in \mathbb{R} : \lim a_n \neq l \end{aligned}$$

and

$$\lim a_n \neq l \Leftrightarrow \overline{\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - l| < \varepsilon}$$

$$\Leftrightarrow \exists \varepsilon > 0 : \overline{\exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - l| < \varepsilon}$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall n_0 \in \mathbb{N}^* : \overline{\forall n \in \mathbb{N}^* - [n_0] : |a_n - l| < \varepsilon}$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall n_0 \in \mathbb{N}^* : \exists n \in \mathbb{N}^* - [n_0] : \overline{|a_n - l| < \varepsilon}$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall n_0 \in \mathbb{N}^* : \exists n \in \mathbb{N}^* - [n_0] : |a_n - l| \geqslant \varepsilon$$

Combining the above statements gives:

$$\boxed{\begin{array}{l} a_n \text{ divergent} \Leftrightarrow \\ \quad \forall l \in \mathbb{R} : \exists \varepsilon > 0 : \forall n_0 \in \mathbb{N}^* : \exists n \in \mathbb{N}^* - [n_0] : |a_n - l| \geqslant \varepsilon \end{array}}$$

which translates:
"a_n is divergent if and only if for all $l \in \mathbb{R}$, there is at least one $\varepsilon > 0$ such that for all $n_0 \in \mathbb{N}^*$, there is at least one $n \in \mathbb{N}^* - [n_0]$ such that $|a_n - l| \geqslant \varepsilon$".

Limits of subsequences

The most useful result involving subsequences is the following:

- $\boxed{\lim a_n = l \Leftrightarrow (\lim a_{2n} = l \wedge \lim a_{2n+1} = l)}$

An immediate consequence of this statement is that:

$$\boxed{\begin{cases} \lim a_{2n} = l_1 \\ \lim a_{2n+1} = l_2 \\ l_1 \neq l_2 \end{cases} \Rightarrow a_n \text{ divergent}}$$

Proof

Assume that $\lim a_{2n} = l_1 \wedge \lim a_{2n+1} = l_2 \wedge l_1 \neq l_2$.
To show that a_n divergent we assume that a_n is convergent. Then:

$$a_n \text{ convergent} \Rightarrow \exists l \in \mathbb{R} : \lim a_n = l.$$

and it follows for a chosen $l \in \mathbb{R}$ that

$$\lim a_n = l \Rightarrow \begin{cases} \lim a_{2n} = l \\ \lim a_{2n+1} = l \end{cases} \Rightarrow \begin{cases} l_1 = l \\ l_2 = l \end{cases} \Rightarrow l_1 = l_2$$

which is a contradiction since $l_1 \neq l_2$. It follows that a_n is divergent. □

- These results can be generalized to a great extent, but we will not be needing the most general formulation. The following generalization is sufficient for our needs:

► $\boxed{(\forall p \in [k] : \lim a_{kn+p-1} = l) \Leftrightarrow \lim a_n = l}$

► $$\boxed{\begin{cases} \exists p, q \in [k] : (\lim a_{kn+p-1} = l_1 \wedge \lim a_{kn+q-1} = l_2) \\ l_1 \neq l_2 \end{cases} \Rightarrow a_n \text{ divergent}}$$

- For $k=2$, the general statements simplify back to the original statements involving $\lim a_{2n}$ and $\lim a_{2n+1}$.
- For $k=3$, we get:

$\lim a_n = l \iff (\lim a_{3n} = l \wedge \lim a_{3n+1} = l \wedge \lim a_{3n+2} = l)$

and also:

$$\left.\begin{array}{l}\lim a_{3n} = l_1 \\ \lim a_{3n+1} = l_2 \\ l_1 \neq l_2\end{array}\right\} \Rightarrow a_n \text{ divergent}$$

$$\left.\begin{array}{l}\lim a_{3n} = l_1 \\ \lim a_{3n+2} = l_2 \\ l_1 \neq l_2\end{array}\right\} \Rightarrow a_n \text{ divergent}$$

$$\left.\begin{array}{l}\lim a_{3n+1} = l_1 \\ \lim a_{3n+2} = l_2 \\ l_1 \neq l_2\end{array}\right\} \Rightarrow a_n \text{ divergent.}$$

(3 combinations)

and so on for larger k.

EXAMPLES

a) Show that $a_n = \sin\left(\frac{n\pi}{2}\right)$ is divergent.

Solution

We note that

$\forall n \in \mathbb{N}^*: \; a_{4n} = \sin\left(\frac{(4n)\pi}{2}\right) = \sin(2n\pi) = \sin(0) = 0$

$\Rightarrow \lim a_{4n} = 0 \qquad (1)$

and

$\forall n \in \mathbb{N}^*: \; a_{4n+1} = \sin\left(\frac{(4n+1)\pi}{2}\right) = \sin\left(\frac{4n\pi}{2} + \frac{\pi}{2}\right) =$

$= \sin(2n\pi + \pi/2) = \sin(\pi/2) = 1$

$\Rightarrow \lim a_{4n+1} = 1 \qquad (2)$

From Eq. (1) and Eq. (2) it follows that a_n divergent.

b) Show that $a_n = (-1)^n \frac{3n+5}{2n+1}$ is divergent.

