SERIES APPROXIMATION OF FUNCTIONS

▼ Power series

• A power series is a series of the form

$$f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n \quad , \forall x \in A$$

with $\forall n \in \mathbb{N}^*: a_n \in \mathbb{R}$ and $x_0 \in \mathbb{R}$. The default domain A is the set of all $x \in \mathbb{R}$ for which the power series converges.

↳ Methodology: To establish convergence of a power series we use the ratio test or root test in conjunction with the absolute convergence test. To establish divergence, we use the absolute ratio test or the absolute root test.

EXAMPLES

a) Find the domain of $f(x) = \sum_{n=1}^{+\infty} \frac{(x-2)^n}{n!}$

Solution: Let $x \in \mathbb{R}$ be given.

Define $\forall n \in \mathbb{N}^*: a_n = \frac{(x-2)^n}{n!}$. Then:

$$\forall n \in \mathbb{N}^*: \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{(x-2)^{n+1}}{(n+1)!}}{\frac{(x-2)^n}{n!}} \right| = \left| \frac{n!\,(x-2)^{n+1}}{(n+1)!\,(x-2)^n} \right| =$$

$$= \left| \frac{n!(x-2)(x-2)^n}{(n+1)n!(x-2)^n} \right| = \left| \frac{x-2}{n+1} \right| = \frac{|x-2|}{n+1}$$

Since:

$$\lim_{t\to+\infty} \frac{1}{t+1} = \lim_{t\to+\infty} \frac{1}{t} = 0 \Rightarrow \lim_{n\in\mathbb{N}^*} \frac{1}{n+1} = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \frac{|x-2|}{n+1} = 0 \Rightarrow \lim \left| \frac{a_{n+1}}{a_n} \right| = 0 \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} \left| \frac{(x-2)^n}{n!} \right| \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{(x-2)^n}{n!} \text{ converges.}$$

It follows that $\forall x\in\mathbb{R} : \left(\sum_{n=1}^{+\infty} \frac{(x-2)^n}{n!} \text{ converges} \right)$ and therefore $A=\mathbb{R}$.

b) Find the domain of $f(x) = \sum_{n=1}^{+\infty} n^2 3^n (x-2)^n$

Solution

Let $x\in\mathbb{R}$ be given, and define $\forall n\in\mathbb{N}^* : a_n = n^2 3^n (x-2)^n$.

Then,

$$\forall n\in\mathbb{N}^* : |a_n|^{1/n} = |n^2 3^n (x-2)^n|^{1/n} = 3\sqrt[n]{n^2}\,|x-2|$$

$$= 3(\sqrt[n]{n})^2 |x-2|$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} |a_n|^{1/n} = \lim_{n\in\mathbb{N}^*} [3(\sqrt[n]{n})^2 |x-2|] = 3|x-2| \lim_{n\in\mathbb{N}^*} (\sqrt[n]{n})^2$$

$$= 3|x-2| (\lim \sqrt[n]{n})^2 = 3|x-2|\cdot 1^2 =$$

$$= 3|x-2|.$$

Since,

$$3|x-2| < 1 \Leftrightarrow |x-2| < 1/3 \Leftrightarrow -1/3 < x-2 < 1/3 \Leftrightarrow$$
$$\Leftrightarrow -1/3+2 < x < 1/3+2 \Leftrightarrow 5/3 < x < 7/3$$

we distinguish between the following cases:

Case 1 : Assume that $x \in (5/3, 7/3)$. Then:

$$5/3 < x < 7/3 \Rightarrow 3|x-2| < 1 \Rightarrow \lim_{n\in\mathbb{N}^*} |a_n|^{1/n} < 1 \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} |n^2 3^n (x-2)^n| \quad \text{converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} n^2 3^n (x-2)^n \quad \text{converges.}$$

Case 2 : Assume that $x \in (-\infty, 5/3) \cup (7/3, +\infty)$.

Then: $3|x-2| > 1 \Rightarrow \lim |a_n|^{1/n} > 1 \Rightarrow$

$$\Rightarrow \sum_{n=1}^{+\infty} n^2 3^n (x-2)^n \quad \text{diverges.}$$

Case 3 : Assume that $x = 7/3$. Then

$$\forall n \in \mathbb{N}^* : a_n = n^2 3^n (x-2)^n = n^2 3^n (7/3 - 6/3)^n = n^2 3^n (1/3)^n$$
$$= n^2.$$

Since $\lim_{x\to+\infty} x^2 = +\infty \Rightarrow a_n$ diverges $\Rightarrow \lim a_n \neq 0 \Rightarrow$

$$\Rightarrow \sum_{n=1}^{+\infty} n^2 3^n (x-2)^n \quad \text{diverges.}$$

Case 4 : Assume that $x = 5/3$

Then :

$$\forall n \in \mathbb{N}^* : a_n = n^2 3^n (x-2)^n = n^2 3^n (5/3-2)^n = n^2 3^n (-1/3)^n$$
$$= n^2 (-1)^n.$$

$$\Rightarrow \forall n \in \mathbb{N}^* : (a_{2n} = (2n)^2 \wedge a_{2n+1} = -(2n+1)^2)$$

Since

$\lim_{t\to+\infty} (4t^2) = +\infty \Rightarrow a_{2n}$ diverges $\Rightarrow a_n$ diverges $\Rightarrow$

$\Rightarrow \lim a_n \neq 0 \Rightarrow \sum_{n=1}^{+\infty} n^2 3^n (x-2)^n$ diverges

It follows that $A = (5/3, 7/3)$.

I Uniform convergence of power series

Motivation

Let $f_n : A \to \mathbb{R}$ be a sequence of functions with $n \in \mathbb{N}^*$ such that ($\forall x \in A : f_n(x)$ converges). We may thus define a new function $f : A \to \mathbb{R}$ as:

$$\forall x \in A : f(x) = \lim_{n \in \mathbb{N}^*} f_n(x)$$

The questions that we wish to address are the following:

a) If ($\forall n \in \mathbb{N}^* : f_n$ continuous on A) can we conclude that (f continuous on A)?

b) If ($\forall n \in \mathbb{N}^* : f_n$ integrable on A) and $I \subseteq A$ is an interval, does it follow that

$$\lim_{n \in \mathbb{N}^*} \int_I f_n(x)dx = \int_I \left[\lim_{n \in \mathbb{N}^*} f_n(x)\right] dx \quad ??$$

c) If ($\forall n \in \mathbb{N}^* : f_n$ differentiable on A), does it follow that

$$\frac{d}{dx} \lim_{n \in \mathbb{N}^*} f_n(x) = \lim_{n \in \mathbb{N}^*} (d/dx) f_n(x) \quad ??$$

The answer to the above questions is NO, in general, as we will illustrate with counterexamples. This will lead us to define the concept of a uniform convergent series. The statements (a) and (b) will be shown to be true if the sequence of functions f_n converges uniformly. (c) holds for power series, but uniform convergence is not sufficient.

COUNTEREXAMPLES

a) Consider the function sequence

$\forall n \in \mathbb{N}^*: \forall x \in [0,1]: f_n(x) = x^n$.

Obviously: $\forall n \in \mathbb{N}^*$: f_n continuous on $[0,1]$.

However:

$$\forall x \in [0,1]: f(x) = \lim_{n \in \mathbb{N}^*} f_n(x) = \lim_{n \in \mathbb{N}^*} x^n = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \in [0,1) \end{cases}$$

which is obviously not continuous at $x = 1$.

So in general, continuity is not always transmitted from f_n to f.

b) Consider the function sequence

$\forall n \in \mathbb{N}^*: \forall x \in [0,1]: f_n(x) = nx(1-x^2)^n$

We have:

$$\int_0^1 f_n(x)\,dx = \int_0^1 nx(1-x^2)^n\,dx = \frac{-n}{2}\int_0^1 (-2x)(1-x^2)^n\,dx =$$

$$= \frac{-n}{2}\int_0^1 (1-x^2)^n(1-x^2)'\,dx =$$

$$= \frac{-n}{2(n+1)}\int_0^1 (n+1)(1-x^2)^n(1-x^2)'\,dx$$

$$= \frac{-n}{2(n+1)}\int_0^1 \left[(1-x^2)^{n+1}\right]'\,dx =$$

$$= \frac{-n}{2(n+1)}\left[(1-x^2)^{n+1}\right]_0^1 = \frac{-n}{2(n+1)}\left[(1-1^2)^{n+1} - (1-0)^{n+1}\right]$$

$$= \frac{n}{2(n+1)}, \quad \forall n \in \mathbb{N}^*$$

Since:

$$\lim_{t\to+\infty} \frac{t}{2(t+1)} = \lim_{t\to+\infty} \frac{t}{2t} = \frac{1}{2} \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \int_0^1 f_n(x)dx = \lim_{n\in\mathbb{N}^*} \frac{n}{2(n+1)} = \boxed{\frac{1}{2}} \qquad (1)$$

However, let $x \in (0,1)$ be given and note that

$$0<x<1 \Rightarrow 0<x^2<1 \Rightarrow 0>-x^2>-1 \Rightarrow -1<-x^2<0$$

$$\Rightarrow 0<1-x^2<1.$$

and also that

$$\forall n\in\mathbb{N}^*: \frac{f_{n+1}(x)}{f_n(x)} = \frac{(n+1)x(1-x^2)^{n+1}}{nx(1-x^2)^n} = \frac{(n+1)(1-x^2)}{n}$$

and therefore, since:

$$\lim_{t\to+\infty}\frac{t+1}{t} = \lim_{t\to+\infty}\frac{t}{t} = 1 \Rightarrow \lim_{n\in\mathbb{N}^*}\frac{n+1}{n} = 1 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*}\frac{f_{n+1}(x)}{f_n(x)} = \lim_{n\in\mathbb{N}^*}\frac{(n+1)(1-x^2)}{n} = (1-x^2)\lim_{n\in\mathbb{N}^*}\frac{n+1}{n}$$

$$= 1-x^2 < 1 \Rightarrow \qquad (2)$$

We also note that $\forall n\in\mathbb{N}^*: \forall x\in(0,1): f_n(x)>0 \qquad (3)$

From Eq.(2) and Eq.(3) it follows that

$$\left(\forall x\in(0,1): \lim_{n\in\mathbb{N}^*} f_n(x) = 0\right) \Rightarrow \int_0^1 \left[\lim_{n\in\mathbb{N}^*} f_n(x)\right]dx = \boxed{0.} \qquad (4)$$

From Eq. (1) and Eq. (4) we have:

$$\begin{cases} \lim\limits_{n\in\mathbb{N}^*} \int_0^1 f_n(x)dx = \frac{1}{2} \\ \int_0^1 \left[\lim\limits_{n\in\mathbb{N}^*} f_n(x)\right] dx = 0 \end{cases} \Rightarrow \lim\limits_{n\in\mathbb{N}^*} \int_0^1 f_n(x)dx \neq \int_0^1 \left[\lim\limits_{n\in\mathbb{N}^*} f_n(x)\right] dx$$

We see, as a result, that the integral and the limit cannot be interchanged.

