EXPONENTIALS AND LOGARITHMS

Definition of powers

Although the concept of raising a number to a power is oftentimes taken for granted, a rigorous definition is not easy to construct and can only be done in multiple steps as follows:

(1) → Powers on $\mathbb{N}$

Let $a \in \mathbb{R}$ and $x \in \mathbb{N}$, noting that $\mathbb{N} = \{0, 1, 2, \dots\}$. We define a^x as follows:

$$\boxed{\begin{cases} a^0 = 1 \\ \forall x \in \mathbb{N}: a^{x+1} = a \cdot a^x \end{cases}}$$

It follows that for $x > 0$:

$$\boxed{a^x = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{x \text{ times}}}$$

Although there is some controversy on whether 0^0 should be presumed to be undefined, no mathematical inconsistencies emerge if we assume that $0^0 = 1$.

An immediate consequence of this definition is that powers satisfy the following properties:

$$\boxed{\begin{array}{l} \forall a\in\mathbb{R}: \forall x_1,x_2\in\mathbb{N}: a^{x_1}a^{x_2} = a^{x_1+x_2} \\ \forall a,b\in\mathbb{R}: \forall x\in\mathbb{N}: (ab)^x = a^x b^x \\ \forall a\in\mathbb{R}: \forall x_1,x_2\in\mathbb{N}: (a^{x_1})^{x_2} = (a^{x_2})^{x_1} = a^{x_1 x_2} \end{array}}$$

→ Our goal is to now expand the definition of powers while preserving the validity of these three fundamental properties

② → Powers on $\mathbb{Z}$

Let $a\in\mathbb{R}-\{0\}$ and $x\in\mathbb{N}-\{0\}$. We define negative integer powers via

$$\boxed{\forall a\in\mathbb{R}-\{0\}: \forall x\in\mathbb{N}-\{0\}: a^{-x} = \frac{1}{a^x}}$$

Note that the fundamental 3 properties continue to hold as follows:

$$\boxed{\begin{array}{l} \forall a\in\mathbb{R}-\{0\}: \forall x_1,x_2\in\mathbb{Z}: a^{x_1}a^{x_2} = a^{x_1+x_2} \\ \forall a,b\in\mathbb{R}-\{0\}: \forall x\in\mathbb{Z}: (ab)^x = a^x b^x \\ \forall a\in\mathbb{R}-\{0\}: \forall x_1,x_2\in\mathbb{Z}: (a^{x_1})^{x_2} = (a^{x_2})^{x_1} = a^{x_1 x_2} \end{array}}$$

We also note that negative powers of 0 cannot be defined. For example, assume that $a = 0^{-x}$ for any $x \in \mathbb{N} - \{0\}$. Then, it follows that

$$a0 = 0^{-x} 0^{x} = 0^{-x+x} = 0^0 = 1$$

which is a contradiction. It is possible, however, to define negative integer powers of any nonzero real number $a \in \mathbb{R} - \{0\}$. We stress that this extension of powers to negative integers is unique No other possible extensions exist that would retain consistency with the three fundamental properties given above.

3 → Powers on $\mathbb{Q}$

Let $a \in (0, +\infty)$ and $x = p/q \in \mathbb{Q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N} - \{0\}$. We define a^x as follows:

- 1 We use the Bolzano and Rolle theorems to show that
  q even $\Rightarrow x^q - a = 0$ has a unique solution on $(0, +\infty)$
  q odd $\Rightarrow x^q - a = 0$ has a unique solution on $\mathbb{R}$
  This unique solution is denoted as $x = \sqrt[q]{a}$ thus defining radicals of order q.
- 2 We then use radicals to define

$$\boxed{a^x = a^{p/q} = [\sqrt[q]{a}]^p}$$

This extended definition continues to satisfy the three fundamental properties of powers as follows:

$$\boxed{\begin{array}{l}\forall a\in(0,+\infty):\forall x_1,x_2\in\mathbb{Q}: a^{x_1}a^{x_2}=a^{x_1+x_2}\\ \forall a,b\in(0,+\infty):\forall x\in\mathbb{Q}: (ab)^x=a^x b^x\\ \forall a\in(0,+\infty):\forall x_1,x_2\in\mathbb{Q}: (a^{x_1})^{x_2}=(a^{x_2})^{x_1}=a^{x_1x_2}\end{array}}$$

This is the only possible definition of rational powers that satisfies the above properties. Extending the definition of rational powers to negative numbers results in inconsistency with the fundamental properties of powers. For example:

$$1=1^{1/2}=[(-1)(-1)]^{1/2}=(-1)^{1/2}(-1)^{1/2}=(-1)^{1/2+1/2}$$
$$=(-1)^1=-1$$

is a contradiction. For this reason, for rational powers a^x, we limit the base a to $a\in(0,+\infty)$.

(4) → Real powers

Let $a\in(0,+\infty)$ and $x\in\mathbb{R}$. The final challenge is to define a^x where the exponent x is an arbitrary real number.

We begin by noting that every real number $x\in\mathbb{R}$ can be approximated by a sequence of rational numbers $x_1,x_2,\dots,x_n,\dots\in\mathbb{Q}$. We then say that $\lim_{n\in\mathbb{N}^*} x_n=x$.

e.g. The number $x=\sqrt{2}$ can be approximated by the following sequence of rational numbers:

$x_1 = 1 \qquad x_4 = 1.414$

$x_2 = 1.4 \qquad x_5 = 1.4142$

$x_3 = 1.41 \qquad x_6 = 1.41421$

and we write $\lim\limits_{n\in\mathbb{N}^*} x_n = \sqrt{2}$.

▸ Limit of sequences

Let a_n be a sequence and let $l\in\mathbb{R}$. We say that

$$\boxed{\begin{array}{l} \lim\limits_{n\in\mathbb{N}^*} a_n = l \Leftrightarrow \forall \varepsilon\in(0,+\infty) : \exists n_0\in\mathbb{N} : \forall n\in\mathbb{N}^*-[n_0] : |a_n - l| < \varepsilon \\ (a_n) \text{ convergent} \Leftrightarrow \exists l\in\mathbb{R} : \lim\limits_{n\in\mathbb{N}^*} a_n = l \end{array}}$$

with $[n_0] = \{1,2,3,\ldots,n_0\}$ and $\mathbb{N}^* = \{1,2,3,\ldots\}$.
We also note that

$$\boxed{\left.\begin{array}{l} \lim\limits_{n\in\mathbb{N}^*} a_n = x \\ \lim\limits_{x\to x_0} f(x) = y_0 \end{array}\right\} \Rightarrow \lim\limits_{n\in\mathbb{N}^*} f(a_n) = l}$$

Let $x_1, x_2, \ldots, x_n, \ldots \in \mathbb{Q}$ be a sequence of rational numbers that approximate $x \in \mathbb{R}$ such that

$$\lim_{n \in \mathbb{N}^*} x_n = x$$

and let $a \in (0, +\infty)$. It can be shown that a^{x_n}, which consists of previously defined rational powers, is a convergent sequence and we define

$$\boxed{a^x = \lim_{n \in \mathbb{N}^*} a^{x_n}}$$

We conclude that real powers satisfy the following properties:

$$\boxed{\begin{array}{l}
\forall x \in \mathbb{R}: \; a^x > 0 \\
\forall x_1, x_2 \in \mathbb{R}: \; a^{x_1} a^{x_2} = a^{x_1 + x_2} \\
\forall x \in \mathbb{R}: \; (ab)^x = a^x b^x \\
\forall x_1, x_2 \in \mathbb{R}: \; (a^{x_1})^{x_2} = (a^{x_2})^{x_1} = a^{x_1 x_2} \\
a > 1 \Rightarrow \begin{cases} a^x > 1 & , \; \forall x \in (0, +\infty) \\ a^x = 1 & , \; \text{for } x = 0 \\ a^x < 1 & , \; \forall x \in (-\infty, 0) \end{cases} \\
0 < a < 1 \Rightarrow \begin{cases} 0 < a^x < 1 & , \; \forall x \in (0, +\infty) \\ a^x = 1 & , \; \text{for } x = 0 \\ a^x > 1 & , \; \forall x \in (-\infty, 0) \end{cases} \\
a > b > 0 \wedge x > 0 \Rightarrow a^x > b^x \\
a > b > 0 \wedge x < 0 \Rightarrow a^x < b^x
\end{array}}$$

Napier's constant

Recall that we defined Napier's constant as

$$\boxed{e = \lim \left(1+\frac{1}{n}\right)^n}$$

We now show that e satisfies the following properties:

1) $\boxed{\lim\limits_{x\to+\infty} \left(1+\frac{1}{x}\right)^x = e}$

Proof

Assume, with no loss of generality, that $x \in (1, +\infty)$.
Define $[x] = \max\{n \in \mathbb{Z} \mid n \leq x\}$.
It follows that $[x] \leq x < [x]+1 \Rightarrow$

$$\Rightarrow \frac{1}{[x]+1} < \frac{1}{x} \leq \frac{1}{[x]}$$

Now, we note that

$$\left(1+\frac{1}{x}\right)^x \geq \left(1+\frac{1}{x}\right)^{[x]} > \left(1+\frac{1}{[x]+1}\right)^{[x]} \quad (1)$$

and

$$\left(1+\frac{1}{x}\right)^x \leqslant \left(1+\frac{1}{[x]}\right)^x < \left(1+\frac{1}{[x]}\right)^{[x]+1} \qquad (2)$$

and

$$\lim_{x\to+\infty}\left(1+\frac{1}{1+[x]}\right)^{[x]} = \lim\left(1+\frac{1}{n+1}\right)^n =$$

$$= \lim\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)} = \frac{\lim\left(1+\frac{1}{n+1}\right)^{n+1}}{\lim\left(1+\frac{1}{n+1}\right)} =$$

$$= \frac{e}{1+0} = e \qquad (3)$$

and

$$\lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]+1} = \lim\left(1+\frac{1}{n}\right)^{n+1} =$$

$$= \lim\left[\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)\right] =$$

$$= \lim\left(1+\frac{1}{n}\right)^n \cdot \lim\left(1+\frac{1}{n}\right) =$$

$$= e\cdot(1+0) = e \qquad (4)$$

Using the squeeze theorem, from (1),(2),(3),(4) we get

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e \qquad \square$$

2) $\boxed{\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e}$

Proof

Let $x = -(y+1) \Leftrightarrow y = -x-1$. Then

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y\to+\infty}\left(1-\frac{1}{y+1}\right)^{-(y+1)} =$$
$$= \lim_{y\to+\infty}\left(\frac{y+1-1}{y+1}\right)^{-(y+1)} = \lim_{y\to+\infty}\left(\frac{y}{y+1}\right)^{-(y+1)}$$
$$= \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y+1} = \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y}\cdot\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)$$
$$= e\cdot(1+0) = e \quad \square$$

3) $\boxed{\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^x = e^a, \ \forall a\in\mathbb{R}}$

Proof

Distinguish three cases:

Case 1: If $a=0$, then

$$\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^x = \lim_{x\to+\infty} 1^x = 1 = e^0 = e^a.$$

Case 2 : If $a>0$, then

$$\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^x=\lim_{x\to+\infty}\left[\left(1+\frac{a}{x}\right)^{x/a}\right]^a=$$

$$=\left[\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^{x/a}\right]^a=$$

$$=\left[\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^y\right]^a=e^a$$

for $y=x/a$.