Solution

We note that

$a_{2n} = (-1)^{2n} \frac{3(2n)+5}{2(2n)+1} = \frac{6n+5}{4n+1}, \; \forall n \in \mathbb{N}^*$

$a_{2n+1} = (-1)^{2n+1} \frac{3(2n+1)+5}{2(2n+1)+1} = (-1) \frac{6n+3+5}{4n+2+1} = \frac{-(6n+8)}{4n+3}, \forall n \in \mathbb{N}^*$

and we now argue that:

$$\lim_{x\to+\infty} \frac{6x+5}{4x+1} = \lim_{x\to+\infty} \frac{6x}{4x} = \frac{6}{4} = \frac{3}{2} \Rightarrow$$

$$\Rightarrow \lim a_{2n} = \lim \frac{6n+5}{4n+1} = \frac{3}{2} \qquad (1)$$

and

$$\lim_{x\to+\infty} \frac{-(6x+8)}{4x+3} = \lim_{x\to+\infty} \frac{-6x}{4x} = \frac{-6}{4} = \frac{-3}{2} \Rightarrow$$

$$\Rightarrow \lim a_{2n+1} = \lim \frac{-(6n+1)}{4n+3} = \frac{-3}{2} \qquad (2)$$

From Eq.(1) and Eq.(2) it follows that a_n is divergent.

c) Show that for $a_n = (-1)^n \frac{3n}{n^2+1}$, $\lim a_n = 0$.

<u>Solution</u>

We note that

$$a_{2n} = (-1)^{2n} \frac{3(2n)}{(2n)^2+1} = \frac{6n}{4n^2+1}, \quad \forall n \in \mathbb{N}^*$$

$$a_{2n+1} = (-1)^{2n+1} \frac{3(2n+1)}{(2n+1)^2+1} = (-1) \frac{6n+3}{4n^2+4n+1+1} =$$

$$= \frac{-(6n+3)}{4n^2+4n+2}, \quad \forall n \in \mathbb{N}^*$$

and also that

$$\lim_{x\to+\infty} \frac{6x}{4x^2+1} = \lim_{x\to+\infty} \frac{6x}{4x^2} = \lim_{x\to+\infty} \frac{6}{4x} = 0 \Rightarrow$$

$$\Rightarrow \lim a_{2n} = \lim \frac{6n}{4n^2+1} = 0 \qquad (1)$$

and

$$\lim_{x\to+\infty} \frac{-(6x+3)}{4x^2+4x+2} = \lim_{x\to+\infty} \frac{-6x}{4x^2} = \lim_{x\to+\infty} \frac{-6}{4x} = 0 \Rightarrow$$

$$\Rightarrow \lim a_{2n+1} = \lim \frac{-(6n+3)}{4n^2+4n+2} = 0 \qquad (2)$$

From Eq. (1) and Eq. (2): $\lim a_n = 0$.

You should know how to state the definitions for limit statements and their negations, as follows:

d) State the definition for $\lim a_n = 3$

Solution

$\lim a_n = 3 \Leftrightarrow \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - 3| < \varepsilon$

or

$\lim a_n = 3$ if and only if for all $\varepsilon > 0$, there is some $n_0 \in \mathbb{N}^*$ such that for all natural numbers n with $n > n_0$ we have $|a_n - 3| < \varepsilon$.

e) State the definition for $\lim a_n \neq 5$

Solution

$\lim a_n \neq 5 \Leftrightarrow \exists \varepsilon > 0 : \forall n_0 \in \mathbb{N}^* : \exists n \in \mathbb{N}^* - [n_0] : |a_n - 5| \geq \varepsilon$

or

$\lim a_n \neq 5$ if and only if there is some $\varepsilon > 0$ such that for all $n_0 \in \mathbb{N}^*$ there is some natural number n with $n > n_0$ such that $|a_n - 5| \geq \varepsilon$.

EXERCISES

③ Show that the following sequences are divergent.

a) $a_n = \dfrac{(-1)^n + (-1)}{2}$

b) $a_n = (-1)^n \dfrac{n+2}{3n}$

c) $a_n = \cos\left(\dfrac{n\pi}{6}\right)$

d) $a_n = \dfrac{2(-2)^n + 2^n}{(-2)^n - 3 \cdot 2^{n-1}}$

e) $a_n = \dfrac{n^2 + (-1)^n n^2}{(n+2)^2}$

f) $a_n = \dfrac{(-1)^n n^2 + 3n - 1}{3n^2 + (-1)^n n + 1}$

④ Determine whether the following sequences are convergent or divergent, and find the limit, if the sequence is convergent

a) $a_n = \dfrac{(-1)^n n \cos(n\pi)}{n+2}$

b) $a_n = (-1)^n \sin\left(\dfrac{n\pi}{4}\right)$

c) $a_n = \sqrt{3n^2 + (-1)^n n} - \sqrt{3n^2 - 2n}$

d) $a_n = (-1)^n \sin\left(\dfrac{n}{2} + (-1)^n n\pi\right)$

e) $a_n = \dfrac{2n[(-1)^n n^2 + 1]}{(-1)^n n^3 + 2}$

f) $a_n = \sin\left(n\pi + \dfrac{2n}{3}\right) + \sin\left(n\pi + \dfrac{4n}{3}\right)$

Bounded sequences

Let a_n be a sequence. We define:

a_n upper bounded $\Leftrightarrow \exists c \in \mathbb{R}: \forall n \in \mathbb{N}^*: a_n \leq c$

a_n lower bounded $\Leftrightarrow \exists c \in \mathbb{R}: \forall n \in \mathbb{N}^*: a_n \geq c$

a_n bounded $\Leftrightarrow \begin{cases} a_n \text{ upper bounded} \\ a_n \text{ lower bounded} \end{cases}$

a_n absolutely bounded $\Leftrightarrow \exists c \in (0, +\infty): \forall n \in \mathbb{N}^*: |a_n| \leq c$

Properties of bounded sequences

The following properties can be derived from the definitions above, for a given sequence a_n.