Uniform convergence definition

Let $f_n : A \to \mathbb{R}$ be a sequence of functions with $n \in \mathbb{N}^*$ and let $f : A \to \mathbb{R}$ be a function. We distinguish between pointwise convergence and uniform convergence as follows:

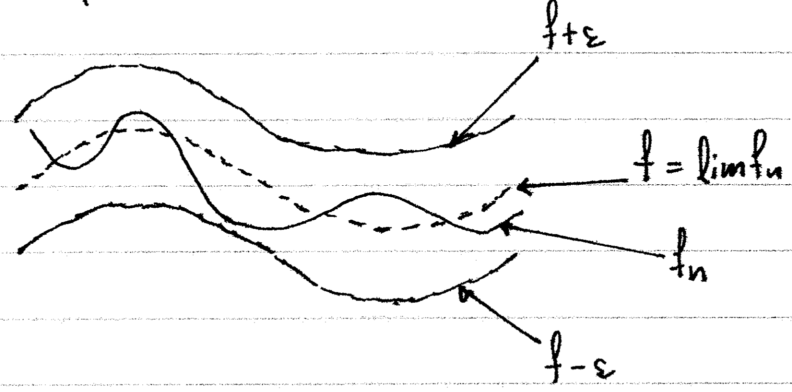
- 1. f_n converges pointwise to f on $A \Leftrightarrow \forall x \in A : \lim_{n \in \mathbb{N}^*} f_n(x) = f(x) \Leftrightarrow$
$\Leftrightarrow \forall x \in A : \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |f_n(x) - f(x)| < \varepsilon$

Note that at the second step we use the definition of the sequence limit.

- 2. f_n converges uniformly to f on $A \Leftrightarrow$
$\Leftrightarrow \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : \forall x \in A : |f_n(x) - f(x)| < \varepsilon$

Note that the only difference between the two definitions is in the location of the $\forall x \in A$ quantifier. Note that similar quantifiers can be swaped (e.g. $\forall x \in A : \forall \varepsilon > 0$ vs. $\forall \varepsilon > 0 : \forall x \in A$) but we cannot swap dissimilar quantifiers (e.g. $\forall x \in A : \exists n_0 \in \mathbb{N}^*$ vs $\exists n_0 \in \mathbb{N}^* : \forall x \in A$) So the two definitions are logically different.

- Interpretation:



For any choice $\varepsilon > 0$, no matter how small, we can find a natural number $n_0 \in \mathbb{N}^*$ such that for all natural numbers $n > n_0$, the entire graph of $f_n(x)$ lies within a distance less than ε from the graph of $f(x)$. In other words, all points of the graph of $f_n(x)$ converge at the same rate towards the graph of $f(x)$. Pointwise convergence only requires each $f_n(x)$ to converge to $f(x)$ eventually, but for some x the convergence can be very fast and for other x excruciatingly slow.

Consequences of Uniform Convergence

(1) Preservation of continuity

Thm: $\boxed{\begin{cases} f_n \text{ converges uniformly to } f \text{ on } A \\ \forall n \in \mathbb{N}^*: f_n \text{ continuous on } x_0 \in A \end{cases} \Rightarrow f \text{ continuous on } x_0}$

Proof

It is sufficient to show that:

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon)$$

Let $\varepsilon > 0$ be given. We need to define an appropriate $\delta > 0$.

Since:

f_n converges uniformly to f on $A \Rightarrow$

$$\Rightarrow \forall \varepsilon_0 > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : \forall p \in A : |f_n(p) - f(p)| < \varepsilon_0$$

We choose $\varepsilon_0 = \varepsilon/3$ and fix $n_0 \in \mathbb{N}^*$ such that

$\forall n \in \mathbb{N}^* - [n_0] : \forall p \in A : |f_n(p) - f(p)| < \varepsilon/3$

We define, for convenience, $N = n_0 + 1$. Then it follows that for $n = N$, we have: $\forall p \in A : |f_N(p) - f(p)| < \varepsilon/3$. (1)

Since f_N continuous on $x_0 \Rightarrow \lim_{x \to x_0} f_N(x) = f_N(x_0) \Rightarrow$

$\Rightarrow \forall \varepsilon_0 > 0 : \exists \delta > 0 : \forall x \in A : (0 < |x - x_0| < \delta \Rightarrow |f_N(x) - f_N(x_0)| < \varepsilon_0)$

Choose $\varepsilon_0 = \varepsilon/3$ and fix $\delta > 0$ such that

$\forall x \in A : (0 < |x - x_0| < \delta \Rightarrow |f_N(x) - f_N(x_0)| < \varepsilon/3)$ (2)

Let $x \in A$ be given and assume that $0 < |x - x_0| < \delta$. Then, it follows, from Eq. (2), that

$|f_N(x) - f_N(x_0)| < \varepsilon/3$ (3)

Furthermore, via Eq. (1),

for $p = x$: $|f_N(x) - f(x)| < \varepsilon/3$ (4)

for $p = x_0$: $|f_N(x_0) - f(x_0)| < \varepsilon/3$ (5)

From Eq. (3), Eq. (4), and Eq. (5):

$$|f(x) - f(x_0)| = |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \leq$$
$$\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)|$$
$$\leq |f_N(x) - f(x)| + \varepsilon/3 + \varepsilon/3 \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

$\Rightarrow |f(x) - f(x_0)| < \varepsilon$.

From the above, it follows that

$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon)$

$\Rightarrow \lim_{x \to x_0} f(x) = f(x_0) \Rightarrow f$ continuous on x_0. $\square$

② Uniform convergence and integration

Thm : We assume that

$$\begin{cases} f_n \text{ converges uniformly to } f \text{ on } [a,b] \\ \forall n\in\mathbb{N}^*: f_n \text{ continuous on } [a,b] \\ \forall n\in\mathbb{N}^*: \forall x\in[a,b]: g_n(x) = \int_a^x f_n(t)\,dt \\ \forall x\in A: g(x) = \int_a^x f(t)\,dt \end{cases}$$

Then: g_n converges uniformly to g on $[a,b]$.

↳ This theorem implies that under the above assumptions, we can interchange the sequence limit and integral as follows:

$$\boxed{\forall x\in[a,b]: \lim_{n\in\mathbb{N}^*} \int_a^x f_n(t)\,dt = \int_a^x \left[\lim_{n\in\mathbb{N}^*} f_n(t)\right] dt}$$

Proof

It is sufficient to show that

$\forall \varepsilon>0: \exists n_0\in\mathbb{N}^*: \forall n\in\mathbb{N}^*-[n_0]: \forall p\in[a,b]: |g_n(p)-g(p)| < \varepsilon.$

Let $\varepsilon>0$ be given. Since

f_n converges uniformly to f on $[a,b] \Rightarrow$

$\Rightarrow \forall \varepsilon_0>0: \exists n_0\in\mathbb{N}^*: \forall n\in\mathbb{N}^*-[n_0]: \forall t\in[a,b]: |f_n(t)-f(t)| < \varepsilon_0$

Choose $\varepsilon_0 = \varepsilon/(b-a)$ and fix $n_0\in\mathbb{N}^*$ such that

$$\forall n\in\mathbb{N}^*-[n_0]: \forall t\in[a,b]: |f_n(t)-f(t)| < \frac{\varepsilon}{b-a}$$

Let $n\in\mathbb{N}^*-[n_0]$ and $p\in[a,b]$ be given. Then:

$$|g_n(p) - g(p)| = \left| \int_a^p f_n(t)\,dt - \int_a^p f(t)\,dt \right| = \left| \int_a^p [f_n(t) - f(t)]\,dt \right|$$

$$\leq \int_a^p |f_n(t) - f(t)|\,dt \leq \int_a^b |f_n(t) - f(t)|\,dt <$$

$$< \int_a^b \frac{\varepsilon}{b-a}\,dt = \frac{\varepsilon}{b-a} \int_a^b dt = \frac{\varepsilon}{b-a} [t]_a^b$$

$$= \frac{\varepsilon}{b-a} (b-a) = \varepsilon \Rightarrow \underline{|g_n(p) - g(p)| < \varepsilon}$$

From the above argument it follows that

$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : \forall p \in [a,b] : |g_n(p) - g(p)| < \varepsilon$

$\Rightarrow$ g_n converges uniformly to g on $[a,b]$.

Sufficient condition for uniform convergence

Thm: Let $f_n : A \to \mathbb{R}$ be a sequence of functions. Then

$$\begin{cases} \forall n \in \mathbb{N}^* : \forall x \in S : 0 \leq |f_n(x)| \leq M_n \\ \sum_{n=1}^{+\infty} M_n \text{ converges} \end{cases} \Rightarrow \sum_{n=1}^{+\infty} f_n(x) \text{ converges uniformly on } S$$

Proof

First, we note that from the comparison test:

$$\begin{cases} \forall n \in \mathbb{N}^* : \forall x \in S : 0 \leq |f_n(x)| \leq M_n \\ \sum_{n=1}^{+\infty} M_n \text{ converges} \end{cases} \Rightarrow \sum_{n=1}^{+\infty} |f_n(x)| \text{ converges} \quad (1)$$

Sufficient to show that

$$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : \forall x \in S : \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{+\infty} f_k(x) \right| < \varepsilon$$

Let $\varepsilon > 0$ be given. Since

$$\sum_{n=1}^{+\infty} M_n \text{ converges} \Rightarrow$$

$$\Rightarrow \forall \varepsilon_0 > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : \left| \sum_{k=1}^{n} M_k - \sum_{k=1}^{+\infty} M_k \right| < \varepsilon_0$$

Choose $\varepsilon_0 = \varepsilon$ and fix $n_0 \in \mathbb{N}^*$ such that

$$\forall n \in \mathbb{N}^* - [n_0] : \left| \sum_{k=n+1}^{+\infty} M_k \right| < \varepsilon$$

Let $n \in \mathbb{N}^* - [n_0]$ and let $x \in S$ be given. Then:

$$\left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{+\infty} f_k(x) \right| = \left| \sum_{k=n+1}^{+\infty} f_k(x) \right| \leq \sum_{k=n+1}^{+\infty} |f_k(x)| \leq$$

$$\leq \sum_{k=n+1}^{+\infty} |f_k(x)| \leq \sum_{k=n+1}^{+\infty} M_k \leq \left| \sum_{k=n+1}^{+\infty} M_k \right| < \varepsilon \Rightarrow$$

$$\Rightarrow \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{+\infty} f_k(x) \right| < \varepsilon$$

From the above argument it follows that

$$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : \forall x \in S : \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{+\infty} f_k(x) \right| < \varepsilon$$

$\Rightarrow \sum_{k=1}^{+\infty} f_k(x)$ converges uniformly on S. $\square$

$\hookrightarrow$ This result is known as the Weierstrass M-test.

Application to series

Given a series $\sum_{n=1}^{+\infty} f_n(x)$ that converges uniformly on $[a,b]$ we may immediately conclude that

$$\boxed{\begin{array}{l} (\forall n\in\mathbb{N}^*: f_n \text{ continuous on } [a,b]) \Rightarrow \sum_{n=1}^{+\infty} f_n(x) \text{ continuous on } [a,b] \\ \forall t\in[a,b]: \int_a^t \left[\sum_{n=1}^{+\infty} f_n(x)\right] dx = \sum_{n=1}^{+\infty} \int_a^t f_n(x)\,dx \end{array}}$$

Unfortunately, uniform convergence is not strong enough to allow a similar result for differentiability such as:

$$\forall x\in[a,b]: \frac{d}{dx}\sum_{k=1}^{+\infty} f_k(x) = \sum_{k=1}^{+\infty} \frac{d}{dx}[f_k(x)] \leftarrow \text{NOT ALWAYS TRUE!!}$$

For example, for $\forall n\in\mathbb{N}^*: \forall x\in\mathbb{R}: f_n(x) = \frac{\sin(nx)}{n^2}$

we will show that

a) $\sum_{n=1}^{+\infty} f_n(x)$ converges uniformly on $\mathbb{R}$; but

b) $\sum_{n=1}^{+\infty} f_n'(x)$ diverges for $x=0$.