Case 3 : If $a<0$, then

$$\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^x=\lim_{x\to+\infty}\left[\left(1+\frac{a}{x}\right)^{x/a}\right]^a=$$

$$=\left[\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^{x/a}\right]^a= \quad (!!)$$

$$=\left[\lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^y\right]^a=$$

$$=e^a \qquad \square$$

It follows from (3) that e^x can be written as the limit of a sequence:

$$\boxed{e^x=\lim\left(1+\frac{x}{n}\right)^n}$$

Recall the Bernoulli inequality:

$$\boxed{1+a>0 \Rightarrow \forall n\in\mathbb{N} : (1+a)^n \geq 1+na}$$

We now use it with (3) to show that:

4) $$\boxed{e^x \geq x+1 \,, \; \forall x\in\mathbb{R}}$$

Proof

Let $x\in\mathbb{R}$ be given. Choose $n_0\in\mathbb{N}$ such that $n_0>-x$. It follows that for

$n>n_0 \Rightarrow n>-x \Rightarrow n+x>0 \Rightarrow 1+\frac{x}{n}>0 \Rightarrow$

$$\Rightarrow \left(1+\frac{x}{n}\right)^n \geq 1+n\cdot\frac{x}{n} = 1+x, \; \forall n>n_0$$

$$\Rightarrow e^x = \lim \left(1+\frac{x}{n}\right)^n \geq 1+x \Rightarrow$$

$$\Rightarrow e^x \geq 1+x. \qquad \square$$

5) $$\boxed{\lim_{x\to 0} \frac{e^x-1}{x} = 1}$$

Proof

Let $x \in (-1,0) \cup (0,1)$ be given. Then

$$e^x \geq x+1 \Rightarrow e^{-x} \geq 1-x \Rightarrow e^x \leq \frac{1}{1-x} \Rightarrow$$

$$\Rightarrow e^x - 1 \leq \frac{1}{1-x} - 1 = \frac{1-1+x}{1-x} = \frac{x}{1-x}$$

and $e^x - 1 \geq (x+1) - 1 = x$. It follows that

$$x \leq e^x - 1 \leq \frac{x}{1-x} \quad (1)$$

Note that $\lim\limits_{x \to 0} \frac{1}{1-x} = \frac{1}{1-0} = 1$ (2).

For $x \in (0,1)$, from (1) we get

$$1 \leq \frac{e^x-1}{x} \leq \frac{1}{1-x} \underset{(2)}{\Rightarrow} \lim_{x \to 0+} \frac{e^x-1}{x} = 1 \quad (3)$$

For $x \in (-1,0)$, from (1) we get

$$1 \geq \frac{e^x-1}{x} \geq \frac{1}{1-x} \underset{(2)}{\Rightarrow} \lim_{x \to 0^-} \frac{e^x-1}{x} = 1 \quad (4)$$

From (3) and (4): $\lim\limits_{x \to 0} \frac{e^x-1}{x} = 1$

Napier's constant – Flowchart

Definition

$$e = \lim \left(1 + \frac{1}{n}\right)^n$$

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \to +\infty} \left(1 + \frac{a}{x}\right)^x = e^a$$

$$\lim \left(1 + \frac{a}{n}\right)^n = e^a$$

$$e^x \geq x + 1, \quad \forall x \in \mathbb{R}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

The natural exponential function

Using the Napier constant e, we define the natural exponential function $\exp : \mathbb{R} \to (0, +\infty)$ such that

$$\forall x \in \mathbb{R} : \ \exp(x) = e^x = \lim_{n \in \mathbb{N}^*} \left(1 + \frac{a}{n}\right)^n$$

Derivative

$$\boxed{\forall x \in \mathbb{R} : [\exp(x)]' = \exp(x)}$$

Proof

Let $x \in \mathbb{R}$ be given. Then,

$$[\exp(x)]' = \lim_{h \to 0} \frac{\exp(x+h) - \exp(x)}{h} =$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x e^h - e^x}{h} =$$

$$= \lim_{h \to 0} \frac{e^x (e^h - 1)}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x \cdot 1 = e^x \qquad \square$$

Using the chain rule, this differentiation rule can be extended to read:

$$\boxed{[e^{g(x)}]' = g'(x) e^{g(x)}}$$

Limits

Since exp is differentiable on $\mathbb{R}$, it is also continuous on $\mathbb{R}$, therefore:

▶ $$\boxed{\forall x_0 \in \mathbb{R} : \lim_{x \to x_0} e^x = e^{x_0}}$$

We can also show that

▶ $$\boxed{\lim_{x \to +\infty} e^x = +\infty \quad \Big| \quad \lim_{x \to -\infty} e^x = 0}$$

Proof

$$\left.\begin{array}{l} \forall x \in \mathbb{R} : e^x \geq x+1 \\ \lim\limits_{x \to +\infty} (x+1) = \lim\limits_{x \to +\infty} x = +\infty \end{array}\right\} \Rightarrow \lim_{x \to +\infty} e^x = +\infty.$$

Let $x \in (-\infty, 0)$ be given. Then

$$e^{-x} \geq (-x)+1 > 0 \Rightarrow 0 < \frac{1}{1-x} < \frac{1}{e^{-x}} \Rightarrow$$

$$\Rightarrow 0 < e^x \leq \frac{1}{1-x}$$

and therefore $\forall x \in (-\infty, 0) : 0 < e^x \leq \dfrac{1}{1-x}$ (1)

Since

$$\lim_{x \to -\infty} \frac{1}{1-x} = \lim_{x \to -\infty} \frac{1}{-x} = 0 \qquad (2)$$

From Eq.(1) and Eq.(2) it follows that $\lim_{x\to-\infty} e^x = 0$. $\square$

Combining those results with the composition theorem, we obtain:

$$\boxed{\begin{array}{l} \lim_{x\to\sigma} g(x) = a \in \mathbb{R} \Rightarrow \lim_{x\to\sigma} e^{g(x)} = e^a \\ \lim_{x\to\sigma} g(x) = +\infty \Rightarrow \lim_{x\to\sigma} e^{g(x)} = +\infty \\ \lim_{x\to\sigma} g(x) = -\infty \Rightarrow \lim_{x\to\sigma} e^{g(x)} = 0 \end{array}}$$

Likewise, via the composition theorem, the result that $\lim_{x\to0} (e^x - 1)/x = 1$, generalizes to:

$$\boxed{\left.\begin{array}{l} \lim_{x\to\sigma} g(x) = 0 \\ \forall x \in N(\sigma,\delta) \cap \mathrm{dom}(g) : g(x) \neq 0 \end{array}\right\} \Rightarrow \lim_{x\to\sigma} \frac{e^{g(x)} - 1}{g(x)} = 1}$$

EXAMPLES

a) Determine the monotonicity and local min/max of the function $f(x) = x^3 \exp(1/x)$.

Solution

- Domain: We require $x \neq 0$, thus $A = \mathbb{R} - \{0\}$
- Derivative

$$f'(x) = [x^3 \exp(1/x)]' =$$
$$= (x^3)' \exp(1/x) + x^3 [\exp(1/x)]'$$
$$= 3x^2 \exp(1/x) + x^3 \exp(1/x)(1/x)'$$
$$= 3x^2 \exp(1/x) + x^3 \exp(1/x)(-1/x^2) =$$
$$= (3x^2 - x) \exp(1/x) = x(3x-1)\exp(1/x).$$

- Monotonicity

x		0		$1/3$	
x	$-$	0	$+$		$+$
$3x-1$	$-$		$-$	0	$+$
$\exp(1/x)$	$+$	$\nparallel$	$+$		$+$
$f'(x)$	$+$	$\nparallel$	$-$	0	$+$
$f(x)$	$\nearrow$	$\nparallel$	$\searrow$	min	$\nearrow$

Local min at $x = 1/3$

Singular point at $x = 0$.

b) $f(x) = e^{-x}\cos x \sin(3x) \leftarrow$ Evaluate $\lim_{x\to+\infty} f(x)$

Solution

Define $b(x) = \cos x \sin(3x)$, $\forall x \in \mathbb{R}$.

