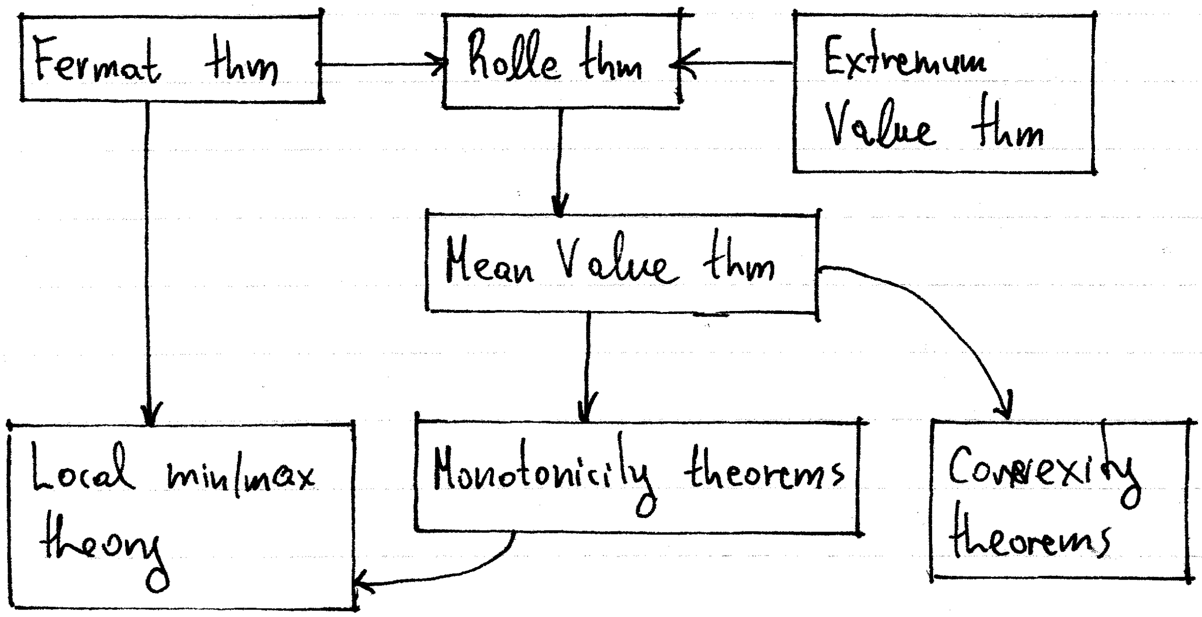
DIFFERENTIAL CALCULUS

Foundation of Differential Calculus

The applications of derivatives are based on a collection of theorems that have the following interdependence amongst themselves



(1) → Fermat theorem

Def: (Interior points)
Let A be a set $A \subseteq \mathbb{R}$. We say that
x_0 interior point of $A \iff \exists\, \delta \in (0,+\infty) : (x_0-\delta, x_0+\delta) \subseteq A$

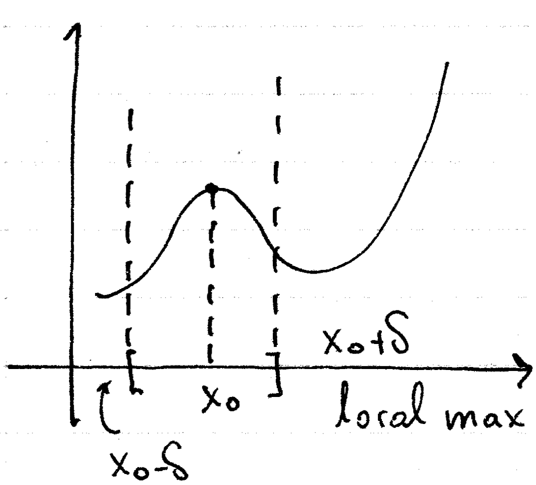
<u>notation</u>: The set of all interior points of a set A is denoted as

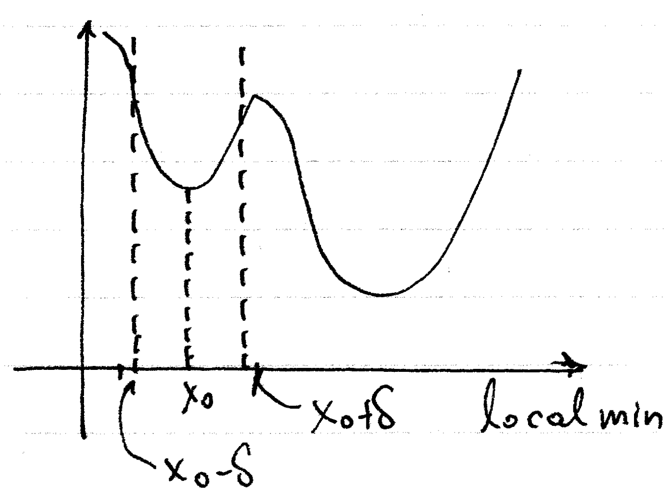
$$\text{int}(A) = \{x_0 \in A \mid x_0 \text{ interior to } A\}$$
$$= \{x_0 \in A \mid \exists \delta \in (0,+\infty) : (x_0-\delta, x_0+\delta) \subseteq A\}$$

▸ In general, given a set defined as a union of intervals, $\text{int}(A)$ can be obtained by changing all closed intervals to open intervals

<u>example</u>: For $A = (1,3] \cup [5,+\infty)$, we have
$$\text{int}(A) = (1,3) \cup (5,+\infty).$$
Consequently, 2 is interior to A but for $x_0 \in \{1,3,5\}$, x_0 is <u>not</u> interior to A.

<u>Def</u>: (<u>Local min/max</u>)
Let $f: A \to \mathbb{R}$ be a function and let $x_0 \in A$.
We say that
a) x_0 <u>local max</u> of $f \iff$
$\iff \exists \delta \in (0,+\infty) : \forall x \in (x_0-\delta, x_0+\delta) \cap A : f(x) \leq f(x_0)$
b) x_0 <u>local min</u> of $f \iff$
$\iff \exists \delta \in (0,+\infty) : \forall x \in (x_0-\delta, x_0+\delta) \cap A : f(x) \geq f(x_0)$

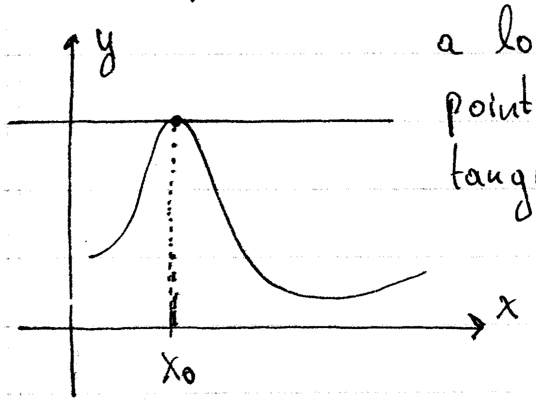




interpretation: A point $x_0 \in A$ is local min of $f: A \to \mathbb{R}$ if and only if $f(x_0)$ is the minimum value of f in a small enough interval around the point x_0. Likewise, a point $x_0 \in A$ is local max of $f: A \to \mathbb{R}$ if and only if $f(x_0)$ is the maximum value of f in a small enough interval around the point x_0.

Thm: (Fermat theorem)
Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ be a function and let $x_0 \in A$.
We have:
$$\begin{cases} x_0 \in \operatorname{int}(A) \\ x_0 \text{ local min or max of } f \\ f \text{ differentiable on } x_0 \end{cases} \Rightarrow f'(x_0) = 0$$

▸ interpretation: If a function is differentiable and has a local max or min at an interior point x_0 of its domain, then the tangent line (ℓ) to the graph of f at the point x_0 is horizontal.



Proof
With no loss of generality, assume that
$$\begin{cases} x_0 \in \operatorname{int}(A) \wedge x_0 \text{ local max of } f \\ f \text{ differentiable on } x_0 \end{cases}$$

It follows that

$x_0 \in \mathrm{int}(A) \Rightarrow \exists \delta_1 \in (0,+\infty): (x_0-\delta_1, x_0+\delta_1) \subseteq A$

x_0 local max of $f \Rightarrow$

$\Rightarrow \exists \delta_2 \in (0,+\infty): \forall x \in (x_0-\delta_2, x_0+\delta_2) \cap A: f(x) \leq f(x_0)$

Choose $\delta_1, \delta_2 \in (0,+\infty)$ such that

$$\begin{cases} (x_0-\delta_1, x_0+\delta_1) \subseteq A \\ \forall x \in (x_0-\delta_2, x_0+\delta_2) \cap A: f(x) \leq f(x_0) \end{cases}$$

Define $\delta = \min\{\delta_1, \delta_2\}$ and define

$$\forall x, x_0 \in A: \lambda(x, x_0) = \frac{f(x)-f(x_0)}{x-x_0}$$

Since

$(x_0-\delta, x_0+\delta) \subseteq (x_0-\delta_1, x_0+\delta_1) \subseteq A \Rightarrow$

$\Rightarrow (x_0-\delta, x_0+\delta) \subseteq A \Rightarrow (x_0-\delta, x_0+\delta) \cap A = (x_0-\delta, x_0+\delta)$

$\Rightarrow \forall x \in (x_0-\delta, x_0+\delta): f(x) \leq f(x_0)$

$\Rightarrow \forall x \in (x_0-\delta, x_0+\delta): f(x) - f(x_0) \leq 0$

$$\Rightarrow \begin{cases} \forall x \in (x_0-\delta, x_0): \lambda(x, x_0) = \dfrac{f(x)-f(x_0)}{x-x_0} \geq 0 & (1) \\ \forall x \in (x_0, x_0+\delta): \lambda(x, x_0) = \dfrac{f(x)-f(x_0)}{x-x_0} \leq 0 & (2) \end{cases}$$

