OTHER INVERSE FUNCTIONS

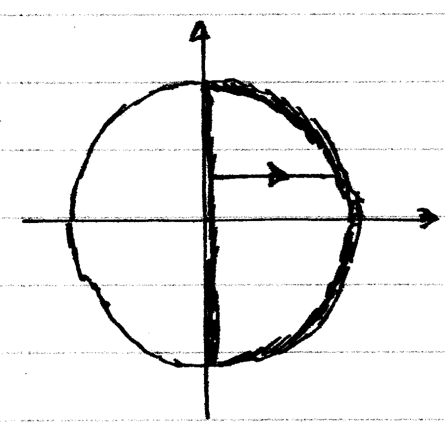
▼ Inverse trigonometric functions

- In general, the trigonometric functions sin, cos, tan, and cot are NOT one-to-one and are NOT invertible. However, by restricting their domain into an appropriate interval, it becomes possible to define inverse trigonometric functions as follows:

▸ notation : Let $f : A \to \mathbb{R}$ be a function and let $B \subseteq A$. We define the restriction $g = f \restriction B$ as follows:

$$\boxed{g = f \restriction B \iff \begin{cases} \forall x \in B : g(x) = f(x) \\ \mathrm{dom}(g) = B \end{cases}}$$

① Inverse of sin $\longrightarrow$ $\boxed{\mathrm{Arcsin} = (\sin \restriction [-\pi/2, \pi/2])^{-1}}$



$$\mathrm{Arcsin}\,x = y \iff \begin{cases} x = \sin y \\ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \end{cases}$$

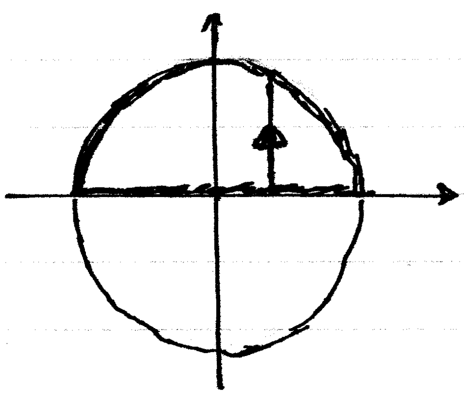
$$\mathrm{dom}(\mathrm{Arcsin}) = [-1, 1]$$

- Properties

a) Arcsin $\nearrow$ $[-1,1]$
b) Arcsin bounded at $[-1,1]$, because
$\forall x \in [-1,1]: |\text{Arcsin}(x)| \leq \pi/2$
c) Arcsin is odd:
$\forall x \in [-1,1]: \text{Arcsin}(-x) = -\text{Arcsin}(x)$
d) Arcsin continuous at $[-1,1]$
* e) Arcsin differentiable at $(-1,1)$ (!!)
with

$$\boxed{\frac{d}{dx}\text{Arcsin}(x) = \frac{1}{\sqrt{1-x^2}}}$$

(2) Inverse of cos $\longrightarrow$ $\boxed{\text{Arccos} = (\cos\restriction[0,\pi])^{-1}}$



$\text{Arccos}\,x = y \Leftrightarrow \begin{cases} x = \cos y \\ 0 \leq y \leq \pi \end{cases}$
$\text{dom}(\text{Arccos}) = [-1,1]$

a) Arccos $\searrow$ $[-1,1]$
b) Arccos bounded at $[-1,1]$:
$\forall x \in [-1,1]: 0 \leq \text{Arccos}(x) \leq \pi$
c) Arccos is neither even nor odd.

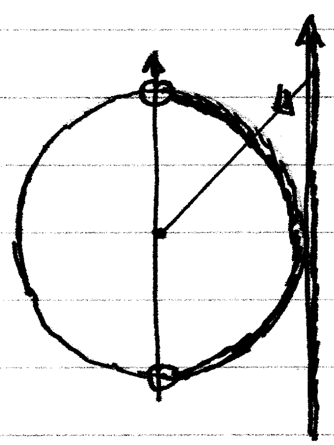
d) Arccos continuous at $[-1,1]$

e) $\boxed{\forall x \in [-1,1] : \operatorname{Arcsin}(x) + \operatorname{Arccos}(x) = \frac{\pi}{2}}$

f) Arccos differentiable at $(-1,1)$ with

$$\boxed{\forall x \in (-1,1) : \frac{d}{dx} \operatorname{Arccos}(x) = -\frac{1}{\sqrt{1-x^2}}}$$

③ Inverse of tan $\longrightarrow$ $\boxed{\operatorname{Arctan} = \left(\tan\restriction(-\pi/2, \pi/2)\right)^{-1}}$



$$\operatorname{Arctan}(x) = y \Leftrightarrow \begin{cases} x = \tan y \\ -\frac{\pi}{2} < y < \frac{\pi}{2} \end{cases}$$

$$\operatorname{dom}(\operatorname{Arctan}) = (-\infty, +\infty) = \mathbb{R}$$

- Properties

a) Arctan $\nearrow \mathbb{R}$

b) Arctan bounded at $\mathbb{R}$ with

$$\boxed{\forall x \in \mathbb{R} : |\operatorname{Arctan}(x)| < \frac{\pi}{2}}$$

c) Arctan is odd with $\boxed{\forall x \in \mathbb{R} : \operatorname{Arctan}(-x) = -\operatorname{Arctan}(x)}$

d) Arctan continuous at $\mathbb{R}$.

