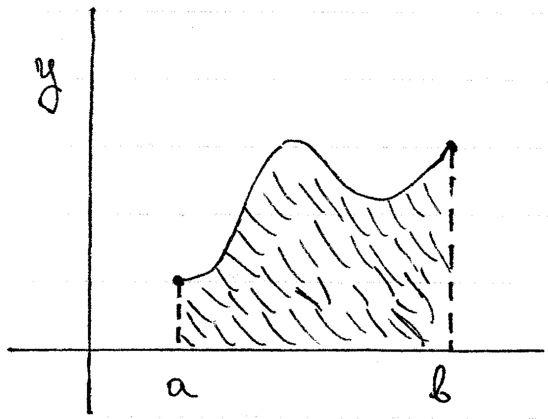
INTEGRAL EQUATIONS

▼ Riemann integral definition



The problem is to calculate the area A between the x-axis, the lines $(l_1): x = a$ and $(l_2): x = b$, and the curve $(c): y = f(x)$. The solution of the problem, according to Riemann, is as follows:

- 1 Divide the interval $[a,b]$ to n equal intervals $[x_{k-1}, x_k]$ with

$$x_k = a + (b-a)(k/n) \quad , \quad \forall k \in [n] \cup \{0\}$$

with $[n] = \{1, 2, \ldots, n\}$.

- 2 Let m_k and M_k be the minimum and maximum value of f in the interval $[x_{k-1}, x_k]$, defined as:

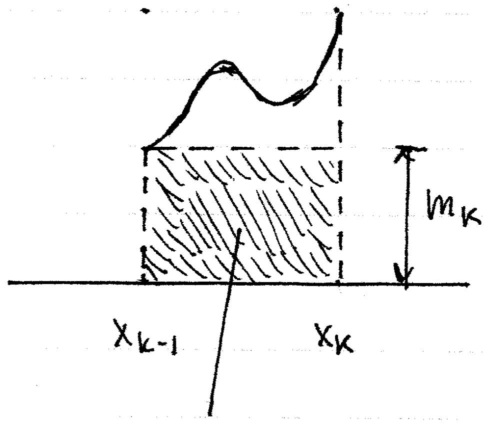
$$m_k(f \mid a,b,n) = \min_{x \in [x_{k-1}, x_k]} f(x) \quad , \quad \forall k \in [n]$$

$$M_k(f \mid a,b,n) = \max_{x \in [x_{k-1}, x_k]} f(x) \quad , \quad \forall k \in [n]$$

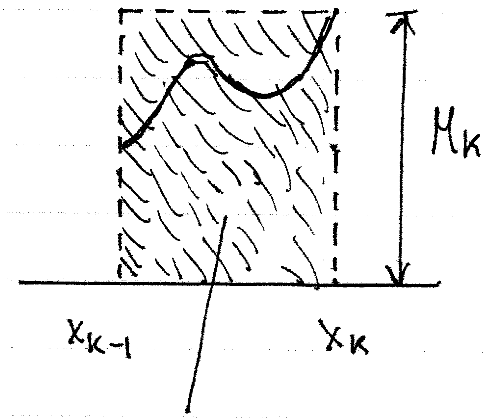
- 3 We form the Riemann sums $L_n(f \mid a,b)$ and $U_n(f \mid a,b)$ given by:

$$L_n(f|a,b) = \sum_{k=1}^{n} m_k(f|a,b,n)(x_k - x_{k-1})$$

$$U_n(f|a,b) = \sum_{k=1}^{n} M_k(f|a,b,n)(x_k - x_{k-1})$$



$m_K(f|a,b,n)(x_K - x_{K-1})$



$M_K(f|a,b,n)(x_K - x_{K-1})$

Obviously, the actual area A will satisfy:
$\forall n \in \mathbb{N} - \{0\}: L_n(f|a,b) \leq A \leq U_n(f|a,b)$

• 4 We prove that

$$\lim_{n\in\mathbb{N}} L_n(f|a,b) = \lim_{n\in\mathbb{N}} U_n(f|a,b) = A$$

$\mapsto$ <u>Riemann integrability</u>

• Let $f: A \to \mathbb{R}$ be a function with $[a,b] \subseteq A$. We say that

$$\boxed{\begin{array}{l} f \text{ integrable} \iff \exists A \in \mathbb{R} : \lim\limits_{n \in \mathbb{N}^*} L_n(f|a,b) = \lim\limits_{n \in \mathbb{N}^*} U_n(f|a,b) = A \\ \text{at } [a,b] \end{array}}$$

- A very important result of Riemann's theory, which is difficult to prove, is that

$$\boxed{f \text{ continuous at } [a,b] \Rightarrow f \text{ integrable at } [a,b]}$$

↦ Integral notation

If f is a function that is integrable at $[a,b]$, then we define:

$$\int_a^b f(x)\,dx = \lim_{n \in \mathbb{N}^*} L_n(f|a,b) = \lim_{n \in \mathbb{N}^*} U_n(f|a,b)$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

Integral evaluation via approximation sequence

- Let $f: A \to \mathbb{R}$ be a function with $[a,b] \subseteq A$ and assume that f is integrable at $[a,b]$. Then:

$$\boxed{\int_a^b f(x)\,dx = \lim_{n \in \mathbb{N}^*} \left[\frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right) \right]}$$

- Note that the existence of the limit above does not imply that f is integrable at $[a,b]$. We need to know ahead of time that f is integrable at $[a,b]$ to apply the above statement.
- Evaluating integrals via approximation sequences requires use of the following basic sums:

$$S_1(n) = \sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$S_2(n) = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$S_3(n) = \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} = [S_1(n)]^2$$

- Proof of $S_1(n)$

We note that $(x+1)^2 = x^2 + 2x + 1, \ \forall x \in \mathbb{R}$

For $x=1$: $\quad 2^2 = 1^2 + 2\cdot 1 + 1$

For $x=2$: $\quad 3^2 = 2^2 + 2\cdot 2 + 1$

For $x=3$: $\quad 4^2 = 3^2 + 2\cdot 3 + 1$

$\ldots\ldots\ldots\ldots\ldots\ldots$

For $x=n$: $\quad (n+1)^2 = n^2 + 2n + 1.$

Adding the above equations gives:

$$\sum_{k=2}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + n \Rightarrow$$

$$\Rightarrow (n+1)^2 + \sum_{k=2}^{n} k^2 = 1^2 + \sum_{k=2}^{n} k^2 + 2S_1(n) + n \Rightarrow$$

$$\Rightarrow (n+1)^2 = 1 + 2S_1(n) + n \Rightarrow$$

$$\Rightarrow 2S_1(n) = (n+1)^2 - n - 1 = (n+1)^2 - (n+1) =$$

$$= (n+1)\left[(n+1) - 1\right] = n(n+1) \Rightarrow$$

$$\Rightarrow S_1(n) = \frac{n(n+1)}{2}$$

↳ The expressions for $S_2(n), S_3(n)$ can be derived similarly using

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1$$

$$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$$

etc...

