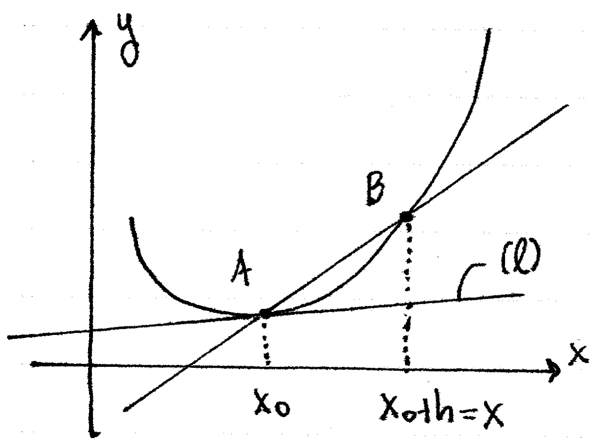
DIFFERENTIAL CALCULUS

Differentiability ~ Tangent line problem



- Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and let $(c): y = f(x)$ be the graph of the function. We assume that f continuous at x_0. Consider the points $A(x_0, f(x_0))$ and $B(x, f(x))$. As B approaches A, the line (AB) approaches a line (ℓ) given by

(ℓ): $y - y_A = a(x - x_A) \Leftrightarrow y - f(x_0) = a(x - x_0)$

with a the slope of the line (ℓ).

- To calculate the slope of (ℓ) we note that the slope of the line (AB) is given by:

$$\lambda(f|x, x_0) = \frac{y_B - y_A}{x_B - x_A} = \frac{f(x) - f(x_0)}{x - x_0}$$

and it follows that

$$a = \lim_{x \to x_0} \lambda(f|x, x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =$$

$$= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

- We will see below that the assumption that f is continuous at x_0 is not sufficient to ensure the existence of the above limit. This motivates the following definition of differentiability:

$$\boxed{f \text{ differentiable at } x_0 \Leftrightarrow \exists\, \ell \in \mathbb{R}: \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \ell}$$

→ <u>Differentiability and continuity</u>

<u>Thm:</u> Let $f: A \to \mathbb{R}$ with $x_0 \in A$. Then:

$$\boxed{f \text{ differentiable at } x_0 \Rightarrow f \text{ continuous at } x_0}$$

<u>Proof</u>

Assume that f differentiable at x_0. Then:

f differentiable at $x_0 \Rightarrow \exists\, \ell \in \mathbb{R}: \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \ell$

and therefore:

$$\lim_{x \to x_0} f(x) = f(x_0) + \lim_{x \to x_0} [f(x) - f(x_0)] =$$

$$= f(x_0) + \lim_{x \to x_0} \left[\frac{f(x)-f(x_0)}{x-x_0} \cdot (x - x_0) \right] =$$

$$= f(x_0) + \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \cdot \lim_{x \to x_0} (x - x_0)$$

$$= f(x_0) + \ell (x_0 - x_0) = f(x_0) + \ell \cdot 0 = f(x_0) \Rightarrow$$

$\Rightarrow f$ continuous at x_0.

- The contrapositive statement reads:
  f NOT continuous at $x_0 \Rightarrow f$ NOT differentiable at x_0.

Corner points

- We say that

$$\boxed{x_0 \text{ corner point of } f \Leftrightarrow \begin{cases} f \text{ continuous at } x_0 \\ f \text{ NOT differentiable at } x_0 \end{cases}}$$

- To show that f NOT differentiable at x_0, we use the negation of the definition of differentiability:

$$\boxed{f \text{ NOT differentiable at } x_0 \Leftrightarrow \forall \ell \in \mathbb{R} : \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \neq \ell}$$

- Corner points can emerge from

a) Sudden change of direction in the graph of the function

b) When the graph of the function becomes momentarily vertical.

EXAMPLE

a) For $f(x)=|x|, \forall x \in \mathbb{R}$ show that $x_0=0$ is a corner point.