Proof

We note that since:

$$\forall x\in\mathbb{R}: |f_n(x)| = \left|\frac{\sin(nx)}{n^2}\right| = \frac{|\sin(nx)|}{n^2} \leq \frac{1}{n^2}$$

we have

$$\left\{\begin{array}{l}\forall x \in \mathbb{R}: 0 \leq |f_n(x)| \leq 1/n^2 \\ \sum_{n=1}^{+\infty} (1/n^2) \text{ converges}\end{array}\right. \Rightarrow \sum_{n=1}^{+\infty} f_n(x) \text{ converges uniformly on } \mathbb{R}.$$

On the other hand:

$$\sum_{n=1}^{+\infty} f_n'(x) = \sum_{n=1}^{+\infty} \left[\frac{d}{dx}\left(\frac{\sin(nx)}{n^2}\right)\right] = \sum_{n=1}^{+\infty} \frac{n\cos(nx)}{n^2} =$$

$$= \sum_{n=1}^{+\infty} \frac{\cos(nx)}{n} \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} f_n'(0) = \sum_{n=1}^{+\infty} \frac{\cos 0}{n} = \sum_{n=1}^{+\infty} \frac{1}{n} \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} f_n'(0) \text{ diverges.} \qquad \square$$

↳ We see that term-by-term differentiation of a series will not always work, even if the original series expansion converges uniformly.

▼ Properties of power series.

We consider a function $f(x)$ defined via a power-series of the form

$$f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n$$

We wish to determine:

1) The domain A of $f(x)$: for which $x \in A$ does the power series converge.
2) The integral of $f(x)$.
3) The derivative of $f(x)$, if it exists.

↦ Radius of convergence

Thm:

a) $\sum_{n=0}^{+\infty} a_n x^n$ converges for $x = b$ $\Rightarrow$ $\sum_{n=0}^{+\infty} a_n x^n$ converges uniformly on $(-|b|, |b|)$

b) $\sum_{n=0}^{+\infty} a_n x^n$ diverges for $x = b$ $\Rightarrow$ $\forall x \in (-\infty, -|b|) \cup (|b|, +\infty)$: $\sum_{n=1}^{+\infty} a_n x^n$ diverges

Proof

a) Let $x \in (-|b|, |b|)$ be given. Since:

$$\sum_{n=0}^{+\infty} a_n b^n \text{ converges} \Rightarrow \lim_{n \in \mathbb{N}^*} a_n b^n = 0 \Rightarrow$$

$$\Rightarrow \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n b^n| < \varepsilon$$

Choose $\varepsilon = 1$ and fix $n_0 \in \mathbb{N}^*$ such that

$\forall n \in \mathbb{N}^* - [n_0] : |a_n b^n| < 1$

Let $n \in \mathbb{N}^* - [n_0]$ be given. Then:

$$|a_n x^n| = \left| \frac{a_n b^n x^n}{b^n} \right| = |a_n b^n| \cdot \left| \frac{x^n}{b^n} \right| < \left| \frac{x^n}{b^n} \right| =$$

$$= \left| \left(\frac{x}{b} \right)^n \right| = \left| \frac{x}{b} \right|^n$$

It follows that

$$\forall n \in \mathbb{N}^* - [n_0] : \forall x \in (-b, b) : 0 \leqslant |a_n x^n| < \left| \frac{x}{b} \right|^n \quad (1)$$

We also know that

$$\forall x \in (-b, b) : \sum_{n=0}^{+\infty} \left| \frac{x}{b} \right|^n \text{ converges} \quad (2)$$

From Eq. (1) and Eq. (2), it follows that $\sum_{n=0}^{+\infty} a_n x^n$ converges uniformly on $(-|b|, |b|)$.

b) Assume that $\sum_{n=1}^{+\infty} a_n b^n$ diverges. Let $x \in (-\infty, -|b|) \cup (|b|, +\infty)$ be given. To show that $\sum_{n=1}^{+\infty} a_n x^n$ diverges, we use proof by contradiction and assume that

$$\sum_{n=1}^{+\infty} a_n x^n \text{ converges} \quad (1)$$

Since:

$$x \in (-\infty, -|b|) \cup (|b|, +\infty) \Rightarrow |x| > |b| \Rightarrow |b| < |x| \Rightarrow b \in (-|x|, |x|) \quad (2)$$

then, from Eq. (1) and Eq. (2) it follows, via the previous result, that

$$\sum_{n=1}^{+\infty} a_n b^n \text{ converges}$$

which is a contradiction, since it is assumed that $\sum_{n=1}^{+\infty} a_n b^n$ diverges. It follows that $\sum_{n=1}^{+\infty} a_n x^n$ diverges □.

↳ Shifting the series around x_0 gives the following generalization:

$$\boxed{\begin{array}{l} \sum_{n=0}^{+\infty} a_n (x-x_0)^n \text{ converges on } \quad x = x_0 + b \Rightarrow \\ \Rightarrow \sum_{n=1}^{+\infty} a_n (x-x_0)^n \text{ converges uniformly on } (x_0 - |b|, x_0 + |b|) \end{array}}$$

↳ <u>Consequences of uniform convergence</u>

- From the above, it immediately follows that given a function $f(x)$ defined as:

$$f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n$$

If $f(x)$ converges at $x = b$, then $f(x)$ converges uniformly on $(x_0 - |b|, x_0 + |b|)$, and we immediately conclude that

a) f is continuous on $(x_0 - |b|, x_0 + |b|)$

b) $\forall c_1, c_2 \in (x_0 - |b|, x_0 + |b|)$: $\int_{c_1}^{c_2} f(x)\,dx = \int_{c_1}^{c_2} \left[\sum_{n=0}^{+\infty} a_n (x-x_0)^n \right] dx$

$$= \sum_{n=0}^{+\infty} \left[\int_{c_1}^{c_2} a_n (x-x_0)^n \, dx \right] = \sum_{n=0}^{+\infty} \left[\frac{a_n (x-x_0)^{n+1}}{n+1} \right]_{x=c_1}^{x=c_2} =$$

$$= \sum_{n=0}^{+\infty} \left[\frac{a_n(c_2-x_0)^{n+1} - a_n(c_1-x_0)^{n+1}}{n+1} \right]$$

- From the proof of the radius of convergence theorem, it is obvious that if we use the comparison test instead of the Weierstrass M-test, we also get the statement:

$$\boxed{\begin{aligned} &\sum_{n=1}^{+\infty} a_n(x-x_0)^n \text{ converges on } x = x_0+b \\ &\Rightarrow \forall x \in (x_0-|b|, x_0+|b|) : \sum_{n=1}^{+\infty} |a_n(x-x_0)^n| \text{ converges} \end{aligned}}$$

- Another immediate consequence of the uniform convergence of a power-series defining a function is that we can integrate term-by-term with an indefinite integral and find the antiderivative of the function:

$$\boxed{\begin{aligned} \int f(x)\,dx &= \int \left[\sum_{n=0}^{+\infty} a_n(x-x_0)^n \right] dx = \sum_{n=0}^{+\infty} \left[\int a_n(x-x_0)^n\,dx \right] = \\ &= \sum_{n=0}^{+\infty} \frac{a_n(x-x_0)^{n+1}}{n+1} + c \end{aligned}}$$

- As we explained before, uniform convergence is not sufficient to justify term-by-term differentiation. However, we will now show that power series can be always differentiated.

Differentiation of a power series

The following theorem allows the term-by-term differentiation of power series

Thm: We define the function

$$\forall x \in [x_0-b, x_0+b] : f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n$$

assuming the corresponding series converges on (x_0-b, x_0+b).

Then:

a) f differentiable on (x_0-b, x_0+b)

b) $\forall x \in (x_0-b, x_0+b) : f'(x) = \sum_{n=1}^{+\infty} n a_n (x-x_0)^{n-1}$

Proof

Let $x \in (x_0-b, x_0+b)$ be given. Since for $x = x_0+b$:

$f(x_0+b) = \sum_{n=1}^{+\infty} a_n b^n$ converges $\Rightarrow \lim_{n\in\mathbb{N}^*} a_n b^n = 0 \Rightarrow$

$\Rightarrow a_n b^n$ bounded $\Rightarrow$

$\Rightarrow a_n b^n$ absolutely bounded

$\Rightarrow \exists M \in (0, +\infty) : \forall n \in \mathbb{N}^* : |a_n b^n| \leq M$

Fix an $M \in (0, +\infty)$ such that: $\forall n \in \mathbb{N}^* : |a_n b^n| \leq M$

Let $n \in \mathbb{N}^*$ be given. Then we have:

$$|n a_n (x-x_0)^{n-1}| = |n a_n| \, |x-x_0|^{n-1} = \frac{n |a_n| \cdot |x-x_0|^n \, |b^n|}{|x-x_0| \cdot |b^n|} =$$

$$= \frac{n}{|x-x_0|} \frac{|x-x_0|^n}{|b|^n} |a_n b^n| \leq \frac{nM}{|x-x_0|} \left| \frac{x-x_0}{b} \right|^n$$

It follows that

$$\forall n \in \mathbb{N}^* : |n a_n (x-x_0)^{n-1}| \leq \frac{nM}{|x-x_0|}\left|\frac{x-x_0}{b}\right|^n = \gamma_n \qquad (1)$$

To show that $\sum_{n=1}^{+\infty} \gamma_n$ converges, we note that

$$\forall n \in \mathbb{N}^* : \frac{\gamma_{n+1}}{\gamma_n} = \frac{\frac{(n+1)M}{|x-x_0|}\left|\frac{x-x_0}{b}\right|^{n+1}}{\frac{nM}{|x-x_0|}\left|\frac{x-x_0}{b}\right|^n} = \frac{n+1}{n}\left|\frac{x-x_0}{b}\right| > 0 \qquad (2)$$

and therefore:

$$\lim_{t\to+\infty} \frac{t+1}{t} = \lim_{t\to+\infty} \frac{t}{t} = 1 \Rightarrow \lim_{n\in\mathbb{N}^*} \frac{n+1}{n} = 1 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \frac{\gamma_{n+1}}{\gamma_n} = \lim_{n\in\mathbb{N}^*}\left[\frac{n+1}{n}\left|\frac{x-x_0}{b}\right|\right] = \left|\frac{x-x_0}{b}\right| \lim_{n\in\mathbb{N}^*} \frac{n+1}{n}$$

$$= \left|\frac{x-x_0}{b}\right|$$

Since $x \in (x_0-b, x_0+b) \Rightarrow x_0 - b < x < x_0 + b \Rightarrow -b < x - x_0 < b$

$$\Rightarrow |x-x_0| < b = |b| \Rightarrow \frac{|x-x_0|}{|b|} < 1 \Rightarrow \left|\frac{x-x_0}{b}\right| < 1$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \frac{\gamma_{n+1}}{\gamma_n} < 1 \qquad (3)$$

From the ratio test, via Eq. (2) and Eq. (3), we conclude that:

$$\sum_{n=1}^{+\infty} \gamma_n \text{ converges} \qquad (4)$$

From Eq. (1) and Eq. (4), via the Weierstrass M-test, it follows that $\sum_{n=1}^{+\infty} na_n(x-x_0)^{n-1}$ converges uniformly on $x\in(x_0-\delta, x_0+\delta)$.