↳ To evaluate the resulting sequence limits, we use the following statement:

$$\lim_{x\to+\infty} f(n) = a \Rightarrow \lim_{n\in\mathbb{N}^*} f(n) = a$$

EXAMPLE

Evaluate via approximation sequences the integral; and show:

$$I = \int_0^a x^2 dx = \frac{a^3}{3}$$

Solution

Define $\forall x \in \mathbb{R} : f(x) = x^2$. Since:

f continuous at $[0,a] \Rightarrow f$ integrable at $[0,a] \Rightarrow$

$$\Rightarrow I = \int_0^a x^2 dx = \lim_{n \in \mathbb{N}^*} \left[\frac{a-0}{n} \sum_{k=1}^{n} f\left(0 + k\frac{a-0}{n}\right) \right] =$$

$$= \lim_{n \in \mathbb{N}^*} \left[\frac{a}{n} \sum_{k=1}^{n} f\left(\frac{ka}{n}\right) \right] = \lim_{n \in \mathbb{N}^*} \left[\frac{a}{n} \sum_{k=1}^{n} \left(\frac{ka}{n}\right)^2 \right]$$

$$= \lim_{n \in \mathbb{N}^*} \left[\left(\frac{a}{n}\right)^3 \sum_{k=1}^{n} k^2 \right] = \lim_{n \in \mathbb{N}^*} \left[\left(\frac{a}{n}\right)^3 S_2(n) \right] =$$

$$= \lim_{n \in \mathbb{N}^*} \left[\frac{a^3}{n^3} \frac{n(n+1)(2n+1)}{6} \right] = \frac{a^3}{6} \lim_{n \in \mathbb{N}^*} \left[\frac{(n+1)(2n+1)}{n^2} \right]$$

$$= \frac{a^3}{6} \lim_{x \to +\infty} \left(\frac{(x+1)(2x+1)}{x^2} \right) = \frac{a^3}{6} \lim_{x \to +\infty} \frac{2x^2}{x^2} =$$

$$= \frac{a^3}{6} \cdot 2 = \frac{a^3}{3}.$$

↳ This method for solving the corresponding geometric problem (the area under a parabola) was initially discovered more than 2000 years ago in Ancient Greece by Archimedes.

Integral of the exponential function

- The integrals of some exponential functions can also be evaluated directly from approximation sequences as follows: First we show that:

$$\boxed{\forall a \in \mathbb{R}-\{1\}: \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}}$$

Proof

Let $a \in \mathbb{R}-\{1\}$ be given. Then

$$(1-a)\sum_{k=0}^{n} a^k = \sum_{k=0}^{n}(a^k - a^{k+1}) = \sum_{k=0}^{n} a^k - \sum_{k=0}^{n} a^{k+1} =$$

$$= \sum_{k=0}^{n} a^k - \sum_{k=1}^{n+1} a^k =$$

$$= a^0 + \sum_{k=1}^{n} a^k - \sum_{k=1}^{n} a^k - a^{n+1} =$$

$$= 1 - a^{n+1} \Rightarrow$$

$$\Rightarrow \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \qquad \square$$

EXAMPLE

Use apporoximation sequences to show that

$$\int_0^a e^x \, dx = e^a - 1, \; \forall a \in (0, +\infty)$$

Solution

Define $\forall x \in \mathbb{R}: f(x) = e^x$. Then

f continuous at $[0,a] \Rightarrow f$ integrable at $[0,a] \Rightarrow$

$$\Rightarrow I = \int_0^a e^x dx = \lim_{n \in \mathbb{N}^*} \left[\frac{a-0}{n} \sum_{k=1}^{n} f\left(0 + k\frac{a-0}{n}\right) \right]$$

$$= \lim_{n \in \mathbb{N}^*} \left[\frac{a}{n} \sum_{k=1}^{n} f\left(\frac{ka}{n}\right) \right] = \lim_{n \in \mathbb{N}^*} \left[\frac{a}{n} \sum_{k=1}^{n} e^{ka/n} \right] =$$

$$= \lim_{n \in \mathbb{N}^*} \left[\frac{a}{n} \sum_{k=1}^{n} (e^{a/n})^k \right]$$

$$= \lim_{n \in \mathbb{N}^*} \left[\frac{a}{n} \left(\sum_{k=0}^{n} (e^{a/n})^k \right) - \frac{a}{n} \right] =$$

$$= \lim_{n \in \mathbb{N}^*} \left[\frac{a}{n} \sum_{k=0}^{n} (e^{a/n})^k \right] = \lim_{n \in \mathbb{N}^*} \left[\frac{a}{n} \frac{1 - e^{a(n+1)/n}}{1 - e^{a/n}} \right]$$

We note that:

$$\lim_{x \to +\infty} \frac{a(x+1)}{x} = \lim_{x \to +\infty} \frac{ax}{x} = a \Rightarrow \lim_{x \to +\infty} \exp\left(\frac{a(x+1)}{x}\right) = e^a$$

$$\Rightarrow \lim_{x \to +\infty} \left[1 - \exp\left(\frac{a(x+1)}{x}\right) \right] = 1 - e^a \qquad (1)$$

and for $y = a/x$, we have $x \to +\infty \Rightarrow y \to 0^+$ with $y \neq 0$ and therefore:

$$\lim_{x \to +\infty} \left[\frac{a}{x} \frac{1}{1 - e^{a/x}} \right] = \lim_{y \to 0^+} \left[\frac{y}{1 - e^y} \right] = \lim_{y \to 0^+} \frac{1}{-e^y}$$

$$= \frac{1}{-e^0} = -1. \qquad (2)$$

From Eq. (1) and Eq. (2) it follows that:

$$I = \lim_{x\to+\infty}\left[\frac{a}{x}\,\frac{1-\exp\left(\frac{a(x+1)}{x}\right)}{1-\exp\left(\frac{a}{x}\right)}\right] =$$

$$= \lim_{x\to+\infty}\left[\frac{a}{x}\,\frac{1}{1-\exp(a/x)}\right]\lim_{x\to+\infty}\left[1-\exp\left(\frac{a(x+1)}{x}\right)\right]$$

$$= (-1)(1-e^{a}) = e^{a}-1.$$

EXERCISES

① Prove the results for the following basic sums:

a) $S_2(n) = \sum_{k=1}^{n} k^2 = 1^2+2^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

b) $S_3(n) = \sum_{k=1}^{n} k^3 = 1^3+2^3+\cdots+n^3 = \dfrac{n^2(n+1)^2}{4} = [S_1(n)]^2$

(Hint: Use $(x+1)^3 = x^3+3x^2+3x+1$
$(x+1)^4 = x^4+4x^3+6x^2+4x+1$)

② Use approximation sequences to show that

a) $\int_a^b c\,dx = c(b-a)$

b) $\int_a^b x\,dx = \dfrac{b^2-a^2}{2}$

c) $\int_a^b x^2 dx = \dfrac{b^3-a^3}{3}$

d) $\int_a^b x^3 dx = \dfrac{b^4-a^4}{4}$

e) $\int_a^b e^x dx = e^b - e^a$

f) $\int_a^b 2^x dx = \dfrac{2^b-2^a}{\ln 2}$

Properties of integrals

① Linearity

Let f, g be integrable at $[a,b]$. Then:

$$\boxed{\begin{aligned} &\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \\ &\forall \lambda \in \mathbb{R}: \int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx \end{aligned}}$$

By induction, these properties give; in general:

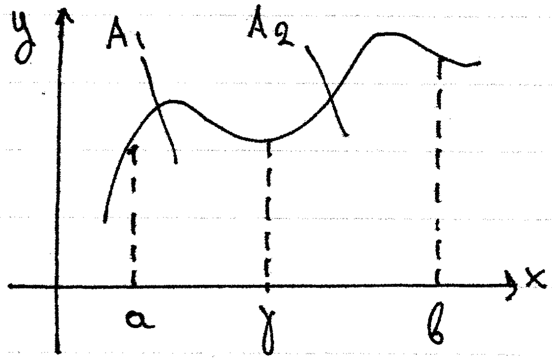
$$\boxed{\left.\begin{aligned} &\forall k \in [n]: f_k \text{ integrable at } [a,b] \\ &\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R} \end{aligned}\right\} \Rightarrow \int_a^b \left[\sum_{k=1}^{n} \lambda_k f_k(x) \right] dx = \sum_{k=1}^{n} \lambda_k \int_a^b f_k(x)\,dx}$$

↳ These properties can be easily proved using approximation sequences.