Solution

Since

$$f(0)=|0|=0 \qquad (1)$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0 \qquad (2)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0 \qquad (3)$$

it follows that

$$\lim_{x\to 0} f(x) = 0 \qquad [\text{from Eq. (2) and Eq. (3)}]$$

$$= f(0) \qquad [\text{from Eq. (1)}]$$

$\Rightarrow f$ continuous at $x_0=0$. (4)

Furthermore:

$$\lambda(f|x,0) = \frac{f(x)-f(0)}{x-0} = \frac{|x|-|0|}{x-0} = \frac{|x|}{x} =$$

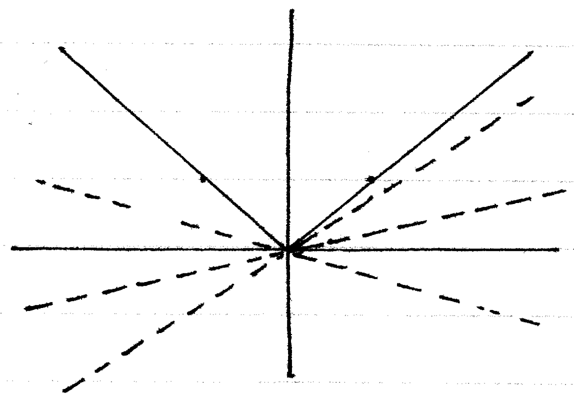
$$= \begin{cases} x/x, & \text{if } x>0 \\ -x/x, & \text{if } x<0 \end{cases} = \begin{cases} 1, & \text{if } x>0 \\ -1, & \text{if } x<0 \end{cases} \Rightarrow$$

$$\Rightarrow \lim_{x\to 0^+} \lambda(f|x,0) = 1 \wedge \lim_{x\to 0^-} \lambda(f|x,0) = -1$$

$$\Rightarrow \lim_{x\to 0} \lambda(f|x,0) \text{ does not exist} \Rightarrow \forall \ell \in \mathbb{R}: \lim \frac{f(x)-f(0)}{x-0} \neq \ell$$

$\Rightarrow f$ not differentiable at $x_0=0$. (5)

From Eq.(4) and Eq.(5): $x_0 = 0$ corner point of f.



From the graph of $f(x) = |x|$, $\forall x \in \mathbb{R}$ we see that the corner point $x_0 = 0$, the function suddenly changes direction. As a result, we cannot draw a unique tangent line at $x_0 = 0$.

b) Show that $f(x) = \sqrt{x}$, $\forall x \in [0, +\infty)$ has a corner point at $x_0 = 0$.

Solution

$$f(0) = \sqrt{0} = 0 \qquad (1)$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \sqrt{x} = \lim_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0 \qquad (2)$$

From Eq.(1) and Eq.(2):

$$\lim_{x \to 0} f(x) = f(0) \Rightarrow f \text{ continuous at } x_0 = 0 \qquad (3)$$

Furthermore:

$$\lambda(f|x,0) = \frac{f(x) - f(0)}{x - 0} = \frac{\sqrt{x} - \sqrt{0}}{x - 0} = \frac{\sqrt{x}}{x} =$$

$$= \frac{\sqrt{x}}{\sqrt{x}\sqrt{x}} = \frac{1}{\sqrt{x}}, \quad \forall x \in (0, +\infty)$$

Since:

$$\begin{cases} \sqrt{x} > 0, \ \forall x \in (0,+\infty) \\ \lim\limits_{x\to 0^+} \sqrt{x} = \sqrt{0} = 0 \end{cases} \Rightarrow \lim_{x\to 0^+} \frac{1}{\sqrt{x}} = +\infty \Rightarrow$$

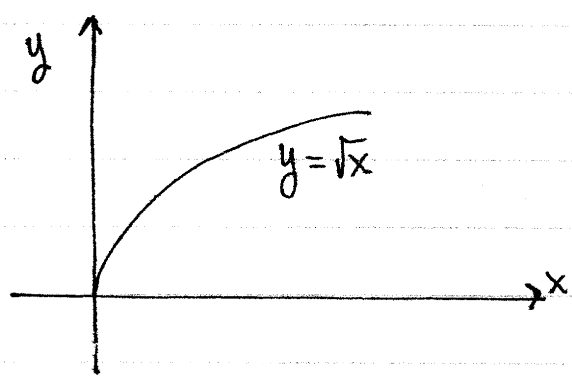
$$\Rightarrow \lim_{x\to 0^+} \lambda(f|x,0) = +\infty$$

$$\Rightarrow \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} \neq \ell, \ \forall \ell \in \mathbb{R}$$

$\Rightarrow f$ not differentiable at $x_0 = 0$. (4)

From Eq. (3) and Eq. (4): $x_0 = 0$ corner point of f.

$\rightarrow$ From the graph of $f(x) = \sqrt{x}$, $\forall x \in [0,+\infty)$ we see that the graph becomes vertical at $x_0 = 0$. This is the second way one may lose differentiability without losing continuity.



c) Consider the function

$$f(x) = \begin{cases} x^3 & , \text{ if } x \in (-\infty, 1] \\ x^2 + x - 1 & , \text{ if } x \in (1, +\infty) \end{cases}$$

Show that f is differentiable at $x_0 = 1$.