$|b(x)| = |\cos x \sin(3x)| = |\cos x||\sin(3x)| \leqslant 1 \cdot 1 = 1$, $\forall x \in \mathbb{R}$

$\Rightarrow$ b bounded on $\mathbb{R}$. (1)

$\lim_{x\to+\infty} e^{-x} = 0$ (2)

From Eq. (1) and Eq. (2), via the zero-bounded theorem, it follows that

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} e^{-x} b(x) = 0$$

c) $f(x) = \dfrac{e^x + 2e^{-x}}{3e^x - e^{-2x}} \leftarrow \lim_{x\to+\infty} f(x)$

Solution

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{e^x + 2e^{-x}}{3e^x - e^{-2x}} = \lim_{x\to+\infty} \frac{e^x[1+2e^{-2x}]}{e^x[3-e^{-3x}]}$$

$$= \lim_{x\to+\infty} \frac{1+2e^{-2x}}{3-e^{-3x}} = \frac{1+2\cdot 0}{3-0} = \frac{1}{3}$$

d) $f(x) = \dfrac{e^{2x} - e^{-2x}}{x} \leftarrow \lim_{x\to 0} f(x)$

Solution

$$f(x) = \frac{e^{2x} - e^{-2x}}{x} = \frac{e^{-2x}\left[e^{4x} - 1\right]}{x} =$$

$$= \left(4e^{-2x}\right) \frac{e^{4x} - 1}{4x} \qquad (1)$$

$$\lim_{x \to 0} \left(4e^{-2x}\right) = 4e^{0} = 4 \qquad (2)$$

$$\left.\begin{array}{l} \lim_{x \to 0} (4x) = 4 \cdot 0 = 0 \\ \forall x \in \mathbb{R} - \{0\} : 4x \neq 0 \end{array}\right\} \Rightarrow \lim_{x \to 0} \frac{e^{4x} - 1}{4x} = 1 \qquad (3)$$

From Eq. (1), Eq. (2), and Eq. (3):

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left[\left(4e^{-2x}\right) \frac{e^{4x} - 1}{4x} \right] = 4 \cdot 1 = 4$$

EXERCISES

① Evaluate the following limits, if they exist.

a) $\lim_{x \to 0} e^{\cos x}$

b) $\lim_{x \to -\infty} e^{x^2-3x}$

c) $\lim_{x \to -\infty} e^{x^3-x^2}$

d) $\lim_{x \to 3^-} \exp\left(\frac{2x+1}{x-3}\right)$

e) $\lim_{x \to 2} \exp\left(\frac{x-3}{(x-2)^2}\right)$

f) $\lim_{x \to +\infty} e^{-2x+1}(\cos 3x + \sin 2x)$

g) $\lim_{x \to -\infty} e^{-x^2}(\sin x \cos x + 1)$

h) $\lim_{x \to +\infty} \frac{2e^{2x}+3}{e^{2x}+5}$

i) $\lim_{x \to +\infty} \frac{e^x + e^{-x} - 2e^{-2x}}{3e^x + 1 + e^{-x}}$

j) $\lim_{x \to -\infty} \exp\left(\frac{x^3+3x}{x^3-3x}\right)$

② Similarly, with the following limits:

a) $\lim_{x \to 0} \frac{e^{2x}-1}{5x}$

b) $\lim_{x \to 0} \frac{e^{3x}-1}{\sin x}$

c) $\lim_{x \to 0} \frac{e^x - e^{-x}}{\tan x}$

d) $\lim_{x \to 0} \frac{e^{ax}-e^{bx}}{x}$, with $a > b \geq 0$

e) $\lim_{x \to 0} \frac{e^{\sin x}-1}{2x}$

f) $\lim_{x \to 0} \frac{e^{\tan x}-1}{3x}$

③ Evaluate and FACTOR the derivatives of the following functions:

a) $f(x) = e^{x+\sin x}$

b) $f(x) = e^{\cot(x^2)}$

c) $f(x) = e^{x\tan x}$

d) $f(x) = e^x(x^2+3x+1)$

e) $f(x) = (x\sin x+1)e^{-x^2}$

f) $f(x) = (x+1)^2(x-2)^3e^{-x^2}$

g) $f(x) = \dfrac{(2x+1)^2e^{-x}}{(x-3)^4}$

h) $f(x) = x^2\sqrt{x+1}\cdot e^x$

④ Analyze the following functions with respect to monotonicity, concavity, and locate all local min/max and inflection points. Show the variation table.

a) $f(x) = x^3e^x$

b) $f(x) = xe^{-x^2}$

c) $f(x) = \dfrac{x^2}{e^x}$

d) $f(x) = xe^{1/x}$

e) $f(x) = \dfrac{e^x-1}{e^x+1}$

f) $f(x) = (2x-1)^3(x+2)^2e^x$

g) $f(x) = \dfrac{(x+1)^2e^{-x}}{(x-1)^2}$

⑤ Use monotonicity to show that

$x>0 \Rightarrow e^x(1+x)>1$

(6) Use the Rolle and Bolzano theorems to show that

a) The equation $e^{2x} - e + 2 = 0$ has a unique solution in $\mathbb{R}$.

b) The equation $x^2 e^{2x} = 1 - x e^x$ has a unique solution in $[0, +\infty)$

c) The equation $e^{2x}(2x-1) + x^4 = 0$ has at least one solution and no more than two solutions in $\mathbb{R}$.

(7) Use the mean-value theorem to show that

$$a < b \Rightarrow e^a(b-a) < e^b - e^a < e^b(b-a)$$

(8) Show that the function $f(x) = e^x / x^n$ with $n \in \mathbb{N} - \{0\}$ and domain $A = (0, +\infty)$ has a unique minimum at $x = n$. Use this result to show that

$$e^x \geqslant \left(\frac{ex}{n}\right)^n , \quad \forall x \in [0, +\infty)$$

(9) Let $f : [0, +\infty) \rightarrow \mathbb{R}$ with $f(0) = 1$ and $\forall x \in [0, +\infty) : f'(x) \geqslant f(x)$.

a) Analyze the function $g(x) = f(x)/e^x$ with respect to monotonicity.

b) Use (a) to show that

$$\forall x \in [0, +\infty) : f(x) \geqslant e^x .$$

Inverse functions

- Let $f: A \to \mathbb{R}$ be a function. We say that:

$$\boxed{f \text{ one-to-one} \Leftrightarrow \forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)}$$

<u>interpretation</u>: A function f is one-to-one if and only if no horizontal line intersects its graph more than once.

<u>Thm</u>:

$$\boxed{\begin{array}{l} f \nearrow A \Rightarrow f \text{ one-to-one} \\ f \searrow A \Rightarrow f \text{ one-to-one} \end{array}}$$

<u>Proof</u>

Assume, without loss of generality that, $f \nearrow A$.
Let $x_1, x_2 \in A$ be given such that $f(x_1) = f(x_2)$.
If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \longleftarrow$ contradiction
If $x_1 > x_2 \Rightarrow f(x_1) > f(x_2) \longleftarrow$ contradiction.
It follows that $x_1 = x_2$. $\square$

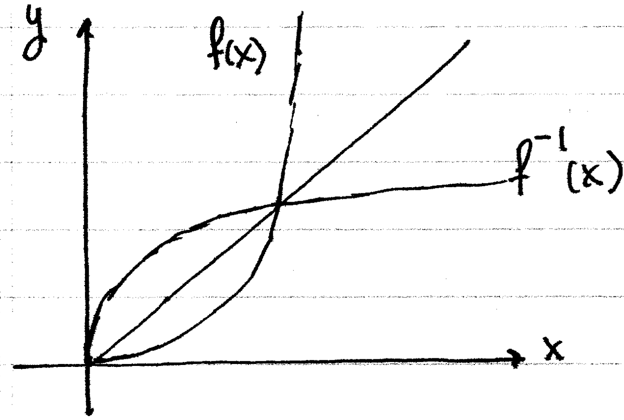
<u>Def</u>: Let $f: A \to \mathbb{R}$ be a one-to-one function. We define the inverse function $f^{-1}: f(A) \to A$ such that

$$\boxed{f^{-1}(x) = y \Leftrightarrow f(y) = x}$$

The immediate consequence of this definition is that

$$\boxed{\begin{array}{l} \forall x \in A : f^{-1}(f(x)) = x \\ \forall x \in f(A) : f(f^{-1}(x)) = x \end{array}}$$

The graph of f^{-1} is the reflection of the graph of f across the line $(\ell): y = x$



<u>Method</u> : To find the inverse of a function $f: A \to \mathbb{R}$ we work as follows:

- 1 We setup the equation
  $f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow \cdots$
- 2 It may be necessary to require restrictions on y to evaluate $f(y)$. If that is the case, then do so.
- 3 Solve for y. During the process, it may be necessary to require restrictions on x to ensure that at least one solution exists. These restrictions define the domain of the inverse function f^{-1}.
- 4 When you show that, under possible restrictions on x, that your equation has a unique solution $y = y_0(x)$, you implicitly prove that both f is one-to-one and that $f^{-1}(x) = y_0(x)$. Thus you have the formula of the inverse function.
- 5 If applicable, check the constraints on y from step 2. They may or may not introduce further restrictions on the variable x and therefore on the domain of the inverse function.

EXAMPLES

a) Find the inverse function of $f(x) = \dfrac{x+3}{2x-5}$

Solution

$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow \dfrac{y+3}{2y-5} = x$ (Require $\underline{2y-5 \neq 0}$)

$\Leftrightarrow y+3 = x(2y-5) \Leftrightarrow y+3 = 2xy - 5x \Leftrightarrow (1-2x)y = -3-5x$ (1)

For $1-2x = 0$: $x = 1/2$, and therefore

(1) $\Leftrightarrow 0y = -3-5\cdot(1/2) \Leftrightarrow 0y = -3-5/2 \longleftarrow$ inconsistent

thus $\underline{x = 1/2 \notin \mathrm{dom}(f^{-1})}$.

For $1-2x \neq 0$:

(1) $\Leftrightarrow y = \dfrac{-3-5x}{1-2x}$

Now we must check the requirement $2y-5 \neq 0$.

We note that:

$$2y-5 = 2\cdot\left(\frac{-3-5x}{1-2x}\right) - 5 = \frac{2(-3-5x)-5}{1-2x} =$$

$$= \frac{-6-10x-5(1-2x)}{1-2x} = \frac{-6-10x-5+10x}{1-2x} =$$

$$= \frac{-11}{1-2x} \neq 0$$

thus $2y-5 \neq 0$ is satisfied.