Define $\forall x\in(x_0-\delta, x_0+\delta)$: $g(x)=\sum_{n=1}^{+\infty} na_n(x-x_0)^{n-1}$

and note that:

$\forall x\in(x_0-\delta,x_0+\delta)$: $\int_{x_0}^{x} g(t)\,dt = \int_{x_0}^{x}\left[\sum_{n=1}^{+\infty} na_n(t-x_0)^{n-1}\right]dt =$

$$=\sum_{n=1}^{+\infty}\left[na_n\int_{x_0}^{x}(t-x_0)^{n-1}\,dt\right]=$$

$$=\sum_{n=1}^{+\infty}\left\{na_n\left[\frac{(t-x_0)^n}{n}\right]_{t=x_0}^{t=x}\right\}=$$

$$=\sum_{n=1}^{+\infty}\left\{a_n\left[(x-x_0)^n-(x_0-x_0)^n\right]\right\}=\sum_{n=1}^{+\infty}a_n(x-x_0)^n=f(x)$$

From the fundamental theorem of calculus it follows that; f is differentiable on $(x_0-\delta, x_0+\delta)$ and

$$f'(x)=\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x)=\sum_{n=1}^{+\infty} na_n(x-x_0)^{n-1}, \forall x\in(x_0-\delta,x_0+\delta).$$

$\square$

Comments on power series differentiation

Consider a function $f(x)$ defined by a convergent power series over a closed interval $[x_0-b, x_0+b]$:

$$\boxed{\forall x \in [x_0-b, x_0+b]: f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n}$$

- Note that the differentiation theorem is only able to give us differentiability in the open interval (x_0-b, x_0+b):

$$\boxed{\forall x \in (x_0-b, x_0+b): f'(x) = \sum_{n=1}^{+\infty} n a_n (x-x_0)^{n-1}}$$

- Because the theorem also gives uniform convergence of the power series of $f'(x)$, we can differentiate $f'(x)$ again term-by-term to obtain the following power series for the second derivative $f''(x)$.

$$\boxed{\forall x \in (x_0-b, x_0+b): f''(x) = \sum_{n=2}^{+\infty} n(n-1) a_n (x-x_0)^{n-2}}$$

- Using proof by induction, the k^{th} derivative $f^{(k)}(x)$ can be shown to be given by the following power series:

$$\boxed{\begin{aligned} \forall k \in \mathbb{N}^*: \forall x \in (x_0-b, x_0+b): f^{(k)}(x) &= \sum_{n=k}^{+\infty} a_n \left[\prod_{l=0}^{k-1} (n-l)\right] (x-x_0)^{n-k} \\ &= \sum_{n=k}^{+\infty} \frac{n!}{(n-k)!} a_n (x-x_0)^{n-k} \end{aligned}}$$

- From the above expression for $f^{(k)}(x)$, using $x = x_0$ we find:

$$f^{(k)}(x_0) = \frac{k!}{(k-k)!} a_k = \frac{k!}{0!} a_k = k! a_k \Rightarrow a_k = \frac{f^{(k)}(x_0)}{k!}$$

It follows that:

$$\boxed{\left[\forall x \in (x_0 - b, x_0 + b) : f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n\right] \Rightarrow \left(\forall n \in \mathbb{N} : a_n = \frac{f^{(n)}(x_0)}{n!}\right)}$$

Because the immediate consequence of this result is that the coefficients a_n are unique (if they exist) it immediately follows that

$$\boxed{\begin{aligned}&\left[\exists c \in [0, +\infty) : \forall x \in (x_0 - c, x_0 + c) : \sum_{n=0}^{+\infty} a_n (x - x_0)^n = \sum_{n=0}^{+\infty} b_n (x - x_0)^n\right] \Leftrightarrow \\ &\Leftrightarrow (\forall n \in \mathbb{N} : a_n = b_n)\end{aligned}}$$

This result in turn can be applied in

a) Solution of differential equations via series expansion

b) The proof of the binomial series.

Taylor expansions

- Let $f: A \to \mathbb{R}$ be a function, let $x_0 \in A$ and assume that $\forall n \in \mathbb{N}^*$: f is n-differentiable on A.
  We define the n^{th}-order Taylor polynomial $T_n(x_0)f$ around x_0 as:

$$\boxed{\forall x \in \mathbb{R} : (T_n(x_0)f)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}$$

- We have already shown that <u>IF</u> $f(x)$ can be represented by a power series around x_0 on an interval (x_0-b, x_0+b) then the Taylor polynomials $T_n(x_0)f$ will converge <u>uniformly</u> to the function $f(x)$ on (x_0-b, x_0+b), and therefore

$$\boxed{\begin{aligned}\forall x \in (x_0-b, x_0+b) : f(x) &= \lim_{n \in \mathbb{N}^*} (T_n(x_0)f)(x) \\ &= \sum_{k=0}^{+\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\end{aligned}}$$

- However, the converse is not true as:
  (a) It is possible for the Taylor series to not converge
  (b) OR, to converge but not converge to $f(x)$.

COUNTEREXAMPLE

Consider the function

$$f(x) = \begin{cases} \exp(-1/x^2) & , \text{ if } x \in \mathbb{R} - \{0\} \\ 0 & , \text{ if } x = 0 \end{cases}$$

Via the limit definition of the derivative and using a recurrence argument we can show that

$$\forall n \in \mathbb{N}^* : f^{(n)}(0) = 0$$

(note: proving this is very challenging, so it is left for you as homework).

It follows that the Taylor series converges on $\mathbb{R}$ as follows:

$$\forall x \in \mathbb{R} : \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n = 0$$

However, we see that, except for $x=0$, the series does not converge to $f(x)$ for $x \in \mathbb{R} - \{0\}$.

↳ To show that a Taylor series converges and converges to the original function $f(x)$, there are three methods:

1) Term-by-term differentiation or integration of a previously established Taylor expansion
2) Directly, via convergence theorems
3) Via special techniques (e.g. the binomial series).

1 → Term-by-term differentiation or integration

We use the geometric series as a point of departure:

$$\boxed{\forall x \in (-1,1): \frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n}$$

to establish the following results:

$$\boxed{\begin{aligned} &\forall x \in (-1,1): \operatorname{Arctan}(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \\ &\forall x \in (-1,1): \ln(1+x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1} \end{aligned}}$$

Proof

a) Let $x \in (-1,1)$ be given. Then:

$$\operatorname{Arctan}(x) = \int_0^x \frac{dt}{1+t^2} = \int_0^x \frac{dt}{1-(-t^2)} \overset{x\in(-1,1)}{=}$$

$$= \int_0^x \left[\sum_{n=0}^{+\infty} (-t^2)^n\right] dt = \sum_{n=0}^{+\infty} \left[\int_0^x (-t^2)^n \, dt\right] =$$

$$= \sum_{n=0}^{+\infty} \left[(-1)^n \int_0^x t^{2n} \, dt\right] = \sum_{n=0}^{+\infty} \left[(-1)^n \left[\frac{t^{2n+1}}{2n+1}\right]_0^x\right]$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

and therefore

$$\forall x \in (-1,1): \operatorname{Arctan}(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

b) Let $x \in (-1,1)$ be given. Then:

$$\ln(1+x) = \ln(1+x) - \ln 1 = \int_0^x [\ln(1+t)]'\,dt = \int_0^x \frac{(1+t)'}{1+t}\,dt$$

$$= \int_0^x \frac{dt}{1+t} = \int_0^x \frac{dt}{1-(-t)} \underset{x \in (-1,1)}{=} \int_0^x \left[\sum_{n=0}^{+\infty} (-t)^n\right] dt$$

$$= \sum_{n=0}^{+\infty} \left[\int_0^x (-t)^n\,dt\right] = \sum_{n=0}^{+\infty} \left[(-1)^n \int_0^x t^n\,dt\right]$$

$$= \sum_{n=0}^{+\infty} \left[(-1)^n \left[\frac{t^{n+1}}{n+1}\right]_0^x\right] = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1} - 0^{n+1}}{n+1}$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad \square$$

↳ These results have the following immediate applications:

1) Calculation of π: $\boxed{\pi = \sum_{n=0}^{+\infty} \frac{4(-1)^n}{2n+1}}$

Proof

Since $\tan(\pi/4) = 1 \Rightarrow \pi/4 = \text{Arctan}(1) \Rightarrow$

$$\Rightarrow \pi = 4\,\text{Arctan}(1) = 4\sum_{n=0}^{+\infty} (-1)^n \frac{1^{2n+1}}{2n+1} = 4\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} =$$

$$= \sum_{n=0}^{+\infty} \frac{4(-1)^n}{2n+1}$$

2) Calculation of logarithm

$\forall x \in (0,2): \ln x = \sum_{n=0}^{+\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$
$\forall x \in [2,+\infty): \ln x = \sum_{n=0}^{+\infty} \frac{1}{n+1} \left(\frac{x-1}{x}\right)^{n+1}$

Proof

The first equation is obtained from the Taylor expansion of $\ln(1+x)$ around $x_0 = 0$, by shifting the variable x.
For the second equation, let $x \in [2,+\infty)$ be given. Then

$$\ln x = -\ln\left(\frac{1}{x}\right) = -\ln\left(1+\frac{1}{x}-1\right) = -\ln\left(1+\frac{1-x}{x}\right) \quad (1)$$

and

$$x \in [2,+\infty) \Rightarrow x > 2 > 0 \Rightarrow 0 < \frac{1}{x} < \frac{1}{2} \Rightarrow$$

$$\Rightarrow -1 < \frac{1}{x} - 1 < \frac{1}{2} - 1 \Rightarrow -1 < \frac{1-x}{x} < \frac{-1}{2}$$

$$\Rightarrow \frac{1-x}{x} \in (-1,1) \quad (2)$$

From Eq.(1) and Eq.(2):

$$\ln x = -\ln\left(1+\frac{1-x}{x}\right) = -\sum_{n=0}^{+\infty} (-1)^n \frac{1}{n+1}\left(\frac{1-x}{x}\right)^{n+1} =$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{n+1}\left(\frac{1-x}{x}\right)^{n+1} = \sum_{n=0}^{+\infty} \frac{1}{n+1}\left(\frac{x-1}{x}\right)^{n+1}$$

EXAMPLES

↦ <u>Method</u> : Taylor expansions of rational functions can be evaluated via geometric series and partial function decomposition

a) Taylor expand $f(x) = \frac{10}{3x^2-14x+8}$ around $x=0$.

<u>Solution</u>

We factor $3x^2-14x+8$:

$$\Delta = b^2-4ac = (-14)^2 - 4\cdot 3\cdot 8 = 196 - 12\cdot 8 = 196-96 = 100 = 10^2$$

$$\Rightarrow x_{1,2} = \frac{-b\pm\sqrt{\Delta}}{2a} = \frac{-(-14)\pm 10}{2\cdot 3} = \frac{14\pm 10}{6} =$$

$$= \begin{cases} 24/6 = 4 \\ 4/6 = 2/3 \end{cases}$$

and therefore $3x^2-14x+8 = 3(x-4)(x-2/3)$. It follows that

$$f(x) = \frac{10}{3x^2-14x+8} = \frac{A}{x-4} + \frac{B}{x-2/3} = \frac{-A}{4-x} + \frac{-B}{2/3-x} =$$

$$= \frac{-A}{4}\,\frac{1}{1-(x/4)} + \frac{-B}{2/3}\,\frac{1}{1-(3x/2)} =$$

$$= \frac{-A}{4}\sum_{k=0}^{+\infty}\left(\frac{x}{4}\right)^k + \frac{-3B}{2}\sum_{k=0}^{+\infty}\left(\frac{3x}{2}\right)^k$$

$$= \sum_{k=0}^{+\infty}\left[\frac{-A}{4\cdot 4^k}x^k + \frac{-3B}{2}\left(\frac{3}{2}\right)^k x^k\right] =$$

$$= \sum_{k=0}^{+\infty} \left[\frac{-A}{4^{k+1}} + \frac{-3^{k+1} B}{2^{k+1}} \right] x^k$$

with

$$A = \left. \frac{10}{3(x-2/3)} \right|_{x=4} = \left. \frac{10}{3x-2} \right|_{x=4} = \frac{10}{3 \cdot 4 - 2} =$$

$$= \frac{10}{12-2} = \frac{10}{10} = 1$$

$$B = \left. \frac{10}{3(x-4)} \right|_{x=2/3} = \frac{10}{3(2/3-4)} = \frac{10}{2-12} =$$

$$= \frac{10}{-10} = -1$$

consequently,

$$f(x) = \sum_{k=0}^{+\infty} \left[\frac{-1}{4^{k+1}} + \frac{-3^{k+1}(-1)}{2^{k+1}} \right] x^k =$$

$$= \sum_{k=0}^{+\infty} \left[\frac{3^{k+1}}{2^{k+1}} - \frac{1}{4^{k+1}} \right] x^k$$

To determine the radius of convergence, we note that the two geometric expansions require

$$\begin{cases} |x/4| < 1 \\ |3x/2| < 1 \end{cases} \Leftrightarrow \begin{cases} |x|/4 < 1 \\ 3|x|/2 < 1 \end{cases} \Leftrightarrow \begin{cases} |x| < 4 \\ |x| < 2/3 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow |x| < 2/3 \Leftrightarrow x \in (-2/3, 2/3)$$

It follows that

$$\forall x \in (-2/3, 2/3) : f(x) = \frac{10}{3x^2 - 14x + 8} = \sum_{k=0}^{+\infty} \left[\frac{3^{k+1}}{2^{k+1}} - \frac{1}{4^{k+1}} \right] x^k$$

↦ <u>Method</u>: Repeating factors on the denominator can be addressed with term-by-term differentiation.

b) Taylor expand $f(x) = \dfrac{x^2}{x^2+4x+4}$ around $x=0$.