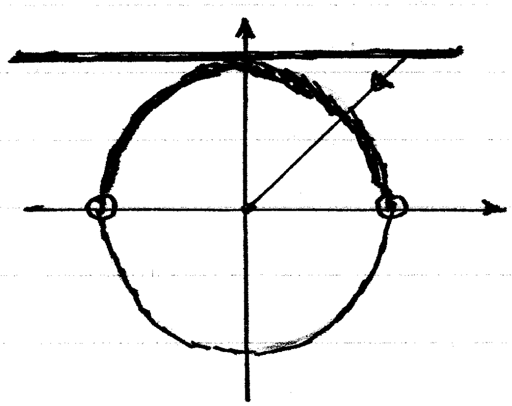
e) Limits to $\pm\infty$:

$\lim\limits_{x \to +\infty} \operatorname{Arctan}(x) = \dfrac{\pi}{2}$	$\lim\limits_{x \to -\infty} \operatorname{Arctan}(x) = -\dfrac{\pi}{2}$

f) Arctan differentiable at $\mathbb{R}$

$$\boxed{\forall x \in \mathbb{R} : \frac{d}{dx} \operatorname{Arctan}(x) = \frac{1}{1+x^2}}$$

④ Inverse of cot $\longrightarrow$ $\boxed{\operatorname{Arccot} = (\cot \restriction (0,\pi))^{-1}}$



$\operatorname{Arccot}(x) = y \Leftrightarrow \begin{cases} x = \cot y \\ 0 < y < \pi \end{cases}$	
$\operatorname{dom}(\operatorname{Arccot}) = \mathbb{R}$	

a) Arccot $\searrow$ $\mathbb{R}$

b) Arccot bounded at $\mathbb{R}$ with

$$\boxed{\forall x \in \mathbb{R} : 0 < \operatorname{Arccot}(x) < \pi}.$$

c) Arccot is not even or odd

d)

$$\boxed{\forall x \in \mathbb{R}: \operatorname{Arctan} x + \operatorname{Arccot} x = \frac{\pi}{2}}$$

e) Limits to $\pm\infty$

$\lim\limits_{x \to +\infty} \operatorname{Arccot}(x) = 0$	$\lim\limits_{x \to -\infty} \operatorname{Arccot}(x) = \pi$

f) Arccot differentiable in $\mathbb{R}$ with

$$\boxed{\forall x \in \mathbb{R}: \frac{d}{dx} \operatorname{Arccot}(x) = \frac{-1}{1+x^2}}$$

Summary of inverse trigonometric functions

Function	Domain	Range	Monotonicity
Arcsin	$[-1,1]$	$[-\pi/2, \pi/2]$	$\nearrow$
Arccos	$[-1,1]$	$[0,\pi]$	$\searrow$
Arctan	$\mathbb{R}$	$[-\pi/2, \pi/2]$	$\nearrow$
Arccot	$\mathbb{R}$	$[0,\pi]$	$\searrow$

$$\frac{d}{dx}\operatorname{Arcsin}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\operatorname{Arctan}(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\operatorname{Arccos}(x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\operatorname{Arccot}(x) = \frac{-1}{1+x^2}$$

$$|\operatorname{Arcsin}(x)| \leq \pi/2, \ \forall x \in [-1,1] \qquad |\operatorname{Arctan}(x)| < \pi/2, \ \forall x \in \mathbb{R}$$

$$0 \leq \operatorname{Arccos}(x) \leq \pi, \ \forall x \in [-1,1] \qquad 0 < \operatorname{Arccot}(x) < \pi, \ \forall x \in \mathbb{R}$$

$$\lim_{x \to +\infty} \operatorname{Arctan}(x) = \frac{\pi}{2} \qquad \lim_{x \to +\infty} \operatorname{Arccot}(x) = 0$$

$$\lim_{x \to -\infty} \operatorname{Arctan}(x) = \frac{-\pi}{2} \qquad \lim_{x \to -\infty} \operatorname{Arccot}(x) = \pi$$

$$\forall x \in [-1,1]: \ \operatorname{Arcsin}(x) + \operatorname{Arccos}(x) = \pi/2$$

$$\forall x \in \mathbb{R}: \ \operatorname{Arctan}(x) + \operatorname{Arccot}(x) = \pi/2$$