1) Equivalence of absolute bounded and bounded

$\boxed{a_n \text{ bounded} \Leftrightarrow a_n \text{ absolutely bounded}}$

2) Convergence and bounded property

$\boxed{a_n \text{ convergent} \Rightarrow a_n \text{ bounded}}$

In general, given a statement $p \Rightarrow q$, the contrapositive statement $\bar{q} \Rightarrow \bar{p}$ is also true. Consequently it follows that

$\boxed{a_n \text{ not bounded} \Rightarrow a_n \text{ divergent}}$

3) Zero-bounded theorem for sequences

$$\boxed{\left.\begin{array}{l}\lim a_n = 0 \\ b_n \text{ bounded}\end{array}\right\} \Rightarrow \lim (a_n b_n) = 0}$$

4) Unbounded sequences

$$\boxed{\left.\begin{array}{l}\forall n \in \mathbb{N}^*: \; a_n \geq f(n) \\ \lim\limits_{x \to +\infty} f(x) = +\infty\end{array}\right\} \Rightarrow a_n \text{ not bounded}}$$

$$\boxed{\left.\begin{array}{l}\forall n \in \mathbb{N}^*: \; a_n \leq f(n) \\ \lim\limits_{x \to +\infty} f(x) = -\infty\end{array}\right\} \Rightarrow a_n \text{ not bounded}}$$

Methodology

- To establish that a sequence is absolutely bounded, we use the following properties of absolute values

$\forall a,b \in \mathbb{R} : |a+b| \leq |a|+|b|$ $\quad$ $\forall a,b \in \mathbb{R} : |ab| = |a||b|$

$\forall a,b \in \mathbb{R} : |a-b| \leq |a|+|b|$ $\quad$ $\forall a \in \mathbb{R} : \forall b \in \mathbb{R}-\{0\} : \left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

as well as the following inequalities

$\forall x \in \mathbb{R} : |\sin x| \leq 1$ $\quad$ $\forall x \in \mathbb{R} : |\mathrm{Arctan}(x)| < \pi/2$

$\forall x \in \mathbb{R} : |\cos x| \leq 1$ $\quad$ $\forall x \in [-1,1] : |\mathrm{Arcsin}(x)| \leq \pi/2$

and the identities

$\forall x \in \mathbb{R} : \mathrm{Arctan}(x) + \mathrm{Arccot}(x) = \pi/2$

$\forall x \in [-1,1] : \mathrm{Arcsin}(x) + \mathrm{Arccos}(x) = \pi/2$

- To show any statement of the form $\forall x \in A : p(x)$ we write:

  Let $x \in A$ be given.
  [Prove $p(x)$]
  It follows that $\forall x \in A : p(x)$

EXAMPLES

a) Show that $a_n = \dfrac{\sin(3\pi n) + \cos(\pi n)}{n^2+1}$ is convergent.

Solution

Define $\forall n \in \mathbb{N}^*: b_n = 1/(n^2+1)$

and $\forall n \in \mathbb{N}^*: c_n = \sin(3\pi n) + \cos(\pi n)$

and note that $\forall n \in \mathbb{N}^*: a_n = b_n c_n$.

We note that:

$$\lim_{x \to +\infty} \frac{1}{x^2+1} = \lim_{x \to +\infty} \frac{1}{x^2} = 0 \Rightarrow \lim b_n = \lim \frac{1}{n^2+1} = 0 \quad (1)$$

Let $n \in \mathbb{N}^*$ be given. Then

$$|c_n| = |\sin(3\pi n) + \cos(\pi n)| \leq |\sin(3n\pi)| + |\cos(\pi n)|$$
$$\leq 1 + 1 = 2$$

It follows that

$(\forall n \in \mathbb{N}^*: |c_n| \leq 2) \Rightarrow c_n$ absolutely bounded

$\Rightarrow c_n$ Bounded (2).