② Chasles theorem

$$\boxed{\left.\begin{aligned} &f \text{ integrable at } I \\ &I \text{ interval} \\ &a, b, \gamma \in I \end{aligned}\right\} \Rightarrow \int_a^b f(x)\,dx = \int_a^\gamma f(x)\,dx + \int_\gamma^b f(x)\,dx}$$

- <u>geometric interpratation</u>



Let

A_1 = area between a, γ

A_2 = area between γ, b

A = area between a, b

Then it is easy to see that

$A = A_1 + A_2$.

Although proving the Chasles theorem geometrically is easy, proving it directly from the Riemann definition requires a lot of more effort.

- <u>generalization</u>: By induction, Chasles theorem can be generalized to give:

$$\boxed{\left.\begin{array}{l} f \text{ integrable at } I \\ I \text{ interval} \\ a, b, \gamma_1, \dots, \gamma_n \in I \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx = \int_a^{\gamma_1} f(x)\,dx + \sum_{k=1}^{n-1} \int_{\gamma_k}^{\gamma_{k+1}} f(x)\,dx + \int_{\gamma_n}^{b} f(x)\,dx}$$

③ <u>Integral bounding</u>

$$\boxed{\left.\begin{array}{l} f \text{ integrable at } [a,b] \\ \forall x \in [a,b] : f(x) \geq m \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx \geq m(b-a)}$$

$$\boxed{\left.\begin{array}{l} f \text{ integrable at } [a,b] \\ \forall x \in [a,b] : f(x) \leq m \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx \leq m(b-a)}$$

Proof

With no loss of generality, assume that f integrable at $[a,b]$ and that $\forall x \in [a,b] : f(x) \geq m$. Then

$\forall x \in [a,b] : f(x) \geq m \Rightarrow$

$$\Rightarrow \forall k \in [n] : f\left(a + k\frac{b-a}{n}\right) \geq m$$

$$\Rightarrow \forall k \in [n] : \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) \geq mn$$

$$\Rightarrow \quad \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) \geq mn\frac{b-a}{n} = m(b-a)$$

$$\Rightarrow I = \int_a^b f(x)\,dx = \lim_{n \in \mathbb{N}^*} \left[\frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\right]$$

$$\geq m(b-a) \Rightarrow$$

$$\Rightarrow \int_a^b f(x)\,dx \geq m(b-a). \qquad \square$$

↳ For $m = 0$, the above statement gives:

$$\left.\begin{array}{l} f \text{ integrable at } [a,b] \\ \forall x \in [a,b] : f(x) \geq 0 \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx \geq 0$$

and an immediate corollary is that

$$\boxed{\left.\begin{array}{l} f, g \text{ integrable at } [a,b] \\ \forall x \in [a,b] : f(x) \geq g(x) \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx}$$

Note that the above results require $a < b$!!

④ Integral Mean Value Theorem

$$\left.\begin{array}{l} f \text{ continuous at } [a,b] \\ g \text{ integrable at } [a,b] \\ \forall x\in[a,b]: g(x)\geq 0 \end{array}\right\} \Rightarrow \exists \xi\in[a,b]: \int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$$

Proof

From the extremum value theorem:

f continuous at $[a,b] \Rightarrow$

$\Rightarrow \exists x_1, x_2 \in [a,b]: \forall x\in[a,b]: f(x_1)\leq f(x)\leq f(x_2)$

Since $\forall x\in[a,b]: g(x)\geq 0$, it follows that

$\forall x\in[a,b]: f(x_1)g(x)\leq f(x)g(x)\leq f(x_2)g(x)$

$$\Rightarrow \int_a^b f(x_1)g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \int_a^b f(x_2)g(x)\,dx$$

$$\Rightarrow f(x_1)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq f(x_2)\int_a^b g(x)dx \qquad (1)$$

We also note that:

$$\forall x\in[a,b]: g(x)\geq 0 \Rightarrow \int_a^b g(x)dx \geq 0$$

and we may therefore distinguish between the following cases:

Case 1: Assume that $\int_a^b g(x)dx = 0$. Then, from Eq. (1):

$$0\leq \int_a^b g(x)dx \leq 0 \Rightarrow \int_a^b g(x)dx = 0$$

$$0 \leqslant \int_a^b f(x)g(x)dx \leqslant 0 \Rightarrow$$

$$\Rightarrow \forall \xi \in [a,b]: \int_a^b f(x)g(x)\,dx = 0 = f(\xi)\cdot 0 = f(\xi)\int_a^b g(x)\,dx$$

$$\Rightarrow \exists \xi \in [a,b]: \int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)\,dx.$$

<u>Case 2</u>: Assume that $\int_a^b g(x)dx > 0$.

Let us define $A = \dfrac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)\,dx}$. From Eq. (1).

$f(x_1) \leqslant A \leqslant f(x_2)$. Assume with no loss of generality that $x_1 < x_2$

Furthermore, from the intermediate value theorem:

$$\left.\begin{array}{l} f \text{ continuous at } [a,b] \\ x_1, x_2 \in [a,b] \end{array}\right\} \Rightarrow [f(x_1), f(x_2)] \subseteq f([x_1,x_2])$$

and it follows that

$$\begin{aligned} f(x_1) \leqslant A \leqslant f(x_2) &\Rightarrow A \in [f(x_1), f(x_2)] \\ &\Rightarrow A \in f([x_1,x_2]) \\ &\Rightarrow \exists \xi \in [x_1,x_2] : f(\xi) = A \\ &\Rightarrow \exists \xi \in [a,b] : f(\xi) = A \\ &\Rightarrow \exists \xi \in [a,b] : \int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx \quad \square \end{aligned}$$

↳ For $\forall x \in [a,b]: g(x) = 1$, the integral mean-value theorem gives:

$$\boxed{f \text{ continuous at } [a,b] \Rightarrow \int_a^b f(x)dx = f(\xi)(b-a)}$$

(5) Absolute value of integral

$$\boxed{\left.\begin{array}{l} f \text{ integrable at } [a,b] \\ |f| \text{ integrable at } [a,b] \end{array}\right\} \Rightarrow \left| \int_a^b f(x)\,dx \right| \leqslant \int_a^b |f(x)|\,dx}$$

Proof

Let us define:

$$A_n = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)$$

$$B_n = \frac{b-a}{n} \sum_{k=1}^{n} \left|f\left(a + k\frac{b-a}{n}\right)\right|$$

Then

$$\left\{\begin{array}{l} f \text{ integrable at } [a,b] \\ |f| \text{ integrable at } [a,b] \end{array}\right. \Rightarrow \lim_{n\in\mathbb{N}^*} A_n = \int_a^b f(x)\,dx \wedge \lim_{n\in\mathbb{N}^*} B_n = \int_a^b |f(x)|\,dx$$

We also note that

$$\forall n\in\mathbb{N}^*: |A_n| = \left| \frac{b-a}{n} \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right) \right| =$$

$$= \left|\frac{b-a}{n}\right| \left| \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right) \right| =$$

$$= \frac{b-a}{n} \left| \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right) \right|$$

$$\leqslant \frac{b-a}{n} \sum_{k=1}^{n} \left| f\left(a+k\frac{b-a}{n}\right) \right| = B_n \Rightarrow$$

$$\Rightarrow \forall n\in\mathbb{N}^*: |A_n| \leqslant B_n \Rightarrow \forall n\in\mathbb{N}^*: -B_n \leqslant A_n \leqslant B_n \Rightarrow$$

$$\Rightarrow -\lim_{n\in\mathbb{N}^*} B_n \leqslant \lim_{n\in\mathbb{N}^*} A_n \leqslant \lim_{n\in\mathbb{N}^*} B_n \Rightarrow \left|\lim_{n\in\mathbb{N}^*} A_n\right| \leqslant \lim_{n\in\mathbb{N}^*} B_n$$

$$\Rightarrow \left| \int_a^b f(x)dx \right| = \left| \lim_{n \in \mathbb{N}^*} A_n \right| \leqslant \lim_{n \in \mathbb{N}^*} B_n = \int_a^b |f(x)|dx$$

$$\Rightarrow \left| \int_a^b f(x)dx \right| \leqslant \int_a^b |f(x)|dx \qquad \square$$