Simplifying expressions with inverse trigonometric functions

To simplify such expressions we let ϑ be the value of the inverse trigonometric function and then we find the appropriate trigonometric function evaluation at ϑ using the following identities:

$\sin^2 x + \cos^2 x = 1$ $1 + \tan^2 x = \frac{1}{\cos^2(x)}$	$\tan x = \frac{\sin x}{\cos x}$
$1 + \cot^2 x = \frac{1}{\sin^2 x}$	$\cot x = \frac{\cos x}{\sin x}$

EXAMPLE

Show $\sin(\text{Arctan} x) = \frac{x}{\sqrt{1+x^2}}$

Solution

Let $\vartheta = \text{Arctan} x \Rightarrow -\pi/2 < x < \pi/2 \wedge \tan\vartheta = x$.
It follows that

$$\sin^2\vartheta = 1-\cos^2\vartheta = 1-\frac{1}{1+\tan^2\vartheta} = 1-\frac{1}{1+x^2}$$

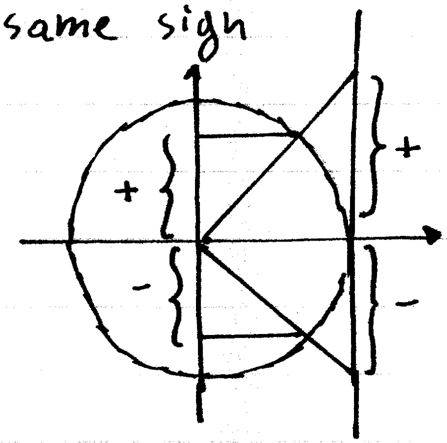
$$= \frac{(1+x^2)-1}{1+x^2} = \frac{x^2}{1+x^2} \Rightarrow \sin\vartheta = \pm\frac{x}{\sqrt{1+x^2}}$$

Since $-\pi/2 \leq \vartheta \leq \pi/2 \Rightarrow$

$\Rightarrow$ $\sin\vartheta$, $\tan\vartheta$ have the same sign (see graph)

$\Rightarrow$ $\sin\vartheta$, x have the same sign

$\Rightarrow$ $\sin\vartheta = \dfrac{x}{\sqrt{1+x^2}}$



↳ Differentiating inverse trigonometric functions

From the chain rule, we have:

$\frac{d}{dx}\text{Arcsin}(f(x))$	$= \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$\frac{d}{dx}\text{Arccos}(f(x))$	$= \frac{-f'(x)}{\sqrt{1-[f(x)]^2}}$
$\frac{d}{dx}\text{Arctan}(f(x))$	$= \frac{f'(x)}{1+[f(x)]^2}$
$\frac{d}{dx}\text{Arccot}(f(x))$	$= \frac{-f'(x)}{1+[f(x)]^2}$

EXERCISES

① Find the default domain for the following functions:

a) $f(x) = \text{Arctan}\left(\sqrt{x^2+x-2}\right)$

b) $f(x) = \text{Arcsin}(3x-2)$

c) $f(x) = \text{Arctan}(e^x)$

d) $f(x) = \text{Arcsin}(e^x)$

e) $f(x) = \text{Arccos}\left(\dfrac{x+1}{2x-1}\right)$

f) $f(x) = \text{Arccos}(\ln x)$

g) $f(x) = \text{Arccos}(x^2+x-2)$

② Show that

a) $\cos(\text{Arctan}\,x) = \dfrac{1}{\sqrt{1+x^2}}$

b) $\sin(\text{Arctan}\,x) = \dfrac{x}{\sqrt{1+x^2}}$

c) $\tan(\text{Arccos}\,x) = \dfrac{\sqrt{1-x^2}}{x}$

d) $\tan(\text{Arcsin}\,x) = \dfrac{x}{\sqrt{1-x^2}}$

e) $\sin(\text{Arccos}\,x) = \sqrt{1-x^2}$

f) $\cos(2\,\text{Arctan}\,x) = \dfrac{1-x^2}{1+x^2}$

g) $\sin(2\,\text{Arctan}\,x) = \dfrac{2x}{1+x^2}$

h) $\sin(2\,\text{Arccos}\,x) = 2x\sqrt{1-x^2}$

③ Evaluate the derivatives of the following functions

a) $f(x) = (x^2+3x+1)\,\text{Arctan}\,x$

b) $f(x) = (1-x^2)\,\text{Arcsin}\,x$

c) $f(x) = (1+x^2)\,\text{Arccot}\,x$

d) $f(x) = \sqrt{1-x^2}\,\text{Arccos}\,x$

④ Similarly for the following functions

a) $f(x) = \text{Arcsin}(1-2x)$

b) $f(x) = \text{Arctan}(\sin\vartheta)$

c) $f(x) = \ln(\text{Arcsin}\, x)$

d) $f(x) = \text{Arctan}(\ln x)$

e) $f(x) = \text{Arcsin}\left(\sqrt{1-x^2}\right)$

f) $f(x) = \text{Arctan}\left(\sqrt{1-x^2}\right)$

g) $f(x) = \text{Arccos}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$

⑤ Evaluate the following limits.