<u>Solution</u>

We note that

$$f(x) = \frac{x^2}{x^2+4x+4} = \frac{x^2}{(x+2)^2} = x^2(x+2)^{-2} =$$

$$= -x^2[(-1)(x+2)^{-2}] = -x^2[(x+2)^{-1}]' =$$

$$= -x^2 \frac{d}{dx}\left[\frac{1}{x+2}\right] = \frac{-x^2}{2}\frac{d}{dx}\left[\frac{1}{1+x/2}\right] =$$

$$= \frac{-x^2}{2}\frac{d}{dx}\left[\frac{1}{1-(-x/2)}\right] = \frac{-x^2}{2}\frac{d}{dx}\left[\sum_{n=0}^{+\infty}\left(\frac{-x}{2}\right)^n\right]$$

$$= \frac{-x^2}{2}\sum_{n=0}^{+\infty}\frac{d}{dx}\left[\frac{(-1)^n}{2^n}x^n\right] = \frac{-x^2}{2}\sum_{n=1}^{+\infty}\frac{(-1)^n}{2^n}nx^{n-1}$$

$$= \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{2^{n+1}}nx^{n+1} = \sum_{n=2}^{+\infty}\frac{(-1)^n(n-1)}{2^n}x^n$$

To determine the radius of convergence, we require that

$|-x/2| < 1 \Leftrightarrow (1/2)|x| < 1 \Leftrightarrow |x| < 2 \Leftrightarrow -2 < x < 2$

$\Leftrightarrow x \in (-2,2)$

and therefore

$$\forall x \in (-2,2): f(x) = \frac{x^2}{x^2+4x+4} = \sum_{n=2}^{+\infty}\frac{(-1)^n(n-1)}{2^n}x^n$$

② → Via convergence theorem

We can estimate the error, when truncating a Taylor expansion and establish convergence via the following theorems:

Thm: Let $f: A \to \mathbb{R}$ be a function with $(x_0-b, x_0+b) \subseteq A$. We assume that

$$\begin{cases} \forall k \in [n+1]: f^{(k)} \text{ differentiable on } (x_0-b, x_0+b) \\ f^{(n+1)} \text{ continuous on } (x_0-b, x_0+b \end{cases}$$

Then:

$$\forall x \in (x_0-b, x_0+b): f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + E_n(f|x,x_0)$$

with $E_n(f|x,x_0) = \dfrac{1}{n!}\displaystyle\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\,dt$

Proof

Let $x \in (x_0-b, x_0+b)$ be given. For $n=1$:

$$E_1(f|x,x_0) = f(x) - \sum_{k=0}^{1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k =$$

$$= f(x) - f(x_0) - f'(x_0)(x-x_0) =$$

$$= \int_{x_0}^{x} f'(t)\,dt - f'(x_0)\int_{x_0}^{x} dt =$$

$$= \int_{x_0}^{x}[f'(t)-f'(x_0)]\,dt = \int_{x_0}^{x}(t-x)'[f'(t)-f'(x_0)]\,dt$$

$$= \Big\{(t-x)[f'(t)-f'(x_0)]\Big\}_{x_0}^{x} - \int_{x_0}^{x}(t-x)[f'(t)-f'(x_0)]'\,dt$$

$$= (x-x)\left[f'(x)-f'(x_0)\right]-(x_0-x)\left[f'(x_0)-f'(x_0)\right]-\int_{x_0}^{x}(t-x)f''(t)dt$$

$$= \int_{x_0}^{x}(x-t)f''(t)dt$$

For $n=m$, we assume that

$$f(x)=\sum_{k=0}^{m}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + E_m(f|x,x_0)$$

$$\text{with } E_m(f|x,x_0)=\frac{1}{m!}\int_{x_0}^{x}(x-t)^m f^{(m+1)}(t)\,dt$$

For $n=m+1$, we have:

$$E_m(f|x,x_0)=\frac{1}{m!}\int_{x_0}^{x}(x-t)^m f^{(m+1)}(t)dt = \frac{1}{m!}\int_{x_0}^{x}\left[\frac{-(x-t)^{m+1}}{m+1}\right]' f^{(m+1)}(t)\,dt$$

$$= \frac{1}{m!}\left[\frac{-(x-t)^{m+1}}{m+1}f^{(m+1)}(t)\right]_{t=x_0}^{t=x} - \frac{1}{m!}\int_{x_0}^{x}\frac{-(x-t)^{m+1}}{m+1}\left[f^{(m+1)}(t)\right]'dt$$

$$= \frac{1}{m!}\left[\frac{-(x-x)^{m+1}}{m+1}f^{(m+1)}(x)\right]-\frac{1}{m!}\left[\frac{-(x-x_0)^{m+1}}{m+1}f^{(m+1)}(x_0)\right]$$

$$+\frac{1}{(m+1)m!}\int_{x_0}^{x}(x-t)^{m+1}f^{(m+2)}(t)\,dt$$

$$= \frac{f^{(m+1)}(x_0)}{(m+1)m!}(x-x_0)^{m+1}+\frac{1}{(m+1)!}\int_{x_0}^{x}(x-t)^{m+1}f^{(m+2)}(t)\,dt$$

$$= \frac{f^{(m+1)}(x_0)}{(m+1)!}(x-x_0)^{m+1}+E_{m+1}(f|x,x_0)$$

and therefore

$$f(x)=\sum_{k=0}^{m}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+E_m(f|x,x_0)=$$

$$= \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(m+1)}(x_0)}{(m+1)!}(x-x_0)^{m+1} + E_{m+1}(f|x,x_0)$$

$$= \sum_{k=0}^{m+1} \frac{f^{(k)}(x_0)}{k!} + E_{m+1}(f|x,x_0)$$

It follows, by induction, that

$$\forall n\in\mathbb{N}^*: \forall x\in(x_0-\delta, x_0+\delta): f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + E_n(f|x,x_0) \quad \square$$

→ The convergence of a Taylor series expansion can be established by the following theorem:

Thm: Let $f: A\to\mathbb{R}$ be a function with $(x_0-\delta, x_0+\delta)\subseteq A$ and assume that

$$\begin{cases} \forall n\in\mathbb{N}^*: f \text{ n-times differentiable on } (x_0-\delta, x_0+\delta) \\ \exists M\in(0,+\infty): \forall n\in\mathbb{N}^*: \forall x\in(x_0-\delta,x_0+\delta): |f^{(n)}(x)|\leq M^n \end{cases}$$

Then:

$$\forall x\in(x_0-\delta,x_0+\delta): f(x) = \sum_{n=1}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Proof

Let $x\in(x_0-\delta,x_0+\delta)$ be given and let $n\in\mathbb{N}^*$ be given. Without loss of generality, we assume that $x>x_0$. Then:

$$|E_n(x)| = \left|\frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\,dt\right| \leq \frac{1}{n!}\int_{x_0}^{x}|(x-t)^n f^{(n+1)}(t)|\,dt$$

$$= \frac{1}{n!}\int_{x_0}^{x}|x-t|^n|f^{(n+1)}(t)|\,dt \leq \frac{1}{n!}\int_{x_0}^{x}|x-t|^n M^{n+1}\,dt$$

$$= \frac{M^{n+1}}{n!}\int_{x_0}^{x}(x-t)^n\,dt = \frac{M^{n+1}}{n!}\left[\frac{(x-t)^{n+1}(-1)}{n+1}\right]_{t=x_0}^{t=x}$$

$$= \frac{M^{n+1}}{n!}\left[\frac{(-1)(x-x)^{n+1}}{n+1} - \frac{(-1)(x-x_0)^{n+1}}{n+1}\right] =$$

$$= \frac{M^{n+1}}{n!}\frac{(x-x_0)^{n+1}}{n+1} = \frac{[M(x-x_0)]^{n+1}}{(n+1)!} \equiv \gamma_n$$

It follows that

$$\forall n \in \mathbb{N}^*: |E_n(x)| \leq \frac{[M(x-x_0)]^{n+1}}{(n+1)!} = \gamma_n$$

and we note that

$$\forall n \in \mathbb{N}^*: \gamma_n > 0$$

and

$$\forall n \in \mathbb{N}^*: \frac{\gamma_{n+1}}{\gamma_n} = \frac{\dfrac{[M(x-x_0)]^{n+2}}{(n+2)!}}{\dfrac{[M(x-x_0)]^{n+1}}{(n+1)!}} = \frac{(n+1)!\,[M(x-x_0)]^{n+2}}{(n+2)!\,[M(x-x_0)]^{n+1}} =$$

$$= \frac{M(x-x_0)}{n+2}$$

We may therefore argue that

$$\lim_{x\to+\infty} \frac{1}{x+2} = \lim_{x\to+\infty}\frac{1}{x} = 0 \Rightarrow \lim_{n\in\mathbb{N}^*}\frac{1}{n+2} = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*}\frac{M(x-x_0)}{n+2} = 0 \Rightarrow \lim_{n\in\mathbb{N}^*}\frac{\gamma_{n+1}}{\gamma_n} = 0 \Rightarrow \lim_{n\in\mathbb{N}^*}\gamma_n = 0$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} E_n(x) = 0 \Rightarrow \lim_{n\in\mathbb{N}^*}\left[f(x) - \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right] = 0$$

$$\Rightarrow f(x) = \sum_{k=0}^{+\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k. \qquad \square$$

We now apply this theorem to establish the convergence of the Taylor series of trigonometric functions and the exponential function.

$$1)\ \boxed{\forall x \in \mathbb{R}: \sin x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}$$

Proof

We define $\forall x \in \mathbb{R}: f(x) = \sin x$

Let $x \in \mathbb{R}$ be given, let $n \in \mathbb{N}$ be given, and note that

$$\begin{cases} f^{(2n)}(x) = (-1)^n \sin x \\ f^{(2n+1)}(x) = (-1)^n \cos x \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} |f^{(2n)}(x)| = |(-1)^n \sin x| = |\sin x| \leq 1 \\ |f^{(2n+1)}(x)| = |(-1)^n \cos x| = |\cos x| \leq 1 \end{cases}$$

It follows that

$(\forall n \in \mathbb{N}: \forall x \in \mathbb{R}: |f^{(n)}(x)| \leq 1 = 1^n) \Rightarrow$

$\Rightarrow$ The Taylor series of $\sin x$ converges on $\mathbb{R}$.