Since f differentiable at x_0

$$f'(x_0) = \lim_{x \to x_0^-} \lambda(x, x_0) \geq 0 \quad , \text{ from Eq. (1)}$$

$$f'(x_0) = \lim_{x \to x_0^+} \lambda(x, x_0) \leq 0 \quad , \text{ from Eq. (2)}$$

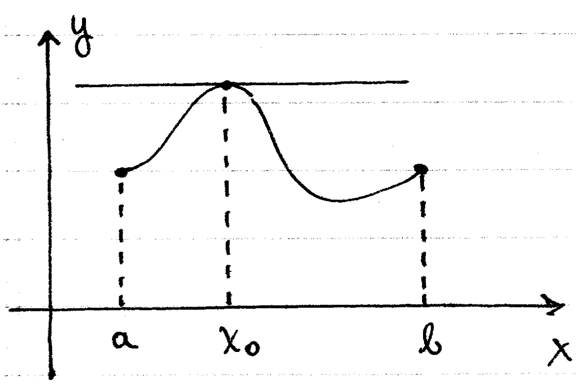
and it follows that $f'(x_0) = 0$. $\square$

② → <u>Rolle theorem</u>

Thm : Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and let $a, b \in A$ with $[a,b] \subseteq A$. Then,

$$\left.\begin{array}{l} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \\ f(a) = f(b) \end{array}\right\} \Rightarrow \exists\, x_0 \in (a,b) : f'(x_0) = 0$$

<u>interpretation</u> :



If a function f is continuous on $[a,b]$ and differentiable on (a,b) and if $f(a) = f(b)$, then there is a point $x_0 \in (a,b)$ where the tangent line to the graph of the function becomes horizontal.

<u>Proof</u>

Assume that

$$\left\{\begin{array}{l} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \\ f(a) = f(b) \end{array}\right.$$

Using the Extremum Value Theorem,

f continuous on $[a,b] \Rightarrow$

$\Rightarrow \exists\, x_1, x_2 \in [a,b] : \forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2)$.

Choose $x_1, x_2 \in [a,b]$ such that

$\forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2)$

We distinguish between the following cases.

Case 1: Assume that $x_1 \in (a,b)$. Then

$(\forall x \in [a,b] : f(x) \geq f(x_1)) \Rightarrow x_1$ local min of f (1)

We also know that

$\begin{cases} x_1 \text{ interior to } (a,b) & (2) \\ f \text{ differentiable on } (a,b) \end{cases}$

From Eq.(1) and Eq.(2), via the Fermat theorem:

$f'(x_1) = 0 \Rightarrow \exists x_0 \in (a,b) : f'(x_0) = 0$ (for $x_0 = x_1$)

Case 2: Assume that $x_2 \in (a,b)$. Then

$(\forall x \in [a,b] : f(x) \leq f(x_2)) \Rightarrow x_2$ local max of f (3)

We also know that

$\begin{cases} x_2 \text{ interior to } (a,b) & (4) \\ f \text{ differentiable on } (a,b) \end{cases}$

From Eq.(3) and Eq.(4), via the Fermat theorem:

$f'(x_2) = 0 \Rightarrow \exists x_0 \in (a,b) : f'(x_0) = 0$ (for $x_0 = x_2$).

Case 3: Assume that $x_1 = a \wedge x_2 = b$.

We define $c = f(a) = f(b)$. Then:

$\forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2)$

$\Rightarrow \forall x \in [a,b] : f(a) \leq f(x) \leq f(b)$

$\Rightarrow \forall x \in [a,b] : c \leq f(x) \leq c$

$\Rightarrow \forall x \in [a,b] : f(x) = c$

$\Rightarrow \forall x \in [a,b] : f'(x) = c$

$\Rightarrow \exists x_0 \in [a,b] : f'(x_0) = c.$

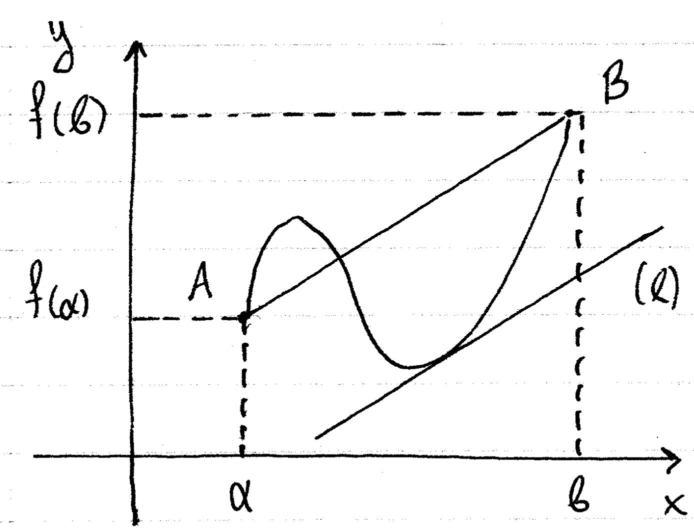
In all cases we conclude that $\exists x_0 \in [a,b] : f'(x_0) = c$.

③ → Mean Value Theorem

Thm: (Lagrange's Mean Value Theorem)
Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ be a function and let $a, b \in A$ such that $[a,b] \subseteq A$. Then

$$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{cases} \Rightarrow \exists\, x_0 \in (a,b) : f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

Interpretation:



If the function f is continuous on $[a,b]$ and differentiable on (a,b), then given the points $A(a, f(a))$ and $B(b, f(b))$ on the graph of f, there is at least one $x_0 \in (a,b)$ such that the tangent line (ℓ) at $x = x_0$ to the graph of f satisfies $(\ell) \parallel (AB)$.

Proof

Assume that

$$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{cases}$$

Define

$\forall x \in [a,b] : F(x) = (a-b)f(x) + [f(b)-f(a)]x + [bf(a) - af(b)]$

and note that

f continuous on $[a,b] \Rightarrow F$ continuous on $[a,b]$ (1)

and

f differentiable on $(a,b) \Rightarrow F$ differentiable on (a,b) (2)

with $\forall x \in (a,b) : F'(x) = (a-b)f'(x) - [f(a)-f(b)]$ (3)

We also have

$$\begin{aligned} F(a) &= (a-b)f(a) + [f(b)-f(a)]a + [bf(a) - af(b)] = \\ &= (a-b)f(a) + af(b) - af(a) + bf(a) - af(b) = \\ &= (a-b-a+b)f(a) + (a-a)f(b) = \\ &= 0f(a) + 0f(b) = 0 \qquad (4) \end{aligned}$$

and

$$\begin{aligned} F(b) &= (a-b)f(b) + [f(b)-f(a)]b + [bf(a) - af(b)] = \\ &= (a-b)f(b) + bf(b) - bf(a) + bf(a) - af(b) = \\ &= (-b+b)f(a) + (a-b+b-a)f(b) = \\ &= 0f(a) + 0f(b) = 0 \qquad (5) \end{aligned}$$

From Eq.(4) and Eq.(5): $F(a) = F(b) = 0$ (6).

From Eq.(1) and Eq.(2) and Eq.(6), via the Rolle theorem:

$$\left\{\begin{array}{l} F \text{ continuous on } [a,b] \\ F \text{ differentiable on } (a,b) \\ F(a) = F(b) \end{array}\right. \Rightarrow \exists x_0 \in (a,b) : F'(x_0) = 0$$

$$\begin{aligned} &\Rightarrow \exists x_0 \in (a,b) : (a-b)f'(x_0) - [f(a)-f(b)] = 0 \\ &\Rightarrow \exists x_0 \in (a,b) : (b-a)f'(x_0) = f(b)-f(a) \\ &\Rightarrow \exists x_0 \in (a,b) : f'(x_0) = \frac{f(b)-f(a)}{b-a} \qquad \square \end{aligned}$$

Remark: During the early development of Calculus, many arguments were based on the concept of the linear approximation

$f(x+\Delta x) \approx f(x) + \Delta x f'(x)$

where Δx is very small relative to x (i.e. $\Delta x \ll x$). The linear approximation assumes that the graph of the function f in the interval $[x, x+\Delta x]$ is approximately a straight line as long as Δx is small enough, and can be therefore represented by a linear function with respect to Δx. The linear approximation can be used to argue, e.g. that if a function has $f'(x) > 0$, then it is increasing from x to $x+\Delta x$. The problem is that such arguments are not rigorous because they are based on a statement that is true only approximately.

According to the Mean Value Theorem, if f satisfies

$$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{cases} \quad \text{with } a = x \text{ and } b = x+\Delta x$$

then we conclude that

$\exists x_0 \in (x, x+\Delta x) : f(x+\Delta x) = f(x) + \Delta x f'(x_0)$

It follows that the linear approximation statement becomes exact if we replace $f'(x)$ with $f'(x_0)$ for some choice of $x_0 \in (x, x+\Delta x)$. This in turn makes it possible to formulate rigorous arguments based on the overall linear approximation concept.

Immediate corollaries of the Mean Value Theorem

The following theorems are immediate consequences of the Mean Value Theorem. We use the assumption that a set $I \subseteq \mathbb{R}$ is an interval, as opposed to a union of disjoint intervals (e.g. $I = [a,b]$ or $I = (a,b]$ or $I = [a,b)$ etc....). A practical definition that encompasses all possibilities is the following:

Def: Let $I \subseteq \mathbb{R}$. We say that

$$I \text{ interval} \Leftrightarrow \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow [x_1, x_2] \subseteq I)$$

We also define the concept of a constant function:

Def: Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $I \subseteq A$. We say that

$$f \text{ constant on } I \Leftrightarrow \forall x_1, x_2 \in I : f(x_1) = f(x_2)$$

We will now show that

Thm: Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $I \subseteq A$. Then:

$$\left\{ \begin{array}{l} I \text{ interval} \\ f \text{ differentiable on } I \\ \forall x \in I : f'(x) = 0 \end{array} \right. \Rightarrow f \text{ constant on } I.$$

Proof

Assume that

$$\begin{cases} I \text{ interval} \\ f \text{ differentiable on } I \\ \forall x \in I : f'(x) = 0 \end{cases}$$

▶ We will show that $\forall x_1, x_2 \in I : f(x_1) = f(x_2)$.

