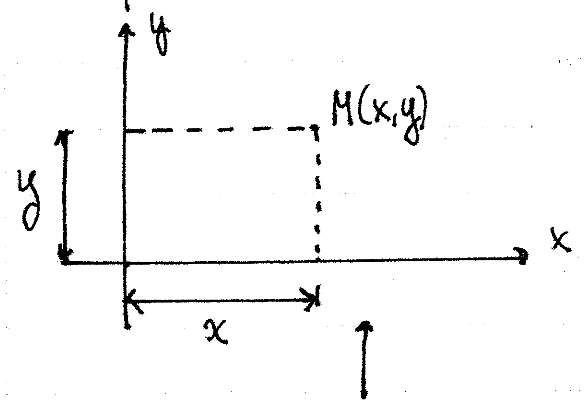
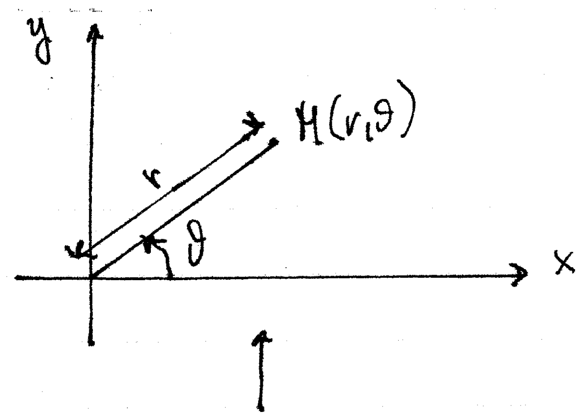
PARAMETRIC CURVES

Definition of parametric curves.

- Parametric curves can be defined either in cartesian or polar coordinates



Cartesian coordinates



Polar coordinates

- The relationship between cartesian coordinates (x,y) and polar coordinates (r,ϑ) is given by:

$$\boxed{\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}}$$

and conversely, by:

$r = \sqrt{x^2+y^2}$	$\vartheta = \begin{cases} \mathrm{Arctan}(y/x) & , \text{ if } x>0 \\ \pi - \mathrm{Arctan}(y/x) & , \text{ if } x<0 \\ \pi/2 & , \text{ if } x=0 \wedge y>0 \\ 3\pi/2 & , \text{ if } x=0 \wedge y<0 \end{cases}$

1 → Cartesian curves

The parametric curve (c) given by:

$$(c): \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a, b]$$

is defined as the set of points:

$$(c) = \{ (f(t), g(t)) \mid t \in [a, b] \}$$

and the corresponding belonging condition is

$$(x, y) \in (c) \iff \exists t \in [a, b] : (x = f(t) \land y = g(t))$$

2 → Polar curves

The parametric curve (c) given by

$$(c): r = f(\theta), \quad \theta \in [a, b]$$

is defined as the cartesian parametric curve

$$(c): \begin{cases} x = f(\theta) \cos\theta \\ y = f(\theta) \sin\theta \end{cases}, \quad \theta \in [a, b]$$

The corresponding set of points is given by

$$(c) = \{ (f(\theta) \cos\theta, f(\theta) \sin\theta) \mid \theta \in [a, b] \}$$

and its belonging condition is given by

$$(x, y) \in (c) \iff \exists \theta \in [a, b] : (x = f(\theta) \cos\theta \land y = f(\theta) \sin\theta)$$

EXAMPLES

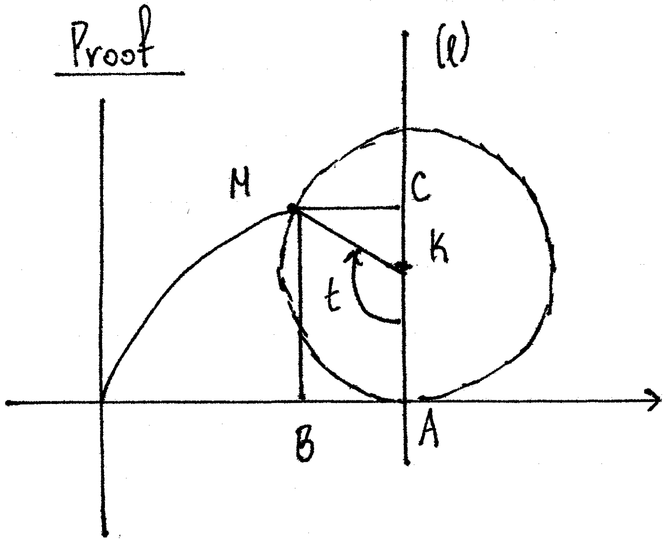
a) The circle with center (x_0, y_0) and radius r is given by:

$$(c): \begin{cases} x = r\cos\vartheta + x_0 \\ y = r\sin\vartheta + y_0 \end{cases}, \quad \vartheta \in [0, 2\pi]$$

b) The cycloid is the curve traced by a fixed point on a circle if the circle "rolls" forward on the x-axis. If the circle has radius a, then the cycloid is given by

$$\boxed{(c): \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, \quad t \in \mathbb{R}}$$

Proof



Let O be the origin, K be the center of the rolling circle, and M a fixed point on the circle. Let A be the point of contact between the circle and the x-axis. Let B the projection of M onto the x-axis.