<u>Solution</u>

Since:

$$\forall x \in (-\infty, 1) : \lambda(f \mid x, 1) = \frac{f(x) - f(1)}{x - 1} = \frac{x^3 - 1^3}{x - 1}$$

$$= \frac{(x-1)(x^2+x+1)}{x-1} = x^2 + x + 1 \Rightarrow$$

$$\Rightarrow \lim_{x \to 1^-} \lambda(f \mid x, 1) = \lim_{x \to 1^-} (x^2 + x + 1) = 1^2 + 1 + 1 = 3. \qquad (1)$$

$$\forall x \in (1, +\infty) : \lambda(f \mid x, 1) = \frac{f(x) - f(1)}{x - 1} = \frac{(x^2 + x - 1) - 1^3}{x - 1} =$$

$$= \frac{x^2 + x - 2}{x - 1} = \frac{(x-1)(x+2)}{x-1} =$$

$$= x + 2 \Rightarrow$$

$$\Rightarrow \lim_{x \to 1^+} \lambda(f \mid x, 1) = \lim_{x \to 1^+} (x + 2) = 1 + 2 = 3 \qquad (2)$$

From Eq. (1) and Eq. (2):

$$\lim_{x \to 1} \lambda(f \mid x, 1) = 3 \Rightarrow \exists a \in \mathbb{R} : \lim_{x \to 1} \lambda(f \mid x, 1) = a$$

$$\Rightarrow f \text{ differentiable at } x_0 = 1.$$

d) Consider the function

$$f(x) = \begin{cases} x^2 [\sin(\pi/x) + \cos(\pi/x)] &, x \in \mathbb{R} - \{0\} \\ 0 &, x = 0 \end{cases}$$

Show that f is differentiable at $x_0 = 0$

Solution

Let $x \in \mathbb{R} - \{0\}$ be given. Then

$$\lambda(f|x,0) = \frac{f(x) - f(0)}{x - 0} = \frac{x^2[\sin(\pi/x) + \cos(\pi/x)] - 0}{x - 0}$$

$$= x[\sin(\pi/x) + \cos(\pi/x)]$$

Define $b(x) = \sin(\pi/x) + \cos(\pi/x)$, $\forall x \in \mathbb{R} - \{0\}$. Then:

$|b(x)| = |\sin(\pi/x) + \cos(\pi/x)| \leq$

$\leq |\sin(\pi/x)| + |\cos(\pi/x)| \leq$

$\leq 1 + 1 = 2$, $\forall x \in \mathbb{R} - \{0\} \Rightarrow$

$\Rightarrow b$ bounded at $\mathbb{R} - \{0\}$. (1)

Also note that $\lim_{x \to 0} x = 0$ (2)

From Eq. (1) and Eq. (2):

$\lim_{x \to 0} x\,b(x) = 0 \Rightarrow \lim_{x \to 0} \lambda(f|x,0) = 0$

$\Rightarrow \exists a \in \mathbb{R} : \lim_{x \to 0} \lambda(f|x,0) = a$

$\Rightarrow f$ differentiable at $x_0 = 0$.

c) Consider the function

$$f(x)=\begin{cases} x^2+2x & , x\in[0,+\infty) \\ ax+b & , x\in(-\infty,0) \end{cases}$$

Find all $a,b\in\mathbb{R}$ for which f differentiable at $x_0=0$.

<u>Solution</u>

We note that

$$\forall x\in(0,+\infty): \lambda(f\,|\,x,0)=\frac{f(x)-f(0)}{x-0}=\frac{(x^2+2x)-(0^2+2\cdot 0)}{x}$$

$$=\frac{x^2+2x}{x}=\frac{x(x+2)}{x}=x+2$$

$$\forall x\in(-\infty,0): \lambda(f\,|\,x,0)=\frac{f(x)-f(0)}{x-0}=\frac{ax+b-0}{x}=\frac{ax+b}{x}$$

$$\lim_{x\to 0^+}\lambda(f\,|\,x,0)=\lim_{x\to 0^+}(x+2)=0+2=2$$

The limit $\lim_{x\to 0^-}\lambda(f\,|\,x,0)$ may or may not exist depending on whether $b=0$ or $b\neq 0$, so we leverage continuity but must do, as a result, a split argument:

($\Rightarrow$): Assume that f differentiable at $x_0=0$. Since:

$$f(0)=0^2+2\cdot 0=0$$

$$\lim_{x\to 0^+}f(x)=0^2+2\cdot 0=0$$

$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}(ax+b)=a\cdot 0+b=b$$

it follows that:

f differentiable at $x_0=0 \Rightarrow f$ continuous at $x_0=0$

$$\Rightarrow \lim_{x\to 0}f(x)=f(0)$$

$$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) \Rightarrow \underline{b=0}$$