Let $x_1, x_2 \in I$ be given and assume with no loss of generality that $x_1 < x_2$. Then

$$\begin{cases} I \text{ interval} \\ x_1, x_2 \in I \wedge x_1 < x_2 \end{cases} \Rightarrow [x_1, x_2] \subseteq I$$

and therefore:

f differentiable on $I \Rightarrow f$ differentiable on $[x_1, x_2] \Rightarrow$

$$\Rightarrow \begin{cases} f \text{ continuous on } [x_1, x_2] \\ f \text{ differentiable on } (x_1, x_2) \end{cases}$$

$$\Rightarrow \exists x_0 \in (x_1, x_2) : f'(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Choose $x_0 \in (x_1, x_2)$ such that $f'(x_0) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

It follows that

$$\begin{aligned} f(x_2) - f(x_1) &= f'(x_0)(x_2 - x_1) \\ &= 0(x_2 - x_1) \qquad [\text{via } \forall x \in I : f'(x) = 0] \\ &= 0 \Rightarrow f(x_1) = f(x_2) \end{aligned}$$

and therefore:

$(\forall x_1, x_2 \in I : f(x_1) = f(x_2)) \Rightarrow$

$\Rightarrow f$ constant on I.

Thm: Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $I \subseteq A$. Then:

$$\begin{cases} I \text{ interval} \\ f, g \text{ differentiable on } I \\ \forall x \in I : f'(x) = g'(x) \end{cases} \Rightarrow \exists c \in \mathbb{R} : \forall x \in I : f(x) = g(x) + c$$

Proof

Assume that

$$\begin{cases} I \text{ interval} & (1) \\ f, g \text{ differentiable on } I \\ \forall x \in I : f'(x) = g'(x) \end{cases}$$

Define $\forall x \in I : h(x) = f(x) - g(x)$. Then

f, g differentiable on $I \Rightarrow h$ differentiable on I (2)

with

$$\forall x \in I : h'(x) = [f(x) - g(x)]' = f'(x) - g'(x)$$
$$= f'(x) - f'(x) = 0 \quad (3)$$

From Eq.(1), Eq.(2), Eq.(3):

h constant on $I \Rightarrow \exists c \in \mathbb{R} : \forall x \in I : h(x) = c$

$\Rightarrow \exists c \in \mathbb{R} : \forall x \in I : f(x) - g(x) = c$

$\Rightarrow \exists c \in \mathbb{R} : \forall x \in I : f(x) = g(x) + c$

Method - Examples

(1) To show that an equation has a unique solution (i.e. $f(x)=0$) in (a,b).

- 1 Use the Bolzano theorem to establish EXISTENCE of a solution $x_0 \in (a,b)$.
- 2 Show that $f'(x) \neq 0$, $\forall x \in (a,b)$
- 3 Assume there are two solutions $x_0, x_1 \in (a,b)$ with $x_0 \neq x_1$ and use the Rolle theorem to reach a contradiction.

EXAMPLES

a) Show that $x^3-3x+1=0$ has a unique solution at $(-1,1)$

Solution

- Existence: Let $f(x) = x^3-3x+1$. Then

$$\left.\begin{aligned} f(-1) &= (-1)^3-3(-1)+1 = -1+3+1=3 \\ f(1) &= 1^3-3\cdot 1+1 = -1 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow f(-1)f(1) = 3\cdot(-1) < 0 \qquad (1)$

f continuous at $[-1,1] \qquad (2)$

From (1) and (2):

$\exists\, x_0 \in (-1,1) : f(x_0) = 0$

• <u>Uniqueness</u> : Assume that the equation is satisfied by $x_0, x_1 \in (-1,1)$ with $x_0 < x_1$

We note that

$f'(x) = (x^3-3x+1)' = 3x^2-3 = 3(x^2-1) < 0, \forall x \in (-1,1) \Rightarrow$

$\Rightarrow f'(x) \neq 0, \forall x \in (-1,1).$ (3)

$$\left.\begin{array}{l} \text{Since } f(x_0) = f(x_1) = 0 \\ f \text{ continuous at } [x_0, x_1] \\ f \text{ differentiable at } (x_0, x_1) \end{array}\right\} \Rightarrow$$

$\Rightarrow \exists x_2 \in (x_0, x_1) : \underline{f'(x_2) = 0.}$

From (3): $f'(x_2) \neq 0$, thus we have a contradiction.

It follows that the solution x_0 is unique.

B) Show that $x^5 + 2x^3 + 7x + 12 = 0$ has a unique solution in $\mathbb{R}$.

<u>Solution</u>

• <u>Existence</u> : Let $f(x) = x^5 + 2x^3 + 7x + 12$.

We note that:

$\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} (x^5 + 2x^3 + 7x + 12) = \lim\limits_{x \to +\infty} x^5 = +\infty \Rightarrow$

$\Rightarrow \exists b \in (0, +\infty) : f(b) > 0$ (1)

and

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} (x^5 + 2x^3 + 7x + 12) = \lim\limits_{x \to -\infty} x^5 = -\infty \Rightarrow$

$\Rightarrow \exists a \in (-\infty, 0) : f(a) < 0$ (2)

From (1) and (2):

$$\left.\begin{array}{l} f(a)f(b)<0 \\ f \text{ continuous at } [a,b] \end{array}\right\} \Rightarrow \exists x_0 \in (a,b) : f(x_0)=0 \Rightarrow$$

$\Rightarrow x_0$ solves the equation.

- <u>Uniqueness</u>: Assume that $x_0, x_1 \in \mathbb{R}$ solve the equation with $x_0 < x_1$. We note that

$$f'(x) = (x^5+2x^3+7x+12)' = 5x^4+6x^2+7 >$$
$$> 5x^4+6x^2 \geq 0, \ \forall x \in \mathbb{R} \Rightarrow$$
$$\Rightarrow \forall x \in \mathbb{R} : f'(x) > 0 \qquad (3)$$

Furthermore:

$$\left.\begin{array}{l} f(x_0)=f(x_1)=0 \\ f \text{ continuous at } [x_0,x_1] \\ f \text{ differentiable at } (x_0,x_1) \end{array}\right\} \Rightarrow \exists x_2 \in (x_0,x_1) : \underline{f'(x_2)=0}.$$

From (3): $\underline{f'(x_2)>0}$, so we have a contradiction. It follows that the equation cannot have more than one solution in $\mathbb{R}$.

↳ In the above solution we have used the statements:

$$\lim_{x\to+\infty} f(x) = +\infty \Rightarrow \exists a \in (0,+\infty) : f(a) > 0$$

$$\lim_{x\to-\infty} f(x) = -\infty \Rightarrow \exists a \in (-\infty,0) : f(a) < 0$$

which are immediate consequences of the limit definition. More generally:

$$\lim_{x\to\sigma} f(x) = L \Rightarrow \exists a \in N(\sigma,\delta) : f(a) \in I(L,\varepsilon)$$

② Inequalities : In general, using the Mean Value Theorem, an inequality satisfied by $f'(x)$ implies an inequality satisfied by $f(x)$.

EXAMPLES

a) Let f be a function differentiable in $\mathbb{R}$. Show that if $\forall x \in \mathbb{R}: 3 \leq f'(x) \leq 5$, then $18 \leq f(8) - f(2) \leq 30$.

Solution

f differentiable in $\mathbb{R} \Rightarrow$ MVT applies on $[2,8] \Rightarrow$

$\Rightarrow \exists x_0 \in (2,8) : f(8) - f(2) = f'(x_0)(8-2) = 6f'(x_0)$ (1)

It follows that

$3 \leq f'(x) \leq 5, \forall x \in \mathbb{R} \Rightarrow 3 \leq f'(x_0) \leq 5 \Rightarrow$

$\Rightarrow 18 \leq 6f'(x_0) \leq 30 \Rightarrow 18 \leq f(8) - f(2) \leq 30.$

↳ Inequalities involving $\boxed{\text{two}}$ variables can be proved via the Mean Value Theorem if it is possible, with or without, some manipulation, to produce an expression of the form $f(b) - f(a)$. Then we can use:

$$f(b) - f(a) = f'(x_0)(b-a)$$

for some $x_0 \in (a,b)$.

b) Show that:

$$0 < a < b < \pi/2 \Rightarrow \frac{a}{b} < \frac{\sin a}{\sin b}$$

<u>Solution</u>

Since $0 < a < b < \pi/2 \Rightarrow b\sin b > 0$ and $ab > 0$.
It follows that

$$\frac{a}{b} < \frac{\sin a}{\sin b} \overset{*}{\Leftrightarrow} \frac{a}{b}(b\sin b) < \frac{\sin a}{\sin b}(b\sin b) \Leftrightarrow$$

$$\Leftrightarrow a\sin b < b\sin a \Leftrightarrow a\sin b - b\sin a < 0 \overset{*}{\Leftrightarrow}$$

$$\Leftrightarrow \frac{a\sin b - b\sin a}{ab} < 0 \Leftrightarrow \frac{\sin b}{b} - \frac{\sin a}{a} < 0 \quad (1).$$