Let C be the projection of M onto the line (ℓ) defined by the points K and A. We note that

$OA = \text{arc } \overset{\frown}{AM} = at$

$MC = MK \sin(\widehat{MKC}) = a \sin(\pi - t) = a\sin t$

$KC = MK \cos(\widehat{MKC}) = a \cos(\pi - t) = -a\cos t$

It follows that if $M(x,y)$ (i.e. the point M has coordinates (x,y)) then:

$x = OB = OA - AB = OA - MC = at - a\sin t = a(t - \sin t)$

$y = BM = AC = AK + KC = a - a\cos t = a(1 - \cos t)$. $\square$

▼ Calculus on parametric evaluations

- Consider a parametric curve (c) given by

$$(c): \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a,b]$$

Then, we define the cartesian derivative dy/dx as a function of the parameter t with $t \in A$ and $A \subseteq [a,b]$ such that

$$\forall t \in A: \left.\frac{dy}{dx}\right|_t = \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{f(t+\Delta t) - f(t)}$$

- Our main result is the following theorem:

> Consider a parametric curve (c) given by
>
> $$(c): \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a,b]$$
>
> and assume that
>
> $$\begin{cases} f, g \text{ differentiable on } [a,b] \\ f', g' \text{ continuous on } [a,b] \\ \forall t \in (a,b): f'(t) \neq 0 \end{cases}$$
>
> Then, it follows that:
>
> a) There is a function $F: f([a,b]) \to \mathbb{R}$ such that
> $$\forall (x,y) \in (c): F(x) = y$$
>
> b) $\forall t \in [a,b]: \left.\frac{dy}{dx}\right|_t = F'(f(t)) = \frac{g'(t)}{f'(t)}$
>
> c) $\forall t_1, t_2 \in [a,b]: \int_{f(t_1)}^{f(t_2)} F(x)\,dx = \int_{t_1}^{t_2} g(t) f'(t)\,dt$

Proof

We need the following preliminary claim:

$(\forall t \in (a,b): f'(t) > 0) \vee (\forall t \in (a,b): f'(t) < 0)$

To show the claim, we assume the opposite statement

$(\exists t \in (a,b): f'(t) < 0) \wedge (\exists t \in (a,b): f'(t) > 0)$

Fix $t_1, t_2 \in (a,b)$ such that $f'(t_1) > 0$ and $f'(t_2) < 0$ and assume, with no loss of generality, that $t_1 < t_2$. Then

$$\left.\begin{array}{l} f'(t_1)f'(t_2) < 0 \\ f' \text{ continuous on } [t_1, t_2] \end{array}\right\} \Rightarrow \exists t_0 \in [t_1, t_2] : f'(t_0) = 0$$

which is a contradiction since $\forall t \in (a,b): f'(t) \neq 0$.

This proves a claim.

- Proof of (a): Assume, with no loss of generality that,

$(\forall t \in (a,b): f'(t) > 0) \Rightarrow f$ strictly increasing on $[a,b]$

$\Rightarrow f$ has an inverse function f^{-1}.

Using the, now established, f^{-1}, we define $F: f([a,b]) \to \mathbb{R}$:

$\forall x \in f([a,b]): F(x) = (g \circ f^{-1})(x)$

We will now show that $\forall (x,y) \in (c): F(x) = y$.

Let $(x,y) \in (c)$ be given. Then $\exists t \in [a,b]: (x = f(t) \wedge y = g(t))$.

Fix a $t \in [a,b]$ such that $x = f(t)$ and $y = g(t)$. Then $t = f^{-1}(x)$ and it follows that

$y = g(t) = g(f^{-1}(x)) = (g \circ f^{-1})(x) = F(x)$.

We have thus shown that $\forall (x,y) \in (c): y = F(x)$.