Thus $f^{-1}(x) = \dfrac{-3-5x}{1-2x}$ with $dom(f^{-1}) = \mathbb{R} - \{1/2\}$.

In the above example we see that the domain of f^{-1} coincides with the widest possible domain. However, this is not always true, as seen in the next example.

b) Find the inverse function of $f(x) = 2 + \dfrac{\sqrt{3x+1}}{3}$

Solution

$$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow 2 + \frac{\sqrt{3y+1}}{3} = x \Leftrightarrow$$

$$\Leftrightarrow 6 + \sqrt{3y+1} = 3x \Leftrightarrow \sqrt{3y+1} = 3x - 6 \Leftrightarrow \sqrt{3y+1} = 3(x-2) \quad (1)$$

Require $3(x-2) \geq 0 \Leftrightarrow x \geq 2$, otherwise equation (1) is inconsistent. For $x \geq 2$:

$$(1) \Leftrightarrow 3y + 1 = 9(x-2)^2 \Leftrightarrow 3y = 9(x-2)^2 - 1 \Leftrightarrow$$

$$\Leftrightarrow y = 3(x-2)^2 - \frac{1}{3}$$

It follows that

$$f^{-1}(x) = 3(x-2)^2 - 1/3 \quad \text{with } dom(f^{-1}) = [2, +\infty)$$

In this example we see that the domain $dom(f^{-1})$ is restricted from the widest possible domain of the polynomial formula for $f^{-1}(x)$ which is $\mathbb{R}$.

Thus, to determine the domain of the inverse function f^{-1}, it is necessary to keep track of all constraints, as I suggested in the methodology.

c) Find the inverse function of $f(x) = 4x - 3$.

<u>Solution</u>

$$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow 4y - 3 = x \Leftrightarrow 4y = x + 3 \Leftrightarrow$$
$$\Leftrightarrow y = \frac{x+3}{4}$$

It follows that:

$f^{-1}(x) = \dfrac{x+3}{4}$ with $\mathrm{dom}(f^{-1}) = \mathbb{R}$ (no constraints).

A property of one-to-one functions

The following property is used later to establish the continuity property of f^{-1}.

$$\boxed{\left.\begin{array}{l} f \text{ one-to-one} \\ I \text{ interval} \\ f \text{ continuous at } I \end{array}\right\} \Rightarrow f \nearrow I \vee f \searrow I}$$

Proof

Assume that not $f \nearrow I$ and not $f \searrow I$. Then there are $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$ such that $f(x_2)$ is not between $f(x_1)$ and $f(x_3)$. Assume, with no loss of generality that $f(x_1) < f(x_3)$. It follows that $f(x_2) \notin [f(x_1), f(x_3)]$. Distinguish two cases:

Case 1: If $\left.\begin{array}{l} f(x_2) < f(x_1) < f(x_3) \\ f \text{ continuous at } [x_2, x_3] \end{array}\right\} \Rightarrow$

$\Rightarrow \exists c \in [x_2, x_3] : f(c) = f(x_1)$ [intermediate value thm]

$\Rightarrow c = x_1$ [f one-to-one]

But $c \geq x_2 > x_1 \Rightarrow c \neq x_1 \longleftarrow$ contradiction.

Case 2: If $\left.\begin{array}{l} f(x_1) < f(x_3) < f(x_2) \\ f \text{ continuous at } [x_1, x_2] \end{array}\right\} \Rightarrow$

$\Rightarrow \exists c \in [x_1, x_2] : f(c) = f(x_3)$ [intermediate value thm]

$\Rightarrow c = x_3$. But $c \leq x_2 < x_3 \Rightarrow c \neq x_3 \longleftarrow$ contradiction.

Thus $f \nearrow I \vee f \searrow I$. $\square$

Properties of inverse functions

1) <u>Monotonicity</u>: Let $f: A \to \mathbb{R}$ be a function.

$$\boxed{\begin{array}{l} f \nearrow A \Rightarrow f^{-1} \nearrow f(A) \\ f \searrow A \Rightarrow f^{-1} \searrow f(A) \end{array}}$$

<u>Proof</u>

Assume, without loss of generality, that $f \nearrow A$.
Let $y_1, y_2 \in f(A)$ be given with $y_1 < y_2$.
Define $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$.
Sufficient to show that $x_1 < x_2$.
Assume that $x_1 \geq x_2$. Then,

$$\begin{aligned} x_1 \geq x_2 &\Rightarrow f(x_1) \geq f(x_2) \quad [\text{because } f \nearrow A] \\ &\Rightarrow f(f^{-1}(y_1)) \geq f(f^{-1}(y_2)) \\ &\Rightarrow y_1 \geq y_2 \leftarrow \text{contradiction.} \end{aligned}$$

It follows that: $x_1 < x_2 \Rightarrow \underline{f^{-1}(y_1) < f^{-1}(y_2)}$

and therefore $f^{-1} \nearrow A$. $\square$

2) <u>Continuity</u> : Let $f: A \to \mathbb{R}$ be a function.

$$\boxed{\left.\begin{array}{l} f \text{ continuous at } A \\ A \text{ interval} \\ f \text{ one-to-one} \end{array}\right\} \Rightarrow f^{-1} \text{ continuous at } f(A)}$$

<u>Proof</u>

$$\text{Since } \left.\begin{array}{l} f \text{ one-to-one} \\ f \text{ continuous at } A \end{array}\right\} \Rightarrow f \uparrow A \vee f \downarrow A$$

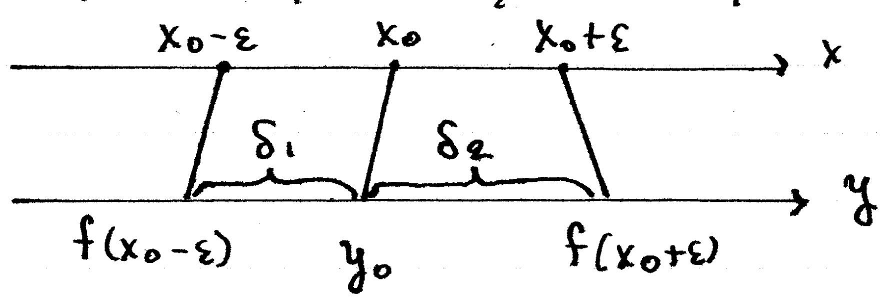
Assume, with no loss of generality, that $f \uparrow A$.
Let $y_0 \in f(A)$ be given and define $x_0 = f^{-1}(y_0)$
To show $\lim\limits_{y \to y_0} f^{-1}(y) = f^{-1}(y_0)$, it is sufficient to show that

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall y \in f(A) : (0 < |y - y_0| < \delta \Rightarrow |f^{-1}(y) - f^{-1}(y_0)| < \varepsilon)$$

Let $\underline{\varepsilon > 0}$ be given such that $(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq A$.

$$\text{Since } \left.\begin{array}{l} f \text{ continuous at } A \\ f \uparrow A \end{array}\right\} \Rightarrow$$

$$\Rightarrow f((x_0 - \varepsilon, x_0 + \varepsilon)) = (f(x_0 - \varepsilon), f(x_0 + \varepsilon)) \Rightarrow$$
$$\Rightarrow f^{-1}((f(x_0 - \varepsilon), f(x_0 + \varepsilon))) = (x_0 - \varepsilon, x_0 + \varepsilon)$$



Let $\delta_1 = y_0 - f(x_0-\varepsilon)$ and
$\delta_2 = f(x_0+\varepsilon) - y_0$ and
$\underline{\delta = \min\{\delta_1, \delta_2\}}$

It follows (see figure) that

$$(y_0-\delta, y_0+\delta) \subseteq (f(x_0-\varepsilon), f(x_0+\varepsilon)) \Rightarrow$$

$$\Rightarrow f^{-1}((y_0-\delta, y_0+\delta)) \subseteq f^{-1}((f(x_0-\varepsilon), f(x_0+\varepsilon)))$$

$$= (x_0-\varepsilon, x_0+\varepsilon)$$

consequently:

$$\underline{0 < |y-y_0| < \delta} \Rightarrow y \in (y_0-\delta, y_0+\delta) \Rightarrow$$

$$\Rightarrow f^{-1}(y) \in (x_0-\varepsilon, x_0+\varepsilon)$$

$$\Rightarrow \underline{|f^{-1}(y) - f^{-1}(y_0)|} = |f^{-1}(y) - x_0| \underline{< \varepsilon}$$

It follows that $\forall y_0 \in f(A): \lim_{y \to y_0} f^{-1}(y) = f^{-1}(y_0) \Rightarrow$

$\Rightarrow f^{-1}$ continuous at $f(A)$ $\square$

3) <u>Differentiability</u> : Let $f: A \to \mathbb{R}$, with $f'(f^{-1}(x)) \neq 0$.

$$\boxed{\left.\begin{array}{l} f \text{ differentiable at } A \\ f \text{ one-to-one} \\ A \text{ union of intervals} \end{array}\right\} \Rightarrow f^{-1} \text{ differentiable at } f(A) \quad \text{with} \quad (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}, \forall x \in f(A)}$$

Proof

Assume with no loss of generality that A is an interval. We note that

$$\left.\begin{array}{l} f \text{ differentiable at } A \Rightarrow f \text{ continuous at } A \\ \qquad\qquad\qquad\qquad\quad f \text{ one-to-one} \end{array}\right\} \Rightarrow$$

$\Rightarrow f^{-1}$ continuous at $f(A)$ [using (2)-previous result]

Let $x_0 \in f(A)$ be given. Define $y_0 = f^{-1}(x_0)$. It follows that $\lim_{x \to x_0} f^{-1}(x) = f^{-1}(x_0) = y_0$.