▸ Summary: Properties of integrals

(1) $(\forall k\in[n]: f$ integrable at $[a,b])$
$\Rightarrow \forall \lambda_1,\lambda_2,\dots,\lambda_n\in\mathbb{R}: \int_a^b \left[\sum_{k=1}^{n}\lambda_k f_k(x)\right]dx = \sum_{k=1}^{n}\lambda_k \int_a^b f_k(x)\,dx$

(2) $\left.\begin{array}{l} f \text{ integrable at } I \\ I \text{ interval} \\ a,b,\gamma_1,\dots,\gamma_n\in I \end{array}\right\} \Rightarrow \int_a^b f(x)dx = \int_a^{\gamma_1} f(x)dx + \sum_{k=1}^{n-1}\int_{\gamma_k}^{\gamma_{k+1}} f(x)dx + \int_{\gamma_n}^{b} f(x)dx$

(3) $\left.\begin{array}{l} f \text{ integrable at } [a,b] \\ \forall x\in[a,b]: f(x)\geq m \end{array}\right\} \Rightarrow \int_a^b f(x)dx \geq m(b-a)$

$\left.\begin{array}{l} f \text{ integrable at } [a,b] \\ \forall x\in[a,b]: f(x)\leq m \end{array}\right\} \Rightarrow \int_a^b f(x)dx \leq m(b-a)$

↳ $\left.\begin{array}{l} f,g \text{ integrable at } [a,b] \\ \forall x\in[a,b]: f(x)\geq g(x) \end{array}\right\} \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx.$

(4) $\left.\begin{array}{l} f \text{ continuous at } [a,b] \\ g \text{ integrable at } [a,b] \\ \forall x\in[a,b]: g(x)\geq 0 \end{array}\right\} \Rightarrow \exists \xi\in[a,b]: \int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$

↳ f continuous at $[a,b] \Rightarrow \exists \xi\in[a,b]: \int_a^b f(x)dx = f(\xi)(b-a)$

(5) $\left.\begin{array}{l} f \text{ integrable at } [a,b] \\ |f| \text{ integrable at } [a,b] \end{array}\right\} \Rightarrow \left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

EXAMPLES

a) Let f be continuous on an interval I and let $a,b,c,d \in I$. Show that:

$$\int_a^b f(x)dx \int_c^d f(x)dx + \int_a^c f(x)dx \int_d^b f(x)dx + \int_a^d f(x)dx \int_b^c f(x)dx = 0$$

Solution

$$I = \int_a^b f(x)dx \int_c^d f(x)dx + \int_a^c f(x)dx \int_d^b f(x)dx + \int_a^d f(x)dx \int_b^c f(x)dx =$$

$$= \int_a^b f(x)dx \int_c^d f(x)dx + \left[\int_a^b f(x)dx + \int_b^c f(x)dx\right]\int_d^b f(x)dx + \int_a^d f(x)dx \int_b^c f(x)dx$$

$$= \int_a^b f(x)dx \left[\int_c^d f(x)dx + \int_d^b f(x)dx\right] + \int_b^c f(x)dx \left[\int_d^b f(x)dx + \int_a^d f(x)dx\right]$$

$$= \int_a^b f(x)dx \int_c^b f(x)dx + \int_b^c f(x)dx \int_a^b f(x)dx =$$

$$= -\int_a^b f(x)dx \int_b^c f(x)dx + \int_a^b f(x)dx \int_b^c f(x)dx = 0.$$

→ The following example is based on the integral properties and the following result, which can be proved directly from approximating sequences:

$$\int_a^b c\,dx = c(b-a).$$

b) Evaluate $I = \int_0^2 \frac{(2x^2+1)dx}{x^2+2} - 3\int_2^0 \frac{dx}{x^2+2}$

Solution

$$I = \int_0^2 \frac{2x^2+1}{x^2+2}\,dx - 3\int_2^0 \frac{dx}{x^2+2} =$$

$$= \int_0^2 \frac{2x^2+1}{x^2+2}\,dx + 3\int_0^2 \frac{dx}{x^2+2} =$$

$$= \int_0^2 \left[\frac{2x^2+1}{x^2+2} + \frac{3}{x^2+2}\right]dx$$

$$= \int_0^2 \frac{2x^2+1+3}{x^2+2}\,dx = \int_0^2 \frac{2x^2+4}{x^2+2}\,dx =$$

$$= \int_0^2 \frac{2(x^2+2)}{x^2+2}\,dx = \int_0^2 2\,dx = 2\cdot 2 = 4.$$

c) Let f be a function that is continuous on $[a,b]$ such that $\forall x \in [a,b]: f(x) > 0$.
Show that $\int_a^b f(x)\,dx > 0$.

Solution

From the integral mean-value theorem:
f continuous at $[a,b] \Rightarrow \exists \xi \in [a,b]: \int_a^b f(x)dx = f(\xi)(b-a)$
Since:

$$(\forall x \in [a,b]: f(x) > 0) \Rightarrow f(\xi) > 0 \Rightarrow f(\xi)(b-a) > 0 \Rightarrow$$

$$\Rightarrow \int_a^b f(x)dx > 0 \qquad \square$$

d) Show that $\lim\limits_{a\to+\infty} \int_a^{a+1} \frac{dx}{x^2+1} = 0$

<u>Solution</u>

Define $\forall x \in \mathbb{R}: f(x) = \frac{1}{x^2+1} \Rightarrow$

$\Rightarrow \forall x \in \mathbb{R}: f'(x) = \frac{-(x^2+1)'}{(x^2+1)^2} = \frac{-2x}{(x^2+1)^2}$

x		0	
$-2x$	$+$	0	$-$
$(x^2+1)^2$	$+$		$-$
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$		$\searrow$

It follows that

$f \searrow (0,+\infty) \Rightarrow \forall a \in (0,+\infty): \forall x \in [a, a+1]: f(a) \geq f(x) \geq f(a+1)$

$\Rightarrow \forall a \in (0,+\infty): f(a)[(a+1)-a] \geq \int_a^{a+1} f(x)\,dx \geq f(a+1)[(a+1)-a]$

$\Rightarrow \forall a \in (0,+\infty): f(a) \geq \int_a^{a+1} f(x)\,dx \geq f(a+1) \qquad (1).$

We also note that

$\lim\limits_{a\to+\infty} f(a) = \lim\limits_{a\to+\infty} \frac{1}{a^2+1} = \lim\limits_{a\to+\infty} \frac{1}{a^2} = 0 \qquad (2)$

$\lim\limits_{a\to+\infty} f(a+1) = \lim\limits_{a\to+\infty} \frac{1}{(a+1)^2+1} = \lim\limits_{a\to+\infty} \frac{1}{a^2} = 0 \qquad (3)$

From Eq. (1) and Eq. (2) and Eq. (3): $\lim\limits_{a\to+\infty} \int_a^{a+1} f(x)\,dx = 0.$

e) Show that

$$\lim_{x\to+\infty}\left[\frac{1}{x^2}\int_0^x \cos t\,[\sin(2t)+\operatorname{Arctan}(3t)]\,dt\right]=0$$