Define $f(x) = \frac{\sin x}{x}$. It follows that:

$$f'(x) = \left(\frac{\sin x}{x}\right)' = \frac{(\sin x)' x - \sin x\,(x)'}{x^2} =$$

$$= \frac{x\cos x - \sin x}{x^2}$$

Since:

$$\left.\begin{array}{l} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } [a,b] \end{array}\right\} \Rightarrow \text{The Mean-Value-Theorem applies on } [a,b] \Rightarrow$$

$$\Rightarrow \exists x_0 \in (a,b) : f(b) - f(a) = f'(x_0)(b-a) \Rightarrow$$

$$\Rightarrow \frac{\sin b}{b} - \frac{\sin a}{a} = f(b) - f(a) = f'(x_0)(b-a) =$$

$$= \frac{x_0\cos x_0 - \sin x_0}{x_0^2} \cdot (b-a) =$$

$$= \frac{(x_0 \cos x_0 - \sin x_0)(b-a)}{x_0^2} \qquad (2)$$

Note that

$a < b \Rightarrow b - a > 0 \qquad (3)$

and $\quad x_0^2 > 0 \qquad (4)$

and

$$\left.\begin{array}{l} |\tan x_0| > |x_0| \\ x_0 \in (0, \pi/2) \end{array}\right\} \Rightarrow \tan x_0 > x_0 \Rightarrow \frac{\sin x_0}{\cos x_0} > x_0 \Rightarrow$$

$$\Rightarrow \sin x_0 > x_0 \cos x_0 \Rightarrow x_0 \cos x_0 - \sin x_0 < 0 \qquad (5)$$

From $(2),(3),(4),(5)$:

$$\frac{\sin b}{b} - \frac{\sin a}{a} < 0 \underset{(1)}{\Longrightarrow} \frac{a}{b} < \frac{\sin a}{\sin b} \qquad \square$$

↳ Note the 3-step process:

- •1 Reduce the inequality to be shown to an equivalent simpler inequality that exposes the $f(b) - f(a)$ expression
- •2 Define $f(x)$ and calculate $f'(x)$.
- •3 Apply the MVT and establish a relation between f and f'.
- •4 Determine if $f'(x_0)$ is positive or negative and backtrack your way back to the original inequality.

↳ Also recall the inequalities:

$|\tan x| > |x|, \ \forall x \in (-\pi/2, 0) \cup (0, \pi/2)$

$|\sin x| < |x|, \ \forall x \in \mathbb{R} - \{0\}$.

EXERCISES

(1) Use the Bolzano and Rolle theorems to show that the following equations have a unique solution in the corresponding sets:

a) $\frac{\cos x}{2} + \frac{1}{(1+x)^2} = 0$ at $(2\pi, 3\pi)$

b) $\cos x = x$ at $(0, \pi)$

c) $x^3 - 9x^2 + 24x - 1 = 0$ at $(0, 2)$

d) $x^5 + x^3 + x = a^2(b-x) + b^2(c-x) + c^2(a-x)$ at $\mathbb{R}$

e) $x^5 + 2x^3 + 7x + 12 = 0$ at $\mathbb{R}$

f) $x^3 - 3x + 1 = 0$ at $(-1, 1)$

(2) Show that the equations $x^2 = x\sin x + \cos x$ has only 2 solutions, that are distinct, in the interval $(-\pi, \pi)$.

(3) Let f be a function that is twice-differentiable in $\mathbb{R}$, with

$$\forall x \in \mathbb{R} : f''(x) \neq 0.$$

Show that the equation $f(x) = 0$ cannot have more than two distinct solutions in $\mathbb{R}$.

(4) Show that the equation $x^n + ax + b = 0$ has
a) at most 2 real solutions when n even
b) no more that 3 real solutions when n is odd.

(5) Show that the equation $x^n + nx + 1 = 0$ has
a) only one real solution when n is odd
b) at most 2 real solutions when n is even.

(6) Use the mean value theorem to prove the following inequalities:

a) $|\sin a - \sin b| \leq |a-b|$ for $a, b \in \mathbb{R}$
b) $n(b-a)a^{n-1} \leq b^n - a^n \leq n(b-a)b^{n-1}$, for $0 < a < b$ and $n \in \mathbb{N}$.
c) $\dfrac{a-b}{\cos^2 b} \leq \tan a - \tan b \leq \dfrac{a-b}{\cos^2 a}$, for $0 < a \leq b < \pi/2$
d) $\sin(a+b) < \sin a + b\cos a$, for $0 < a < a+b < \pi/2$
e) $\dfrac{\tan a}{\tan b} < \dfrac{b}{a}$, for $0 < a < b < \pi/2$
(Hint: Use $f(x) = x\tan x$)

(7) Let f be a function continues with $[a,b]$, differentiable in (a,b), and with $f(a) = f(b)$. Show that there exist $c_1, c_2 \in (a,b)$ such that $f'(c_1) + f'(c_2) = 0$.

V Monotonicity and local min/max

Derivatives can be used to determine whether a function f is increasing or decreasing in specific intervals I. We give the following definitions for functions that are strictly increasing or decreasing or weakly increasing or decreasing on some interval

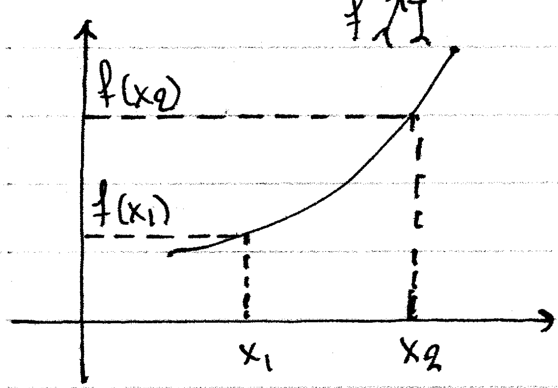
$\boxed{\begin{array}{l} \underline{\text{Def}}: \text{Let } f: A \to \mathbb{R} \text{ be a function and let } I \subseteq A \text{ be an interval. Then:} \\ f \nearrow\!\!\!\nearrow I \Leftrightarrow \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2)) \\ f \searrow\!\!\!\searrow I \Leftrightarrow \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow f(x_1) > f(x_2)) \\ f \nearrow I \Leftrightarrow \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)) \\ f \searrow I \Leftrightarrow \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)) \end{array}}$

<u>terminology</u>:

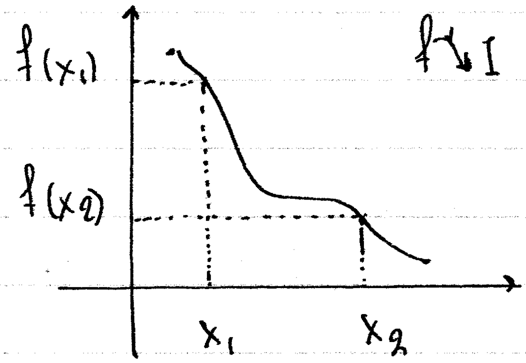
$f \nearrow\!\!\!\nearrow I \longrightarrow f$ strictly increasing on I
$f \searrow\!\!\!\searrow I \longrightarrow f$ strictly decreasing on I
$f \nearrow I \longrightarrow f$ increasing on I
$f \searrow I \longrightarrow f$ decreasing on I

<u>interpretation</u>

$f \nearrow\!\!\!\nearrow I$



If $f \nearrow\!\!\!\nearrow I$ then for any $x_1, x_2 \in I$, if $x_1 < x_2$ then $f(x_1) < f(x_2)$. Similar statements apply to the other 3 definitions



Preliminaries

Recall that the general characterization of an interval $I \subseteq \mathbb{R}$ is given by the equivalence

$$I \text{ interval} \Leftrightarrow \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow [x_1, x_2] \subseteq I)$$

In the context of discussing the Fermat theorem, we also introduced the interior $\text{int}(A)$ of a set A as:

$$\text{int}(A) = \{x_0 \in A \mid \exists \delta \in (0, +\infty) : (x_0 - \delta, x_0 + \delta) \subseteq A\}$$

Applying this definition to intervals, it is easy to show that:

$$(A = [a,b] \vee A = [a,b) \vee A = (a,b] \vee A = (a,b)) \Rightarrow \text{int}(A) = (a,b)$$

$$(A = [a,+\infty) \vee A = (a,+\infty)) \Rightarrow \text{int}(A) = (a,+\infty)$$

$$(A = (-\infty, a] \vee A = (-\infty, a)) \Rightarrow \text{int}(A) = (-\infty, a)$$

An important property of the interior, needed to prove the main theorems, is:

> **Lemma:** Let A, B be two sets such that $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$.
> Then:
> $A \subseteq B \Rightarrow \text{int}(A) \subseteq \text{int}(B)$.

① Monotonicity theorem

Thm: Let $f: A \to \mathbb{R}$ be a function and let $I \subseteq A$ be an interval. Then:

$$\begin{cases} f \text{ continuous on } I \\ f \text{ differentiable on } \operatorname{int}(I) \\ \forall x \in \operatorname{int}(I): f'(x) > 0 \end{cases} \Rightarrow f \nearrow I$$

$$\begin{cases} f \text{ continuous on } I \\ f \text{ differentiable on } \operatorname{int}(I) \\ \forall x \in \operatorname{int}(I): f'(x) < 0 \end{cases} \Rightarrow f \searrow I$$

Remark: Note that differentiability implies continuity. Also, note that there are 8 possible types of intervals, and therefore 16 corresponding statements. For example, for $I = (a,b]$ we obtain:

$$\begin{cases} f \text{ differentiable on } (a,b) \\ f \text{ continuous on } x_0 = b \\ \forall x \in (a,b): f'(x) > 0 \end{cases} \Rightarrow f \nearrow (a,b]$$

There are 15 other similar statements.