From Eq. (1) and Eq. (2):

$$\begin{cases} \lim b_n = 0 \\ c_n \text{ bounded} \end{cases} \Rightarrow \lim (b_n c_n) = 0 \Rightarrow \lim a_n = 0$$

$\Rightarrow a_n$ convergent

b) Show that $a_n = (-1)^n \operatorname{Arccos}((-1)^n/3)\sin(\pi/n)$ is convergent

Solution

We define:

$\forall n \in \mathbb{N}^*: b_n = (-1)^n \operatorname{Arccos}((-1)^n/3)$

$\forall n \in \mathbb{N}^*: c_n = \sin(\pi/n)$

Let $n \in \mathbb{N}^*$ be given. Then:

$$|b_n| = |(-1)^n \operatorname{Arccos}((-1)^n/3)| = |(-1)^n||\operatorname{Arccos}((-1)^n/3)|$$
$$= |\operatorname{Arccos}((-1)^n/3)| = |\pi/2 - \operatorname{Arcsin}((-1)^n/3)|$$
$$\leq |\pi/2| + |\operatorname{Arcsin}((-1)^n/3)| \leq \pi/2 + \pi/2 = \pi$$

and therefore:

$$(\forall n \in \mathbb{N}^*: |b_n| \leq \pi) \Rightarrow b_n \text{ absolutely bounded}$$
$$\Rightarrow b_n \text{ bounded} \quad (1)$$

We also note that:

$$\lim_{x \to +\infty} \frac{\pi}{x} = 0 \Rightarrow \lim_{x \to +\infty} \sin\left(\frac{\pi}{x}\right) = \sin(0) = 0 \Rightarrow$$
$$\Rightarrow \lim c_n = \lim \sin(\pi/n) = 0 \quad (2)$$

From Eq.(2) and Eq.(3):

$$\begin{cases} \lim c_n = 0 \\ b_n \text{ bounded} \end{cases} \Rightarrow \lim(b_n c_n) = 0 \Rightarrow \lim a_n = 0$$
$$\Rightarrow a_n \text{ convergent.}$$

↳ To show that a sequence is bounded
1) We show it is absolutely bounded ; OR
2) We show it is convergent.

c) Show that the sequence

$$a_n = \frac{n\cos(n\pi/4) - n^2\sin(n\pi)}{n^3+1}, \quad \forall n \in \mathbb{N}^*$$

is bounded.

Solution

Let $n \in \mathbb{N}^*$ be given. Then:

$$|a_n| = \left| \frac{n\cos(n\pi/4) - n^2\sin(n\pi)}{n^3+1} \right| = \frac{|n\cos(n\pi/4) - n^2\sin(n\pi)|}{|n^3+1|}$$

$$\leq \frac{|n\cos(n\pi/4)| + |n^2\sin(n\pi)|}{n^3+1} = \frac{n|\cos(n\pi/4)| + n^2|\sin(n\pi)|}{n^3+1}$$

$$\leq \frac{n+n^2}{n^3+1} \leq \frac{n^2+n^2}{n^3+1} = \frac{2n^2}{n^3+1} \leq \frac{2n^2}{n^3} =$$

$$= \frac{2}{n} \leq 2$$

and it follows that

$$(\forall n \in \mathbb{N}^* : |a_n| \leq 2) \Rightarrow a_n \text{ absolutely bounded}$$
$$\Rightarrow a_n \text{ bounded.} \quad \square$$

↳ Note that to find an upper bound for a fraction with positive numerator and denominator, we can increase the fraction by increasing the numerator or by decreasing the denominator, using $n \geq 1$.

d) Show that the sequence

$$\forall n \in \mathbb{N}^*: \quad a_n = \frac{n^2 - n + 1}{n^2 + n + 1}$$

is bounded.

Solution

1st method

Since

$$\lim_{x \to +\infty} \frac{x^2 - x + 1}{x^2 + x + 1} = \lim_{x \to +\infty} \frac{x^2}{x^2} = 1 \Rightarrow$$

$\Rightarrow \lim a_n = 1 \Rightarrow a_n$ convergence $\Rightarrow a_n$ bounded.

2nd method

Let $n \in \mathbb{N}^*$ be given. Then:

$$|a_n| = \left| \frac{n^2 - n + 1}{n^2 + n + 1} \right| = \frac{|n^2 - n + 1|}{|n^2 + n + 1|} = \frac{|n^2 - n + 1|}{n^2 + n + 1} \leqslant$$

$$\leqslant \frac{|n^2| + |n| + 1}{n^2 + n + 1} = \frac{n^2 + n + 1}{n^2 + n + 1} = 1$$

and therefore:

$(\forall n \in \mathbb{N}^*: |a_n| \leqslant 1) \Rightarrow a_n$ absolutely bounded

$\Rightarrow a_n$ bounded. $\square$

e) Show that the sequence

$$\forall n \in \mathbb{N}^*: a_n = \frac{n^2+1}{5n+1+n\cos(\pi n/3)}$$

is <u>not</u> bounded.

<u>Solution</u>

Let $n \in \mathbb{N}^*$ be given. Then:

$$a_n = \frac{n^2+1}{5n+1+n\cos(\pi n/3)} \geq \frac{n^2+1}{5n+n+n\cos(n\pi/3)} =$$

$$= \frac{n^2+1}{6n+n\cos(n\pi/3)} \geq \frac{n^2+1}{6n+n} = \frac{n^2+1}{7n} \geq$$

$$\geq \frac{n^2}{7n} = \frac{n}{7}$$

and therefore: $\forall n \in \mathbb{N}^*: a_n \geq n/7$ (1)

We also have: $\lim_{x \to +\infty} \frac{x}{7} = +\infty$ (2)

From Eq.(1) and Eq.(2) it follows that a_n not bounded.