Solution

Define:

$$\forall x\in\mathbb{R}:\ f(x)=\frac{1}{x^2}\int_0^x \cos t\,[\sin(2t)+\operatorname{Arctan}(3t)]\,dt$$

and note that

$$\begin{aligned}
\forall x\in\mathbb{R}:\ |f(x)| &= \left|\frac{1}{x^2}\int_0^x \cos t\,[\sin(2t)+\operatorname{Arctan}(3t)]\,dt\right| \\
&= \left|\frac{1}{x^2}\right|\left|\int_0^x \cos t\,[\sin(2t)+\operatorname{Arctan}(3t)]\,dt\right| \\
&= \frac{1}{x^2}\left|\int_0^x \cos t\,[\sin(2t)+\operatorname{Arctan}(3t)]\,dt\right| \\
&\leqslant \frac{1}{x^2}\int_0^x |\cos t\,(\sin(2t)+\operatorname{Arctan}(3t))|\,dt \\
&= \frac{1}{x^2}\int_0^x |\cos t|\cdot|\sin(2t)+\operatorname{Arctan}(3t)|\,dt \\
&\leqslant \frac{1}{x^2}\int_0^x |\sin(2t)+\operatorname{Arctan}(3t)|\,dt \\
&\leqslant \frac{1}{x^2}\int_0^x (|\sin(2t)|+|\operatorname{Arctan}(3t)|)\,dt \\
&\leqslant \frac{1}{x^2}\int_0^x (1+\pi/2)\,dt = \frac{1+\pi/2}{x^2}\int_0^x dt
\end{aligned}$$

$$= \frac{1+\pi/2}{x^2}(x-0) = \frac{\pi+2}{2x} \Rightarrow$$

$$\Rightarrow \forall x \in \mathbb{R} : |f(x)| \leqslant \frac{\pi+2}{2x} \quad (1).$$

and

$$\lim_{x\to+\infty} \frac{\pi+2}{x} = (\pi+2)\lim_{x\to+\infty}\frac{1}{x} = (\pi+2)\cdot 0 = 0 \quad (2).$$

From Eq.(1) and Eq.(2):

$$\left\{\begin{array}{l} \forall x \in \mathbb{R}: |f(x)| \leqslant \dfrac{\pi+2}{2x} \\ \lim\limits_{x\to+\infty} \dfrac{\pi+2}{2x} = 0 \end{array}\right. \Rightarrow \lim_{x\to+\infty} f(x) = 0.$$

↳ We have used the bounds:

$\forall x \in \mathbb{R} : |\sin x| \leqslant 1$

$\forall x \in \mathbb{R} : |\cos x| \leqslant 1$

$\forall x \in \mathbb{R} : |\mathrm{Arctan}\, x| \leqslant \pi/2$

combined with the theorem:

$$\left.\begin{array}{l} \forall x \in N(\sigma,\delta) : |f(x)| \leqslant g(x) \\ \lim\limits_{x\to\sigma} g(x) = 0 \end{array}\right\} \Rightarrow \lim_{x\to\sigma} f(x) = 0.$$

EXERCISES

(3) Show the following statements:

a) $\forall a,b \in (0,+\infty)$: $\displaystyle\int_a^b \frac{dx}{x^3+1} + \int_a^b \frac{x^3\,dx}{x^3+1} = b-a$

b) $\displaystyle\int_1^2 \frac{dx}{x^2+4x} + \int_2^3 \frac{dx}{x^2+4x} + \int_3^1 \frac{dx}{x^2+4x} = 0$

c) Let f be integrable on an interval I and let $a,b,c,d \in I$. Then:

$$\int_a^b f(x)\,dx + \int_c^d f(x)\,dx = \int_c^b f(x)\,dx + \int_a^d f(x)\,dx$$

$$\int_a^b f(x)\,dx \int_c^d f(x)\,dx - \int_a^c f(x)\,dx \int_b^d f(x)\,dx = \int_a^b f(x)\,dx \int_c^b f(x)\,dx$$

d) $\begin{cases} f \text{ continuous at } [a,b] \\ \forall x \in [a,b] : f(x) < 0 \end{cases} \Rightarrow \displaystyle\int_a^b f(x)\,dx < 0$

e) $\begin{cases} f \text{ continuous at } [a,b] \\ \displaystyle\int_a^b f(x)\,dx > 0 \end{cases} \Rightarrow \exists \xi \in [a,b] : f(\xi) > 0$

f) $\forall a,b \in \mathbb{R}$: $0 < a < b \Rightarrow \dfrac{b-a}{b^2} \leqslant \displaystyle\int_a^b \frac{dx}{x^2} \leqslant \frac{b-a}{a^2}$

g) $0 \leqslant \displaystyle\int_1^e \frac{\ln x}{x}\,dx \leqslant \frac{e-1}{e}$

h) $0 \leqslant \displaystyle\int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx \leqslant \frac{e^2-1}{e^2+1}$

i) $0 \leqslant \int_{\pi/6}^{\pi/2} \frac{1-\sin x}{\sin x}\, dx \leqslant \frac{\pi}{3}$

j) $\lim\limits_{x \to +\infty} \int_{x-1}^{x+1} \frac{dt}{\sqrt{t^2+1}} = 0$

k) $\lim\limits_{x \to +\infty} \int_{x^2}^{x^2+1} \operatorname{Arctan}(t)\, dt = 0$

l) $\lim\limits_{x \to +\infty} \int_{x}^{2x} \cos x \sin(1/t)\, dt = 0$ (Hint: Use zero-bounded theorem).

m) $\lim\limits_{x \to +\infty} \left[\frac{1}{x^3} \int_0^{x^2} (\sin t \sin(2t) + \cos^3(3t)]\, dt \right] = 0$

n) $\lim\limits_{x \to +\infty} \left[\frac{1}{2x+1} \int_0^{\sqrt{x}} \cos(2t)(1+\operatorname{Arctan}(t))\, dt \right] = 0$

o) $\lim\limits_{x \to +\infty} \left[\frac{1}{x^2+3x} \int_0^{x} \frac{3+\sin t}{2+\cos t}\, dt \right] = 0$

Fundamental theorem of calculus

Before proving the fundamental theorem of calculus we have to use the definition of the integral to prove directly that

$$\int_a^b c\,dx = c(b-a), \quad \forall a,b,c \in \mathbb{R}.$$

We also use the property that

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

① Fundamental Theorem. I

$$\boxed{\left.\begin{array}{l} f \text{ continuous at } [a,b] \\ F(x) = \int_a^x f(t)dt, \ \forall x \in [a,b] \end{array}\right\} \Rightarrow \left\{\begin{array}{l} F \text{ differentiable in } [a,b] \\ F'(x) = f(x), \ \forall x \in [a,b] \end{array}\right.}$$

Proof

Let $x_0 \in [a,b]$ be given. It is sufficient to show that

$$\lim_{x \to x_0} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0)$$

Let $\varepsilon>0$ be given. Since f continuous at $[a,b] \Rightarrow$
$\Rightarrow \exists \delta>0 : \forall t \in (x_0-\delta, x_0+\delta) : |f(t)-f(x_0)|<\varepsilon$. (1)
Let $x \in (x_0-\delta, x_0+\delta)$ be given. It follows that:

$$\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)=\frac{1}{x-x_0}\left[\int_a^x f(t)dt-\int_a^{x_0} f(t)dt\right]-f(x_0)$$
$$=\frac{1}{x-x_0}\int_{x_0}^{x} f(t)\,dt-\frac{f(x_0)}{x-x_0}\int_{x_0}^{x} dt=$$
$$=\frac{1}{x-x_0}\int_{x_0}^{x} f(t)dt-\frac{1}{x-x_0}\int_{x_0}^{x} f(x_0)dt=$$
$$=\frac{1}{x-x_0}\left[\int_{x_0}^{x} f(t)dt-\int_{x_0}^{x} f(x_0)dt\right]=$$
$$=\frac{1}{x-x_0}\int_{x_0}^{x}[f(t)-f(x_0)]\,dt \Rightarrow$$

$$\Rightarrow \left|\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right|=\left|\frac{1}{x-x_0}\int_{x_0}^{x}[f(t)-f(x_0)]\,dt\right|=$$
$$=\frac{1}{|x-x_0|}\left|\int_{x_0}^{x}[f(t)-f(x_0)]dt\right|\leqslant$$
$$\leqslant\frac{1}{|x-x_0|}\int_{x_0}^{x}|f(t)-f(x_0)|\,dt\leqslant$$
$$\leqslant\frac{1}{|x-x_0|}\int_{x_0}^{x}\varepsilon\,dt=\frac{\varepsilon}{|x-x_0|}\cdot(x-x_0)\leqslant$$
$$\leqslant\frac{\varepsilon}{|x-x_0|}|x-x_0|=\varepsilon,\ \forall x\in(x_0-\delta,x_0+\delta)$$