Proof

Assume that, given an interval I:

$$\begin{cases} f \text{ continuous on } I \\ f \text{ differentiable on } \operatorname{int}(I) \\ \forall x \in \operatorname{int}(I): f'(x) > 0 \end{cases}$$

Let $x_1, x_2 \in I$ be given and assume that $x_1 < x_2$. Then:

$$\begin{cases} x_1 < x_2 \wedge I \text{ interval} \\ f \text{ continuous on } I \end{cases} \Rightarrow \begin{cases} [x_1, x_2] \subseteq I \\ f \text{ continuous on } I \end{cases} \Rightarrow$$

$$\Rightarrow f \text{ continuous on } [x_1, x_2] \quad (1)$$

and

$$\begin{cases} [x_1, x_2] \subseteq I \\ f \text{ differentiable on } \operatorname{int}(I) \end{cases} \Rightarrow \begin{cases} \operatorname{int}([x_1, x_2]) \subseteq \operatorname{int}(I) \\ f \text{ differentiable on } \operatorname{int}(I) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} (x_1, x_2) \subseteq \operatorname{Int}(I) \\ f \text{ differentiable on } I \end{cases}$$

$$\Rightarrow f \text{ differentiable on } (x_1, x_2) \quad (2)$$

From Eq. (1) and Eq. (2), via the Mean Value Theorem, we have:

$$\exists x_0 \in (x_1, x_2) : f'(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Choose some $x_0 \in (x_1, x_2)$ such that $f'(x_0) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

Then:

$$\begin{aligned} x_0 \in (x_1, x_2) &\Rightarrow x_0 \in \operatorname{int}([x_1, x_2]) \\ &\Rightarrow x_0 \in \operatorname{int}(I) \quad [\text{via } [x_1, x_2] \subseteq I \text{ and lemma}] \\ &\Rightarrow f'(x_0) > 0 \quad [\text{hypothesis}] \\ &\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \\ &\Rightarrow f(x_2) - f(x_1) > 0 \quad [\text{via } x_2 - x_1 > 0] \\ &\Rightarrow f(x_1) < f(x_2) \end{aligned}$$

We have thus shown:

$$(\forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2))) \Rightarrow f \nearrow I$$

A similar argument proves the second statement.

② Local min/max — 1st derivative test

Thm: Let $f: A \to \mathbb{R}$ be a function, let $\delta \in (0, +\infty)$, and let $x_0 \in \mathrm{int}(A)$ such that

$$\begin{cases} f \text{ differentiable on } (x_0-\delta, x_0) \cup (x_0, x_0+\delta) \\ f \text{ continuous on } x_0 \end{cases}$$

Then, it follows that

$$\begin{cases} \forall x \in (x_0-\delta, x_0) : f'(x) < 0 \\ \forall x \in (x_0, x_0+\delta) : f'(x) > 0 \end{cases} \Rightarrow x_0 \text{ local minimum of } f$$

$$\begin{cases} \forall x \in (x_0-\delta, x_0) : f'(x) > 0 \\ \forall x \in (x_0, x_0+\delta) : f'(x) < 0 \end{cases} \Rightarrow x_0 \text{ local maximum of } f$$

$$\begin{cases} \forall x \in (x_0-\delta, x_0) : f'(x) < 0 \\ \forall x \in (x_0, x_0+\delta) : f'(x) < 0 \end{cases} \Rightarrow x_0 \text{ \underline{not} local min/max of } f$$

$$\begin{cases} \forall x \in (x_0-\delta, x_0) : f'(x) > 0 \\ \forall x \in (x_0, x_0+\delta) : f'(x) > 0 \end{cases} \Rightarrow x_0 \text{ \underline{not} local min/max of } f.$$

Remark: Fermat theorem can be used to eliminate all points that are not local min/max, leaving us with a small set of candidates, that we call critical points. We use the following contrapositive form of the Fermat theorem:

$$\begin{cases} x_0 \in \mathrm{int}(A) \\ f \text{ differentiable on } x_0 \\ f'(x_0) \neq 0 \end{cases} \Rightarrow x_0 \text{ not local min or max}$$

It eliminates most points $x_0 \in A$ from the domain A of the function f. The surviving points that require further investigation are:

$$\boxed{\begin{array}{l} \text{a) All } x_0 \text{ not interior to } A \text{ (i.e. } x_0 \notin \operatorname{int}(A)) \\ \text{b) All } x_0 \text{ such that } f \text{ not differentiable on } x_0 \\ \text{c) All } x_0 \text{ with } f'(x_0) = 0. \end{array}}$$

These are the <u>critical points</u> of f.

<u>Remark</u>: The above theorem handles all points $x_0 \in \operatorname{int}(A)$ where f continuous on x_0. It covers most critical points for cases (b) and (c). However, if f not continuous on x_0 OR if x_0 not interior to A, then the definition of local min/max needs to be used directly, to determine what happens.

<u>Proof</u>

We will prove the first statement. Assume that

$$\begin{cases} f \text{ differentiable on } (x_0-\delta, x_0) \cup (x_0, x_0+\delta) \\ f \text{ continuous on } x_0 \\ \forall x \in (x_0-\delta, x_0) : f'(x) < 0 \\ \forall x \in (x_0, x_0+\delta) : f'(x) \geq 0 \end{cases}$$

It is sufficient to show that $\forall x \in (x_0-\delta, x_0+\delta) : f(x) \geq f(x_0)$.

Note that from the preceding theorem:

$$\begin{cases} f \text{ differentiable on } (x_0-\delta, x_0) \\ f \text{ continuous on } x_0 \\ \forall x \in (x_0-\delta, x_0) : f'(x) < 0 \end{cases} \Longrightarrow$$

$$\Rightarrow \begin{cases} f \text{ differentiable on } (x_0-\delta, x_0) \\ f \text{ continuous on } (x_0-\delta, x_0] \\ \forall x \in (x_0-\delta, x_0): f'(x) < 0 \end{cases} \Rightarrow \underline{f \searrow (x_0-\delta, x_0]}$$

and

$$\begin{cases} f \text{ differentiable on } (x_0, x_0+\delta) \\ f \text{ continuous on } x_0 \\ \forall x \in (x_0, x_0+\delta): f'(x) > 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f \text{ differentiable on } (x_0, x_0+\delta) \\ f \text{ continuous on } [x_0, x_0+\delta) \\ \forall x \in (x_0, x_0+\delta): f'(x) > 0 \end{cases} \Rightarrow \underline{f \nearrow [x_0, x_0+\delta)}$$

Let $x \in (x_0-\delta, x_0+\delta)$ be given. We distinguish between the following cases:

<u>Case 1</u>: Assume $x = x_0$. Then

$$f(x) = f(x_0) \Rightarrow f(x) \geq f(x_0)$$

<u>Case 2</u>: Assume $x \in (x_0-\delta, x_0)$. Then:

$$\begin{aligned} x \in (x_0-\delta, x_0) &\Rightarrow x < x_0 \\ &\Rightarrow f(x) > f(x_0) \quad [\text{via } f \searrow (x_0-\delta, x_0]] \\ &\Rightarrow f(x) \geq f(x_0) \end{aligned}$$

<u>Case 3</u>: Assume $x \in (x_0, x_0+\delta)$. Then:

$$\begin{aligned} x \in (x_0, x_0+\delta) &\Rightarrow x > x_0 \\ &\Rightarrow f(x) > f(x_0) \quad [\text{via } f \nearrow [x_0, x_0+\delta)] \\ &\Rightarrow f(x) \geq f(x_0). \end{aligned}$$

From the above argument, it follows that

$$\forall x \in (x_0-\delta, x_0+\delta): f(x) \geq f(x_0)$$

$\Rightarrow x_0$ local min of f.

Similar proofs apply to the other 3 statements.

▶ Methodology: In general, to find the monotonicity and local max/min of a function,

we work as follows:

- 1 Find the domain A of f (if not given)
- 2 Calculate and factor $f'(x)$. Check whether the domain of f' includes all points of A.
- 3 Make a table of signs for $f'(x)$.
- 4 Use the sign table to deduce the monotonicity of f.
- 5 Use the monotonicity to deduce the local min/max.

Critical points

Recall that possible local min/max can be found only among the following critical points:

1) Points x_0 with $f'(x_0) \overset{?}{=} 0$
2) Endpoints of closed intervals of A (i.e. $x_0 \notin \text{int}(A)$)
   example: For $A = [1,2) \cup [3,+\infty)$
   possible local max/min may exist at $x_0 = 1$ and $x_0 = 3$.
3) Points x_0 where $f'(x_0)$ is not defined (i.e. f not differentiable on x_0).

EXAMPLE

a) Determine the monotonicity and local min/max for the function

$\forall x \in \mathbb{R}: f(x) = (x-1)^2(x+2)^3$

Solution

• Domain: No restrictions, so domain of f is $A = \mathbb{R}$.

$\forall x \in \mathbb{R}: f'(x) = \left[(x-1)^2(x+2)^3\right]' =$

$$= \left[(x-1)^2\right]'(x+2)^3 + (x-1)^2\left[(x+2)^3\right]'$$
$$= 2(x-1)(x-1)'(x+2)^3 + (x-1)^2\, 3(x+2)^2(x+2)' =$$
$$= 2(x-1)(x+2)^3 + 3(x-1)^2(x+2)^2 =$$
$$= (x-1)(x+2)^2\left[2(x+2) + 3(x-1)\right] =$$
$$= (x-1)(x+2)^2(2x+4+3x-3) =$$
$$= (x-1)(x+2)^2(5x+1)$$

Since:

x		-2		$-1/5$		1	
$x-1$	$-$		$-$		$-$	0	$+$
$(x+2)^2$	$+$	0	$+$		$+$		$+$
$5x+1$	$-$		$-$	0	$+$		$+$
$f'(x)$	$+$	0	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$		$\nearrow$	max	$\searrow$	min	$\nearrow$

$f \nearrow$ on $(-\infty, -2)$, $(-2, -1/5)$, and $(1, +\infty)$

$f \searrow$ on $(-1/5, 1)$

local max at $x_0 = -1/5$ and local min at $x_0 = 1$.

▸ Note that $x_0 = -2$ is NOT local min or max!

EXAMPLE

Determine the monotonicity and the local min/max for the function $f(x) = x\sqrt{1-2x}$

Solution

• Domain of f.