and also:

$$\begin{cases} \forall n \in \mathbb{N}: f^{(2n)}(0) = (-1)^n \sin 0 = 0 \\ \forall n \in \mathbb{N}: f^{(2n+1)}(0) = (-1)^n \cos 0 = (-1)^n \end{cases}$$

and therefore:

$$\forall x \in \mathbb{R}: f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n =$$

$$= \sum_{n=0}^{+\infty} \left[\frac{f^{(2n)}(0)}{(2n)!} x^{2n} + \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} \right]$$

$$= \sum_{n=0}^{+\infty} \left[0x^{2n} + \frac{(-1)^n}{(2n+1)!} x^{2n+1} \right] = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$2)\ \boxed{\forall x \in \mathbb{R}: \cos x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

Proof

We define $\forall x \in \mathbb{R}: \quad f(x) = \cos x$

Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$ be given, and note that

$$\begin{cases} f^{(2n)}(x) = (-1)^n \cos x \\ f^{(2n+1)}(x) = (-1)^n [-\sin x] = (-1)^{n+1} \sin x \end{cases} \Longrightarrow$$

$$\Rightarrow \begin{cases} |f^{(2n)}(x)| = |(-1)^n \cos x| = |\cos x| \leq 1 \\ |f^{(2n+1)}(x)| = |(-1)^{n+1} \sin x| = |\sin x| \leq 1 \end{cases}$$

It follows that

$(\forall n \in \mathbb{N}: \forall x \in \mathbb{R}: |f^{(n)}(x)| \leq 1 = 1^n) \Rightarrow$

$\Rightarrow$ The Taylor series of $\cos x$ converges on $\mathbb{R}$.

and also

$$\begin{cases} \forall n \in \mathbb{N}: f^{(2n)}(0) = (-1)^n \cos 0 = (-1)^n \\ \forall n \in \mathbb{N}: f^{(2n+1)}(0) = (-1)^{n+1} \sin 0 = (-1)^{n+1} \cdot 0 = 0 \end{cases}$$

and therefore:

$$\forall x \in \mathbb{R}: f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n =$$

$$= \sum_{n=0}^{+\infty} \left[\frac{f^{(2n)}(0)}{(2n)!} x^{2n} + \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} \right]$$

$$= \sum_{n=0}^{+\infty} \left[\frac{(-1)^n}{(2n)!} x^{2n} + 0 x^{2n+1} \right] = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

3) $\boxed{\forall x \in \mathbb{R}: \ e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$

<u>Proof</u>

We define $\forall x \in \mathbb{R}: f(x) = e^x$

Let $a \in (0, +\infty)$ be given. Let $x \in (-a, a)$ and $n \in \mathbb{N}$ be given, and note that

$f^{(n)}(x) = e^x \Rightarrow$

$\Rightarrow |f^{(n)}(x)| = |e^x| = e^x < e^a < e^{na} = (e^a)^n$

It follows that

$(\forall n \in \mathbb{N} : \forall x \in (-a, a) : |f^{(n)}(x)| < (e^a)^n) \Rightarrow$

$\Rightarrow$ The Taylor series of $f(x) = e^x$ converges on $(-a, a)$ for all $a \in (0, +\infty) \Rightarrow$

$\Rightarrow$ The Taylor series of $f(x) = e^x$ converges on $\mathbb{R}$. (1)

Since

$(\forall n \in \mathbb{N} : f^{(n)}(0) = e^0 = 1) \overset{(1)}{\Rightarrow}$

$$\Rightarrow \forall x \in \mathbb{R} : f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \qquad \square$$

EXAMPLES

a) Show that $\forall x \in \mathbb{R}: \sin x \sin(3x) = \sum_{n=0}^{+\infty} (-1)^n \frac{2^{2n-1}(1-2^{2n})}{(2n)!} x^{2n}$

Solution

$$\forall x \in \mathbb{R}: \sin x \cdot \sin(3x) = (1/2)\left[\cos(x-3x) - \cos(x+3x)\right] =$$
$$= (1/2)\left[\cos(-2x) - \cos(4x)\right] = (1/2)\left[\cos(2x) - \cos(4x)\right]$$
$$= \frac{1}{2}\left[\sum_{n=0}^{+\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} - \sum_{n=0}^{+\infty} (-1)^n \frac{(4x)^{2n}}{(2n)!}\right] =$$
$$= \frac{1}{2}\sum_{n=0}^{+\infty}\left[\frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} - \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!}\right] =$$
$$= \frac{1}{2}\sum_{n=0}^{+\infty} (-1)^n \frac{(2^{2n} - 4^{2n})}{(2n)!} x^{2n} =$$
$$= \sum_{n=0}^{+\infty} (-1)^n \frac{2^{2n}(1-2^{2n})}{2(2n)!} x^{2n} =$$
$$= \sum_{n=0}^{+\infty} (-1)^n \frac{2^{2n-1}(1-2^{2n})}{(2n)!} x^{2n}$$

b) Consider the error function:

$$\forall x \in \mathbb{R}: \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

Show that

$$\forall x \in \mathbb{R}: \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)\, n!} x^{2n+1}$$

Solution

$$\forall x \in \mathbb{R}: \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\,dt = \frac{2}{\sqrt{\pi}} \int_0^x \left[\sum_{n=0}^{+\infty} \frac{(-t^2)^n}{n!} \right] dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x \left[\sum_{n=0}^{+\infty} \frac{(-1)^n t^{2n}}{n!} \right] dt =$$

$$= \frac{2}{\sqrt{\pi}} \cdot \sum_{n=0}^{+\infty} \left[\int_0^x \frac{(-1)^n t^{2n}}{n!}\,dt \right] =$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \left[\frac{(-1)^n}{n!} \int_0^x t^{2n}\,dt \right] =$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \left[\frac{(-1)^n}{n!} \left[\frac{t^{2n+1}}{2n+1} \right]_{t=0}^{t=x} \right] =$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \left[\frac{(-1)^n}{n!} \left(\frac{x^{2n+1} - 0}{2n+1} \right) \right] =$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)\, n!} x^{2n+1}$$

c) Consider the Fresnel integral function

$$S(x) = \int_0^x \sin(t^2)\,dt \,, \quad \forall x \in \mathbb{R}$$

Show that

$$\forall x \in \mathbb{R} = S(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(4n+3)(2n+1)!} x^{4n+3}$$

<u>Solution</u>

$$\forall x \in \mathbb{R}: S(x) = \int_0^x \sin(t^2)\,dt = \int_0^x \left[\sum_{n=0}^{+\infty} (-1)^n \frac{(t^2)^{2n+1}}{(2n+1)!} \right] dt$$

$$= \int_0^x \left[\sum_{n=0}^{+\infty} (-1)^n \frac{t^{4n+2}}{(2n+1)!} \right] dt = \sum_{n=0}^{+\infty} \left[\int_0^x (-1)^n \frac{t^{4n+2}}{(2n+1)!} dt \right]$$

$$= \sum_{n=0}^{+\infty} \left[\frac{(-1)^n}{(2n+1)!} \int_0^x t^{4n+2} dt \right] = \sum_{n=0}^{+\infty} \left[\frac{(-1)^n}{(2n+1)!} \frac{x^{4n+3}}{4n+3} \right]$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(4n+3)(2n+1)!} x^{4n+3}$$

↳ The previous two examples show that Taylor series expansions can be used to evaluate challenging integrals that cannot be reduced in terms of elementary functions.

3 → Binomial series

This is a generalization of Newton's binomial expansion. We begin with a preliminary discussion of the binomial expansion. We then provide the generalization to binomial series.

- The Newton binomial expansion reads:

$$\boxed{\begin{aligned} &\forall a,b\in\mathbb{R}:\forall n\in\mathbb{N}^*: (a+b)^n = \sum_{k=0}^{n}\binom{n}{k} a^{n-k} b^k && (1)\\ &\text{with } \forall n,k\in\mathbb{N}: \binom{n}{k} = \frac{n!}{k!(n-k)!} && (2)\end{aligned}}$$

- We can also show that

$$\boxed{\begin{aligned} &\forall n\in\mathbb{N}: \binom{n}{0}=1 \wedge \binom{n}{n}=1 && (3)\\ &\forall n,k\in\mathbb{N}^*: \binom{n+1}{k} = \binom{n}{k}+\binom{n}{k-1} && (4)\end{aligned}}$$

- From Eq.(3) and Eq.(4) we immediately conclude that for all $n,k\in\mathbb{N}$, $\binom{n}{k}$ is an integer, and that it can be evaluated recursively via Pascal's triangle:

$n=1:\ 1\ \ 1$	$(a+b)^1 = a+b$
$n=2:\ 1\ \ 2\ \ 1$	$(a+b)^2 = a^2+2ab+b^2$
$n=3:\ 1\ \ 3\ \ 3\ \ 1$	$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$
$n=4:\ 1\ \ 4\ \ 6\ \ 4\ \ 1$	$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$
$n=5:\ 1\ \ 5\ \ 10\ \ 10\ \ 5\ \ 1$	$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$

- To establish the binomial expansion, we prove Eq.(3) and Eq.(4) and then use them to prove Eq.(1) by induction.

Proof of Eq.(3) and Eq.(4)

Let $n \in \mathbb{N}$ be given. Then

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{0!\,n!} = \frac{1}{0!} = \frac{1}{1} = 1$$

$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{1}{(n-n)!} = \frac{1}{0!} = \frac{1}{1} = 1$$

Let $n, k \in \mathbb{N}^*$ be given. Then:

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-(k-1))!}$$

$$= \frac{n!}{k(k-1)!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} =$$

$$= \frac{n!}{k(k-1)!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)(n-k)!} =$$

$$= \frac{n!}{(k-1)!(n-k)!}\left[\frac{1}{k} + \frac{1}{n-k+1}\right]$$

$$= \frac{n!}{(k-1)!(n-k)!}\left[\frac{(n-k+1)+k}{k(n-k+1)}\right] =$$

$$= \frac{n!}{(k-1)!(n-k)!}\,\frac{n+1}{k(n-k+1)} = \frac{(n+1)!}{k!(n-k+1)} =$$

$$= \frac{(n+1)!}{k!((n+1)-k)!} = \binom{n+1}{k}$$

Proof of Eq.(1)

Let $a, b \in \mathbb{R}$ be given.

For $n = 1$:

$$(a+b)^1 = a+b = a^1 b^0 + a^0 b^1 = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1 =$$

$$= \sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k$$

For $n=m$, we assume that

$$(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k$$

For $n=m+1$, we will show that

$$(a+b)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} a^{m+1-k} b^k$$

We have:

$$(a+b)^{m+1} = (a+b)(a+b)^m = (a+b) \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k$$

$$= \sum_{k=0}^{m} \binom{m}{k} (a+b) a^{m-k} b^k = \sum_{k=0}^{m} \binom{m}{k} (a^{m-k+1} b^k + a^{m-k} b^{k+1})$$

$$= \sum_{k=0}^{m} \binom{m}{k} a^{m-k+1} b^k + \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^{k+1}$$

$$= \sum_{k=0}^{m} \binom{m}{k} a^{(m+1)-k} b^k + \sum_{k=1}^{m+1} \binom{m}{k-1} a^{m-(k-1)} b^{(k-1)+1}$$

$$= \sum_{k=0}^{m} \binom{m}{k} a^{(m+1)-k} b^k + \sum_{k=1}^{m+1} \binom{m}{k-1} a^{(m+1)-k} b^k$$

$$= \binom{m}{0} a^{m+1} + \sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1} \right] a^{(m+1)-k} b^k + \binom{m}{m} b^{m+1}$$

$$= \binom{m+1}{0} a^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} a^{(m+1)-k} b^k + \binom{m+1}{m+1} b^{m+1}$$

$$= \sum_{k=0}^{m+1} \binom{m+1}{k} a^{(m+1)-k} b^k .$$

- The binomial expansion can be generalized for real exponents, yielding the <u>binomial series</u>, according to the following theorem:

<u>Thm:</u> $\forall a \in \mathbb{R} : \forall x \in (-1,1) : (1+x)^a = \sum_{n=0}^{+\infty} \binom{a}{n} x^n$

with $\forall a \in \mathbb{R} : \binom{a}{0} = 1$, and
$\forall a \in \mathbb{R} : \forall n \in \mathbb{N}^* \binom{a}{n} = \prod_{k=1}^{n} \left(\frac{a+1-k}{k} \right) =$
$= \frac{a(a-1)(a-2)\cdots(a+1-n)}{n!}$

<u>Proof</u>

Let $a \in \mathbb{R}$ be given and define $\forall x \in (-1,1) : f(x) = (1+x)^a$
Let $n \in \mathbb{N}^*$ be given and note that $f(0) = (1+0)^a = 1$ and

$f'(x) = a(1+x)^{a-1}$

$f''(x) = a(a-1)(1+x)^{a-2}$

$\vdots$

$f^{(n)}(x) = a(a-1)\dots(a+1-n)(1+x)^{a-n}$.