EXERCISES

⑤ Show that the following sequences are bounded:

a) $a_n = (-7)^n$

b) $a_n = \dfrac{n+1}{n^2+8}$

c) $a_n = \dfrac{n\cos(\pi n/3) + \sin(\pi n/4)}{n^2+3n+1}$

d) $a_n = \dfrac{1}{n}\sin\left(\dfrac{\pi n}{2}\right)$

e) $a_n = \dfrac{5\sin(3\pi n)}{4n}$

f) $a_n = \dfrac{n+5}{n+2}$

g) $a_n = \sqrt{n^2+n} - n$

h) $a_n = \dfrac{\sqrt{n+1} + \sqrt{n+2}}{\sqrt{n+3}}$

⑥ Show that the following sequences are NOT bounded:

a) $a_n = \dfrac{4n^2+1}{3n}$

b) $a_n = \dfrac{n^2+2}{n+2}$

c) $a_n = -4n^2+3n+1$

d) $a_n = 2n^3-n+1$

e) $a_n = \dfrac{n^2}{2n+\cos^2(\pi n)}$

f) $a_n = \dfrac{2n^2+5}{3n+n\sin(\pi n)}$

⑦ Use the zero-bounded theorem to evaluate the limits of the following sequences.

a) $a_n = (-1)^n \operatorname{Arccot}\left(1+\sin(\pi^n n)\right) \ln\left(\dfrac{n+1}{n+3}\right)$

b) $a_n = \dfrac{\sin(n\pi/3) + \cos(n\pi/4)}{n^3+3n+1}$

c) $a_n = \cos(\pi n)\left[\sqrt{n+1} - \sqrt{n}\right]$

d) $a_n = \cos(\pi n/4)\left[(-1)^n + \sin(n\pi/3)\right]\left[\sqrt{n^2+3} - \sqrt{n^2+1}\right]$

e) $a_n = (-1)^n \cos\left(\frac{\pi n}{4}\right) \sin\left(\frac{1}{n}\right)$

f) $a_n = \left[\sin\left(\frac{n\pi}{3}\right) + \cos\left(\frac{n\pi}{4}\right)\right] \ln\left(\frac{n^2+3n+1}{n^2+2n+4}\right)$

Recursive sequences and monotonicity.

- Let a_n be a sequence. Then we introduce the following definitions:

$$\boxed{\begin{array}{l} a_n \text{ increasing} \Leftrightarrow \forall n \in \mathbb{N}^*: a_{n+1} \geq a_n \\ a_n \text{ decreasing} \Leftrightarrow \forall n \in \mathbb{N}^*: a_{n+1} \leq a_n \end{array}}$$

- From the least upper bound axiom we can prove the following statements

$$\boxed{\left.\begin{array}{l} a_n \text{ increasing} \\ a_n \text{ upper bounded} \end{array}\right\} \Rightarrow a_n \text{ convergent}}$$

$$\boxed{\left.\begin{array}{l} a_n \text{ decreasing} \\ a_n \text{ lower bounded} \end{array}\right\} \Rightarrow a_n \text{ convergent}}$$

↳ Methodology

These results can be used to evaluate the limit of a recursive sequence of the form

$$\begin{cases} a_0 = c \\ a_{n+1} = f(a_n) \end{cases}$$

- 1 We use the method of induction to show that a_n is increasing or decreasing.
- 2 Likewise we show that a_n is upper or lower bounded.
- 3 We conclude that a_n is convergent and define $x = \lim a_n$.
- 4 We show that $x = f(x)$ and solve for x.

  a) If there is a unique solution then that solution is the limit

b) If there are multiple solutions then we use bounds to eliminate all but one solution.

EXAMPLE

Evaluate the limit $\lim a_n$ of the sequence a_n given by

$$\begin{cases} a_1 = 3 \\ a_{n+1} = \sqrt{a_n + 7} \end{cases}$$

Solution

• Monotonicity

We note that $a_2 = \sqrt{a_1 + 7} = \sqrt{3+7} = \sqrt{10} > \sqrt{9} = 3 = a_1$

$\Rightarrow a_2 > a_1$. We will thus show that a_n increasing.

Assume that for $n = k$: $a_{k+1} > a_k > 0$ (1)

We will show that for $n = k+1$: $a_{k+2} > a_{k+1} > 0$

From Eq. (1):

$$a_{k+1} > a_k > 0 \Rightarrow a_{k+1} + 7 > a_k + 7 > 7$$
$$\Rightarrow \sqrt{a_{k+1} + 7} > \sqrt{a_k + 7} > \sqrt{7}$$
$$\Rightarrow a_{k+2} > a_{k+1} > 0.$$

By induction, it follows that

$(\forall n \in \mathbb{N}^*: a_{n+1} > a_n) \Rightarrow a_n$ increasing (2)

• Bounds

We will show that $\forall n \in \mathbb{N}^*: a_n < 4$.

For $n=1$: $a_1 = 3 < 4$.

Assume that for $n=k$: $\underline{0 < a_k < 4}$. (3)

For $n=k+1$ we will show that: $0 < a_{k+1} < 4$.