and consequently:

$$(f^{-1})'(x_0) = \lim_{x \to x_0} \frac{f^{-1}(x) - f^{-1}(x_0)}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{f^{-1}(x) - f^{-1}(x_0)}{f(f^{-1}(x)) - f(f^{-1}(x_0))} =$$

$$= \lim_{x \to x_0} \frac{f^{-1}(x) - y_0}{f(f^{-1}(x)) - f(y_0)} \underset{\uparrow}{=}$$

$$\lim_{x \to x_0} f^{-1}(x) = f^{-1}(x_0) = y_0$$

$$= \lim_{y \to y_0} \frac{y - y_0}{f(y) - f(y_0)} = \frac{1}{\lim_{y \to y_0} \frac{f(y) - f(y_0)}{y - y_0}} =$$

$$= \frac{1}{f'(y_0)} = \frac{1}{f'(f^{-1}(x_0))} \qquad \square$$

Properties of inverse functions - Flowchart

$$\left.\begin{array}{l} f \text{ one-to-one} \\ A \text{ interval} \\ f \text{ continuous at } A \end{array}\right\} \Rightarrow f \nearrow A \vee f \searrow A$$

$$f \nearrow A \Rightarrow f^{-1} \nearrow f(A)$$
$$f \searrow A \Rightarrow f^{-1} \searrow f(A)$$

$$\left.\begin{array}{l} f \text{ continous at } A \\ f \text{ one-to-one} \\ A \text{ interval} \end{array}\right\} \Rightarrow f^{-1} \text{ continuous at } f(A)$$

$$\left.\begin{array}{l} f \text{ differentiable at } A \\ f \text{ one-to-one} \\ A \text{ interval} \\ f'(x) \neq 0 \,,\ \forall x \in A \end{array}\right\} \Rightarrow f^{-1} \text{ differentiable at } f(A) \text{ with}$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

EXANPLE

Given the function $f(x) = x^3+x+1$, evaluate $(f^{-1})'(1)$

Solution

We note that

$\forall x \in \mathbb{R} : f'(x) = (x^3+x+1)' = 3x^2+1 \geq 1 > 0$

$\Rightarrow (\forall x \in \mathbb{R} : f'(x) > 0) \Rightarrow f \nearrow \mathbb{R} \Rightarrow f$ one-to-one.

therefore f^{-1} can be defined. Since:

$f^{-1}(1) = x \Leftrightarrow f(x) = 1 \Leftrightarrow x^3+x+1 = 1 \Leftrightarrow$

$\Leftrightarrow x^3+x=0 \Leftrightarrow x(x^2+1)=0 \Leftrightarrow$

$\Leftrightarrow x=0 \vee x^2+1=0 \Leftrightarrow x=0$

it follows that $f^{-1}(1)=0$, and therefore

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} =$$

$$= \frac{1}{3 \cdot 0^2+1} = 1$$

EXERCISES

(10) Show that the following functions are one-to-one and find their inverse

a) $f(x) = 3x+2$

b) $f(x) = 1-2x$

c) $f(x) = 2x^3+7$

d) $f(x) = 3+\sqrt{x-1}$

e) $f(x) = -1-\sqrt{2-3x}$

f) $f(x) = \dfrac{2}{3x}$

g) $f(x) = \dfrac{2x+3}{x-2}$

h) $f(x) = \dfrac{x+4}{3x-1}$

(11) Use monotonicity to show that the following functions are one-to-one and then calculate $(f^{-1})'(a)$ at the value of a given below:

a) $f(x) = e^x + (x-2)(x+2)$ at $a = e-3$ (with $A_f = (0, +\infty)$)

b) $f(x) = x^3+4x+2$ at $a=2$

c) $f(x) = x(\tan^2 x+1)+1$, $\forall x \in [0, \pi/2)$ at $a=1$

d) $f(x) = (x^2+1)e^x$ at $a = 2e$

(12) Let $f : \mathbb{R} \to \mathbb{R}$ be a one-to-one function which is differentiable in $\mathbb{R}$. Let $g : \mathbb{R} \to \mathbb{R}$ be the function $g(x) = af(x)+b$ with $a \neq 0$. Show that

$$(g^{-1})'(x) = \frac{1}{a}(f^{-1})'(x).$$

▼ The natural logarithm

The natural logarithm function is defined as the inverse of the natural exponential function exp.
Since:

$[\exp(x)]' = e^x > 0,\ \forall x \in \mathbb{R} \Rightarrow \exp \uparrow \mathbb{R} \Rightarrow$

$\Rightarrow \exp$ one-to-one

it follows that exp has an inverse function that we will denote: $\ln = \exp^{-1}$.

▸ Domain and definition of ln

Given a known x, we have:

$\ln x = y \Leftrightarrow \exp(y) = x \Leftrightarrow e^y = x \Leftrightarrow e^y - x = 0. \quad (1)$

We claim that Eq.(1) has no solution if $x \in (-\infty, 0]$ and a unique solution if $x \in (0, +\infty)$. It follows that the domain of ln is

$$\mathrm{dom}(\ln) = (0, +\infty)$$

and consequently; ln is defined via:

$$\boxed{\forall x \in (0, +\infty): \ \ln x = y \Leftrightarrow e^y = x}$$

Proof of claim

Since

$\forall x \in (-\infty, 0]: \ e^y - x \geq e^y > 0$

$\Rightarrow \forall x \in (-\infty, 0]: \ e^y - x \neq 0$

$\Rightarrow$ The equation (1) has no solution with respect to y for all $x \in (-\infty, 0]$.

Let $x \in (0, +\infty)$ be given and define $g(y) = e^y - x$. Then,
$$\lim_{y \to -\infty} g(y) = \lim_{y \to -\infty} (e^y - x) = 0 - x = -x < 0 \Rightarrow$$
$$\Rightarrow \exists a \in (-\infty, 0) : g(a) < 0$$
and
$$\lim_{y \to +\infty} g(y) = \lim_{y \to +\infty} (e^y - x) = +\infty \Rightarrow$$
$$\Rightarrow \exists b \in (0, +\infty) : g(b) > 0$$
Choose $a \in (-\infty, 0)$ and $b \in (0, +\infty)$ such that $g(a) < 0$ and $g(b) > 0$. Then:
$$\left.\begin{array}{l} g(a)g(b) < 0 \\ g \text{ continuous on } [a, b] \end{array}\right\} \Rightarrow \exists x_0 \in (a, b) : g(x_0) = 0$$
$\Rightarrow$ The equation (1) has at least one solution $y_0 \in \mathbb{R}$.
We will now show this solution is unique. To show a contradiction, assume that $y_0, y_1 \in \mathbb{R}$ are solutions to equation (1). Then it follows that:
$$\left.\begin{array}{l} g(y_0) = g(y_1) = 0 \\ g \text{ continuous on } [y_0, y_1] \\ g \text{ differentiable on } (y_0, y_1) \end{array}\right\} \Rightarrow \exists z \in (y_0, y_1) : g'(z) = 0$$
$$\Rightarrow \exists z \in (y_0, y_1) : \exp(z) = 0$$
which is a contradiction, since
$$\forall z \in \mathbb{R} : g'(z) = \exp(z) > 0$$
We conclude that Eq. (1) has a unique solution on $\mathbb{R}$. $\square$

▸ Algebraic identities

Since $e^0 = 1 \Rightarrow \boxed{\ln 1 = 0}$

$e^1 = e \Rightarrow \boxed{\ln e = 1}$

Furthermore:

$$\boxed{\begin{array}{l} \forall x_1, x_2 \in (0, +\infty): \ln(x_1 x_2) = \ln x_1 + \ln x_2 \\ \forall x_1, x_2 \in (0, +\infty): \ln(x_1 / x_2) = \ln x_1 - \ln x_2 \\ \forall x \in (0, +\infty): \forall a \in \mathbb{R}: \ln(x^a) = a \ln x \end{array}}$$

Proof

Let $x_1, x_2 \in (0, +\infty)$ be given. Then:

$$\begin{aligned} \ln(x_1 x_2) &= \ln(\exp(\ln x_1) \exp(\ln x_2)) \\ &= \ln(\exp(\ln x_1 + \ln x_2)) \\ &= \ln x_1 + \ln x_2 \end{aligned}$$

and

$$\ln(x_1 / x_2) = \ln\left(\frac{\exp(\ln x_1)}{\exp(\ln x_2)}\right) = \ln(\exp(\ln x_1 - \ln x_2))$$

$$= \ln x_1 - \ln x_2 .$$

Let $x \in (0, +\infty)$ and $a \in \mathbb{R}$ be given. Then,

$$\ln(x^a) = \ln(\exp(\ln x)^a) = \ln(\exp(a \ln x)) =$$

$$= a \ln x \qquad \square$$

▸ **Derivative**

$$\boxed{(\ln x)' = \frac{1}{x}, \quad \forall x \in (0,+\infty)}$$

Proof

Define $f(x) = \exp(x)$, $\forall x \in \mathbb{R}$.

Note that $f'(x) = [\exp(x)]' = \exp(x)$, $\forall x \in \mathbb{R}$

Since:

$$\left.\begin{array}{l} f \text{ differentiable on } \mathbb{R} \\ f \text{ one-to-one} \end{array}\right\} \Rightarrow \begin{array}{l} f^{-1} \text{ differentiable} \\ \text{on } (0,+\infty) \end{array}$$

with

$$(\ln x)' = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\ln x)} =$$

$$= \frac{1}{\exp(\ln x)} = \frac{1}{x}$$

$\square$

Via the chain rule, we get:

$$\boxed{[\ln(f(x))]' = \frac{f'(x)}{f(x)}}$$

▸ **Generalization**

$\frac{d}{dx} \ln\lvert x\rvert = \frac{1}{x}$	$\forall x \in \mathbb{R} - \{0\}$
$\frac{d}{dx} \ln\lvert f(x)\rvert = \frac{f'(x)}{f(x)}$	

Monotonicity and convexity

Since for $f(x) = \ln x$:

$\forall x \in (0, +\infty): f'(x) = 1/x > 0 \Rightarrow$ $f \nearrow (0,+\infty)$

$\forall x \in (0, +\infty): f''(x) = -1/x^2 < 0$ $\Rightarrow$ f convex down on $(0,+\infty)$

It follows that f is also <u>one-to-one</u>. It follows that

$$\boxed{\begin{array}{l} \ln x_1 = \ln x_2 \Leftrightarrow x_1 = x_2 > 0 \\ \ln x_1 > \ln x_2 \Leftrightarrow x_1 > x_2 > 0 \end{array}}$$

We may use these properties to solve equations and inequalities involving exponentials and logarithms.