For $b=0$:

$$\lim_{x\to 0^-} \lambda(f(x,0)) = \lim_{x\to 0^-} \frac{ax+0}{x} = \lim_{x\to 0^-} a = a$$

and therefore:

f differentiable at $x_0=0 \Rightarrow \exists\, \ell \in \mathbb{R} : \lim_{x\to 0} \lambda(f(x,0)) = \ell$

$$\Rightarrow \lim_{x\to 0^-} \lambda(f(x,0)) = \lim_{x\to 0^+} \lambda(f(x,0))$$

$$\Rightarrow a = 2.$$

We have thus shown that

f differentiable at $x_0=0 \Rightarrow (a=2 \wedge b=0)$

$(\Leftarrow)$: Assume that $a=2 \wedge b=0$. Then:

$$a=2\wedge b=0 \Rightarrow \forall x \in (-\infty,0) : \lambda(f(x,0)) = \frac{2x+0}{x} = \frac{2x}{x} = 2$$

$$\Rightarrow \lim_{x\to 0^-} \lambda(f(x,0)) = 2 = \lim_{x\to 0^+} \lambda(f(x,0))$$

$$\Rightarrow \lim_{x\to 0} \lambda(f(x,0)) = 2 \Rightarrow \exists\, \ell \in \mathbb{R} : \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow f \text{ differentiable at } x_0=0.$$

We have thus shown that:

f differentiable at $x_0=0 \Leftrightarrow a=2 \wedge b=0$. $\square$

↳ Note that a direct argument of the form

f differentiable at $x_0=0 \Leftrightarrow \cdots \Leftrightarrow \cdots \Leftrightarrow$

$\Leftrightarrow a=2 \wedge b=0$

is not possible if we wish to use continuity.

Consequently the forward ($\Rightarrow$) and backward ($\Leftarrow$) arguments need to be done separately.

EXERCISES

(1) Show that the function

$$f(x)=\begin{cases} x^2+4x, & x\in[0,+\infty) \\ x^2-4x, & x\in(-\infty,0) \end{cases}$$

is continuous on $\mathbb{R}$ but not differentiable at $x_0=0$.

(2) Show that the function

$$f(x)=(x+|x|)^2, \quad \forall x\in\mathbb{R}$$

is continuous and differentiable at $x_0=0$

(3) Define the function

$$f(x)=\begin{cases} x\sin(2x)\cos(\pi/x)[1+\sin(\pi/x)], & \text{if } x\in\mathbb{R}-\{0\} \\ 0, & \text{if } x=0. \end{cases}$$

Show that f is differentiable at $x_0=0$.

(4) Let $f: A\to\mathbb{R}$ be a function and define

$\forall x\in A: g(x)=xf(x)$

Show that: f continuous at $x=0 \Rightarrow g$ differentiable at $x=0$.

(5) Find all $a,b\in\mathbb{R}$ such that the following functions are differentiable at x_0

a) $$f(x)=\begin{cases} ax+b, & x\in(-\infty,3) \\ x^2, & x\in[3,+\infty) \end{cases} \quad \text{at } x_0=3$$

b) $f(x) = \begin{cases} ax^2 + 2bx & , \text{ if } x \in [1, +\infty) \\ bx - a & , \text{ if } x \in (-\infty, 1) \end{cases}$ at $x_0 = 1$.

6) Let $f: A \to \mathbb{R}$ be a function and define $g: A \to \mathbb{R}$ such that

$\forall x \in A: g(x) = |f(x)|$

Show that:

$\begin{cases} f \text{ differentiable at } x_0 \in A \\ f(x_0) \neq 0 \end{cases} \Rightarrow g$ differentiable at x_0.

(Hint: We write:

$$\lambda(g|x, x_0) = \ldots = \frac{(|f(x)| - |f(x_0)|)(|f(x)| + |f(x_0)|)}{(x - x_0)(|f(x)| + |f(x_0)|)}$$

$= \ldots$

use $|x|^2 = x^2$ and take it from there)