From Eq. (3):

$$0 < a_k < 4 \Rightarrow 7 < a_k + 7 < 4+7 \Rightarrow \sqrt{7} < \sqrt{a_k+7} < \sqrt{11}$$

$$\Rightarrow \sqrt{7} < a_{k+1} < \sqrt{11} \Rightarrow 0 < a_{k+1} < 4$$

By induction it follows that

$(\forall n \in \mathbb{N}^*: a_n < 4) \Rightarrow \underline{a_n \text{ upper Bounded}}$ (4)

From Eq. (2) and Eq. (4):

$$\begin{cases} a_n \text{ increasing} \\ a_n \text{ upper Bounded} \end{cases} \Rightarrow a_n \text{ convergent} \Rightarrow \exists x \in \mathbb{R}: \lim a_n = x.$$

Since:

$$x = \lim a_{n+1} = \lim \sqrt{a_n + 7} = \sqrt{\lim a_n + 7} = \sqrt{x+7}$$

we require that $\underline{x \geq 0}$ and note that

$$x = \sqrt{x+7} \Leftrightarrow x^2 = x + 7 \Leftrightarrow x^2 - x - 7 = 0$$

$$\Delta = (-1)^2 - 4 \cdot 1 \cdot (-7) = 1 + 28 = 29 \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{-(-1) \pm \sqrt{29}}{2 \cdot 1} = \frac{1 \pm \sqrt{29}}{2} = \begin{cases} \dfrac{1-\sqrt{29}}{2} \leftarrow \text{rejected} \\ \dfrac{1+\sqrt{29}}{2} \leftarrow \text{accepted} \end{cases}$$

It follows that

$$\lim a_n = \frac{1+\sqrt{29}}{2}$$

↳ The negative solution is rejected because it violates the requirement $x \geq 0$.

EXERCISES

(8) Show that the following sequences converge and find their limit.

a) $(a_n): \begin{cases} a_1 = 1 \\ a_{n+1} = \sqrt{1+a_n} \end{cases}$

b) $(a_n): \begin{cases} a_1 = 3 \\ a_{n+1} = \dfrac{3a_n - 4}{5} \end{cases}$

c) $(a_n): \begin{cases} a_1 = 2 \\ a_{n+1} = \dfrac{2a_n - 3}{4} \end{cases}$

d) $(a_n): \begin{cases} a_1 = 1/4 \\ a_{n+1} = \dfrac{1}{2} a_n^2 + \dfrac{1}{8} \end{cases}$

e) $(a_n): \begin{cases} a_1 = 1 \\ a_{n+1} = (1/3) a_n + 2 \end{cases}$

f) $(a_n): \begin{cases} a_1 = 0 \\ a_{n+1} = \dfrac{3a_n + 1}{4} \end{cases}$

g) $(a_n): \begin{cases} a_1 = 3 \\ a_{n+1} = (1/5)(a_n^2 + 4) \end{cases}$

h) $(a_n): \begin{cases} a_1 = 2 \\ a_{n+1} = \sqrt{1+2a_n} - 1 \end{cases}$

i) $(a_n): \begin{cases} a_1 = 2 \\ a_{n+1} = \sqrt{a_n + 6} \end{cases}$

j) $(a_n): \begin{cases} a_1 = 3 \\ a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{2}{a_n}\right) \end{cases}$

k) $(a_n): \begin{cases} a_1 = 1 \\ a_{n+1} = \sqrt{3a_n + 4} \end{cases}$

V Convergence and order

We consider two sequences a_n, b_n. We can show from the sequence limit definition that

$$\boxed{\begin{array}{ll} (1) & \left.\begin{array}{l}\forall n \in \mathbb{N}^* : a_n \geq 0 \\ a_n \text{ convergent}\end{array}\right\} \Rightarrow \lim a_n \geq 0 \\ (2) & \lim a_n > 0 \Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : a_n > 0 \\ (3) & \left.\begin{array}{l}\forall n \in \mathbb{N}^* : |a_n| \leq |b_n| \\ \lim b_n = 0\end{array}\right\} \Rightarrow \lim a_n = 0 \end{array}}$$

Note that while statement (2) holds for strict inequalities, statement (1) cannot be generalized into using only strict inequalities. For example, the sequence

$$\forall n \in \mathbb{N}^* : \quad a_n = 1/n$$

satisfies the statement

$$\left\{\begin{array}{l}\forall n \in \mathbb{N}^* : a_n > 0 \\ a_n \text{ convergent}\end{array}\right. \qquad (1)$$

however, it is easy to show that $\lim a_n = 0$. Therefore, given Eq. (1), the best we can do in general is to conclude that $\lim a_n \geq 0$.

Squeeze theorem

$$\boxed{\left.\begin{array}{l} b_n, c_n \text{ convergent} \\ \forall n \in \mathbb{N}^* : b_n \leq a_n \leq c_n \\ \lim b_n = \lim c_n = l \end{array}\right\} \Rightarrow \lim a_n = l}$$

Proof

Let $n \in \mathbb{N}^*$ be given. Then:

$b_n \leq a_n \leq c_n \Rightarrow 0 \leq a_n - b_n \leq c_n - b_n \Rightarrow |a_n - b_n| \leq |c_n - b_n|$

and it follows that

$$\forall n \in \mathbb{N}^*: |a_n - b_n| \leq |c_n - b_n| \qquad (1)$$

We also note that

$$\lim (c_n - b_n) = \lim c_n - \lim b_n = \ell - \ell = 0 \qquad (2)$$

From Eq. (1) and Eq. (2), it follows that $\lim (a_n - b_n) = 0$ and therefore:

$$\lim a_n = \lim (a_n - b_n + b_n) = \lim (a_n - b_n) + \lim b_n$$
$$= 0 + \ell = \ell. \quad \square$$