We require $1-2x \geq 0 \Leftrightarrow -2x \geq -1 \Leftrightarrow x \leq 1/2$

and therefore the domain of f is $A = (-\infty, 1/2]$

• Derivative of f

$$f'(x) = \left[x\sqrt{1-2x}\,\right]' =$$

$$= (x)'\sqrt{1-2x} + x\left(\sqrt{1-2x}\,\right)' =$$

$$= \sqrt{1-2x} + x\,\frac{(1-2x)'}{2\sqrt{1-2x}} =$$

$$= \sqrt{1-2x} + \frac{-2x}{2\sqrt{1-2x}} = \sqrt{1-2x} - \frac{x}{\sqrt{1-2x}} =$$

$$= \frac{(\sqrt{1-2x}\,)^2 - x}{\sqrt{1-2x}} = \frac{(1-2x)-x}{\sqrt{1-2x}} = \frac{1-3x}{\sqrt{1-2x}}, \forall x \in (-\infty, 1/2).$$

• Monotonicity and local min/max

x		$1/3$		$1/2$	
$1-3x$	$+$	0	$-$		$-$
$\sqrt{1-2x}$	$+$		$+$	0	
f'	$+$	0	$-$		
f	$\nearrow$	max	$\searrow$	min	

$f \nearrow (-\infty, 1/3)$

$f \searrow (1/3, 1/2)$

local max at $x_0 = 1/3$

local min at $x_0 = 1/2$.

EXAMPLE

Determine the monotonicity and local min/max of the function $f(x) = \dfrac{\sqrt{4-x^2}}{(x+1)^2}$

Solution

- Domain: We require $\begin{cases} 4-x^2 \geq 0 \\ x+1 \neq 0 \end{cases} \Leftrightarrow \begin{cases} (2-x)(2+x) \geq 0 \\ x \neq -1 \end{cases}$ (1)

x		-2		$+2$	
$2-x$	$+$		$+$	0	$-$
$2+x$	$-$	0	$+$		$+$
	$-$	0	$+$	0	$-$

and therefore

$$\text{Eq.}(1) \Leftrightarrow \begin{cases} x \in [-2,2] \\ x \neq -1 \end{cases} \Leftrightarrow x \in [-2,2] - \{-1\}$$

so the domain of f is:

$A = [-2,2] - \{-1\} = [-2,-1) \cup (-1,2]$.

- Derivative

$$f'(x) = \left[\frac{\sqrt{4-x^2}}{(x+1)^2}\right]' = \frac{(\sqrt{4-x^2})'(x+1)^2 - \sqrt{4-x^2}\,[(x+1)^2]'}{(x+1)^4}$$

$$= \frac{1}{(x+1)^4}\left[\frac{(4-x^2)'}{2\sqrt{4-x^2}}(x+1)^2 - \sqrt{4-x^2}\,2(x+1)(x+1)'\right]$$

$$= \frac{1}{(x+1)^4}\left[\frac{-2x}{2\sqrt{4-x^2}}(x+1)^2 - 2(x+1)\sqrt{4-x^2}\right] =$$

$$= \frac{1}{(x+1)^4}\left[\frac{-x(x+1)^2}{\sqrt{4-x^2}} - 2(x+1)\sqrt{4-x^2}\right] =$$

$$= \frac{1}{(x+1)^4}\left[\frac{-x(x+1)^2 - 2(x+1)\left(\sqrt{4-x^2}\right)^2}{\sqrt{4-x^2}}\right] =$$

$$= \frac{1}{(x+1)^4}\,\frac{(x+1)\left[-x(x+1) - 2(4-x^2)\right]}{\sqrt{4-x^2}} =$$

$$= \frac{1}{(x+1)^3}\,\frac{-x^2-x-8+2x^2}{\sqrt{4-x^2}} = \frac{x^2-x-8}{(x+1)^3\sqrt{4-x^2}}, \ \forall x \in (-2,2)-\{-1\}$$

$(x+1)^3$ has zero at -1

$4-x^2$ has zeroes $2, -2$

For x^2-x-8:

$$\Delta = (-1)^2 - 4\cdot 1\cdot(-8) = 1+32 = 33 \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{-(-1)\pm\sqrt{33}}{2\cdot 1} = \frac{1\pm\sqrt{33}}{2} = \begin{cases} (1+\sqrt{33})/2 = x_1 \\ (1-\sqrt{33})/2 = x_2 \end{cases}$$

and note that

$$x_1 - 2 = \frac{1+\sqrt{33}}{2} - 2 = \frac{1+\sqrt{33}-4}{2} = \frac{\sqrt{33}-3}{2} =$$

$$= \frac{\sqrt{33}-\sqrt{9}}{2} > 0 \Rightarrow x_1 > 2$$

and

$$x_2 - (-2) = \frac{1-\sqrt{33}}{2} - (-2) = \frac{1-\sqrt{33}+4}{2} =$$

$$= \frac{5-\sqrt{33}}{2} = \frac{\sqrt{25}-\sqrt{33}}{2} < 0 \Rightarrow x_2 < -2.$$

Thus: $\frac{1-\sqrt{33}}{2} < -2 < -1 < 2 < \frac{1+\sqrt{33}}{2}$

- Monotonicity

x		$(1-\sqrt{33})/2$		-2		-1		2		$(1+\sqrt{33})/2$	
$\sqrt{4-x^2}$				0	$+$		$+$	0			
$(x+1)^3$	$-$		$-$		$-$	0	$+$		$+$		$+$
x^2-x-8	$+$	0	$-$		$-$		$-$		$-$	0	$+$
$f'(x)$				‖	$+$	‖	$-$	‖			
$f(x)$					$\nearrow$	‖	$\searrow$				
				min				min			

$f \nearrow (-2,-1)$
$f \searrow (-1,2)$
$x_0=-2$ local min
$x_0=2$ local min

↳ Note that

a) $-2,2 \in A$ but $f'(-2)$, $f'(2)$ are not defined.
Never theless, the function f has local min/max at $-2,2$!

b) f has singularity at $x_0=-1$, so there is no local min/max on that point.

(2) → <u>2nd derivative test</u>

Thm: Let $f : A \to \mathbb{R}$ and $x_0 \in \text{int}(A)$ and $\delta \in (0, +\infty)$ such that

$$\begin{cases} f \text{ twice differentiable on } x_0 \\ f \text{ differentiable on } (x_0-\delta, x_0) \cup (x_0, x_0+\delta) \end{cases}$$

Then:

$f'(x_0) = 0 \wedge f''(x_0) > 0 \Rightarrow x_0$ local min of f

$f'(x_0) = 0 \wedge f''(x_0) < 0 \Rightarrow x_0$ local max of f

<u>Remark</u>: The second derivative test has the major disadvantage that if $f'(x_0) = 0 \wedge f''(x_0) = 0$, then the test is inconclusive and cannot classify x_0 as local maximum or minimum. Furthermore, in some problems calculating f'' may involve a lot of computation. An example that easily breaks the 2nd derivative test is

$f(x) = (x-1)^4, \ \forall x \in \mathbb{R}$

Then,

$f'(x) = 4(x-1)^3 (x-1)' = 4(x-1)^3$

$f''(x) = 12(x-1)^2 (x-1)' = 12(x-1)^2$

and $f'(1) = 0 \wedge f''(1) = 0$, so $x_0 = 1$ cannot be classified as local minimum and maximum. On the other hand, using the first derivative test:

x		1	
$4(x-1)^3$	$-$	0	$+$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	min	$\nearrow$

thus:

$x_0 = 1$ is local minimum

Remark: Consequently, the 2nd derivative test is not recommended for polynomials, rational functions, and functions with square roots. It does afford us however with a practical advantage for trigonometric functions that can have an infinite number of critical points.

Proof

Assume that

$$\begin{cases} f \text{ twice differentiable on } x_0 \\ f \text{ differentiable on } (x_0-\delta, x_0) \cup (x_0, x_0+\delta) \\ f'(x_0) = 0 \wedge f''(x_0) > 0 \end{cases}$$

Since f differentiable on $(x_0-\delta, x_0) \cup (x_0, x_0+\delta)$, we can consider the following limit:

$$\lim_{x \to x_0} \frac{f'(x)}{x-x_0} = \lim_{x \to x_0} \frac{f'(x) - f'(0)}{x-x_0} \quad [\text{via } f'(x_0) = 0]$$

$$= f''(x_0) \quad [\text{via } f \text{ twice differentiable on } x_0]$$

$$> 0 \quad [\text{via } f''(x_0) > 0]$$

$$\Rightarrow \exists\, \delta_1 \in (0, +\infty) : \forall x \in N(x_0; \delta_1) : \frac{f'(x)}{x-x_0} > 0$$

Choose a $\delta_1 \in (0, +\infty)$ that satisfies the above statement and define $\delta_2 = \min\{\delta, \delta_1\}$. It follows that

$$\begin{cases} \forall x \in (x_0-\delta_2, x_0): (f'(x)/(x-x_0) > 0 \wedge x-x_0 < 0) \\ \forall x \in (x_0, x_0+\delta_2): (f'(x)/(x-x_0) > 0 \wedge x-x_0 > 0) \end{cases}$$

$$\Rightarrow \begin{cases} \forall x \in (x_0-\delta_2, x_0): f'(x) < 0 \\ \forall x \in (x_0, x_0+\delta_2): f'(x) > 0 \end{cases} \quad (1)$$

and

f differentiable on $(x_0-\delta_2, x_0) \cup (x_0, x_0+\delta_2)$ (2)

f twice differentiable on $x_0 \Rightarrow f$ continuous on x_0 (3)

From Eq. (1), Eq. (2), Eq. (3), using the first derivative test, x_0 local minimum of f.

▸ The other case can be proved by a similar argument.