- Proof of (b)

Let $t \in [a,b]$ be given. Then

$g'(t) = (g \circ id([a,b]))'(t) = (g \circ (f^{-1} \circ f))'(t) = ((g \circ f^{-1}) \circ f)'(t) =$

$$= (F\circ f)'(t) = F'(f(t))\,f'(t) \Rightarrow F'(f(t)) = \frac{g'(t)}{f'(t)}$$

and

$$\left.\frac{dy}{dx}\right|_t = \lim_{\Delta t\to 0} \frac{g(t+\Delta t)-g(t)}{f(t+\Delta t)-f(t)} = \lim_{\Delta t\to 0} \frac{\left(\dfrac{g(t+\Delta t)-g(t)}{\Delta t}\right)}{\left(\dfrac{f(t+\Delta t)-f(t)}{\Delta t}\right)} =$$

$$= \frac{\displaystyle\lim_{\Delta t\to 0} \frac{g(t+\Delta t)-g(t)}{\Delta t}}{\displaystyle\lim_{\Delta t\to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t}} = \frac{g'(t)}{f'(t)}$$

It follows that

$$\forall t\in[a,b]: \left.\frac{dy}{dx}\right|_t = F'(f(t)) = \frac{g'(t)}{f'(t)}$$

• <u>Proof of (c)</u>

Let $t_1, t_2 \in [a,b]$ be given. Then:

$$I = \int_{f(t_1)}^{f(t_2)} F(x)\,dx = \int_{f(t_1)}^{f(t_2)} (g\circ f^{-1})(x)\,dx = \int_{f(t_1)}^{f(t_2)} g(f^{-1}(x))\,dx$$

Let $t = f^{-1}(x)$. Then: $x = f(t) \Rightarrow dx = f'(t)\,dt$.

For $x = f(t_1) \Rightarrow t = f^{-1}(f(t_1)) = t_1$

For $x = f(t_2) \Rightarrow t = f^{-1}(f(t_2)) = t_2$

It follows that

$$I = \int_{t_1}^{t_2} g(t)\cdot f'(t)\,dt$$

and therefore

$$\forall t_1, t_2 \in [a,b]: \int_{f(t_1)}^{f(t_2)} F(x)\,dx = \int_{t_1}^{t_2} g(t)\,f'(t)\,dt$$

$\square$

Methodology: Derivatives dy/dx and d^2y/dx^2

The previous result for calculating dy/dx can be rewritten as:

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

Applying this result onto itself gives the second derivative d^2y/dx^2 which is given by:

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}}$$

EXAMPLES

(a) Consider the parametric curve

$$(c): \begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}, \quad t \in \mathbb{R}$$

a) Find all points where (c) has a horizontal tangent.

b) Find the concavity of (c) in terms of $t \in \mathbb{R}$.

Solution

a) $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(d/dt)(t^3-3t)}{(d/dt)t^2} = \frac{3t^2-3}{2t} =$$

$$= \frac{3(t^2-1)}{2t} = \frac{3(t-1)(t+1)}{2t}$$

We have a horizontal tangent at $t \Leftrightarrow dy/dx = 0 \Leftrightarrow$

$$\Leftrightarrow \frac{3(t-1)(t+1)}{2t} = 0 \Leftrightarrow 3(t-1)(t+1) = 0 \Leftrightarrow$$

$$\Leftrightarrow t-1=0 \vee t+1=0 \Leftrightarrow t=1 \vee t=-1$$

For $t=1$: $x = t^2 = 1^2 \wedge y = t^3 - 3t = 1^3 - 3 \cdot 1 = 1 - 3 = -2$

thus $(x,y) = (1,-2)$

For $t=-1$: $x = t^2 = (-1)^2 = 1 \wedge y = t^3 - 3t = (-1)^3 - 3(-1) = -1+3 = 2$

thus $(x,y) = (1,2)$

B)

$$\frac{d^2y}{dx^2} = \frac{(d/dt)(dy/dx)}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{(d/dt)t^2} =$$

$$= \frac{1}{2t} \cdot \frac{(3t^2-3)'(2t) - (3t^2-3)(2t)'}{4t^2} =$$

$$= \frac{(6t)(2t) - (3t^2-3)2}{8t^3} = \frac{12t^2 - 6t^2 + 6}{8t^3} =$$

$$= \frac{6t^2+6}{8t^3} = \frac{3t^2+3}{4t^3} = \frac{3(t^2+1)}{4t^3}$$

Sign table:

t		0	
$3(t^2+1)$	$+$		$+$
$4t^3$	$-$	0	$+$
d^2y/dx^2	$-$	$\nparallel$	$+$
$y=F(x)$	$\cap$		$\cup$

and therefore (c) is concave down for $t \in (-\infty, 0)$ and (c) is concave up for $t \in (0, +\infty)$.