We also note that

$$e^0 = 1 \Rightarrow \boxed{\ln 1 = 0}$$

and therefore

$$\boxed{\begin{array}{l} \ln x > 0 \Leftrightarrow x > 1 \\ \ln x < 0 \Leftrightarrow 0 < x < 1 \end{array}}$$

EXAMPLES

→ When solving equations, inequalities or determining the domain of functions, the expression $\ln(f(x))$ should result in the restriction $f(x) > 0$.

a) Solve the equation $\ln(\ln(2x+1)) = 5$

<u>Solution</u>

We require $\begin{cases} 2x+1 > 0 \\ \ln(2x+1) > 0 \end{cases}$

and note that

$2x+1 > 0 \Leftrightarrow 2x > -1 \Leftrightarrow x > -1/2 \Leftrightarrow x \in (-1/2, +\infty)$

$\ln(2x+1) > 0 \Leftrightarrow \ln(2x+1) > \ln 1 \Leftrightarrow 2x+1 > 1 \Leftrightarrow$

$\Leftrightarrow 2x > 0 \Leftrightarrow x > 0 \Leftrightarrow x \in (0, +\infty)$

thus the domain of the equation is

$A = (-1/2, +\infty) \cap (0, +\infty) = (0, +\infty)$.

We have:

$\ln(\ln(2x+1)) = 5 \Leftrightarrow \ln(2x+1) = e^5 \Leftrightarrow$

$\Leftrightarrow 2x+1 = \exp(e^5) \Leftrightarrow 2x = \exp(e^5) - 1$

$\Leftrightarrow x = (1/2)[\exp(e^5) - 1]$

which is accepted since:

$x = (1/2)[\exp(e^5) - 1] \geq (1/2)[e^5 + 1 - 1] = e^5/2$

$\geq (5+1)/2 > 0 \Rightarrow x \in (0, +\infty)$.

b) Solve the equation $\ln(x-1) + \ln(x+1) = 2$

Solution

We require

$$\begin{cases} x-1>0 \\ x+1>0 \end{cases} \Leftrightarrow \begin{cases} x>1 \\ x>-1 \end{cases} \Leftrightarrow x>1 \Leftrightarrow x \in (1,+\infty)$$

thus the domain of the equation is $A=(1,+\infty)$.

It follows that

$$\ln(x-1)+\ln(x+1)=2 \Leftrightarrow \ln[(x-1)(x+1)] = \ln e^2 \Leftrightarrow$$

$$\Leftrightarrow (x-1)(x+1)=e^2 \Leftrightarrow x^2-1=e^2 \Leftrightarrow x^2=e^2+1$$

$$\Leftrightarrow x=\sqrt{e^2+1} \in (1,+\infty) \vee x=-\sqrt{1+e^2} \notin (1,+\infty)$$

$$\Leftrightarrow x=\sqrt{e^2+1}$$

The solution $x=-\sqrt{1+e^2}$ is rejected.

c) Solve the inequality $3^{x+1} < 5^{2x+1}$

Solution

There are no restrictions. Therefore

$$3^{x+1} < 5^{2x+1} \Leftrightarrow \ln(3^{x+1}) < \ln(5^{2x+1}) \Leftrightarrow$$

$$\Leftrightarrow (x+1)\ln 3 = (2x+1)\ln 5 \Leftrightarrow$$

$$\Leftrightarrow (\ln 3)x + \ln 3 = (2\ln 5)x + \ln 5 \Leftrightarrow$$

$$\Leftrightarrow (\ln 3 - 2\ln 5)x = \ln 5 - \ln 3 \Leftrightarrow$$

$$\Leftrightarrow x = \frac{\ln 5 - \ln 3}{\ln 3 - 2\ln 5}$$

d) Find the monotonicity and local min/max of

$$f(x) = \ln\left(\frac{x^2-1}{x^2+1}\right)$$

Solution

• Domain

We require $\frac{x^2-1}{x^2+1} > 0 \Leftrightarrow \frac{(x-1)(x+1)}{x^2+1} > 0 \Leftrightarrow$

x		-1		$+1$	
$x-1$	$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$
x^2+1	$+$		$+$		$+$
	$+$	0	$-$	0	$+$

$\Leftrightarrow x \in (-\infty,-1) \cup (1,+\infty)$

It follows that the domain of f is $A = (-\infty,-1)\cup(1,+\infty)$

• Derivative

$$f'(x) = \left[\ln\left(\frac{x^2-1}{x^2+1}\right)\right]' = \frac{x^2+1}{x^2-1}\left(\frac{x^2-1}{x^2+1}\right)' =$$

$$= \frac{x^2+1}{x^2-1}\,\frac{(x^2-1)'(x^2+1)-(x^2-1)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{x^2+1}{x^2-1}\,\frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}$$

$$= \frac{2x(x^2+1-x^2+1)}{(x^2-1)(x^2+1)} = \frac{4x}{(x^2-1)(x^2+1)}$$

$$= \frac{4x}{(x-1)(x+1)(x^2+1)}$$

- <u>Monotonicity</u>

x		-1		0		1	
$4x$	$-$		$-$	0	$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$		$+$
x^2+1	$+$		$+$		$+$		$+$
$f'(x)$	$-$	‖		0		‖	$+$
$f(x)$	↘	‖				‖	↗

No local min or max.

Limits with ln function

From continuity of ln it follows that

$$\boxed{\forall x_0 \in (0,+\infty): \lim_{x \to x_0} \ln x = \ln x_0}$$

We can also show, using the definition of the limit, that:

$$\lim_{x \to +\infty} \ln x = +\infty$$

$$\lim_{x \to 0^+} \ln x = -\infty$$

To evaluate the limit of $\ln(f(x))$ we use the composition theorem to show that

$$\lim_{x \to \sigma} f(x) = a > 0 \Rightarrow \lim_{x \to \sigma} \ln(f(x)) = \ln a$$

$$\lim_{x \to \sigma} f(x) = +\infty \Rightarrow \lim_{x \to \sigma} \ln(f(x)) = +\infty$$

$$\left.\begin{array}{l} \lim_{x \to \sigma} f(x) = 0 \\ \forall x \in N(\sigma,\delta) \cap \mathrm{dom}(f): f(x) > 0 \end{array}\right\} \Rightarrow \lim_{x \to \sigma} \ln(f(x)) = -\infty$$

which we can use to evaluate limits of functions involving natural logarithms.

EXAMPLES

a) $f(x) = \ln(\sin x + \cos x) \leftarrow \lim\limits_{x \to 0} f(x)$

Solution

$$\lim_{x \to 0} (\sin x + \cos x) = \sin 0 + \cos 0 = 0 + 1 = 1 \Rightarrow$$

$$\Rightarrow \lim_{x \to 0} f(x) = \lim_{x \to 0} \ln(\sin x + \cos x) = \ln 1 = 0$$

b) $f(x) = \ln(\ln(3x+1)) \leftarrow \lim\limits_{x \to 0^+} f(x)$

Solution

$$\lim_{x \to 0^+} (3x+1) = 3 \cdot 0 + 1 = 1 \Rightarrow$$

$$\Rightarrow \lim_{x \to 0^+} \ln(3x+1) = \ln 1 = 0 \qquad (1)$$

Let $x \in (0,1)$ be given. Then:

$x > 0 \Rightarrow 3x > 0 \Rightarrow 3x + 1 > 1 \Rightarrow \ln(3x+1) > \ln 1$

$\Rightarrow \ln(3x+1) > 0$

and therefore: $\forall x \in (0,1) : \ln(3x+1) > 0 \qquad (2)$

From Eq. (1) and Eq. (2):

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \ln(\ln(3x+1)) = -\infty.$$

c) $f(x) = \ln(2x^2+3x-1) - \ln(x^2-x+4) \leftarrow \lim\limits_{x\to+\infty} f(x)$

Solution

Since,

$$f(x) = \ln(2x^2+3x-1) - \ln(x^2-x+4) =$$

$$= \ln\left(\frac{2x^2+3x-1}{x^2-x+1}\right)$$

it follows that

$$\lim_{x\to+\infty} \frac{2x^2+3x-1}{x^2-x+1} = \lim_{x\to+\infty} \frac{2x^2}{x^2} = 2 \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \ln\left(\frac{2x^2+3x-1}{x^2-x+1}\right) = \ln 2$$

EXERCISES

(13) Find the default domain for the following functions:

a) $f(x) = \ln\left(\frac{x^2-4}{x^2-1}\right)$

b) $f(x) = \ln(x^2-4) - \ln(x^2-1)$

c) $f(x) = \frac{\ln(x^2+3x+2)}{\ln(1-x)}$

(14) Solve the following equations with respect to x. (First find the domain of the equations)

a) $\ln(4x-1) = 2\ln 2 + \ln(x^2-1)$

b) $\frac{1}{2}\ln(x+2) + \ln(\sqrt{x+3}) = 1 + \ln 3$

c) $2\ln x - \ln(x+1) = \ln 4 - \ln 3$

d) $\ln(\ln(3x+1)) = 0$

e) $\ln(\ln(x^2-x-3)) = 0$

(15) Similarly with the equations

a) $3^{x+1} = 5^{2x-1}$

b) $(\sqrt{2})^{x+1} = (\sqrt{3})^{1-2x}$

c) $(1+\sqrt{5})^{2x+3} = (2+\sqrt{5})^{x-4}$

(16) Evaluate and simplify the derivatives of the following functions

a) $f(x) = e^x \ln x$

b) $f(x) = \sqrt{\ln x}$

c) $f(x) = \ln(x^2+x+1)$

d) $f(x) = x^2 \ln(2x-1)$

e) $f(x) = (2+\ln(3x-1))^3$

f) $f(x) = [\ln(x^2+5)]^2$

g) $f(x) = [\ln(x^2+1)^2]$

h) $f(x) = \ln\left(\dfrac{x^2+2}{1-x}\right)$

i) $f(x) = \dfrac{1+\ln x}{1-\ln x}$

j) $f(x) = \dfrac{\ln x}{e^x(x^2+1)}$

(17) Use the mean value theorem to show that:

a) $0 < b \leq a \Rightarrow \dfrac{a-b}{a} \leq \ln\left(\dfrac{a}{b}\right) \leq \dfrac{a-b}{b}$

b) $\forall x \in (0,+\infty): \dfrac{x}{x+1} \leq \ln(1+x) \leq x$

c) $x > 1 \Rightarrow \dfrac{2}{x+1} < \ln\left(\dfrac{x+1}{x-1}\right) < \dfrac{2}{x-1}$

d) $0 < a < b < \pi/2 \Rightarrow (b-a)\tan a < \ln\left(\dfrac{\cos a}{\cos b}\right) < (b-a)\tan b$

(18) Show that the equation

$$\ln\left(\frac{1}{x}-1\right) + \frac{1}{x-1} = 0$$

has a unique solution in the interval $(1/5, 1/4)$, using Bolzano and Rolle theorems.