We have thus shown that

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in (x_0 - \delta, x_0 + \delta) :$$
$$: \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon \Rightarrow$$

$$\Rightarrow \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0) \Rightarrow$$

$\Rightarrow$ F differentiable at x_0 with $F'(x_0) = f(x_0)$. $\square$

↳ In Leibnitz notation the theorem reads:

$$\boxed{\frac{d}{dx} \int_c^x f(t)\,dt = f(x)}$$

↳ Combining the fundamental theorem of calculus with the chain rule gives the following more general differentiation rule.

$$\boxed{\frac{d}{dx} \int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))\,b'(x) - f(a(x))\,a'(x)}$$

EXERCISES

④ Evaluate and simplify the derivatives of the following functions

a) $f(x) = \int_1^x (t^2 \ln t + e^{-t})\, dt$

b) $f(x) = \int_x^3 t^2 e^t (t-1)^3\, dt$

c) $f(x) = \int_0^{1/x} t \ln t\, dt$

d) $f(x) = \int_{x^2-1}^{x^2+1} \frac{t^2-1}{t^2+1}\, dt$

e) $f(x) = \int_{\ln x}^{\ln(x^2)} t e^{-t}\, dt$

⑤ Show that

a) $\frac{d}{dx} \int_{\cos x}^{\sin x} \operatorname{Arcsin}(t)\, dt = x(\cos x - \sin x) + \frac{\pi}{2} \sin x$

b) $\frac{d}{dx} \int_0^{\cos x} \operatorname{Arccos}(2t^2-1)\, dt = -2x \sin x$

c) $\frac{d}{dx} \int_0^{\tan x} \operatorname{Arctan}\left(\frac{2t}{1+t^2}\right) dt = 2x(1+\tan^2 x)$

⑥ Show that the function $f(x) = \int_0^{\sin x} \operatorname{Arccos}(t)\,dt$ has a local minimum at $x=0$.

⑦ Analyze the function

$$f(x) = \int_x^{1-x} \frac{dt}{1+t^2}$$

with respect to monotonicity and concavity. Locate the local minimum and maximum points and the inflection points.

⑧ Use the fundamental theorem of calculus and L'Hospital's theorem to evaluate the following limits

a) $\lim\limits_{x\to 0} \frac{1}{x} \int_0^{x} \cosh(t)\,dt$

b) $\lim\limits_{x\to 0} \frac{1+\cos x}{1-\cos x} \int_0^{x} \tan(3t)\,dt$

c) $\lim\limits_{x\to 0} \frac{\sin x}{x^2} \int_1^{e^x} \ln(t)\,dt$

d) $\lim\limits_{x\to e} \frac{1}{\ln(\ln x)} \int_1^{\ln x} e^{2t}(t+1)\,dt$

e) $\lim\limits_{x\to 0} \frac{\tan x}{x^3} \int_0^{x} dt \int_1^{\cos t} ds\, \operatorname{Arcsin}(s)$

② Fundamental theorem of calculus II

$$\boxed{\left.\begin{array}{l} F \text{ differentiable at } [a,b] \\ F'(x) = f(x),\ \forall x \in [a,b] \\ f \text{ continuous at } [a,b] \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)}$$

Proof

$$\forall x \in [a,b] : \frac{d}{dx}\int_a^x f(t)\,dt = f(x) = \frac{dF(x)}{dx} \Rightarrow$$

$$\Rightarrow \exists\, c \in \mathbb{R} : \int_a^x f(t)\,dt = F(x) + c \quad , \ \forall x \in [a,b]$$

For $x = a$:

$$F(a) + c = \int_a^a f(t)\,dt = 0 \Rightarrow c = -F(a)$$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a). \quad \square$$

$\updownarrow \rightarrow$ Equivalently

$$\boxed{\int_a^b f'(x)\,dx = f(b) - f(a)}$$

→ The FTC II motivates the definition of the indefinite integral:

$$\boxed{\int f'(x)\,dx = f(x) + c}$$

→ <u>Integration formulas</u>

1) $$\int x^a\,dx = \begin{cases} \dfrac{x^{a+1}}{a+1} + c &, \text{ if } a \neq -1 \\ \ln|x| + c &, \text{ if } a = -1 \end{cases}$$

▸ Special cases:

a) $\int dx = x + c$ b) $\int \dfrac{dx}{\sqrt{x}} = 2\sqrt{x} + c$

2) $\int \sin x\,dx = -\cos x + c$ 6) $\int a^x\,dx = \dfrac{a^x}{\ln a} + c$, if $a \neq 1$

3) $\int \cos x\,dx = \sin x + c$ → $\int e^x\,dx = e^x + c$

4) $\int \dfrac{dx}{\cos^2 x} = \tan x + c$ 7) $\int \dfrac{dx}{1+x^2} = \operatorname{Arctan} x + c$

5) $\int \dfrac{dx}{\sin^2 x} = -\cot x + c$ 8) $\int \dfrac{dx}{\sqrt{1-x^2}} = \operatorname{Arcsin} x + c$

EXAMPLES

1) $I = \int_1^2 \frac{x+1}{x^3}\,dx = \int_1^2 (x^{-2}+x^{-3})\,dx =$

$$= \int_1^2 x^{-2}dx + \int_1^2 x^{-3}dx = \left[\frac{x^{-1}}{-1}\right]_1^2 + \left[\frac{x^{-2}}{-2}\right]_1^2 =$$

$$= \left[-\frac{1}{x}\right]_1^2 + \left[-\frac{1}{2x^2}\right]_1^2 =$$

$$= \left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right) + \left(-\frac{1}{2\cdot 2^2}\right) - \left(\frac{-1}{2\cdot 1^2}\right) =$$

$$= -\frac{1}{2} + 1 - \frac{1}{8} + \frac{1}{2} = 1 - \frac{1}{8} = \frac{7}{8}$$

2) $I = \int_0^{\pi/4} \frac{1+\cos^3 x}{\cos^2 x}\,dx = \int_0^{\pi/4} \frac{dx}{\cos^2 x} + \int_0^{\pi/4} \cos x\,dx$

$$= \left[\tan x\right]_0^{\pi/4} + \left[\sin x\right]_0^{\pi/4} =$$

$$= \tan\left(\frac{\pi}{4}\right) - \tan 0 + \sin\left(\frac{\pi}{4}\right) - \sin 0 =$$

$$= 1 + \frac{\sqrt{2}}{2} = \frac{2+\sqrt{2}}{2}$$

3) $I = \int_{-1}^{1} \frac{dx}{x^3} \longrightarrow$ $f(x) = 1/x^3$ NOT continuous at $[-1,1]$ thus we cannot apply the FTC II !!