8) Consider the polar curve $(c): r = 1+\sin\vartheta$. Evaluate the derivative dy/dx and find all points where the tangent line to (c) is horizontal.

Solution

We note that

$$(c): \begin{cases} x = (1+\sin\vartheta)\cos\vartheta \\ y = (1+\sin\vartheta)\sin\vartheta \end{cases}, \quad \vartheta \in [0, 2\pi).$$

Then:

$$dx/d\vartheta = [(1+\sin\vartheta)\cos\vartheta]' = (1+\sin\vartheta)'\cos\vartheta + (1+\sin\vartheta)(\cos\vartheta)' =$$
$$= \cos\vartheta\cos\vartheta + (1+\sin\vartheta)(-\sin\vartheta) = \cos^2\vartheta - \sin\vartheta - \sin^2\vartheta$$
$$= 1-\sin^2\vartheta - \sin\vartheta - \sin^2\vartheta = (1-\sin\vartheta)(1+\sin\vartheta) - \sin\vartheta(1+\sin\vartheta) =$$
$$= (1-\sin\vartheta-\sin\vartheta)(1+\sin\vartheta) = (1-2\sin\vartheta)(1+\sin\vartheta)$$

and

$$dy/d\vartheta = [(1+\sin\vartheta)\sin\vartheta]' = (1+\sin\vartheta)'\sin\vartheta + (1+\sin\vartheta)(\sin\vartheta)' =$$
$$= \cos\vartheta\sin\vartheta + (1+\sin\vartheta)\cos\vartheta = \cos\vartheta[\sin\vartheta + (1+\sin\vartheta)]$$
$$= \cos\vartheta(2\sin\vartheta + 1).$$

and therefore

$$\frac{dy}{dx} = \frac{dy/d\vartheta}{dx/d\vartheta} = \frac{\cos\vartheta(2\sin\vartheta+1)}{(1-2\sin\vartheta)(1+\sin\vartheta)}$$

The tangent line to (c) at ϑ is horizontal $\Leftrightarrow dy/dx = 0$

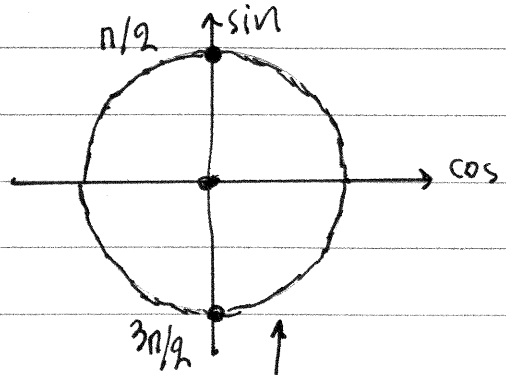
$$\Leftrightarrow \frac{\cos\vartheta(2\sin\vartheta+1)}{(1-2\sin\vartheta)(1+\sin\vartheta)} = 0 \leftarrow \text{Require } (1-2\sin\vartheta)(1+\sin\vartheta) \neq 0$$

$$\Leftrightarrow \cos\vartheta(2\sin\vartheta+1) = 0 \Leftrightarrow \cos\vartheta = 0 \lor 2\sin\vartheta + 1 = 0$$

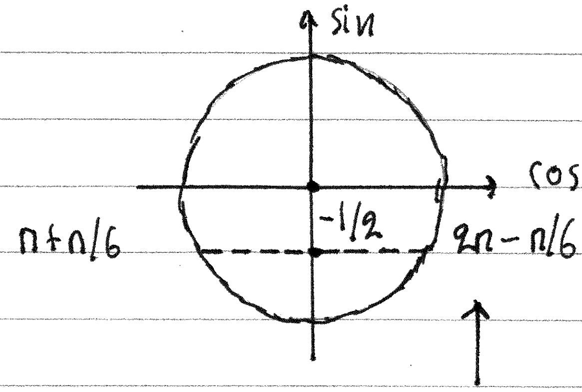
$$\Leftrightarrow \cos\vartheta = 0 \lor \sin\vartheta = \frac{-1}{2} = \sin\left(\frac{-\pi}{6}\right) \Leftrightarrow$$

$\Leftrightarrow (\vartheta = \pi/2 \vee \vartheta = 3\pi/2) \vee (\vartheta = 2\pi - \pi/6 \vee \vartheta = \pi + \pi/6)$

$\Leftrightarrow \vartheta = \pi/2 \vee \vartheta = 3\pi/2 \vee \vartheta = 11\pi/6 \vee \vartheta = 7\pi/6$