a) $\lim\limits_{x\to+\infty} \text{Arcsin}\left(\dfrac{x^2}{x^2+x+1}\right)$

b) $\lim\limits_{x\to-\infty} \text{Arctan}\left(\dfrac{x^2\sqrt{3}}{(x+3)^2}\right)$

c) $\lim\limits_{x\to 3^-} \text{Arctan}\left(\dfrac{x^2+1}{x^2-9}\right)$

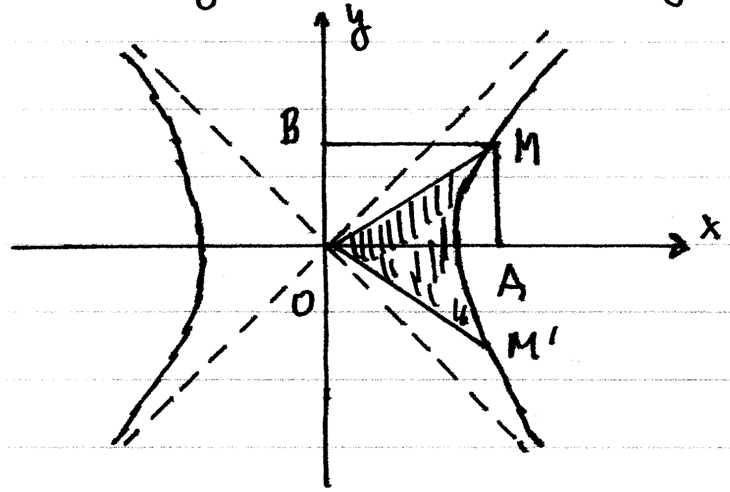
d) $\lim\limits_{x\to 1^+} \text{Arccot}\left(\dfrac{1-2x^2}{x^2+4x+3}\right)$

e) $\lim\limits_{x\to-\infty} \text{Arctan}\left((x+2)^2(x-3)^3\right)$

f) $\lim\limits_{x\to 0^+} \text{Arctan}(\ln(\sin x))$

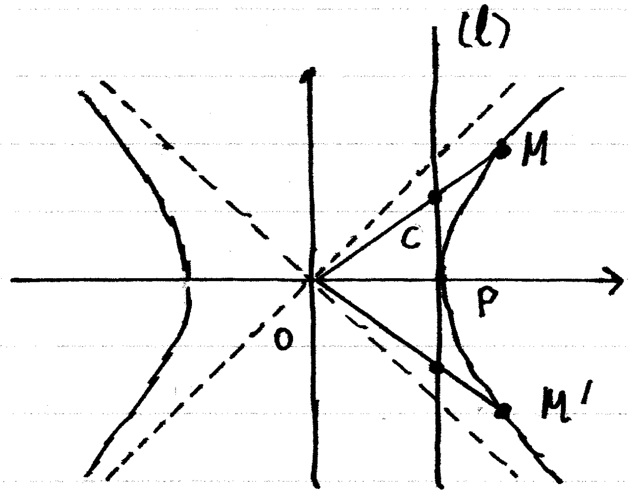
Hyperbolic functions

- They are defined geometrically in terms of the hyperbola $(c): x^2 - y^2 = 1$



$x =$ the area between the hyperbola and OM and its mirror image OM'.
Project M to A and B.
For $x < 0$ we use the left branch of the hyperbola. Thus we define

$$\boxed{\begin{aligned} \sinh(x) &= \overline{OB} \\ \cosh(x) &= \overline{OA} \end{aligned}}$$



To define tanh, for $x > 0$, consider the line $(\ell): x = 1$. Let C be the point where OM intersects (ℓ). For $x < 0$ we use the left branch, the line $(\ell): x = -1$ and place the point M below the x-axis. Then we define

$$\boxed{\tanh(x) = \overline{PC}}$$

with P the point where (ℓ) intersects x-axis.