→ The 2nd derivative test is usually avoided because
a) Calculating $f''(x)$ may be tedious
b) If $f''(x_0)=0$, then the test is inconclusive (i.e. totally worthless)

An important EXCEPTION is with trigonometric functions where we have to deal with infinite sets of local min or max points.

EXAMPLE

Find the local min and local max of the function $f(x) = x - \sin(2x)$.

Solution

- 1 No constraints, so domain of f is $A=\mathbb{R}$.
- 2 We have

$$f'(x) = 1-\cos(2x)(2x)' = 1-2\cos(2x), \ \forall x \in \mathbb{R}$$
$$f''(x) = [1-2\cos(2x)]' = +2\sin(2x)(2x)' = 4\sin(2x), \forall x \in \mathbb{R}$$

- 3 All possible local min/max are zeroes of f', since f differentiable on $\mathbb{R}$. Note that

$$f'(x)=0 \Leftrightarrow 1-2\cos(2x)=0 \Leftrightarrow 2\cos(2x)=1 \Leftrightarrow$$
$$\Leftrightarrow \cos(2x) = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right) \Leftrightarrow$$
$$\Leftrightarrow \exists k \in \mathbb{Z} : (2x = 2k\pi + \pi/3 \vee 2x = 2k\pi - \pi/3)$$
$$\Leftrightarrow \exists k \in \mathbb{Z} : (x = k\pi + \pi/6 \vee x = k\pi - \pi/6)$$

$\Leftrightarrow x \in \{k\pi + \pi/6 \, , \, k\pi - \pi/6 \mid k \in \mathbb{Z}\}$

For $x_0 = k\pi \pm \pi/6$:

$$f''(x_0) = 4\sin\left[2(k\pi \pm \pi/6)\right] = 4\sin(2k\pi \pm \pi/3) =$$

$$= 4\sin(\pm\pi/3) = \pm 4\sin(\pi/3) = \pm 4(\sqrt{3}/2) = \pm 2\sqrt{3}$$

and therefore:

for $x_0 = k\pi + \pi/6$; $k \in \mathbb{Z}$:

$f'(x_0) = 0 \wedge f''(x_0) = 2\sqrt{3} > 0 \Rightarrow \forall k \in \mathbb{Z}: x_0 = k\pi + \pi/6$ local min

for $x_0 = k\pi - \pi/6$; $k \in \mathbb{Z}$:

$f'(x_0) = 0 \wedge f''(x_0) = -2\sqrt{3} < 0 \Rightarrow$

$\Rightarrow \forall k \in \mathbb{Z}: x_0 = k\pi - \pi/6$ local max.

EXERCISES

⑧ Analyze the following functions with respect to monotonicity

a) $f(x) = x^3 - 6x$

b) $f(x) = \dfrac{2x+3}{x-2}$

c) $f(x) = \dfrac{x^2+1}{x-1}$

d) $f(x) = x^4 - x^2 + 1$

e) $f(x) = \sqrt{x^2-1}$

f) $f(x) = \sqrt{x^2}$

g) $f(x) = \dfrac{x^3}{x^2-1}$

h) $f(x) = \dfrac{x}{4-x^2}$

⑨ Analyze the following functions with respect to monotonicity and find all local min and max points.

a) $f(x) = x^3 - 9x^2 + 5$

b) $f(x) = 2 - 3x^4$

c) $f(x) = (x-2)^4 + 3$

d) $f(x) = (x-5)^3$

e) $f(x) = x + 1/x$

f) $f(x) = \dfrac{x+1}{x^2-2}$

g) $f(x) = x\sqrt{4-x^2}$

h) $f(x) = x^2\sqrt{2x-1}$

i) $f(x) = \dfrac{x^2-3x+2}{x^2+2x+1}$

j) $f(x) = (x-1)^3(2x+1)^2$

k) $f(x) = \dfrac{(3x+1)^2}{(x-2)^3}$

(10) Similarly for the following functions:

a) $f(x) = (3x+2)^3 (x-2)^4$

b) $f(x) = x^2 (x^2-1)^2$

c) $f(x) = \dfrac{\sqrt{x+1}}{2x-1}$

d) $f(x) = \dfrac{3x+5}{\sqrt{4x+1}}$

e) $f(x) = \sin^2 x$

f) $f(x) = 2\sin x + \cos 2x$

g) $f(x) = 2\sin^2 x - 2\sin x + 3$

(11) Use monotonicity to show that

a) $\ln(1+x) \leq x - x^2/2 + x^3/3 \ , \ \forall x \in [0, +\infty)$

b) $\dfrac{1}{3}\tan x + \dfrac{2}{3}\sin x > x \ , \ \forall x \in (0, \pi/2)$

c) $\dfrac{\sin x}{x} \geq \dfrac{2}{\pi} \ , \ \forall x \in (0, \pi/2]$

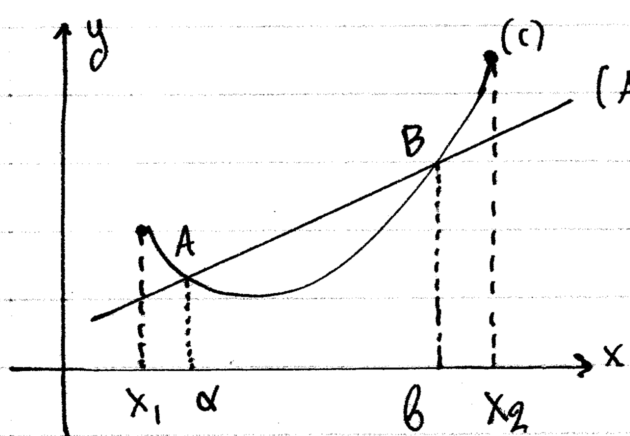
d) $x\sin x + \cos x > 1 \ , \ \forall x \in (0, \pi/2]$

e) $\sin x \geq x - x^3/6 \ , \ \forall x \in [0, +\infty)$

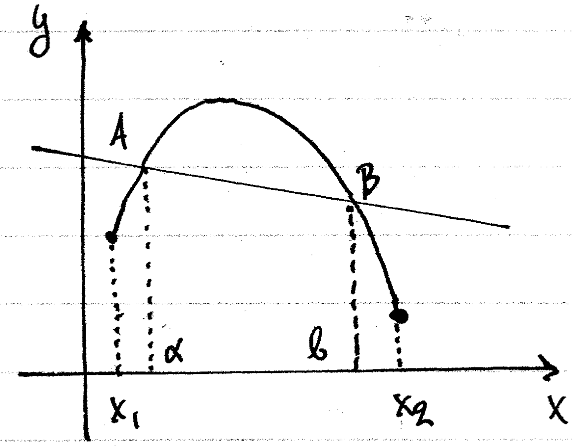
Convexity

The general definition of convexity can be stated geometrically as follows:

Def: (Geometric definition of convexity)
Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and let $[x_1, x_2] \subseteq A$ be an interval. We say that
a) f convex up on $[x_1, x_2] \iff$ for any points $a, b \in [x_1, x_2]$ with $a < b$, the secant line (AB) passing through $A(a, f(a))$ and $B(b, f(b))$ is ABOVE the curve $(c): y = f(x)$ for all $x \in (a, b)$
b) f convex down on $[x_1, x_2] \iff$ for any points $a, b \in [x_1, x_2]$ with $a < b$, the secant line (AB) passing through $A(a, f(a))$ and $B(b, f(b))$ is BELOW the curve $(c): y = f(x)$ for all $x \in (a, b)$.



f convex up on $[x_1, x_2]$



f convex down on $[x_1, x_2]$.

To rewrite an equivalent geometric definition, we note that for all $t\in(0,1)$, a parametric representation of the line segment AB is given by

$$x = a + t(b-a)$$

and

$$y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a) = f(a) + \frac{f(b)-f(a)}{b-a}t(b-a) =$$

$$= f(a) + t[f(b)-f(a)] = (1-t)f(a) + tf(b)$$

and therefore

$$AB : \begin{cases} x = a + t(b-a) \\ y = (1-t)f(a) + tf(b) \end{cases}, \forall t\in[0,1]$$

Consequently, the claims can be rewritten as follows:

AB is above $(c) \iff \forall t\in(0,1): f(a+t(b-a)) < (1-t)f(a) + tf(b)$

AB is below $(c) \iff \forall t\in(0,1): f(a+t(b-a)) > (1-t)f(a) + tf(b)$

We may therefore write the following equivalent, definition algebraic as follows:

Def: (Algebraic definition of convexity)

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $[x_1, x_2] \subseteq A$. We say that

f convex up on $[x_1, x_2] \iff$

$$\iff \forall a\in[x_1,x_2): \forall b\in(a,x_2]: \forall t\in(0,1):$$
$$: f(a+t(b-a)) < (1-t)f(a) + tf(b)$$

f convex down on $[x_1, x_2] \iff$

$$\iff \forall a\in[x_1,x_2): \forall b\in(a,x_2]: \forall t\in(0,1):$$
$$: f(a+t(b-a)) > (1-t)f(a) + tf(b)$$

• Properties of convexity

① → Convexity and monotonicity of f'

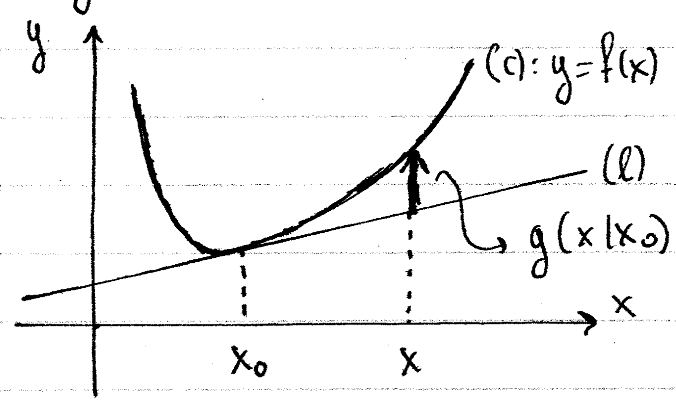
The definition of convexity does not require the function f to be differentiable or even continuous. However, if the function f is in fact differentiable, then we can show that:

> **Thm**: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and $[x_1, x_2] \subseteq A$ such that f differentiable on $[x_1, x_2]$. Then:
>
> f convex up on $[x_1, x_2] \Leftrightarrow f' \nearrow [x_1, x_2]$
>
> f convex down on $[x_1, x_2] \Leftrightarrow f' \searrow [x_1, x_2]$

② → Tangent line characterization of convexity

Given the function f, let us define $g(x|x_0)$ as:

$$g(x|x_0) = f(x) - [f'(x_0)(x - x_0) + f(x_0)]$$



The interpretation of $g(x|x_0)$ is that it measures the difference in y coordinates between a point $(x, f(x))$ on the graph of the function f and a point with the same x coordinate on the tangent line of the function f

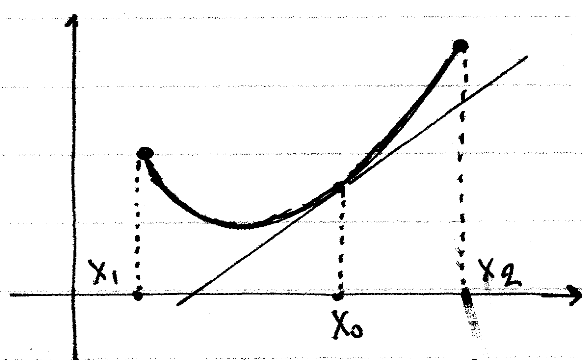
with contact point chosen at $(x_0, f(x_0))$. It follows that:

(a) The graph (c) of the function f is ABOVE the tangent line (ℓ) if and only if
$$\forall x \in [x_1, x_2] - \{x_0\} : g(x|x_0) > 0$$

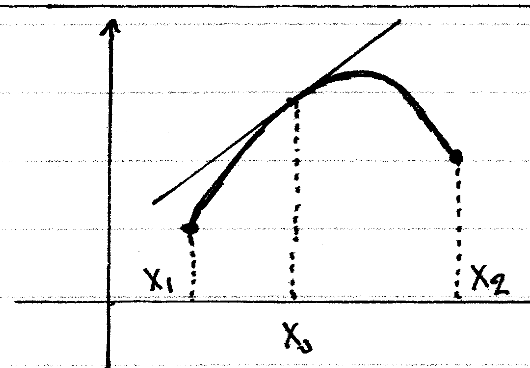
(b) The graph (c) of the function f is BELOW the tangent line (ℓ) if and only if
$$\forall x \in [x_1, x_2] - \{x_0\} : g(x|x_0) < 0$$

The corresponding theorem gives an equivalent characterization of convexity for differentiable functions:

Thm: Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $[x_1, x_2] \subseteq A$. Then
$$f \text{ convex up on } [x_1, x_2] \Leftrightarrow \forall x, x_0 \in [x_1, x_2] : (x \neq x_0 \Rightarrow g(x|x_0) > 0)$$
$$f \text{ convex down on } [x_1, x_2] \Leftrightarrow \forall x, x_0 \in [x_1, x_2] : (x \neq x_0 \Rightarrow g(x|x_0) < 0)$$



f convex up on $[x_1, x_2]$



f convex down on $[x_1, x_2]$.

(3) → Convexity and the 2nd derivative

Thm: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $[x_1, x_2] \subseteq A$.

Then:

$$\left\{\begin{array}{l} f \text{ twice differentiable on } [x_1, x_2] \\ \forall x \in (x_1, x_2): f''(x) > 0 \end{array}\right. \Rightarrow f \text{ convex up on } [x_1, x_2]$$

$$\left\{\begin{array}{l} f \text{ twice differentiable on } [x_1, x_2] \\ \forall x \in (x_1, x_2): f''(x) < 0 \end{array}\right. \Rightarrow f \text{ convex down on } [x_1, x_2]$$

▸ Method for determining convexity

- 1. Calculate and FACTOR $f''(x)$
- 2. Make a sign table for $f''(x)$
- 3. Indicate convexity on table
- 4. Inflection points arises at the points where the convexity changes.

▸ Variation table

The variation table indicates the shape of the curve of the function in more detail.

- 1. We make a monotonicity and convexity table separately.
- 2. We merge the two tables into a table of the form

x	
f'	
f''	
f	

where for f we use the notations: ⤴ ↱ ⤵ ↳
and where we indicate both the local min/max and the inflection points.

EXAMPLE

Determine the monotonicity, convexity, and make a variation table for the function

$$f(x) = \frac{x^3}{x^2-1}$$

Solution

- Domain.

We require $x^2-1 \neq 0$. Note that

$$x^2-1=0 \Leftrightarrow (x-1)(x+1)=0 \Leftrightarrow x-1=0 \lor x+1=0 \Leftrightarrow$$
$$\Leftrightarrow x=1 \lor x=-1 \Leftrightarrow x \in \{-1,1\}$$

and therefore the domain of f is $A = \mathbb{R} - \{-1,1\}$.

- Derivatives.

We have

$$f'(x) = \left[\frac{x^3}{x^2-1}\right]' = \frac{(x^3)'(x^2-1) - x^3(x^2-1)'}{(x^2-1)^2} =$$
$$= \frac{3x^2(x^2-1) - x^3(2x)}{(x-1)^2(x+1)^2} = \frac{3x^4-3x^2-2x^4}{(x-1)^2(x+1)^2} =$$
$$= \frac{x^4-3x^2}{(x-1)^2(x+1)^2} = \frac{x^2(x^2-3)}{(x-1)^2(x+1)^2} =$$
$$= \frac{x^2(x-\sqrt{3})(x+\sqrt{3})}{(x-1)^2(x+1)^2}, \quad \forall x \in \mathbb{R} - \{-1,1\}$$

and

$$f''(x)=\left[\frac{x^4-3x^2}{(x-1)^2(x+1)^2}\right]'=\left[\frac{x^4-3x^2}{(x^2-1)^2}\right]'=$$

$$=\frac{(x^4-3x^2)'(x^2-1)^2-(x^4-3x^2)\left[(x^2-1)^2\right]'}{(x^2-1)^4}$$

$$=\frac{(4x^3-6x)(x^2-1)^2-(x^4-3x^2)\,2\,(x^2-1)(x^2-1)'}{(x^2-1)^4}$$

$$=\frac{(x^2-1)\left[(4x^3-6x)(x^2-1)-2(x^4-3x^2)(2x)\right]}{(x^2-1)^4}$$

$$=\frac{4x^5-4x^3-6x^3+6x-4x^5+12x^3}{(x^2-1)^3}$$

$$=\frac{(4-4)x^5+(-4-6+12)x^3+6x}{(x-1)^3(x+1)^3}$$

$$=\frac{2x^3+6x}{(x-1)^3(x+1)^3}=\frac{2x(x^2+3)}{(x-1)^3(x+1)^3},\ \forall x\in\mathbb{R}-\{1,-1\}$$

- Monotonicity

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
x^2	+		+		+	0	+		+		+
$x-\sqrt{3}$	−		−		−		−		−	0	+
$x+\sqrt{3}$	−	0	+		+		+		+		+
$(x-1)^2$	+		+		+		+	0	+		+
$(x+1)^2$	+		+	0	+		+		+		+
$f'(x)$	+	0	−	ǁ	−	0	−	ǁ	−	0	+
$f(x)$	↗	max	↘ $-\infty$	ǁ	$+\infty$ ↘		↘ $-\infty$	ǁ	$+\infty$ ↘	min	↗

Local max at $x_0 = -\sqrt{3}$

Local min at $x_0 = \sqrt{3}$

- Convexity

x		-1		0		1	
$2x$	$-$		$-$	0	$+$		$+$
x^2+3	$+$		$+$		$+$		$+$
$(x-1)^3$	$-$		$-$		$-$	0	$+$
$(x+1)^3$	$-$	0	$+$		$+$		$+$
$f''(x)$	$-$	$\parallel$	$+$	0	$-$	$\parallel$	$+$
$f(x)$	$\cap$		$\cup$	infl.	$\cap$		$\cup$

- Variation table

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
$f'(x)$	$+$	0	$-$	$\parallel$	$-$	0	$-$	$\parallel$	$-$	0	$+$
$f''(x)$	$-$		$-$	$\parallel$	$+$	0	$-$	$\parallel$	$+$		$+$
$f(x)$	$\nearrow$	max	$\searrow -\infty$	$\parallel$	$+\infty \searrow$	infl.	$\searrow -\infty$	$\parallel$	$+\infty \searrow$	min	$\nearrow$
				vertical asymptote				vertical asymptote			

EXERCISES

(12) Analyze the following functions with respect to monotonicity and convexity and build the variation table. Locate all local min/max and all inflection points.

a) $f(x) = 4x^3 - 8x^2 + 2$

b) $f(x) = x^3 - 2x^2 + x - 5$

c) $f(x) = x^3 - 6x^2 - 15x$

d) $f(x) = x^4 - 6x^2$

e) $f(x) = (x-2)^5 + 3x + 1$

f) $f(x) = \dfrac{x^3}{x^2 - 1}$

g) $f(x) = \dfrac{x^2 - x}{x^2 + 1}$

h) $f(x) = \dfrac{x^3 - 9x}{x^2 - 1}$

(13) Show that the inflection points of the function

$$f(x) = \frac{a - x}{x^2 + a^2}$$

are all on the same line, for $a \neq 0$.

(14) Find all $a \in \mathbb{R}$ such that the line tangent to the graph of the function $f(x) = x^3 - ax^2$ at its inflection point passes through the point $(0,0)$ (i.e. the origin).