$\cos\vartheta = 0$



$\sin\vartheta = -1/2 = \sin(-\pi/6)$

From these solutions, we need to reject all solutions that violate the requirement

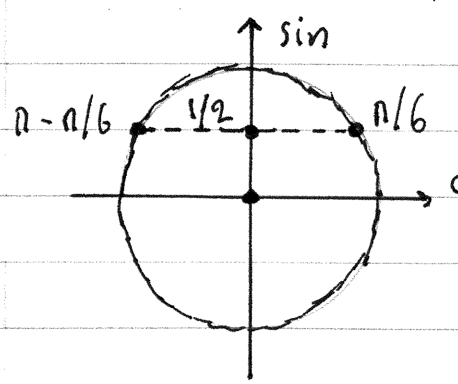
$(1 - 2\sin\vartheta)(1 + \sin\vartheta) \neq 0$

therefore, we solve the equation:

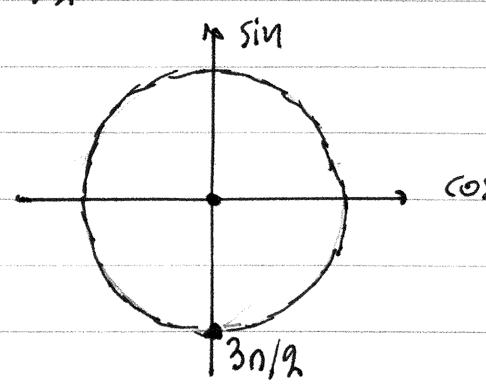
$(1 - 2\sin\vartheta)(1 + \sin\vartheta) = 0 \Leftrightarrow 1 - 2\sin\vartheta = 0 \vee 1 + \sin\vartheta = 0$

$\Leftrightarrow \sin\vartheta = 1/2 = \sin(\pi/6) \vee \sin\vartheta = -1 = \sin(3\pi/2)$

$\Leftrightarrow (\vartheta = \pi/6 \vee \vartheta = 5\pi/6) \vee \vartheta = 3\pi/2$



$\sin\vartheta = 1/2 = \sin(\pi/6)$



$\sin\vartheta = -1 = \sin(3\pi/2)$

It follows that the solution $\vartheta = 3\pi/2$ is rejected and all other solutions are accepted. We conclude that:

$dy/dx = 0 \Leftrightarrow \vartheta \in \{\pi/2, 11\pi/6, 7\pi/6\}$.

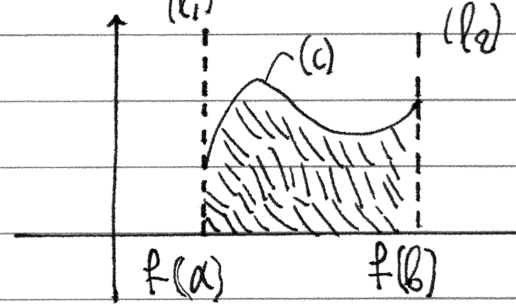
Methodology: Area under the curve (c).

From the previous theorem, the area between the curve (c) defined as

$$(c): \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a, b]$$

the x-axis, and the lines $(l_1): x = f(a)$ and $(l_2): x = f(b)$ is given by

$$\boxed{A = \int_a^b g(t) f'(t)\,dt}$$



EXAMPLE

Find the area under the curve (c) (the cycloid) defined as

$$(c): \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, \quad t \in [0, 2\pi]$$

from $t = 0$ to $t = 2\pi$.

Solution

Define $\forall t \in [0, 2\pi]: \begin{cases} f(t) = a(t - \sin t) \\ g(t) = a(1 - \cos t) \end{cases}$

Then the area under (c) from $t = 0$ to $t = 2\pi$ is:

$$A = \int_0^{2\pi} g(t) f'(t)\,dt = \int_0^{2\pi} a(1-\cos t)[a(t - \sin t)]'\,dt =$$

$$= \int_0^{2\pi} a(1-\cos t)\, a(1 - \cos t)\,dt = a^2 \int_0^{2\pi} (1 - \cos t)^2\,dt =$$

$$= a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\, dt =$$

$$= a^2 \int_0^{2\pi} (1 - 2\cos t)\, dt + a^2 \int_0^{2\pi} \frac{1 + \cos(2t)}{2}\, dt$$

$$= a^2 \Big[t - 2\sin t \Big]_0^{2\pi} + \frac{a^2}{2} \Big[t + \frac{\sin(2t)}{2} \Big]_0^{2\pi} =$$