Derivative function

- Let $f: A \to \mathbb{R}$ be a function and let $S \subseteq A$. We say that
  f differentiable at $S \Leftrightarrow \forall x_0 \in S: f$ differentiable at x_0
- If $f: A \to \mathbb{R}$ is differentiable at S, then we define the derivative function $f': S \to \mathbb{R}$ as:
  $$\forall x_0 \in S: f'(x_0) = \lim_{x \to x_0} \lambda(f \mid x, x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$
- The notation $f'(x)$ is attributed to Newton. The Leibnitz notation of the derivative is:
  $$\frac{df}{dx} = f' \quad \text{and} \quad \left.\frac{df}{dx}\right|_{x = x_0} = f'(x_0)$$
- If f' is also differentiable at S, then the derivative of f' is denoted as f'' and is called the <u>2nd derivative</u> of f. Likewise we define
  $$f'' = \frac{df'}{dx} = \frac{d^2 f}{dx^2}$$
  $$f''' = \frac{df''}{dx} = \frac{d^3 f}{dx^3}$$

  Beyond the 3rd derivative, we use the notation $f^{(4)}, f^{(5)}, \dots, f^{(n)}$ and write:
  $$f^{(n)} = \frac{df^{(n-1)}}{dx} = \frac{d^n f}{dx^n}$$

- If we can define $f^{(n)}$ at x_0 we say that f is n-times differentiable at x_0. Likewise, for $S \subseteq A$, we say that f n-times differentiable at $S \Leftrightarrow \forall x_0 \in S : f$ n-times differentiable at x_0.

Derivatives of basic functions

① $$\boxed{f(x) = ax + b, \forall x \in \mathbb{R} \Rightarrow f'(x) = a, \forall x \in \mathbb{R}}$$

Proof

Since

$$\forall x, x_0 \in \mathbb{R}: \lambda(f \mid x, x_0) = \frac{f(x) - f(x_0)}{x - x_0} = \frac{(ax+b) - (ax_0+b)}{x - x_0}$$

$$= \frac{ax - ax_0}{x - x_0} = \frac{a(x - x_0)}{x - x_0} = a \Rightarrow$$

$$\Rightarrow \forall x_0 \in \mathbb{R}: f'(x_0) = \lim_{x \to x_0} \lambda(f \mid x, x_0) = a. \qquad \square$$

For the next result we use the identity

$$\boxed{\forall a, b \in \mathbb{R} : \forall n \in \mathbb{N} - \{0\}: a^n - b^n = (a-b) \sum_{k=0}^{n-1} (a^{n-k-1} b^k)}$$

Note that:

$n=2$: $a^2 - b^2 = (a-b)(a+b)$

$n=3$: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$n=4$: $a^4 - b^4 = (a-b)(a^3 + a^2 b + ab^2 + b^3)$

Proof

$$(a-b)\sum_{k=0}^{n-1} a^{n-k-1} b^k = \sum_{k=0}^{n-1} (a-b) a^{n-k-1} b^k =$$

$$= \sum_{k=0}^{n-1} (a^{n-k} b^k - a^{n-k-1} b^{k+1}) =$$

$$= \sum_{k=0}^{n-1} a^{n-k} b^k - \sum_{k=0}^{n-1} a^{n-k-1} b^{k+1} =$$

$$= a^n + \sum_{k=1}^{n-1} a^{n-k} b^k - \sum_{k=0}^{n-2} a^{n-k-1} b^{k+1} - a^{n-(n-1)-1} b^{(n-1)+1}$$

$$= a^n + \sum_{k=1}^{n-1} a^{n-k} b^k - \sum_{k=1}^{n-1} a^{n-k} b^k - b^n =$$

$$= a^n - b^n \qquad \square$$

(2) $\boxed{f(x) = ax^n, \ \forall x \in \mathbb{R} \Rightarrow f'(x) = nax^{n-1}, \ \forall x \in \mathbb{R}}$

Proof

Since:

$$\lambda(f \mid x, x_0) = \frac{f(x) - f(x_0)}{x - x_0} = \frac{ax^n - ax_0^n}{x - x_0} = \frac{a(x^n - x_0^n)}{x - x_0} =$$

$$= \frac{a(x - x_0) \sum_{k=0}^{n-1} x^{n-k-1} x_0^k}{x - x_0} =$$

$$= a \sum_{k=0}^{n-1} x^{n-k-1} x_0^k \Longrightarrow$$

$$\Rightarrow f'(x_0) = \lim_{x \to x_0} \lambda(f|x,x_0) = \lim_{x \to x_0} \left[a \sum_{k=0}^{n-1} x^{n-k-1} x_0^k \right] =$$

$$= a \lim_{x \to x_0} \sum_{k=0}^{n-1} x^{n-k-1} x_0^k = a \sum_{k=0}^{n-1} \lim_{x \to x_0} (x^{n-k-1} x_0^k)$$

$$= a \sum_{k=0}^{n-1} x_0^{n-k-1} x_0^k = a \sum_{k=0}^{n-1} x_0^{n-1} = a n x_0^{n-1} =$$

$$= n a x_0^{n-1}, \quad \forall x_0 \in \mathbb{R}. \qquad \square$$

(3) $\boxed{f(x) = \sqrt{x}, \ \forall x \in [0, +\infty) \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}, \ \forall x \in (0, +\infty)}$