EXERCISES

(9) Evaluate the following integrals:

a) $I = \int_{-1}^{1} x^3 dx$ b) $I = \int_{-2}^{0} (x + e^x) dx$ c) $I = \int_{2}^{1} \sqrt{x}\, dx$

d) $I = \int_{1}^{2} \frac{dx}{\sqrt{x}}$ e) $I = \int_{2}^{3} x\sqrt{x}\, dx$ f) $I = \int_{0}^{\pi/4} \sin x\, dx$

g) $I = \int_{-\pi/6}^{\pi/6} \frac{dx}{\cos^2 x}$ h) $I = \int_{\pi/6}^{\pi/3} \frac{dx}{\sin^2 x}$

i) $I = \int_{0}^{2} (3^x + \sqrt{x}) dx$ j) $I = \int_{-\sqrt{3}}^{\sqrt{3}/3} \frac{dx}{1+x^2}$

k) $I = \int_{1/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$ l) $I = \int_{0}^{2} (3^x + 4^x) dx$

(10) Evaluate the following integrals

a) $I = \int_{1}^{2} x(x+2)^2 dx$ b) $I = \int_{1}^{3} x^2(\sqrt{x} + 1) dx$

c) $I = \int_{-\pi/3}^{2\pi/3} \sin(x/2)\cos(x/2)\, dx$ d) $I = \int_{0}^{1} \frac{x^2+2}{x^2+1} dx$

e) $I = \int_0^{1/2} \frac{\sqrt{1-x^2}}{(1+x)(1-x)} \, dx$

f) $I = \int_0^{\pi/4} (2\cos^2(x/2)-1) \, dx$ g) $I = \int_0^3 \frac{2^x}{e^x} \, dx$

h) $I = \int_1^2 2^x (3^x - 5^x) \, dx$ i) $I = \int_0^1 \frac{3^x + 4^x}{5^x} \, dx$

Method of substitution

The method of substitution is based on the following theorem:

<u>Thm</u>:

$$\left.\begin{array}{l} g \text{ differentiable at } [a,b] \\ g' \text{ continuous at } [a,b] \\ f \text{ continuous at } g([a,b]) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(y)\, dy$$

Remark: If we let $y = g(x)$ then the substitution theorem implies the formal relationship:

$$dy = g'(x)\, dx .$$

<u>Proof</u>

Let $F(x) = \int_{g(a)}^{x} f(t)\,dt$. It follows that $F'(x) = f(x)$.

and thus:

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_a^b F'(g(x))\, g'(x)\,dx =$$

$$= \int_a^b [F(g(x))]'\, dx =$$

$$= F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} F'(y)\,dy =$$

$$= \int_{g(a)}^{g(b)} f(y)\,dy \qquad \square$$

$\hookrightarrow$ <u>Immediate consequences</u>

1) $\displaystyle\int \sin(ax)\,dx = \frac{-\cos(ax)}{a} + c$

2) $\displaystyle\int \cos(ax)\,dx = \frac{\sin(ax)}{a} + c$

3) $\displaystyle\int a^{bx}\,dx = \frac{a^{bx}}{b\ln a} + c$

$\hookrightarrow \displaystyle\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$

EXAMPLES

1) $I = \int_0^{\pi/2} 2^{\cos 3x} \sin(3x)\,dx$

Let $y = g(x) = \cos(3x) \Rightarrow \begin{cases} dy = -3\sin(3x)\,dx \\ g(0) = \cos(0) = 1 \\ g(\pi/2) = \cos(3\pi/2) = 0 \end{cases} \Rightarrow$

$$\Rightarrow I = \int_1^0 2^y (-1/3)\,dy = (-1/3)\int_1^0 2^y\,dy =$$

$$= -\frac{1}{3}\left[\frac{2^y}{\ln 2}\right]_1^0 = -\frac{1}{3}\,\frac{2^0 - 2^1}{\ln 2} =$$

$$= -\frac{1}{3}\,\frac{1-2}{\ln 2} = \frac{1}{3\ln 2}$$

2) $I = \int (2x-1)e^{x^2-x}\,dx$

Let $y = x^2 - x \Rightarrow dy = (2x-1)\,dx \Rightarrow$

$$\Rightarrow I = \int e^y\,dy = e^y + c \underset{\uparrow}{=} e^{x^2-x} + c.$$

backsubstitution.

↳ With indefinite integrals we work similarly as with definite integrals but it is necessary to use backsubstitution to return to the original variable.

Methodology

1) $\boxed{I = \int \frac{f'(x)}{f(x)}\,dx} \longrightarrow$ Let $y = f(x)$

EXAMPLE

$$I = \int_0^2 \frac{2x}{x^2+1}\,dx$$

Let $y = g(x) = x^2+1 \Rightarrow \begin{cases} dy = 2x\,dx \\ g(0) = 0^2+1 = 1 \\ g(2) = 2^2+1 = 5 \end{cases} \Rightarrow$

$$\Rightarrow I = \int_1^5 \frac{dy}{y} = \Big[\ln|y|\Big]_1^5 = \ln 5 - \ln 1 = \ln 5$$

2) $\boxed{I = \int \frac{f'(x)}{\sqrt{f(x)}}\,dx} \longrightarrow$ Let $y = f(x)$.

EXAMPLE

$$I = \int_0^{\pi/8} \frac{\sin 2x}{\sqrt{1+\cos 2x}}\,dx$$

Let $y = g(x) = 1 + \cos 2x \Rightarrow \begin{cases} dy = -2\sin 2x\, dx \\ g(0) = 1 + \cos 0 = 1 + 1 = 2 \\ g(\pi/8) = 1 + \cos(\pi/4) = \\ \quad = 1 + \sqrt{2}/2 \end{cases}$

$$\Rightarrow I = -\frac{1}{2}\int_0^{\pi/8} \frac{-2\sin 2x}{\sqrt{1+\cos 2x}}\, dx =$$

$$= \frac{-1}{2}\int_2^{1+\sqrt{2}/2} \frac{dy}{\sqrt{y}} = \frac{-1}{2}\left[2\sqrt{y}\right]_2^{1+\sqrt{2}/2} =$$

$$= \frac{-1}{2}\left[2\sqrt{1+\frac{\sqrt{2}}{2}} - 2\sqrt{2}\right] =$$

$$= -\sqrt{1+\frac{\sqrt{2}}{2}} + \sqrt{2}$$

3) $\boxed{I = \int f(ax+b)\, dx}$ $\longrightarrow$ Let $y = ax + b$

EXAMPLE

$$I = \int_0^1 \frac{dx}{(2x+1)^4}$$

Let $y = g(x) = 2x + 1 \Rightarrow \begin{cases} dy = 2dx \Rightarrow dx = (1/2)\, dy \\ g(0) = 2 \cdot 0 + 1 = 1 \quad \Rightarrow \\ g(2) = 2 \cdot 1 + 1 = 3 \end{cases}$

$$\Rightarrow I = \int_1^3 \frac{(1/2)\, dy}{y^4} = \frac{1}{2}\left[\frac{y^{-3}}{-3}\right]_1^3 = \frac{1}{2}\left[\frac{-1}{3y^3}\right]_1^3 =$$

$$= \frac{1}{2} \cdot \left(\frac{-1}{3}\right)\left(\frac{1}{3^3} - \frac{1}{1^3}\right) =$$

$$= \frac{-1}{6}\left(\frac{1}{27} - 1\right) = \frac{-1}{6}\left(-\frac{26}{27}\right) =$$

$$= \frac{13}{3 \cdot 27} = \frac{13}{81}$$

4) $\boxed{I = \int F(x, \sqrt{ax+b})\, dx}$

$\updownarrow \rightarrow$ Let $y = \sqrt{ax+b} \Leftrightarrow y^2 = ax+b \Leftrightarrow ax = y^2 - b$

$\Leftrightarrow x = \dfrac{y^2 - b}{a}$

It follows that $dx = \dfrac{2}{a} y\, dy$ and therefore

$$I = \int F\left(\frac{y^2 - b}{a}, y\right) \frac{2}{a} y\, dy$$

The new integral does not have radicals.