$$= a^2 \Big[(2\pi - 2\sin(2\pi)) - (0 - 2\sin 0) \Big] + \frac{a^2}{2} \Big[\Big(2\pi + \frac{\sin(4\pi)}{2}\Big) - \Big(0 + \frac{\sin 0}{2}\Big) \Big]$$

$$= a^2 \Big[(2\pi - 0) - (0 - 0) \Big] + \frac{a^2}{2} \Big[(2\pi + 0) - (0 + 0) \Big] =$$

$$= 2\pi a^2 + \pi a^2 = 3\pi a^2.$$

Arclength of parametric curve

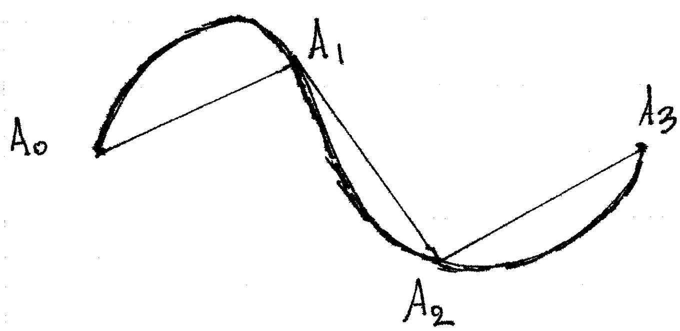
- Consider a curve (c) given by

$$(c): \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a,b]$$

We subdivide the interval $[a,b]$ into ~~se~~ equal subintervals $[t_{k-1}, t_k]$ with $k \in [n]$ such that $a = t_0 < t_1 < \cdots < t_n = b$ with $\forall k \in \{0\} \cup [n] : t_k = a + (b-a)(k/n)$.

We can the approximate the length of the curve (c) with the length of a polygonal line passing through the points $A_k(f(t_k), g(t_k))$ with $k \in \{0\} \cup [n]$, whose length is given by:

$$\boxed{L_n(c) = \sum_{k=1}^{n} \sqrt{(f(t_k) - f(t_{k-1}))^2 + (g(t_k) - g(t_{k-1}))^2}}$$



Def: We say that the curve (c) is <u>rectifiable</u> if and only if the sequence $L_n(c)$ converges. If it does, then the <u>arclength</u> $\ell(c)$ of (c) is defined as:

$$\ell(c) = \lim_{n \in \mathbb{N}^*} L_n(c)$$

Thm: Let (c) be the curve defined as

$$(c): \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a,b]$$

and assume that:

$$\begin{cases} f, g \text{ differentiable on } [a,b] \\ f', g' \text{ continuous on } [a,b] \end{cases}$$

Then (c) is rectifiable with arclength given by

$$\ell(c) = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Proof

The polygonal approximation to the arclength of the curve (c) is given by:

$$\forall n \in \mathbb{N}^*: L_n(c) = \sum_{k=1}^{n} \sqrt{[f(t_n(k)) - f(t_n(k-1))]^2 + [g(t_n(k)) - g(t_n(k-1))]^2}$$

with

$$\forall n \in \mathbb{N}^*: \forall k \in \{0\} \cup [n]: t_n(k) = a + (b-a)(k/n)$$

Let $n \in \mathbb{N}^*$ be given. Using the mean-value theorem, we have:

$$f, g \text{ differentiable on } [a,b] \Rightarrow \forall k \in [n]: \begin{cases} f \text{ differentiable on } [t_n(k-1), t_n(k)] \\ g \text{ differentiable on } [t_n(k-1), t_n(k)] \end{cases}$$

$$\Rightarrow \forall k \in [n]: \exists \tau_n(k) \in (t_n(k-1), t_n(k)): \begin{cases} f(t_n(k)) - f(t_n(k-1)) = f'(\tau_n(k))(t_n(k) - t_n(k-1)) \\ g(t_n(k)) - g(t_n(k-1)) = g'(\tau_n(k))(t_n(k) - t_n(k-1)) \end{cases}$$

We note that $t_n(k) - t_n(k-1) = (b-a)/n$, and therefore:

$$\ell(C) = \lim_{n\in\mathbb{N}^*} L_n(C) =$$

$$= \lim_{n\in\mathbb{N}^*} \sum_{k=1}^{n} \sqrt{[f(t_n(k)) - f(t_n(k-1))]^2 + [g(t_n(k)) - g(t_n(k-1))]^2}$$

$$= \lim_{n\in\mathbb{N}^*} \sum_{k=1}^{n} \sqrt{[f'(\tau_n(k))(b-a)/n]^2 + [g'(\tau_n(k))(b-a)/n]^2}$$