Proof

$$\forall x, x_0 \in [0, +\infty) : \lambda(f|x,x_0) = \frac{f(x) - f(x_0)}{x - x_0} = \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} =$$

$$= \frac{\sqrt{x} - \sqrt{x_0}}{(\sqrt{x})^2 - (\sqrt{x_0})^2} = \frac{\sqrt{x} - \sqrt{x_0}}{(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})} = \frac{1}{\sqrt{x} + \sqrt{x_0}} \Rightarrow$$

$$\Rightarrow \forall x_0 \in (0, +\infty) : f'(x_0) = \lim_{x \to x_0} \lambda(f|x,x_0) = \lim_{x \to x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}} =$$

$$= \frac{1}{\sqrt{x_0} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}. \qquad \square$$

↳ Note that, as was shown previously, although the function $f(x) = \sqrt{x}$ is defined at $x = 0$, it is <u>not</u> differentiable at $x = 0$.

Basic differentiation rules

Let f, g be functions differentiable at a set $A \subseteq \mathbb{R}$ and let $a \in \mathbb{R}$. Then:

$$\boxed{\begin{array}{l} h(x) = f(x)+g(x), \forall x \in A \Rightarrow h'(x) = f'(x)+g'(x), \forall x \in A \\ h(x) = af(x), \forall x \in A \Rightarrow h'(x) = af'(x), \forall x \in A \\ h(x) = f(x)g(x), \forall x \in A \Rightarrow h'(x) = f'(x)g(x)+f(x)g'(x), \forall x \in A \end{array}}$$

Proof

a) Assume that $h(x) = f(x)+g(x), \forall x \in A$. Then

$$\forall x, x_0 \in A : \lambda(h|x,x_0) = \frac{h(x)-h(x_0)}{x-x_0} =$$

$$= \frac{[f(x)+g(x)]-[f(x_0)+g(x_0)]}{x-x_0} =$$

$$= \frac{[f(x)-f(x_0)]+[g(x)-g(x_0)]}{x-x_0} =$$

$$= \frac{f(x)-f(x_0)}{x-x_0} + \frac{g(x)-g(x_0)}{x-x_0} = \lambda(f|x,x_0)+\lambda(g|x,x_0)$$

$$\Rightarrow \forall x_0 \in A : h'(x_0) = \lim_{x \to x_0} \lambda(h|x,x_0) =$$

$$= \lim_{x \to x_0} [\lambda(f|x,x_0)+\lambda(g|x,x_0)]$$

$$= \lim_{x \to x_0} \lambda(f|x,x_0) + \lim_{x \to x_0} \lambda(g|x,x_0)$$

$$= f'(x_0)+g'(x_0).$$

b) Assume that $h(x) = af(x)$, $\forall x \in A$. Then

$$\forall x, x_0 \in A: \ \lambda(h|x,x_0) = \frac{h(x)-h(x_0)}{x-x_0} = \frac{af(x)-af(x_0)}{x-x_0} =$$

$$= \frac{a[f(x)-f(x_0)]}{x-x_0} = a\lambda(f|x,x_0) \Rightarrow$$

$$\Rightarrow \forall x_0 \in A: \ h'(x_0) = \lim_{x\to x_0} \lambda(h|x,x_0) = \lim_{x\to x_0}[a\lambda(f|x,x_0)]$$

$$= a\lim_{x\to x_0}\lambda(f|x,x_0) = af'(x_0)$$

c) Assume that $h(x) = f(x)g(x)$, $\forall x \in A$. Then

$$\forall x, x_0 \in A: \ \lambda(h|x,x_0) = \frac{h(x)-h(x_0)}{x-x_0} = \frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} =$$

$$= \frac{f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0} =$$

$$= \frac{g(x)[f(x)-f(x_0)]+f(x_0)[g(x)-g(x_0)]}{x-x_0} =$$

$$= f(x_0)\frac{g(x)-g(x_0)}{x-x_0} + g(x)\frac{f(x)-f(x_0)}{x-x_0} =$$

$$= f(x_0)\lambda(g|x,x_0) + \lambda(f|x,x_0)g(x)$$

We note that:

g differentiable at $x_0 \Rightarrow g$ continuous at x_0

$$\Rightarrow \lim_{x\to x_0} g(x) = g(x_0)$$

and therefore:

$$\forall x_0 \in A: h'(x_0) = \lim_{x \to x_0} \lambda(h|x, x_0) =$$

$$= \lim_{x \to x_0} [f(x_0)\lambda(g|x, x_0) + \lambda(f|x, x_0)g(x)]$$

$$= f(x_0) \lim_{x \to x_0} \lambda(g|x, x_0) + \lim_{x \to x_0} \lambda(f|x, x_0) \lim_{x \to x_0} g(x)$$

$$= f(x_0)g'(x_0) + f'(x_0)g(x_0)$$

$$= f'(x_0)g(x_0) + f(x_0)g'(x_0). \qquad \square$$

EXAMPLES

a) Evaluate f' and f'' for the function
$f(x) = x^4 + 2x^3 + 5x^2$, $\forall x \in \mathbb{R}$.

Solution

$$\begin{aligned}
f'(x) &= (x^4 + 2x^3 + 5x^2)' \\
&= (x^4)' + (2x^3)' + (5x^2)' && [\text{addition rule}] \\
&= 4x^3 + 2(3x^2) + 5(2x) && [\,(x^n)' = nx^{n-1}\,] \\
&= 4x^3 + 6x^2 + 10x, && \forall x \in \mathbb{R}
\end{aligned}$$

and

$$\begin{aligned}
f''(x) &= (4x^3 + 6x^2 + 10x)' \\
&= (4x^3)' + (6x^2)' + (10x)' \\
&= 4(3x^2) + 6(2x) + 10 \\
&= 12x^2 + 12x + 10, \quad \forall x \in \mathbb{R}.
\end{aligned}$$

↦ In evaluating derivatives of polynomials it is ok to skip steps, including the addition rule, and evaluate the derivative quickly. However, you should NEVER skip the application of the product rule.

↦ Tangent line : The tangent line (ℓ) to the graph $(c): y = f(x)$ of a function f at $x = x_0$ is given by $(\ell): y - f(x_0) = f'(x_0)(x - x_0)$

b) Find the tangent line to the graph of

$$f(x) = (x^2+2x)\sqrt{x} \quad , \forall x \in [0,+\infty)$$

at $x_0 = 2$.

Solution

Note that

$$f'(x) = [(x^2+2x)\sqrt{x}\,]' =$$

$$= (x^2+2x)'\sqrt{x} + (x^2+2x)(\sqrt{x})' =$$

$$= (2x+2)\sqrt{x} + (x^2+2x)\left(\frac{1}{2\sqrt{x}}\right) =$$

$$= \frac{(2x+2)\sqrt{x}\,(2\sqrt{x}) + (x^2+2x)}{2\sqrt{x}} = \frac{2x(2x+2)+(x^2+2x)}{2\sqrt{x}}$$

$$= \frac{4x^2+4x+x^2+2x}{2\sqrt{x}} = \frac{5x^2+6x}{2\sqrt{x}} = \frac{x(5x+6)}{2\sqrt{x}} =$$

$$= \frac{1}{2}\sqrt{x}\,(5x+6) \quad , \forall x \in (0,+\infty)$$

and therefore

$$f(2) = (2^2+2\cdot 2)\sqrt{2} = (4+4)\sqrt{2} = 8\sqrt{2}$$

$$f'(2) = \frac{1}{2}\sqrt{2}\,(5\cdot 2+6) = \frac{\sqrt{2}}{2}\cdot 16 = 8\sqrt{2}$$

It follows that

$$(\ell): y - f(2) = f'(2)(x-2) \Leftrightarrow y - 8\sqrt{2} = 8\sqrt{2}\,(x-2) \Leftrightarrow$$

$$\Leftrightarrow y - 8\sqrt{2} = 8\sqrt{2}\,x - (8\sqrt{2})\cdot 2 \Leftrightarrow 8\sqrt{2}\,x - y - 8\sqrt{2} = 0$$

and therefore

$$(\ell): 8\sqrt{2}\,x - y - 8\sqrt{2} = 0.$$

Derivatives of multiple formula functions

If a function f follows a given formula at a closed interval $[a,b]$, then the corresponding side limit of $\lambda(f|x,x_0)$ with $x \to x_0^+$ or $x \to x_0^-$ at $x_0 = a$ and $x_0 = b$ can be evaluated directly from the differentiation rules. The same is true for intervals of the form $[a,+\infty)$ or $(-\infty,a]$ at $x_0 = a$, and also for points in the interior of the interval. HOWEVER, at the boundary points $x_0 = a$ or $x_0 = b$ of OPEN intervals of the form (a,b) or $(a,+\infty)$ or $(-\infty,a)$, etc., we have to use the limit definition directly.