EXAMPLE

$$I = \int_1^2 x \sqrt{3x+2}\, dx$$

Let $y = \sqrt{3x+2} = g(x) \Rightarrow \begin{cases} g(1) = \sqrt{3+2} = \sqrt{5} \\ g(2) = \sqrt{6+2} = \sqrt{8} \end{cases}$

and since

$$y = \sqrt{3x+2} \Leftrightarrow y^2 = 3x+2 \Leftrightarrow 3x = y^2 - 2 \Leftrightarrow$$
$$\Leftrightarrow x = \frac{y^2-2}{3}$$

we have $dx = (1/3)\, 2y\, dy = (2/3)\, y\, dy$.
Thus

$$I = \int_{\sqrt{5}}^{\sqrt{8}} \frac{y^2-2}{3}\, y \cdot (2/3)\, y\, dy = \frac{2}{9} \int_{\sqrt{5}}^{\sqrt{8}} y^2 (y^2-2)\, dy =$$
$$= \frac{2}{9} \int_{\sqrt{5}}^{\sqrt{8}} (y^4 - 2y^2)\, dy = \frac{2}{9} \left[\frac{y^5}{5} - \frac{2y^3}{3} \right]_{\sqrt{5}}^{\sqrt{8}} =$$
$$= \frac{2}{9} \left[\frac{(\sqrt{8})^5 - (\sqrt{5})^5}{5} - \frac{2(\sqrt{8})^3 - 2(\sqrt{5})^3}{3} \right]$$
$$= \frac{2}{9} \left[\frac{8^2\sqrt{8} - 5^2\sqrt{5}}{5} - \frac{2 \cdot 8\sqrt{8} - 2 \cdot 5\sqrt{5}}{3} \right]$$
$$= \frac{2}{9} \left[\frac{64\sqrt{8} - 25\sqrt{5}}{5} - \frac{16\sqrt{8} - 10\sqrt{5}}{3} \right]$$
$$= \frac{2}{9} \left[\left(\frac{64}{5} - \frac{16}{3} \right) \sqrt{8} + \left(\frac{10}{3} - \frac{25}{5} \right) \sqrt{5} \right]$$
$$= \cdots = \frac{224}{135} \sqrt{8} - \frac{10}{27} \sqrt{5} = \frac{448}{135} \sqrt{2} - \frac{10}{27} \sqrt{5}.$$

In some problems human creativity is needed to "see" the correct substitution

EXAMPLE

$$I = \int x^5 \sqrt{1-x^2}\, dx$$

Let $y = 1 - x^2 \Rightarrow dy = -2x\,dx$ and since
$x^2 = 1 - y^2 \Rightarrow x^4 = (1-y^2)^2$.
It follows that

$$I = \int (1-y^2)^2 \sqrt{y} \cdot (-1/2)\, dy =$$

$$= -\frac{1}{2} \int \sqrt{y}\,(1 - 2y^2 + y^4)\, dy =$$

$$= -\frac{1}{2} \int (\sqrt{y} - 2y^2\sqrt{y} + y^4\sqrt{y})\, dy =$$

$$= -\frac{1}{2} \left[\frac{y\sqrt{y}}{3/2} - 2\,\frac{y^3\sqrt{y}}{7/2} + \frac{y^5\sqrt{y}}{11/2} \right] + c$$

$$= -\frac{y\sqrt{y}}{3} + \frac{y^3\sqrt{y}}{7} - \frac{y^5\sqrt{y}}{11} + c$$

$$= -y\sqrt{y} \left[\frac{1}{3} - \frac{y^2}{7} + \frac{y^4}{11} \right] + c$$

$$= -y\sqrt{y} \cdot \frac{77 - 33y^2 + 21y^4}{232} + c$$

$$= -(1-x^2)\sqrt{1-x^2}\left[\frac{77-33(1-x^2)^2+21(1-x^2)^4}{232}\right]+c$$

$$= \frac{-1}{232}(1-x^2)(77-33(1-x^2)^2+21(1-x^2)^4)\sqrt{1-x^2}+c$$

EXERCISES

(11) Evaluate the following integrals.

a) $\displaystyle\int_0^{-10} e^{x+5}\,dx$

b) $\displaystyle\int_0^1 \frac{3x^2}{\sqrt{x^3+1}}\,dx$

c) $\displaystyle\int_{-1}^0 (2x+3)e^{x^2+3x+2}\,dx$

d) $\displaystyle\int_0^1 (5x-3)^5\,dx$

e) $\displaystyle\int_1^2 \sqrt[3]{x+1}\,dx$

f) $\displaystyle\int_0^1 \frac{dx}{3x+1}$

g) $\displaystyle\int_0^{\pi/2} \cos(3x-\pi/2)\,dx$

h) $\displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{\cos^2(-4x)}$

i) $\displaystyle\int_0^{e-1} \sin(x+1)\,dx$

j) $\displaystyle\int_0^{\sqrt[3]{2}} x^2 e^{x^3}\,dx$

k) $\displaystyle\int_0^{\sqrt{\pi}/2} x\sin(4x^2)\,dx$

l) $\displaystyle\int_{-1}^1 \frac{2x-1}{x^2-x+1}\,dx$

m) $\displaystyle\int_0^{\pi} \cos^2 x \sin x\, 2^{\cos^3 x}\,dx$

n) $\displaystyle\int_1^e \frac{e^{\ln x}}{x}\,dx$

o) $\displaystyle\int_0^{\pi/4} \frac{e^{\tan x}}{\cos^2 x}\,dx.$

(12) Evaluate the following integrals.

a) $\displaystyle\int_0^{\sqrt{3}} \frac{2x+1}{x^2+1}\,dx$

b) $\displaystyle\int \frac{\tan x}{\cos^2 x}\,dx$

c) $\displaystyle\int e^{\cos x} \sin x\,dx$

d) $\displaystyle\int_0^{3/4} \frac{\sin\sqrt{1-x}}{\sqrt{1-x}}\,dx$

e) $\displaystyle\int \frac{\sin(\ln(4x^2))}{x}\,dx$

f) $\displaystyle\int \frac{3e^{2x}}{\sqrt{1-e^{2x}}}\,dx$

g) $\displaystyle\int_0^{\pi/4} \frac{\sin x}{1+\cos^2 x}\,dx$

h) $\displaystyle\int_0^{\pi/6} 2^{\cos x} \cdot \sin x\,dx$

i) $\displaystyle\int_0^{\pi/3} \tan x\,dx$

j) $\displaystyle\int \frac{x^2\cos(x^3-2)}{\sin^2(x^3-2)}\,dx$

k) $\displaystyle\int_0^{\sqrt{3}/6} \frac{\exp(\operatorname{Arctan}(2x))}{1+4x^2}\,dx$

(13) Consider the integrals

$$I_1 = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx, \quad I_2 = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx.$$

Show that $I_1 = I_2 = \pi/4$.

(14) Evaluate the following integrals

a) $\displaystyle\int_0^1 x\sqrt{x+1}\,dx$

b) $\displaystyle\int_0^{5/3} \frac{x\,dx}{\sqrt{3x+4}}$

c) $\displaystyle\int \frac{x^2+3x}{\sqrt{x+4}}$

d) $\displaystyle\int \frac{dx}{x-\sqrt{x}}$

e) $\displaystyle\int_1^{5/2} x^2\sqrt{2x-1}\,dx$

f) $\displaystyle\int_0^2 \frac{x}{1+\sqrt{x+1}}\,dx$