(19) Analyze the following functions with respect to monotonicity, convexity, find all local min/max, find all inflection points, and show the variation table.

a) $f(x) = \ln(1-x^2)$
b) $f(x) = \dfrac{x}{\ln x}$
c) $f(x) = \ln(x-1) - x$
d) $f(x) = x\ln(1/x)$
e) $f(x) = \ln x - x$
f) $f(x) = \dfrac{2-\ln x}{x}$
g) $f(x) = x^2+3x-\ln(x^2)$
h) $f(x) = \ln\left(\dfrac{x+2}{x-2}\right)$

(20) Analyze the function $f(x) = (\ln x)/x$ with respect to monotonicity and show that f has a global maximun.
Then show that $e^{\pi} > \pi^{e}$.

(21) Analyze the function
$$f(x) = \frac{\ln(x-1)}{\ln x}$$
with respect to monotonicity. Then show that
a) $\ln(e-1)\ln(e+1) < 1$
b) $\ln(e^{\pi}-1)\ln(e^{\pi}+1) < \pi^2$.

(22) Evaluate the following limits, if they exist.

a) $\lim\limits_{x \to 1} \ln(|\ln x|)$

b) $\lim\limits_{x \to -\infty} [2\ln(3x^2+1) - \ln(x^4-1)]$

c) $\lim\limits_{x \to +\infty} [3\ln(2x+1) - 2\ln(3x+1)]$

d) $\lim\limits_{x \to +\infty} [2\ln(x+1) - \ln(x-3)]$

e) $\lim\limits_{x \to 6^+} [\ln(\sqrt{x+3} - 3) - \ln(x-6)]$

f) $\lim\limits_{x \to 0^+} [\ln(\sqrt{x^2+1} - 1) - \ln x]$

g) $\lim\limits_{x \to +\infty} [\ln(1 + \ln(x^3-1) - \ln(x-1) - \ln(x^2+x+1))]$

The general exponential function

- Let $a \in (0, +\infty)$. The general exponential function is defined as
$$f(x) = a^x, \quad \forall x \in \mathbb{R}.$$
- All properties of the general exponential function are inherented from the natural exponential function and the natural logarithmic function, via the following key statement:

$$\boxed{\forall a \in (0, +\infty) : \forall x \in \mathbb{R} : a^x = \exp(x \ln a)}$$

Proof

Let $a \in (0, +\infty)$ and $x \in \mathbb{R}$ be given. Then:

$$\begin{aligned} a^x &= [\exp(\ln a)]^x = \\ &= [e^{\ln a}]^x = e^{x \ln a} \\ &= \exp(x \ln a) \end{aligned}$$
$\square$

- **Derivatives with respect to x**

$$\boxed{\begin{aligned} (a^x)' &= a^x \ln a \\ (x^a)' &= a x^{a-1} \end{aligned}}$$

Proof

$$(a^x)' = [\exp(x\ln a)]' = \exp(x\ln a)\,(x\ln a)'$$
$$= \exp(x\ln a)\,(\ln a) = a^x \ln a$$

and

$$(x^a)' = [\exp(a\ln x)]' = \exp(a\ln x)\,(a\ln x)' =$$
$$= \exp(a\ln x)\,(ax^{-1}) =$$
$$= x^a\,(ax^{-1}) = ax^{a-1} \qquad \square$$

- Combining these differentiation rules with the chain rule gives the following more powerful differentiation rules:

$$(a^x)' = a^x \ln a \longrightarrow \boxed{(a^{g(x)})' = a^{g(x)} \ln a}$$
$$(x^a)' = ax^{a-1} \longrightarrow \boxed{([g(x)]^a)' = a[g(x)]^{a-1} g'(x)}$$

- We also note that

$$[f(x)^{g(x)}]' = [\exp(g(x)\ln(f(x)))]' =$$
$$= \exp(g(x)\ln(f(x)))\,[g(x)\ln(f(x))]' =$$
$$= f(x)^{g(x)}\,[g'(x)\ln(f(x)) + g(x)\,[\ln(f(x))]'] =$$
$$= f(x)^{g(x)}\,[g'(x)\ln(f(x)) + \frac{g(x)f'(x)}{f(x)}]$$

and therefore

$$\boxed{[f(x)^{g(x)}]' = f(x)^{g(x)}\left[g'(x)\ln f(x) + \frac{g(x)f'(x)}{f(x)}\right]}$$

Limits

As a consequence of differentiability, we have

$$\boxed{\forall x_0 \in \mathbb{R} : \lim_{x \to x_0} a^x = a^{x_0}}$$

For limits $x \to +\infty$ or $x \to -\infty$, we note that

$0 < a < 1 \Rightarrow \ln a < 0$

$a > 1 \Rightarrow \ln a > 0$

and combine this with the identity

$a^x = \exp(x \ln a)$

We conclude that

$$\boxed{\lim_{x \to +\infty} a^x = \begin{cases} +\infty & , \text{ if } a > 1 \\ 0 & , \text{ if } 0 < a < 1 \end{cases}}$$

$$\boxed{\lim_{x \to -\infty} a^x = \begin{cases} 0 & , \text{ if } a > 1 \\ +\infty & , \text{ if } 0 < a < 1 \end{cases}}$$

EXAMPLES

a) Find and simplify the derivative of
$f(x) = \sqrt[3]{x}\,(x+1)$

Solution

$$f'(x) = [\sqrt[3]{x}\,(x+1)]' = [x^{1/3}(x+1)]' =$$
$$= (x^{4/3} + x^{1/3}]' = (4/3)x^{4/3-1} + (1/3)x^{1/3-1}$$
$$= (4/3)x^{1/3} + (1/3)x^{-2/3} =$$
$$= (1/3)x^{-2/3}[4x+1] =$$
$$= (1/3)\frac{4x+1}{(\sqrt[3]{x})^2} = \frac{4x+1}{3(\sqrt[3]{x})^2}$$

b) Find and simplify the derivative of
$f(x) = x^{\sin x}$

Solution

$$f'(x) = (x^{\sin x})' = [\exp(\sin x \ln x)]' =$$
$$= \exp(\sin x \ln x)(\sin x \ln x)' =$$
$$= x^{\sin x}[(\sin x)'\ln x + \sin x(\ln x)'] =$$
$$= x^{\sin x}[\cos x \ln x + \sin x \cdot (1/x)]$$
$$= x^{\sin x - 1}[x\cos x \ln x + \sin x].$$

c) Given the function $f(x) = (1+x)^{\sin(1/x)}$, evaluate $\lim_{x\to 0} f(x)$.

Solution

We note that

$f(x) = (1+x)^{\sin(1/x)} = \exp(\ln(1+x)\sin(1/x))$

and we have:

$$\lim_{x\to 0}(1+x) = 1+0 = 1 \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow \lim_{x\to 0} \ln(1+x) = \ln 1 = 0 \\ \forall x \in (-1,0)\cup(0,1) : |\sin(1/x)| \leqslant 1 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \lim_{x\to 0}[\ln(1+x)\sin(1/x)] = 0 \Rightarrow$$

$$\Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0} \exp(\ln(1+x)\sin(1/x)) =$$

$$= \exp(0) = 1.$$

EXERCISES

(23) Find the default domain of the following functions:

a) $f(x) = (2x-1)^{\sin x}$
b) $f(x) = (x^3+1)^{x-1}$
c) $f(x) = \left(\dfrac{x^2-1}{x^2+3x+2}\right)^x$
d) $f(x) = \left(\dfrac{x^2+9x+18}{x-1}\right)^{\ln x}$

(24) Evaluate and simplify the derivatives of the following functions:

a) $f(x) = x^x$
b) $f(x) = (\cos x)^{\sin(2x)}$
c) $f(x) = x^{1/x}$
d) $f(x) = x^{\sqrt{x}}$
e) $f(x) = \left(1+\frac{1}{x}\right)^x$
f) $f(x) = \sin(x^x)$
g) $f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

(25) Same with the following functions

a) $f(x) = 2^x \ln x$
b) $f(x) = \dfrac{2^x+3^x}{3^x+5^x}$
c) $f(x) = \cos(2^x)$
d) $f(x) = \tan(3^x+7^x)$
e) $f(x) = 3^x(2x+1)^3$
f) $f(x) = \dfrac{x^2+3x+2}{5^x}$

(26) Show that the equation $3^x + 4^x = 5^x$ has a unique solution in $\mathbb{R}$.