→ Application to limits of sequences $a_n = \sqrt[n]{f(n)}$

Recall that we have already shown that:

$$\boxed{\begin{aligned} &\forall a \in (0, +\infty): \lim \sqrt[n]{a} = 1 \\ &\lim \sqrt[n]{n} = 1 \end{aligned}}$$

Using the squeeze theorem, we show that

$$\boxed{\begin{cases} \forall n \in \mathbb{N}^*: a_n > 0 \\ \lim a_n = \ell > 0 \end{cases} \Rightarrow \lim \sqrt[n]{a_n} = 1}$$

Proof

$\lim a_n = l > 0 \Rightarrow \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - l| < \varepsilon$

$\Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - l| < l/2$

For the existing $n_0 \in \mathbb{N}^*$, let $n \in \mathbb{N}^* - [n_0]$ be given. Then

$|a_n - l| < l/2 \Rightarrow -l/2 < a_n - l < l/2 \Rightarrow l/2 < a_n < 3l/2$

$\Rightarrow \sqrt[n]{l/2} < \sqrt[n]{a_n} < \sqrt[n]{3l/2}$

and therefore $\forall n \in \mathbb{N}^* - [n_0] : \sqrt[n]{l/2} < \sqrt[n]{a_n} < \sqrt[n]{3l/2}$ (1)

We also note that

$l/2 > 0 \Rightarrow \lim \sqrt[n]{l/2} = 1$ (2)

$3l/2 > 0 \Rightarrow \lim \sqrt[n]{3l/2} = 1$ (3)

From Eq. (1), Eq. (2), Eq. (3) via the squeeze theorem we conclude that

$\lim \sqrt[n]{a_n} = 1$ □

↳ D'Alembert theorem

$$\boxed{\left.\begin{array}{l} \forall n \in \mathbb{N}^* : a_n > 0 \\ \lim \dfrac{a_{n+1}}{a_n} = l < 1 \end{array}\right\} \Rightarrow \lim a_n = 0}$$

Proof

We note that

$$\lim \left(\frac{a_{n+1} - a_n}{a_n} \right) = \lim \left(\frac{a_{n+1}}{a_n} - 1 \right) = \left(\lim \frac{a_{n+1}}{a_n} \right) - 1$$

$$= l - 1 < 1 - 1 = 0 \Rightarrow \lim \left(\frac{a_{n+1} - a_n}{a_n} \right) < 0$$

$\Rightarrow \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: \dfrac{a_{n+1} - a_n}{a_n} < 0$

$\Rightarrow \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: a_{n+1} - a_n < 0$
$\Rightarrow \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: a_{n+1} < a_n$
For the existing $n_0 \in \mathbb{N}^*$, it follows that
a_{n+n_0} decreasing (1)
We also note that:
$\begin{cases} a_{n+n_0} \text{ decreasing} \\ \forall n \in \mathbb{N}^*: a_n > 0 \end{cases} \Rightarrow \begin{cases} a_{n+n_0} \text{ decreasing} \\ a_n \text{ lower bounded} \end{cases} \Rightarrow$

$\Rightarrow a_{n+n_0}$ convergent
$\Rightarrow a_n$ convergent
$\Rightarrow \exists A \in \mathbb{R}: \lim a_n = A.$
Since $(\forall n \in \mathbb{N}^*: a_n > 0) \Rightarrow \lim a_n \geq 0 \Rightarrow A \geq 0$, to show that $A = 0$, we will assume that $A > 0$. Then:
$$\lim \frac{a_{n+1}}{a_n} = \frac{\lim a_{n+1}}{\lim a_n} = \frac{A}{A} = 1$$
which is a contradiction, since we have assumed that
$$\lim \frac{a_{n+1}}{a_n} < 1$$
It follows that: $A = 0 \Rightarrow \lim a_n = 0.$ $\square$

EXAMPLES

a) Evaluate the limit of $a_n = \sqrt[n]{\frac{3n^2+2}{n^2+1}}$

Solution

We note that

$$\lim_{x\to+\infty} \frac{3x^2+2}{x^2+1} = \lim_{x\to+\infty} \frac{3x^2}{x^2} = 3 \Rightarrow \lim_{n\in\mathbb{N}^*} \frac{3n^2+2}{n^2+1} = 3 \quad (1)$$

and

$$\forall n\in\mathbb{N}^*: \frac{3n^2+2}{n^2+1} > 0 \qquad (2)$$

From Eq.(1) and Eq.(2): $\lim_{n\in\mathbb{N}^*} a_n = \lim_{n\in\mathbb{N}^*} \sqrt[n]{\frac{3n^2+2}{n^2+1}} = 1$

b) Evaluate the limit of $a_n = \sqrt[n]{n^2+3n+1}$

Solution

We note that

$$a_n = \sqrt[n]{n^2+3n+1} = \sqrt[n]{n^2}\,\sqrt[n]{1+3n^{-1}+n^{-2}} =$$
$$= (\sqrt[n]{n})^2\,\sqrt[n]{1+3n^{-1}+n^{-2}}$$