It follows that

$f^{(n)}(0) = a(a-1)\dots(a+1-n)(1+0)^{a-n} =$
$= a(a-1)\cdots(a+1-n)$, $\forall n \in \mathbb{N}^*$

and therefore the Taylor expansion of $f(x)$ is given by:

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{+\infty} \frac{a(a-1)\cdots(a+1-n)}{n!} x^n =$$

$$= \sum_{n=0}^{+\infty} \binom{a}{n} x^n.$$

To establish the convergence of this Taylor series on $(-1,1)$ we will use the ratio test and the absolute convergence test. Let $a\in\mathbb{R}$ and $x\in(-1,1)$ be given and define:

$$\forall n\in\mathbb{N}^*: \gamma_n = \binom{a}{n} x^n$$

It follows that

$$\forall n\in\mathbb{N}^*: \left|\frac{\gamma_{n+1}}{\gamma_n}\right| = \left|\frac{\binom{a}{n+1} x^{n+1}}{\binom{a}{n} x^n}\right| = \left|\frac{x^{n+1}\prod_{k=1}^{n+1}\left(\frac{a+1-k}{k}\right)}{x^n\prod_{k=1}^{n}\left(\frac{a+1-k}{k}\right)}\right| =$$

$$= \left|\frac{x\cdot\frac{a+1-(n+1)}{n+1}\prod_{k=1}^{n}\left(\frac{a+1-k}{k}\right)}{\prod_{k=1}^{n}\left(\frac{a+1-k}{k}\right)}\right|$$

$$= \left|\frac{x(a+1-n-1)}{n+1}\right| = |x|\frac{|a-n|}{n+1}$$

Since:

$$\lim_{t\to+\infty}\frac{|a-t|}{t+1} = \lim_{t\to+\infty}\frac{-(a-t)}{t+1} = \lim_{t\to+\infty}\frac{t-a}{t+1} = \lim_{t\to+\infty}\frac{t}{t} = 1$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*}\frac{|a-n|}{n+1} = 1 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*}\left|\frac{\gamma_{n+1}}{\gamma_n}\right| = \lim_{n\in\mathbb{N}^*}\left[|x|\frac{|a-n|}{n+1}\right] = |x|$$

we conclude that

$$x \in (-1,1) \Rightarrow |x| < 1 \Rightarrow \lim_{n \in \mathbb{N}^*} \left| \frac{\gamma_{n+1}}{\gamma_n} \right| < 1 \Rightarrow \sum_{n=0}^{+\infty} |\gamma_n| \text{ converges}$$

$$\Rightarrow \sum_{n=0}^{+\infty} \gamma_n \text{ converges} \Rightarrow \sum_{n=0}^{+\infty} \binom{a}{n} x^n \text{ converges.} \quad \square$$

Factorial and double factorial

Binomial series should be simplified and written in terms of the factorial $n!$ or the double factorial $n!!$ and we recall the corresponding definitions below:

1) Factorial

$$\boxed{\begin{aligned} 0! &= 1 \\ n! &= \prod_{k=1}^{n} k = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \end{aligned}}$$

2) Double factorial

$$\boxed{\begin{aligned} 0!! &= 1 \\ (2n)!! &= \prod_{k=1}^{n} (2k) = 2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n) \\ (2n+1)!! &= \prod_{k=0}^{n} (2k+1) = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1) \end{aligned}}$$

Double factorials can be simplified to factorials using the following identities:

$$\boxed{(2n)!! = 2^n n!} \qquad \boxed{(2n-1)!! = \frac{(2n)!}{2^n n!}}$$

EXAMPLE

a) Write the Taylor expansion of

$\forall x \in (-\infty, 2): \quad f(x) = \dfrac{1}{\sqrt{2-x}}$

around $x=0$, and find the radius of convergence

Solution

We have:

$$f(x) = \frac{1}{\sqrt{2-x}} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1-(x/2)}} = \frac{1}{\sqrt{2}}\left[1+\left(\frac{-x}{2}\right)\right]^{-1/2} =$$

$$= \frac{1}{\sqrt{2}} \sum_{n=0}^{+\infty} \binom{-1/2}{n}\left(\frac{-x}{2}\right)^n = \frac{1}{\sqrt{2}} \sum_{n=0}^{+\infty} \binom{-1/2}{n} \frac{(-1)^n}{2^n} x^n$$

Since.

$$\forall n \in \mathbb{N}^*: \binom{-1/2}{n} = \prod_{k=1}^{n} \frac{-1/2+1-k}{k} = \prod_{k=1}^{n} \frac{1/2-k}{k} = \prod_{k=1}^{n} \frac{1-2k}{2k}$$

$$= \frac{1}{(2n)!!} \prod_{k=1}^{n} (1-2k) = \frac{(-1)^n}{2^n n!} \prod_{k=1}^{n} (2k-1)$$

$$= \frac{(-1)^n}{2^n n!} \cdot 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) = \frac{(-1)^n (2n-1)!!}{2^n n!}$$

it follows that

$$f(x) = \frac{1}{\sqrt{2}}\left[1 + \sum_{n=1}^{+\infty} \binom{-1/2}{n} \frac{(-1)^n}{2^n} x^n\right] =$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \sum_{n=1}^{+\infty} \frac{(-1)^n (2n-1)!!}{2^n n!} \frac{(-1)^n}{2^n} x^n$$

$$= \frac{\sqrt{2}}{2} + \sum_{n=1}^{+\infty} \frac{(2n-1)!! \sqrt{2}}{2^{2n+1} n!} x^n$$

To determine the convergence radius, we note that the Binomial series requires

$$\left|\frac{-x}{2}\right| < 1 \Leftrightarrow \frac{|-x|}{2} < 1 \Leftrightarrow \frac{|x|}{2} < 1 \Leftrightarrow |x| < 2$$

$$\Leftrightarrow -2 < x < 2 \Leftrightarrow x \in (-2, 2)$$

and therefore:

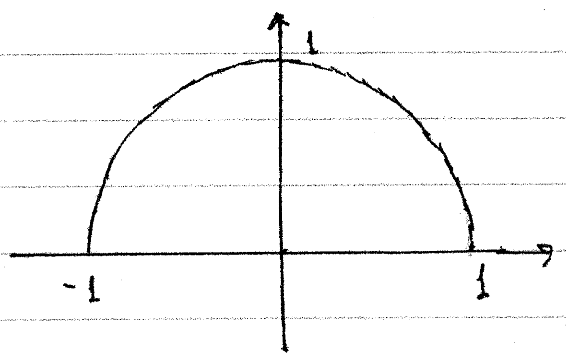
$$\forall x \in (-2,2): \frac{1}{\sqrt{2-x}} = \frac{\sqrt{2}}{2} + \sum_{n=1}^{+\infty} \frac{(2n-1)!!\sqrt{2}}{2^{2n+1}\, n!} x^n .$$

8) Yet another series for π

We will show that $$\pi = 4 - \frac{2}{3} - \sum_{n=2}^{+\infty} \frac{(2n-3)!!}{2^{n-2}(2n+1)n!}$$

Solution

We note that the function
$$\forall x \in [-1,1] : f(x) = \sqrt{1-x^2}$$
graphs a half-circle with radius 1, because
$$x^2 + [f(x)]^2 = x^2 + (\sqrt{1-x})^2 = x^2 + (1-x^2) = 1 \,,\ \forall x \in [-1,1]$$
and $\forall x \in [-1,1] : f(x) \geq 0$. Consequently the area over $[-1,1]$ is $\pi/2$, and over $[0,1]$ the area under the graph of $f(x)$ is $\pi/4$. It follows that



$$\frac{\pi}{4} = \int_0^1 \sqrt{1-x^2}\, dx \qquad (1)$$

Note that if we attempt to evaluate this integral using trigonometric substitution, this will only confirm Eq.(1). On the other hand, if we evaluate the integral by term-by-term integration of the Taylor series of $f(x)$, this will give us a series approximation for π.

$$\sqrt{1-x^2} = (1+(-x^2))^{1/2} = \sum_{n=0}^{+\infty} \binom{1/2}{n} (-x^2)^n = \sum_{n=0}^{+\infty} (-1)^n \binom{1/2}{n} x^{2n}, \forall x \in [-1,1]$$

with

$$\forall n\in\mathbb{N}^*: \binom{1/2}{n} = \prod_{k=1}^{n}\left(\frac{1/2+1-k}{k}\right) = \prod_{k=1}^{n}\frac{3/2-k}{k} =$$

$$= \prod_{k=1}^{n}\frac{3-2k}{2k} = (-1)^n\prod_{k=1}^{n}\frac{2k-3}{2k} =$$

$$= \frac{(-1)^n}{(2n)!!}\prod_{k=1}^{n}(2k-3) = \frac{(-1)^n\left[(-1)\cdot 1\cdot 3\cdot\ldots\cdot(2n-3)\right]}{2^n n!}$$

$$= \begin{cases}\dfrac{(-1)^{n+1}}{2^n n!}, & \text{if } n=1\\[2ex] \dfrac{(-1)^{n+1}(2n-3)!!}{2^n n!}, & \text{if } n>1\end{cases} =$$

$$= \begin{cases}\dfrac{1}{2}, & \text{if } n=1\\[2ex] \dfrac{(-1)^{n+1}(2n-3)!!}{2^n n!}, & \text{if } n>1\end{cases}$$