$$= \lim_{n\in\mathbb{N}^*} \sum_{k=1}^{n} \sqrt{(b-a)^2/n^2}\sqrt{[f'(\tau_n(k))]^2 + [g'(\tau_n(k))]^2}$$

$$= \lim_{n\in\mathbb{N}^*} \left[\frac{b-a}{n} \sum_{k=1}^{n} \sqrt{[f'(\tau_n(k))]^2 + [g'(\tau_n(k))]^2}\right]$$

$$= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

The very last step follows from the assumption that f', g' continuous on $[a,b]$ which implies that

$$h(t) = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

is integrable on $[a,b]$. $\square$

EXAMPLE

Find the arclength of the cycloid (c) given by

$$(c): \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, \quad t \in [0, 2\pi]$$

Solution

Since

$$\forall t \in [0, 2\pi]: \ dx/dt = a\,(d/dt)(t - \sin t) = a(1 - \cos t)$$

$$\forall t \in [0, 2\pi]: \ dy/dt = a\,(d/dt)(1 - \cos t) = + a \sin t$$

it follows that

$$\forall t \in [0, 2\pi]: \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2(1 - \cos t)^2 + (+a \sin t)^2 =$$

$$= a^2(1 - 2\cos t + \cos^2 t) + a^2 \sin^2 t$$

$$= a^2 - 2a^2 \cos t + a^2(\cos^2 t + \sin^2 t)$$

$$= a^2 - 2a^2 \cos t + a^2 = 2a^2 - 2a^2 \cos t =$$

$$= 2a^2(1 - \cos t) = 4a^2 \sin^2(t/2)$$

and therefore:

$$l(c) = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{4a^2 \sin^2(t/2)}\, dt$$

$$= \int_0^{2\pi} 2|a|\,|\sin(t/2)|\, dt = 2|a| \int_0^{2\pi} \sin(t/2)\, dt =$$

$$= 2|a| \left[\frac{-\cos(t/2)}{1/2}\right]_0^{2\pi} = 4|a|\left[-\cos(t/2)\right]_0^{2\pi} =$$

$$= 4|a|\left[(-\cos(2\pi/2)) - (-\cos 0)\right] = 4|a|\left[-\cos(\pi) + \cos 0\right]$$

$$= 4|a|\left[-(-1) + 1\right] = 4|a|\left[1 + 1\right] = 8|a|$$

Arclength of a polar curve

Thm: Consider a polar curve $(c): r = f(\vartheta)$, $\vartheta \in [a,b]$ and assume that

$$\begin{cases} f \text{ differentiable on } [a,b] \\ f' \text{ continuous on } [a,b] \end{cases}$$

Then

$$\ell(c) = \int_a^b \sqrt{[f(\vartheta)]^2 + [f'(\vartheta)]^2}\, d\vartheta$$

Proof

Since, by definition

$$(c): \begin{cases} x = f(\vartheta)\cos\vartheta \\ y = f(\vartheta)\sin\vartheta \end{cases}, \quad \vartheta \in [a,b]$$

it follows that for any $\vartheta \in [a,b]$:

$$dx/d\vartheta = (d/d\vartheta)[f(\vartheta)\cos\vartheta] = f'(\vartheta)\cos\vartheta + f(\vartheta)(\cos\vartheta)' =$$
$$= f'(\vartheta)\cos\vartheta + f(\vartheta)[-\sin\vartheta] = f'(\vartheta)\cos\vartheta - f(\vartheta)\sin\vartheta$$

$$dy/d\vartheta = (d/d\vartheta)[f(\vartheta)\sin\vartheta] = f'(\vartheta)\sin\vartheta + f(\vartheta)(\sin\vartheta)' =$$
$$= f'(\vartheta)\sin\vartheta + f(\vartheta)\cos\vartheta$$

and therefore

$$(dx/d\vartheta)^2 + (dy/d\vartheta)^2 = [f'(\vartheta)\cos\vartheta - f(\vartheta)\sin\vartheta]^2 + [f'(\vartheta)\sin\vartheta + f(\vartheta)\cos\vartheta]^2$$
$$= [f'(\vartheta)]^2\cos^2\vartheta - 2f(\vartheta)f'(\vartheta)\sin\vartheta\cos\vartheta + [f(\vartheta)]^2\sin^2\vartheta$$
$$+ [f'(\vartheta)]^2\sin^2\vartheta + 2f(\vartheta)f'(\vartheta)\sin\vartheta\cos\vartheta + [f(\vartheta)]^2\cos^2\vartheta =$$
$$= [f'(\vartheta)]^2(\cos^2\vartheta + \sin^2\vartheta) + [f(\vartheta)]^2(\cos^2\vartheta + \sin^2\vartheta) =$$
$$= [f'(\vartheta)]^2 + [f(\vartheta)]^2.$$

It follows that the arclength of (c) is given by:

$$\ell(c) = \int_a^b \sqrt{(dx/d\vartheta)^2 + (dy/d\vartheta)^2}\, d\vartheta =$$

$$= \int_a^b \sqrt{[f(\vartheta)]^2 + [f'(\vartheta)]^2}\, d\vartheta \qquad \square$$

EXAMPLE

Consider the curve (c): $r = 1+\sin\vartheta$, $\vartheta\in[0,2\pi]$. Evaluate the arclength of (c).