Since:

$$\left.\begin{aligned} &\lim_{n\in\mathbb{N}^*} (1+3n^{-1}+n^{-2}) = 1 + \lim_{n\in\mathbb{N}^*} (3/n) + \lim_{n\in\mathbb{N}^*} (3/n^2) = \\ &\qquad = 1+0+0 = 1 \\ &\forall n\in\mathbb{N}^*: 1+3n^{-1}+n^{-2} > 0 \end{aligned}\right\} \Rightarrow \lim_{n\in\mathbb{N}^*} \sqrt[n]{1+3n^{-1}+n^{-2}} = 1$$

it follows that:

$$\lim_{n\in\mathbb{N}^*} a_n = \lim_{n\in\mathbb{N}^*} \left[(\sqrt[n]{n})^2 \sqrt[n]{1+3n^{-1}+n^{-2}} \right] =$$

$$= \left[\lim_{n\in\mathbb{N}^*} \sqrt[n]{n}\right]^2 \lim_{n\in\mathbb{N}^*} \left[\sqrt[n]{1+3n^{-1}+n^{-2}}\right]$$

$$= 1^2 1 = 1$$

c) Evaluate the limit of $a_n = \sqrt[n]{3^n+7^n}$

Solution

We note that

$$a_n = \sqrt[n]{3^n+7^n} = \sqrt[n]{7^n[(3/7)^n+1]} = \sqrt[n]{7^n}\sqrt[n]{1+(3/7)^n}$$
$$= 7\sqrt[n]{1+(3/7)^n}, \quad \forall n\in\mathbb{N}^*$$

and since

$$\lim_{n\in\mathbb{N}^*} [1+(3/7)^n] = 1+\lim_{n\in\mathbb{N}^*} (3/7)^n = 1+0 = 1 \qquad (1)$$

$$\forall n\in\mathbb{N}^*: \ 1+(3/7)^n > 0 \qquad (2)$$

it follows from Eq.(1) and Eq.(2) that

$$\lim_{n\in\mathbb{N}^*} a_n = 7\lim_{n\in\mathbb{N}^*} \sqrt[n]{1+(3/7)^n} = 7\cdot 1 = 7.$$

d) Evaluate the limit of $a_n = \dfrac{2^n}{n!}$

Solution

We note that $\forall n\in\mathbb{N}^*: a_n > 0 \qquad (1)$

and

$$\forall n\in\mathbb{N}^*: \frac{a_{n+1}}{a_n} = \frac{\left[\dfrac{2^{n+1}}{(n+1)!}\right]}{\dfrac{2^n}{n!}} = \frac{2^{n+1}n!}{2^n(n+1)!} = \frac{2n!}{(n+1)n!}$$

$$= \frac{2}{n+1} \qquad (2)$$

From Eq. (2), since

$$\lim_{x\to+\infty} \frac{2}{x+1} = \lim_{x\to+\infty} \frac{2}{x} = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \frac{a_{n+1}}{a_n} = \lim_{n\in\mathbb{N}^*} \frac{2}{n+1} = 0 < 1 \qquad (3)$$

From Eq. (1) and Eq. (3): $\lim a_n = 0$

e) Evaluate the limit of $a_n = \dfrac{n!}{n^n}$

Solution

We note that

$\forall n \in \mathbb{N}^*: a_n > 0 \qquad (1)$

and

$$\forall n \in \mathbb{N}^*: \frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{n^n (n+1)!}{n!\,(n+1)^{n+1}} =$$

$$= \frac{n^n (n+1)\, n!}{n!\,(n+1)^n (n+1)} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n =$$

$$= \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \frac{a_{n+1}}{a_n} = \lim_{n\in\mathbb{N}^*} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{\lim_{n\in\mathbb{N}^*} (1+1/n)^n} = \frac{1}{e} < 1 \qquad (2)$$

From Eq. (1) and Eq. (2): $\lim_{n\in\mathbb{N}^*} a_n = 0$

EXERCISES

(9) Evaluate the limits of the following sequences:

a) $a_n = \sqrt[n]{n^2+1}$

b) $a_n = \sqrt[n]{5n^2+2n}$

c) $a_n = \sqrt[n]{2n^3-n+5}$

d) $a_n = \sqrt[n]{2^n+3^n+5^n}$

e) $a_n = \sqrt[n]{2^n+4^n+9^n}$

f) $a_n = \sqrt[n]{3+1/n}$

g) $a_n = \sqrt[n]{\dfrac{3n^2+n+1}{6n^2+5n-2}}$

h) $a_n = \sqrt[n]{\dfrac{7n+1}{3n+2}}$

(10) Similarly, evaluate the limits of the following sequences.

a) $a_n = \dfrac{n}{3^n}$

b) $a_n = \dfrac{n}{2^n}$

c) $a_n = \dfrac{n!}{n^n}$

d) $a_n = \dfrac{n^3}{6^n}$

e) $a_n = \dfrac{3^n}{n!}$

f) $a_n = \dfrac{n^{n+1}+2^n n!}{(2n)^n}$

g) $a_n = \dfrac{5^n+n^3}{6^n+n^2}$

h) $a_n = \dfrac{4^n n!}{(9n)^n}$