Algebraic properties

Function	Domain	Range	Parity
$\sinh(x) = \dfrac{e^x - e^{-x}}{2}$	$\mathbb{R}$	$\mathbb{R}$	odd
$\cosh(x) = \dfrac{e^x + e^{-x}}{2}$	$\mathbb{R}$	$[1, +\infty)$	even
$\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	$\mathbb{R}$	$(-1, 1)$	odd

Variation Table

x		0	
$\sinh(x)$	$-\infty$ ↗		↗ $+\infty$
$\cosh(x)$	$+\infty$ ↘		↗ $+\infty$
$\tanh(x)$	-1 ↗		↗ 1

Derivatives

$\dfrac{d}{dx}\sinh(x) = \cosh(x)$
$\dfrac{d}{dx}\cosh(x) = \sinh(x)$
$\dfrac{d}{dx}\tanh(x) = 1/(\cosh x)^2 = 1 - \tanh^2 x$

Identities

$\sinh(-x) = -\sinh(x)$ $\cosh(-x) = \cosh(x)$ $\cosh^2 x - \sinh^2 x = 1$ $\dfrac{1}{\cosh^2 x} = 1 - \tanh^2 x$	$\sinh(x+y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x)$ $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

Inverse hyperbolic functions

Function	Domain	Range
$\sinh^{-1}(x) = \ln(x+\sqrt{x^2+1})$	$\mathbb{R}$	$\mathbb{R}$
$\cosh^{-1}(x) = \ln(x+\sqrt{x^2-1})$	$[1,+\infty)$	$[0,+\infty)$ (!!)
$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$(-1,1)$	$\mathbb{R}$

- Note that cosh is not "1-1" (because cosh is even), thus to define $\cosh^{-1}$ we restrict the domain of cosh to $[0,+\infty)$ before inverting. Thus

$$\cosh^{-1} = (\cosh \restriction [0,+\infty))^{-1}.$$

Derivatives of inverse trigonometric functions

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}, \quad \forall x \in \mathbb{R}$$

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}, \quad \underline{\underline{\forall x \in (-1,+\infty)}} (!!)$$

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}, \quad \forall x \in (-1,1)$$

- Note that $\cosh^{-1}$ is not differentiable over its entire domain (not differentiable at $x=-1$)

EXERCISES

⑥ Prove, by definition, that

a) $\sinh(x+y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x)$

b) $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

c) $\cosh^2 x - \sinh^2 x = 1.$

Then use these identities to show that

d) $\sinh(x+y) + \sinh(x-y) = 2\sinh(x)\cosh(x)$

e) $\cosh(x+y) + \cosh(x-y) = 2\cosh(x)\cosh(y)$

f) $\cosh(x+y) - \cosh(x-y) = 2\sinh(x)\sinh(y)$

g) $\sinh(x+y)\sinh(x-y) = \sinh^2 x - \sinh^2 y$

h) $\cosh(x+y)\cosh(x-y) = \cosh^2 x + \sinh^2 y$

⑦ Show that:

a) $\cosh(2x) = \cosh^4 x - \sinh^4 x$

b) $\sinh^2(x)[1 - \tanh^2(x)] = \tanh^2(x)$

c) $(\cosh x + \sinh x)^n = \cosh(nx) + \sinh(nx)$

⑧ Use the properties of logarithms to show that

a) $\sinh^{-1}(x) + \sinh^{-1}(-x) = 0$

b) $\tanh^{-1}(x) + \tanh^{-1}(-x) = 0$

⑨ Prove the results given for the derivatives of the inverse hyperbolic functions $\sinh^{-1}(x)$, $\cosh^{-1}(x)$, $\tanh^{-1}(x)$.

(10) Show that

a) $\lim\limits_{x \to -1^+} \tanh^{-1}(x) = -\infty$

b) $\lim\limits_{x \to 1^-} \tanh^{-1}(x) = +\infty$

c) $\lim\limits_{x \to +\infty} \cosh^{-1}(x) = +\infty$

d) $\lim\limits_{x \to +\infty} \sinh^{-1}(x) = +\infty$

e) $\lim\limits_{x \to +\infty} \left[\cosh^{-1}(x) - \sinh^{-1}(x)\right] = 0$

f) $\lim\limits_{x \to -1^+} \sinh^{-1}(\tanh^{-1}(x)) = -\infty$

(11) Use the identity $e^{ix} = \cos x + i\sin x$ to show that

a) $\sin x = -i\sinh(ix)$

b) $\cos x = \cosh(ix)$

c) $\tan x = \tanh(ix)$

(Here: $i = \sqrt{-1}$).

↳ This exercise establishes the relationship between trigonometric functions and hyperbolic functions.

De L'Hospital's theorem

- The De L'Hospital theorem provides an additional method for resolving indeterminate forms $0/0$ and ∞/∞.