Solution

Define $\forall\vartheta\in[0,2\pi]: f(\vartheta) = 1+\sin\vartheta$. Then:

$\forall\vartheta\in[0,2\pi]: f'(\vartheta) = \cos\vartheta$.

and therefore

$$[f(\vartheta)]^2+[f'(\vartheta)]^2 = (1+\sin\vartheta)^2+\cos^2\vartheta = 1+2\sin\vartheta+\sin^2\vartheta+\cos^2\vartheta =$$

$$= 1+2\sin\vartheta+1 = 2+2\sin\vartheta = 2(1+\sin\vartheta) =$$

$$= 2[1+\cos(\pi/2-\vartheta)] = 4\cos^2(\pi/4-\vartheta/2),\ \forall\vartheta\in[0,2\pi].$$

It follows that

$$\ell(c) = \int_0^{2\pi}\sqrt{[f(\vartheta)]^2+[f'(\vartheta)]^2}\,d\vartheta = \int_0^{2\pi}\sqrt{4\cos^2(\pi/4-\vartheta/2)}\,d\vartheta =$$

$$= \int_0^{2\pi} 2\left|\cos(\pi/4-\vartheta/2)\right| d\vartheta$$

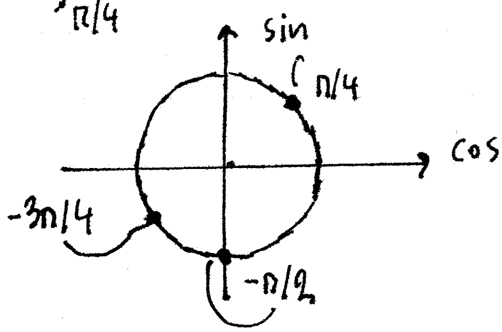
Let $\varphi = \pi/4-\vartheta/2 \Rightarrow d\varphi = (-1/2)\,d\vartheta \Rightarrow d\vartheta = -2\,d\varphi$

For $\vartheta=0$: $\varphi = \pi/4-0/2 = \pi/4$

For $\vartheta=2\pi$: $\varphi = \pi/4-(2\pi)/2 = \pi/4-\pi = -3\pi/4$

and therefore

$$\ell(c) = \int_{\pi/4}^{-3\pi/4} 2|\cos\varphi|(-2)\,d\varphi = 4\int_{-3\pi/4}^{\pi/4}|\cos\varphi|\,d\varphi$$



Note that:

$\forall\varphi\in[-3\pi/4,-\pi/2]: \cos\varphi\leq 0$

$\forall\varphi\in[-\pi/2,\pi/4]: \cos\varphi\geq 0$

Removing the absolute value gives:

$$l(c) = -4\int_{-3\pi/4}^{-\pi/2} \cos\varphi \, d\varphi + 4\int_{-\pi/2}^{\pi/4} \cos\varphi \, d\varphi = -4\Big[\sin\varphi\Big]_{-3\pi/4}^{-\pi/2} + 4\Big[\sin\varphi\Big]_{-\pi/2}^{\pi/4} =$$

$$= -4\left[\sin\left(\frac{-\pi}{2}\right) - \sin\left(\frac{-3\pi}{4}\right)\right] + 4\left[\sin\left(\frac{\pi}{4}\right) - \sin\left(\frac{-\pi}{2}\right)\right] =$$

$$= -4\left[-\sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{3\pi}{4}\right)\right] + 4\left[\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right)\right] =$$

$$= -4\left[-1 + \sin(\pi - \pi/4)\right] + 4\left[\frac{\sqrt{2}}{2} + 1\right] =$$

$$= -4\left[-1 + \sin(\pi/4)\right] + 2\sqrt{2} + 4 =$$

$$= -4\left[-1 + \frac{\sqrt{2}}{2}\right] + 2\sqrt{2} + 4 = 4 - 2\sqrt{2} + 2\sqrt{2} + 4 = 8$$