It follows that

$$\frac{\pi}{4} = \int_0^1\sqrt{1-x^2}\,dx = \int_0^1\left[\sum_{n=0}^{+\infty}(-1)^n\binom{1/2}{n}x^{2n}\right]dx =$$

$$= \sum_{n=0}^{+\infty}\left[(-1)^n\binom{1/2}{n}\int_0^1 x^{2n}\,dx\right] = \sum_{n=0}^{+\infty}\left[(-1)^n\binom{1/2}{n}\left[\frac{x^{2n+1}}{2n+1}\right]_0^1\right] =$$

$$= \sum_{n=0}^{+\infty}\left[\frac{(-1)^n}{2n+1}\binom{1/2}{n}\right] =$$

$$= \frac{(-1)^0}{2\cdot 0+1}\binom{1/2}{0} + \frac{(-1)^1}{2\cdot 1+1}\binom{1/2}{1} + \sum_{n=2}^{+\infty}\frac{(-1)^n}{2n+1}\binom{1/2}{n} =$$

$$= \frac{1}{1}\cdot 1 + \frac{-1}{3}\,\frac{1}{2} + \sum_{n=2}^{+\infty}\frac{(-1)^n}{2n+1}\,\frac{(-1)^{n+1}(2n-3)!!}{2^n n!} =$$

$$= 1 - \frac{1}{6} + \sum_{n=2}^{+\infty} \frac{(-1)^{2n+1}(2n-3)!!}{(2n+1)2^n n!} =$$

$$= 1 - \frac{1}{6} - \sum_{n=2}^{+\infty} \frac{(2n-3)!!}{(2n+1)2^n n!} \Rightarrow$$

$$\Rightarrow \pi = 4 - \frac{2}{3} - \sum_{n=2}^{+\infty} \frac{4(2n-3)!!}{(2n+1)2^n n!} =$$

$$= 4 - \frac{2}{3} - \sum_{n=2}^{+\infty} \frac{(2n-3)!!}{2^{n-2}(2n+1)n!}$$

$\hookrightarrow$ Unfortunately this series converges slowly. Using 6 terms:

$$a_6 = 4 - \frac{2}{3} - \frac{1!!}{2^0 \cdot 5 \cdot 2!} - \frac{3!!}{2^1 \cdot 7 \cdot 3!} - \frac{5!!}{2^2 \cdot 9 \cdot 4!} - \frac{7!!}{2^3 \cdot 11 \cdot 5!}$$

$$= 4 - \frac{2}{3} - \frac{1}{10} - \frac{3 \cdot 1}{2 \cdot 7 \cdot 3 \cdot 2 \cdot 1} - \frac{5 \cdot 3}{4 \cdot 9 \cdot 4 \cdot 3 \cdot 2 \cdot 1} -$$

$$- \frac{7 \cdot 5 \cdot 3}{8 \cdot 11 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} =$$

$$= 4 - \frac{2}{3} - \frac{1}{10} - \frac{1}{28} - \frac{5}{288} - \frac{7}{704}$$

$$= \frac{703049}{221760} = 3.1703\ldots \longleftarrow \text{First decimal converged}$$

4 → Product of series

Given power series for two functions $f(x)$ and $g(x)$, we may calculate the power series for $f(x)g(x)$ according to the following theory.

Def: Let a_n, b_n be two sequences with $n \in \mathbb{N}$. We define the Cauchy product $c_n = (a*b)_n$ as:

$$\boxed{\forall n \in \mathbb{N}: (a*b)_n = \sum_{k=0}^{n} a_k b_{n-k}}$$

Thm: (Merten's theorem)

Let a_n, b_n be two sequences with $n \in \mathbb{N}$. Then:

$$\left\{ \begin{array}{l} \sum_{n=0}^{+\infty} |a_n| \text{ converges} \\ \sum_{n=0}^{+\infty} b_n \text{ converges} \end{array} \right. \Rightarrow \left[\sum_{n=0}^{+\infty} a_n\right]\left[\sum_{n=0}^{+\infty} b_n\right] = \sum_{n=0}^{+\infty} (a*b)_n$$

It is important to note that the assumption

$$\left\{ \begin{array}{l} \sum_{n=0}^{+\infty} a_n \text{ converges} \\ \sum_{n=0}^{+\infty} b_n \text{ converges} \end{array} \right.$$

is not sufficiently strong to derive the conclusion of Merten's theorem. For example, using:

$\forall n \in \mathbb{N} : a_n = b_n = \dfrac{(-1)^n}{\sqrt{1+n}}$

we can prove, via the alternating test that $\sum_{n=0}^{+\infty} a_n$ and $\sum_{n=0}^{+\infty} b_n$ converges, but that $\sum_{n=0}^{+\infty} (a*b)_n$ diverges. Details are given in the counterexample below.

↳ Since any convergent power series converges absolutely (and also uniformly), we can always use Merten's theorem to multiply power series. It follows that given an interval $A = (x_0 - r, x_0 + r)$ as well as:

$$\forall x \in A : \left(f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n \wedge g(x) = \sum_{n=0}^{+\infty} b_n (x-x_0)^n \right)$$

using Merten's theorem, it follows that

$$\forall x \in A : f(x) g(x) = \left[\sum_{n=0}^{+\infty} a_n (x-x_0)^n \right] \left[\sum_{n=0}^{+\infty} b_n (x-x_0)^n \right] =$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} [a_k (x-x_0)^k][b_{n-k} (x-x_0)^{n-k}] \right]$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} a_k b_{n-k} (x-x_0)^n \right]$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} a_k b_{n-k} \right] (x-x_0)^n$$

and therefore:

$$\boxed{\left[\sum_{n=0}^{+\infty} a_n (x-x_0)^n \right] \left[\sum_{n=0}^{+\infty} b_n (x-x_0)^n \right] = \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} a_k b_{n-k} \right] (x-x_0)^n}$$

EXAMPLE

a) Write the Taylor expansion for $f(x) = e^x \cos x$ around $x=0$ and find the radius of convergence.

Solution

$$f(x) = e^x \cos x = \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right] =$$

$$= \left[\sum_{n=0}^{+\infty}\left(\frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!}\right)\right]\left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right]$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right] + \left[\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right]$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{2k}}{(2k)!} (-1)^{n-k} \frac{x^{2n-2k}}{(2n-2k)!}\right] +$$

$$+ \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{2k+1}}{(2k+1)!} (-1)^{n-k} \frac{x^{2n-2k}}{(2n-2k)!}\right]$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(2k)!\,(2n-2k)!}\right] x^{2n}$$

$$+ \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(2k+1)!\,(2n-2k)!}\right] x^{2n+1}, \quad \forall x \in \mathbb{R}.$$

The reason why the above series converges on $\mathbb{R}$ is because the series expansions of both e^x and $\cos x$ converge on $\mathbb{R}$.

↳ Note that due to the fact that the expansion of $\cos x$ has only even terms, it becomes necessary for $f(x)$ to separate the even from the odd powers of x.

b) Consider the sequence $\forall n \in \mathbb{N}^*: a_n = \dfrac{(-1)^n}{\sqrt{1+n}}$

and show that

$$\sum_{n=0}^{+\infty} a_n \text{ converges} \wedge \sum_{n=0}^{+\infty} (a*a)_n \text{ diverges.}$$

Solution

- To show that $\sum_{n=0}^{+\infty} a_n$ converges:

Define $\forall n \in \mathbb{N}^*: b_n = \dfrac{1}{\sqrt{1+n}}$

We note that $\forall n \in \mathbb{N}^* : b_n > 0$ (1)

and

$$\lim_{x \to +\infty} (1+x) = \lim_{x \to +\infty} x = +\infty \Rightarrow \lim_{x \to +\infty} \sqrt{1+x} = +\infty \Rightarrow \lim_{x \to +\infty} \frac{1}{\sqrt{1+x}} = 0$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} b_n = \lim_{n \in \mathbb{N}^*} \frac{1}{\sqrt{1+n}} = 0 \quad (2)$$

We claim that b_n decreasing.

Let $n \in \mathbb{N}^*$ be given. Then:

$$n+1 > n \Rightarrow (n+1)+1 > n+1 > 0 \Rightarrow \sqrt{(n+1)+1} > \sqrt{n+1} > 0$$

$$\Rightarrow \frac{1}{\sqrt{(n+1)+1}} < \frac{1}{\sqrt{n+1}} \Rightarrow b_{n+1} < b_n$$

and it follows that

$(\forall n \in \mathbb{N}^* : b_{n+1} < b_n) \Rightarrow b_n$ decreasing (3)

From Eq.(1), Eq.(2), Eq.(3):

$$\sum_{n=0}^{+\infty} (-1)^n b_n = \sum_{n=0}^{+\infty} \frac{(-1)^n}{\sqrt{1+n}} \text{ converges.}$$

- To show that $\sum_{n=0}^{+\infty} (a*a)_n$ diverges:

$$\forall n \in \mathbb{N}: |(a*a)_n| = \left|\sum_{k=0}^{n} a_k a_{n-k}\right| = \left|\sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}}\right|$$

$$= \left|\sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{k+1}\,\sqrt{n-k+1}}\right| = \left|(-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\,\sqrt{n-k+1}}\right|$$

$$= \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\,\sqrt{n-k+1}} \geqslant \sum_{k=0}^{n} \frac{1}{\sqrt{n+1}\,\sqrt{n+1}} =$$

$$= \sum_{k=0}^{n} \frac{1}{n+1} = \frac{n+1}{n+1} = 1$$

$$\Rightarrow \left(\forall n \in \mathbb{N}: |(a*a)_n| \geq 1\right) \Rightarrow \lim_{n\in\mathbb{N}} (a*a)_n \neq 0 \Rightarrow$$

$$\Rightarrow \sum_{n=0}^{+\infty} (a*a)_n \text{ diverges.}$$

↳ This counterexample demonstrates that absolute convergence is needed for at least one of the two convergent series in order to guarantee the convergence of the series defined via the Cauchy product.

c) Write the Taylor expansion of

$$\forall x \in \mathbb{R} - \{-1/2\} : f(x) = \frac{e^x}{2x+1}$$

around $x=0$, and find the radius of convergence.

Solution

We have:

$$f(x) = \frac{e^x}{2x+1} = e^x \cdot \frac{1}{1-(-2x)} = \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-2x)^n\right] =$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-2)^n x^n\right] =$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{1}{k!} \cdot (-2)^{n-k}\right] x^n =$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-2)^{n-k}}{k!}\right] x^n$$

For convergence, we require

$$|-2x| < 1 \Leftrightarrow |2x| < 1 \Leftrightarrow 2|x| < 1 \Leftrightarrow |x| < 1/2 \Leftrightarrow$$

$$\Leftrightarrow x \in (-1/2, +1/2)$$

d) Write the Taylor expansion of

$\forall x \in (-1, +\infty): f(x) = \dfrac{\ln(x+1)}{x^2+2x+1}$

around $x=0$ and find the radius of convergence.

Solution

Since:

$$f(x) = \frac{\ln(x+1)}{x^2+2x+1} = \frac{\ln(x+1)}{(x+1)^2} = [\ln(x+1)](x+1)^{-2} =$$

$$= \left[\sum_{n=1}^{+\infty} \frac{(-1)^{n+1} x^n}{n}\right]\left[\sum_{n=0}^{+\infty} \binom{-2}{n} x^n\right] =$$

$$= x\left[\sum_{n=1}^{+\infty} \frac{(-1)^{n+1} x^{n-1}}{n}\right]\left[\sum_{n=0}^{+\infty} \binom{-2}{n} x^n\right]$$

$$= x\left[\sum_{n=0}^{+\infty} \frac{(-1)^{n+2} x^n}{n+1}\right]\left[\sum_{n=0}^{+\infty} \binom{-2}{n} x^n\right]$$

$$= x\left[\sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} \binom{-2}{k} \frac{(-1)^{n-k}}{n-k+1}\right) x^n\right]$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \binom{-2}{k} \frac{(-1)^{n-k}}{n-k+1}\right] x^{n+1}$$

with

$$\binom{-2}{k} = \prod_{a=1}^{k} \frac{-2+1-a}{a} = \prod_{a=1}^{k} \frac{-1-a}{a} = (-1)^k \prod_{a=1}^{k} \frac{a+1}{a}$$

$$= (-1)^k \frac{2 \cdot 3 \cdot \ldots \cdot (k+1)}{1 \cdot 2 \cdot \ldots \cdot k} = \frac{(-1)^k (k+1)!}{k!} =$$

$$= \frac{(-1)^k (k+1) k!}{k!} = (-1)^k (k+1)$$

it follows that

$$f(x) = \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} (-1)^k (k+1) \frac{(-1)^{n-k}}{n-k+1} \right] x^{n+1} =$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^n (k+1)}{n-k+1} \right] x^{n+1} =$$

$$= \sum_{n=0}^{+\infty} (-1)^n \left[\sum_{k=0}^{n} \frac{k+1}{n-k+1} \right] x^{n+1}$$

Convergence requires $x \in (-1, 1)$ for the expansion of $\ln(x+1)$ and $(x+1)^{-2}$.