- <u>Thm</u>: Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ be two functions that satisfy the following conditions:

1) f, g differentiable at $N(x_0, \delta)$
2) $g'(x) \neq 0, \forall x \in N(x_0, \delta)$
3) $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \lor \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \pm\infty$
4) $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l \in \mathbb{R} \lor \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \pm\infty$

Then, it follows that

$$\boxed{\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

↳ In using this theorem, we show condition (4) retroactively by successfully evaluating the limit. However, if application of De L'Hospital gives a limit that does not exist, then this shows that applying De L'Hospital is not properly justified, and the limit may or may not exist.

EXAMPLES

- $\boxed{0/0 \text{ Form}}$

a) $$\lim_{x\to 1} \frac{x-1-\ln x}{(x-1)\ln x} = \left(\frac{0}{0}\right) \overset{\text{L'Hospital}}{=} \lim_{x\to 1} \frac{[x-1-\ln x]'}{[(x-1)\ln x]'} =$$

$$= \lim_{x\to 1} \frac{1-0-\frac{1}{x}}{(x-1)'\ln x + (x-1)(\ln x)'} =$$

$$= \lim_{x\to 1} \frac{1-\frac{1}{x}}{\ln x + \frac{x-1}{x}} = \left(\frac{0}{0}\right) \underset{\text{L'Hospital.}}{=}$$

$$= \lim_{x\to 1} \frac{0-\left(-\frac{1}{x^2}\right)}{\frac{1}{x} + \frac{(x-1)'x-(x-1)(x)'}{x^2}} =$$

$$= \lim_{x\to 1} \frac{\frac{1}{x^2}}{\frac{1}{x} + \frac{x-(x-1)}{x^2}} = \lim_{x\to 1} \frac{\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}} =$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

b) $\lim\limits_{x\to 0} \dfrac{2\sin x - \sin(2x)}{x - \sin x} = \left(\dfrac{0}{0}\right) = \lim\limits_{x\to 0} \dfrac{2\cos x - 2\cos(2x)}{1-\cos x}$

$= \left(\dfrac{0}{0}\right) = \lim\limits_{x\to 0} \dfrac{-2\sin x + 4\sin 2x}{\sin x} = \left(\dfrac{0}{0}\right)$

$= \lim\limits_{x\to 0} \dfrac{-2\cos x + 8\cos 2x}{\cos x} = \dfrac{-2+8}{1} = 6.$

↳ We apply De L'Hospital 3 times!

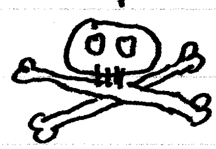
• $\boxed{\text{Form } \infty/\infty}$

c) $\lim\limits_{x\to +\infty} \dfrac{\ln(1+e^x)}{x+1} = \left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to +\infty} \dfrac{\frac{(1+e^x)'}{1+e^x}}{1} =$

$= \lim\limits_{x\to +\infty} \dfrac{e^x}{e^x+1} = \left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to +\infty} \dfrac{e^x}{e^x} = 1.$

• <u>Failure of De L'Hospital:</u>

d) $\lim\limits_{x\to \infty} \dfrac{x+\sin x}{x} = \left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to \infty} \dfrac{1+\cos x}{1} =$



(wrong step)
(condition 4 fails)

$= \lim\limits_{x\to \infty} (1+\cos x)$ ← does not exist

However: $f(x) = \dfrac{x+\sin x}{x} = 1 + \dfrac{\sin x}{x}$ and since

$$\left.\begin{array}{l}\sin x \text{ bounded at } \mathbb{R} \\ \lim\limits_{x\to+\infty} \frac{1}{x} = 0\end{array}\right\} \Rightarrow \lim_{x\to+\infty} \frac{\sin x}{x} = 0 \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(1 + \frac{\sin x}{x}\right) = 1+0 = 1.$$

so the limit does in fact exist!

- <u>Application</u> : $\boxed{\lim\limits_{x\to+\infty}[f(x)+f'(x)] = \ell \Rightarrow \begin{cases}\lim\limits_{x\to+\infty} f(x) = \ell \\ \lim\limits_{x\to+\infty} f'(x) = 0\end{cases}}$

<u>Proof</u>

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{e^x f(x)}{e^x} = \lim_{x\to+\infty} \frac{(e^x)' f(x) + e^x f'(x)}{(e^x)'} =$$

$$= \lim_{x\to+\infty} \frac{e^x[f(x)+f'(x)]}{e^x} =$$

$$= \lim_{x\to+\infty} [f(x)+f'(x)] = \ell.$$

Thus: $\lim\limits_{x\to+\infty} f'(x) = \lim\limits_{x\to+\infty} [(f(x)+f'(x)) - f(x)] =$

$$= \lim_{x\to+\infty}(f(x)+f'(x)) - \lim_{x\to+\infty} f(x) =$$

$$= \ell - \ell = 0. \quad \square$$

Other indeterminent forms

Other indeterminent forms can be reduced to to the forms 0/0 and ∞/∞.