(27) Use the mean-value theorem to prove that if $a > b$ and $x > 1$, then
$$(a-b)x^a \ln x < (x^a - x^b) < (a-b)x^b \ln x$$

The general logarithmic function

The general logarithmic function is defined as the inverse of the general exponential function

$f(x) = a^x,\ \forall x \in \mathbb{R}$

with $a \in (0,1) \cup (1,+\infty)$. We note that

$a \in (0,1) \cup (1,+\infty) \Rightarrow \ln a \neq 0 \Rightarrow$

$\Rightarrow (\forall x \in \mathbb{R} : f'(x) = a^x \ln a > 0) \vee (\forall x \in \mathbb{R} : f'(x) = a^x \ln a < 0)$

$\Rightarrow f \nearrow \mathbb{R} \vee f \searrow \mathbb{R} \Rightarrow f$ one-to-one $\Rightarrow$

$\Rightarrow f$ has an inverse

We define the general logarithmic function $\log_a$ as $\log_a = f^{-1}$, with f as defined above.

Note that for $a = 1$, we have $f(x) = 1^x = 1,\ \forall x \in \mathbb{R}$ which is not one-to-one, therefore has no inverse, therefore $\log_a$ cannot be defined for $a = 1$.

Domain and definition of $\log_a$

We will show that, given $a \in (0,1) \cup (1,+\infty)$

$$\boxed{\forall x \in (0,+\infty) : \log_a x = \frac{\ln x}{\ln a}}$$

with $\text{dom}(\log_a) = (0,+\infty)$. It follows that the theory of the natural logarithmic function $\ln$ can be used to handle all mathematical problems that involve general logarithms.

<u>Proof</u>

Given any x, y, we note that:

$$\log_a x = y \iff a^y = x$$
$$\iff \exp(y \ln a) = x \leftarrow \text{Require } x > 0$$
$$\iff y \ln a = \ln x$$
$$\iff y = \frac{\ln x}{\ln a}$$

The requirement $x > 0$ indicates that the domain of $\log_a$ is the set $A = (0, +\infty)$, and we conclude that

$$\forall x \in (0, +\infty): \log_a x = \frac{\ln x}{\ln a} \qquad \square$$

→ <u>Immediate consequences</u>

$$\boxed{\begin{array}{l} \log_a 1 = 0, \quad \forall a \in (0,1) \cup (1, +\infty) \\ \log_a a = 1, \quad \forall a \in (0,1) \cup (1, +\infty) \end{array}}$$

Decimal logarithms

For base $a = 10$, we define the decimal logarithm function $\log$ as:

$$\boxed{\forall x \in (0, +\infty) : \log x = \log_{10} x = \frac{\ln x}{\ln 10}}$$

and note that:

$\log 1 = 0$	$\log 10000 = 4$
$\log 10 = 1$	$\log 100000 = 5$
$\log 100 = 2$	$\log 1000000 = 6$
$\log 1000 = 3$	$\log 10000000 = 7$
etc	

We see that $\log x$ gives the order of magnitude of x in the decimal system (i.e. the number of zeroes for $x = 10^n$).

Domains with logarithm function

To find the domain of

$$f(x) = \log_{a(x)} g(x) = \frac{\ln g(x)}{\ln a(x)}$$

we require: $\begin{cases} g(x) > 0 \\ a(x) > 0 \\ a(x) \neq 1 \end{cases}$

EXAMPLE

For $f(x) = \log_{3x-1}(x^2+3x+2)$

we require

$3x-1 > 0$ and $x^2+3x+2 > 0$ and $3x-1 \neq 1$. (1)

Since

$3x-1 > 0 \Leftrightarrow 3x > 1 \Leftrightarrow x > 1/3$, and

$x^2+3x+2 > 0 \Leftrightarrow (x+2)(x+1) > 0 \Leftrightarrow x \in (-\infty, -2) \cup (-1, +\infty)$

x		-2		-1	
x^2+3x+2	$+$	0	$-$	0	$+$

$3x-1 \neq 1 \Leftrightarrow 3x \neq 2 \Leftrightarrow x \neq 2/3$.

Thus the domain is

$$A = [(1/3, +\infty) \cap [(-\infty, -2) \cup (-1, +\infty)]] - \{2/3\}$$

$$= (1/3, +\infty) - \{2/3\} = (1/3, 2/3) \cup (2/3, +\infty)$$

Derivatives with logarithms

From the chain rule, we easily get:

$$\boxed{[\log_a f(x)]' = \frac{f'(x)}{f(x)\ln a}}$$

For the most general case:

$$[\log_{a(x)} f(x)]' = \left[\frac{\ln f(x)}{\ln a(x)}\right]' =$$

$$= \frac{(\ln f(x))'(\ln a(x)) - (\ln f(x))(\ln a(x))'}{[\ln a(x)]^2} =$$

$$= \frac{\frac{f'(x)}{f(x)}\ln a(x) - \frac{a'(x)}{a(x)}\ln f(x)}{[\ln a(x)]^2} =$$

$$= \frac{a(x) f'(x)\ln a(x) - f(x) a'(x)\ln f(x)}{f(x)\, a(x)\, [\ln a(x)]^2}$$

EXAMPLES

a) Find and simplify the derivative of
$f(x) = \log_4 (x^2+3x+2)$
Solution

$$f'(x) = [\log_4 (x^2+3x+2)]' = \left[\frac{\ln(x^2+3x+2)}{\ln 4}\right]' =$$

$$= \frac{1}{\ln 4} \frac{(x^2+3x+2)'}{x^2+3x+2} = \frac{2x+3}{(x^2+3x+2)\ln 4}$$

b) Find and simplify the derivative of
$f(x) = \log_{2x-1} (\sin x)$.
Solution

$$f'(x) = [\log_{2x-1} (\sin x)]' = \left[\frac{\ln(\sin x)}{\ln(2x-1)}\right]' =$$

$$= \frac{[\ln(\sin x)]' \ln(2x-1) - \ln(\sin x)[\ln(2x-1)]'}{[\ln(2x-1)]^2}$$

$$= \frac{1}{[\ln(2x-1)]^2}\left[\frac{(\sin x)'}{\sin x} \ln(2x-1) - \ln(\sin x) \frac{(2x-1)'}{2x-1}\right]$$

$$= \frac{1}{[\ln(2x-1)]^2}\left[\frac{\cos x}{\sin x} \ln(2x-1) - \ln(\sin x) \frac{2}{2x-1}\right]$$

$$= \frac{(2x-1)\ln(2x-1)\cos x - 2\ln(\sin x)\sin x}{(2x-1)(\sin x)[\ln(2x-1)]^2}$$

▶ Limits with logarithms

EXAMPLES

a) $f(x) = \log_{1/3}(x^2+1) \leftarrow \lim_{x\to+\infty} f(x)$

Solution

Since
$$f(x) = \log_{1/3}(x^2+1) = \frac{\ln(x^2+1)}{\ln(1/3)} = \frac{\ln(x^2+1)}{-\ln 3}$$

we have
$$\lim_{x\to+\infty}(x^2+1) = \lim_{x\to+\infty} x^2 = +\infty \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} \ln(x^2+1) = +\infty \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\ln(x^2+1)}{-\ln 3} = -\infty$$ □

b) $f(x) = \log_x(2x^2+1) - \log_x(x^2-1) \leftarrow \lim_{x\to+\infty} f(x)$

Solution

Since
$$f(x) = \log_x(2x^2+1) - \log_x(x^2-1) =$$

$$= \frac{\ln(2x^2+1)}{\ln x} - \frac{\ln(x^2-1)}{\ln x} =$$

$$= \frac{\ln(2x^2+1) - \ln(x^2-1)}{\ln x} =$$

$$= \frac{1}{\ln x} \ln\left(\frac{2x^2+1}{x^2-1}\right)$$

we have

$$\lim_{x\to+\infty} \frac{2x^2+1}{x^2-1} = \lim_{x\to+\infty} \frac{2x^2}{x^2} = 2 \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} \ln\left(\frac{2x^2+1}{x^2-1}\right) = \ln 2$$

and

$$\lim_{x\to+\infty} \ln x = +\infty \Rightarrow \lim_{x\to+\infty} \frac{1}{\ln x} = 0$$

and therefore

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left[\frac{1}{\ln x} \ln\left(\frac{2x^2+1}{x^2-1}\right)\right]$$

$$= 0 \cdot \ln 2 = 0$$

EXERCISES

(28) Find the default domain for the following functions:

a) $f(x) = \log_3(x^5 - x^3)$

b) $f(x) = \log_{x+1}(x^2-4)$

c) $f(x) = \log_{2x}(x^2+2x)$

d) $f(x) = \log_{x-1}(x+1) + \log_{x+1}(x-1)$

e) $f(x) = \log_{x^2-4}(x^2+3x+2)$

(29) Find and simplify the derivatives of the following functions:

a) $f(x) = \log_{e^2}(x^2+1)$

b) $f(x) = \log_x 3$

c) $f(x) = \log_x(\cos x)$

d) $f(x) = \log_{x+1}(x-1)$

e) $f(x) = \log_x(3x)$

f) $f(x) = \log_x 3$

(30) Show that, for $a, b, c \in (0,1) \cup (1, +\infty)$:

a) $\log_a\left(\frac{1}{b^5}\right)\log_b(a^2) = -10$

b) $\log_a(bc) = \dfrac{1}{\log_b a} + \dfrac{1}{\log_c a}$

c) $\log_{ab}(c) = \dfrac{\log_b(c)}{1+\log_b(a)}$

(31) Evaluate the following limits, if they exist:

a) $\lim\limits_{x \to \pi/2^-} \log_3(\cos x)$

b) $\lim\limits_{x \to +\infty} [2\log_{1/2}(x^2+2x) - \log_{1/2}(x^2-2x)]$

c) $\lim\limits_{x \to 0^+} [\log_{x+2}(\sin x) - \log_{x+2}(x)]$

d) $\lim\limits_{x \to 0^+} [\log_{3x+2}(\tan(2x)) - \log_{3x+2}(\tan(3x))]$

e) $\lim\limits_{x \to +\infty} [\log_x(x+\sin x) - \log_x(x)]$

(32) Solve the following equations with respect to x:

a) $\log_x 2 + \log_2 x = \dfrac{5}{3}$

b) $\log_x 256 = (\log_x 4)^2 + 3$

c) $\log_3 x \log_9 x = 2$

d) $2(\log_x 8)^2 + \log_x 64 + \log_x 8 = 9.$