- $\boxed{\text{Form } \infty-\infty}$: Simplify and reduce to 0/0 or ∞/∞.

a) $\lim\limits_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=(\infty-\infty)=\lim\limits_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}=\left(\frac{0}{0}\right)$

$=\lim\limits_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^4}\,\frac{x^2}{\sin^2 x}\right)=$

$=\lim\limits_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^4}\right)\lim\limits_{x\to 0}\left(\frac{x}{\sin x}\right)^2=$

$=\lim\limits_{x\to 0}\frac{x^2-\sin^2 x}{x^4}\underset{\uparrow *}{=}\lim\limits_{x\to 0}\frac{2x-2\sin x\cos x}{4x^3}=$

$=\lim\limits_{x\to 0}\frac{2x-\sin 2x}{4x^3}=\left(\frac{0}{0}\right)=\lim\limits_{x\to 0}\frac{2-2\cos 2x}{12x^2}=$

$=\lim\limits_{x\to 0}\frac{1-\cos 2x}{6x^2}=\left(\frac{0}{0}\right)=\lim\limits_{x\to 0}\frac{\sin 2x}{6x}=\left(\frac{0}{0}\right)$

$=\lim\limits_{x\to 0}\frac{2\cos 2x}{6}=\frac{2\cos 0}{6}=\frac{1}{3}.$

1 → As can be seen from this example, De L'Hospital's theorem supplements our previous techniques for evaluating trigonometric limits but does <u>not</u> render them obsolete!

• $\boxed{\text{Form } 0 \cdot \infty}$ ⟶ It occurs for a function of the form $f(x) = g_1(x) g_2(x)$ with $\lim_{x \to \sigma} g_1(x) = 0$ and $\lim_{x \to \sigma} g_2(x) = \pm\infty$

We use the following technique:

$$f(x) = g_1(x) g_2(x) = \frac{g_1(x)}{1/g_2(x)} = \left(\frac{0}{0}\right)$$

and apply De L'Hospital. Or: $f(x) = \dfrac{g_2(x)}{1/g_1(x)} = \left(\dfrac{\infty}{\infty}\right)$

EXAMPLE

6) $\lim_{x \to 0^+} (\tan x \cdot \ln x) = (0 \cdot (-\infty)) = \lim_{x \to 0^+} \dfrac{\ln x}{\dfrac{1}{\tan x}} = \left(\dfrac{\infty}{\infty}\right)$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{\tan^2 x} (\tan x)'} =$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{\tan^2 x} \frac{1}{\cos^2 x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}}$$

$$= \lim_{x \to 0^+} \frac{-\sin^2 x}{x} = \lim_{x \to 0^+} \frac{\sin x}{x} \cdot \lim_{x \to 0^+} (-\sin x)$$

$$= 1 \cdot (-0) = 0.$$

• $\boxed{\text{Form } 0^0,\ \infty^0,\ 1^\infty}$ ⟶ Occurs from functions of the form $f(x) = a(x)^{b(x)}$.

We use the following technique:

$$\lim_{x\to\sigma} f(x) = \lim_{x\to\sigma} a(x)^{b(x)} = \lim_{x\to\sigma} \exp\left(b(x)\ln a(ax)\right)$$

$$= \exp\left(\lim_{x\to\sigma} b(x)\ln a(x)\right) = \dots$$

The resulting limit is $0/0$, ∞/∞, $0\cdot\infty$.

EXAMPLE

a) $$\lim_{x\to 0}\left(\tan\frac{x}{2}\right)^{1/\ln x} = (0^0) = \lim_{x\to 0}\exp\left(\frac{1}{\ln x}\ln\left(\tan\frac{x}{2}\right)\right)$$

$$= \exp\left[\lim_{x\to 0}\frac{\ln(\tan(x/2))}{\ln x}\right] = \left(\frac{\infty}{\infty}\right)$$

$$= \exp\left[\lim_{x\to 0}\frac{\frac{1}{\tan(x/2)}\left(\tan(x/2)\right)'}{1/x}\right] =$$

$$= \exp\left[\lim_{x\to 0}\frac{\frac{1}{\tan(x/2)}\,\frac{1}{\cos^2(x/2)}\,\frac{1}{2}}{1/x}\right]$$

$$= \exp\left[\lim_{x\to 0}\frac{x}{2\sin(x/2)\cos(x/2)}\right] =$$

$$= \exp\left[\lim_{x\to 0} \frac{x/2}{\sin(x/2)} \lim_{x\to 0} \frac{1}{\cos(x/2)}\right] =$$

$$= \exp\left[1 \cdot \frac{1}{\cos 0}\right] = e^1 = e .$$

$$\text{б)} \lim_{x\to+\infty} \left(\frac{x+2}{x+1}\right)^x = \lim_{x\to+\infty} \exp\left[x \ln\left(\frac{x+2}{x+1}\right)\right] =$$

$$= \exp\left[\lim_{x\to+\infty} x \ln\left(\frac{x+2}{x+1}\right)\right] =$$

$$= \exp\left[\lim_{x\to+\infty} \frac{\ln\left(\frac{x+2}{x+1}\right)}{1/x}\right] = \left(\frac{0}{0}\right)$$

$$= \exp\left[\lim_{x\to+\infty} \frac{\frac{x+1}{x+2}\left(\frac{x+2}{x+1}\right)'}{-1/x^2}\right] =$$

$$= \exp\left[\lim_{x\to+\infty} \left(-x^2 \frac{x+1}{x+2} \frac{(x+2)'(x+1)-(x+2)(x+1)'}{(x+1)^2}\right)\right]$$

$$= \exp\left[\lim_{x\to+\infty} \left(-x^2 \frac{x+1}{x+2} \frac{(x+1)-(x+2)}{(x+1)^2}\right)\right] =$$

$$= \exp\left[\lim_{x\to+\infty} \left(-x^2 \frac{x+1}{x+2} \frac{-1}{(x+1)^2}\right)\right] =$$

$$= \exp\left[\lim_{x\to+\infty}\left(\frac{x^2}{(x+1)(x+2)}\right)\right] = \exp\left[\lim_{x\to+\infty}\frac{x^2}{x^2}\right]$$

$$= \exp(1) = e.$$

EXERCISES

(12) Evaluate the following limits:

1) $\lim\limits_{x\to\pi/2} \dfrac{\cos x}{2x-\pi}$

2) $\lim\limits_{x\to 0} \dfrac{\cos 3x - 1}{x^2}$

3) $\lim\limits_{x\to 0} \dfrac{e^x + e^{-x} - 2\cos x}{1-\cos x}$

4) $\lim\limits_{x\to 0} \dfrac{1-\cos x - \ln(\cos x)}{x^2}$

5) $\lim\limits_{x\to +\infty} \dfrac{\ln^2 x}{x}$

6) $\lim\limits_{x\to 0} \dfrac{\cos x - \cos 2x}{x^2}$

7) $\lim\limits_{x\to +\infty} \dfrac{e^x}{x+\ln x}$

8) $\lim\limits_{x\to 0^+} \left(\dfrac{1}{1-\cos x} - \dfrac{1}{x}\right)$

9) $\lim\limits_{x\to 0^+} \left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$

10) $\lim\limits_{x\to 1^+} \left(\dfrac{1}{x-1} - \dfrac{1}{\ln x}\right)$

11) $\lim\limits_{x\to 0^+} x^x$

12) $\lim\limits_{x\to +\infty} (\ln x)^{1/x}$

13) $\lim\limits_{x\to 0^+} x^{\sin x}$

14) $\lim\limits_{x\to +\infty} \left[x^2 \ln\left(\cos\left(\dfrac{\pi}{x}\right)\right)\right]$

15) $\lim\limits_{x\to 0} \left(\dfrac{3e^x - e^{-x}}{2}\right)^{1/x}$

16) $\lim\limits_{x\to 0} (\sin x \cdot \ln x)$

17) $\lim\limits_{x\to 0^+} (\sin x)^{\sin x}$

18) $\lim\limits_{x\to 0^+} (\sin x)^x$

19) $\lim\limits_{x\to 1^+} \left(\dfrac{1}{x-1}\right)^{x-1}$

20) $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2} - \cot^2 x\right)$

21) $\lim\limits_{x\to 0^+} (1+x^2)^{\cot x}$

22) $\lim\limits_{x\to +\infty} \left(1+\dfrac{3}{x}\right)^x$

(13) Consider the limit $\lim_{x \to 0} \frac{x^2 \sin(1/x)}{\sin x}$

a) Show that the first 3 conditions of the De L'Hospital theorem are satisfied but that the 4th condition is violated. (Hint: Use without proof the statement that $\lim_{x \to \pm\infty} \cos x$ does not exist)

b) Use other methods to show that

$$\lim_{x \to 0} \frac{x^2 \sin(1/x)}{\sin x} = 0$$

(14) Let f be a function differentiable in $(-a, a)$ with $a > 0$ and $f(0) = f'(0) = 1$ and $\forall x \in (-a, a): f(x) > 0$. Show that

$$\lim_{x \to 0} [f(x)]^{1/x} = e$$

(15) Let f be a function that is twice differentiable in $\mathbb{R}$ with f' continuous in $\mathbb{R}$ and $f(0) = f'(0) = 0$. Let g be defined as

$$g(x) = \begin{cases} \frac{f(x)}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Show that

a) g differentiable in $\mathbb{R}$

b) g' continuous in $\mathbb{R}$.