c) Evaluate the derivative of the function

$$f(x) = \begin{cases} 3x^2+x & , \text{ if } x \in (-\infty,1] \\ 2(x-1)^2+3 & , \text{ if } x \in (1,+\infty). \end{cases}$$

<u>Solution</u>

- We note that

$$\forall x \in (-\infty,1): f(x) = 3x^2+x \Rightarrow \forall x \in (-\infty,1): f'(x) = 6x+1$$

$$\forall x \in (1,+\infty): f(x) = 2(x-1)^2+3 = 2(x^2-2x+1)+3 =$$
$$= 2x^2-4x+2+3 = 2x^2-4x+5 \Rightarrow$$
$$\Rightarrow \forall x \in (1,+\infty): f'(x) = 4x-4$$

- At $x=1$:

$$\forall x \in (-\infty,1]: f(x) = 3x^2+x \Rightarrow$$
$$\Rightarrow \lim_{x \to 1^-} \lambda(f|x,1) = \frac{d}{dx}(3x^2+x)\Big|_{x=1} = (6x+1)\Big|_{x=1} =$$

$$= 6\cdot 1+1 = 7. \qquad (1)$$

and

$$\forall x\in(1,+\infty): \lambda(f|x,1) = \frac{f(x)-f(1)}{x-1} = \frac{[2(x-1)^2+3]-(3\cdot 1^2+1)}{x-1}$$

$$= \frac{2(x-1)^2+3-4}{x-1} = \frac{2(x-1)^2-1}{x-1} =$$

$$= 2(x-1)+\frac{-1}{x-1}.$$

Since
$$\left.\begin{array}{l}\lim\limits_{x\to 1^+} 2(x-1) = 2(1-1) = 0 \\ \lim\limits_{x\to 1^+} \dfrac{-1}{x-1} = -\infty\end{array}\right\} \Rightarrow \lim_{x\to 1^+} \lambda(f|x,1) = -\infty \qquad (2)$$

From Eq.(1) and Eq.(2):

$$\lim_{x\to 1^-} \lambda(f|x,1) \neq \lim_{x\to 1^+} \lambda(f|x,1) \Rightarrow$$

$$\Rightarrow \lim_{x\to 1} \lambda(f|x,1) \text{ does not exist } \Rightarrow$$

$$\Rightarrow \forall \ell\in\mathbb{R}: \lim_{x\to 1} \lambda(f|x,1) \neq \ell \Rightarrow$$

$\Rightarrow f$ not differentiable at $x=1$.

It follows that

$$f'(x) = \begin{cases} 6x+1, & x\in(-\infty,1) \\ 4x-4, & x\in(1,+\infty)\end{cases}$$

↳ Sometimes, but not always, lack of continuity can be used to deduce lack of differentiability

2nd method

Since

$$\left.\begin{array}{l}\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} [2(x-1)^2+3] = 2(1-1)^2+3 = 3 \\ \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (3x^2+x) = 3\cdot 1^2+1 = 3+1 = 4\end{array}\right\} \Rightarrow$$

$\Rightarrow \lim\limits_{x\to 1^+} f(x) \neq \lim\limits_{x\to 1^-} f(x) \Rightarrow \lim\limits_{x\to 1} f(x)$ does not exist $\Rightarrow$

$\Rightarrow f$ not continuous at $x=1 \Rightarrow$

$\Rightarrow f$ not differentiable at $x=1$.

d) Evaluate the derivative of the function

$$f(x) = \begin{cases} x^2 - x - 1 & , x \in (1, +\infty) \\ x - 2 & , x \in (-\infty, 1] \end{cases}$$

Solution

Since

$\forall x \in (1, +\infty): f(x) = x^2 - x - 1 \Rightarrow \forall x \in (1, +\infty): f'(x) = 2x - 1$

$\forall x \in (-\infty, 1): f(x) = x - 2 \Rightarrow \forall x \in (-\infty, 1): f'(x) = 1$

At $x = 1$:

$\forall x \in (-\infty, 1]: f(x) = x - 2 \Rightarrow$

$$\Rightarrow \lim_{x \to 1^-} \lambda(f|x,1) = \frac{d}{dx}(x-2)\Big|_{x=1} = 1\Big|_{x=1} = 1 \qquad (1)$$

and

$$\forall x \in (1, +\infty): \lambda(f|x,1) = \frac{f(x) - f(1)}{x-1} = \frac{(x^2 - x - 1) - (1-2)}{x-1} =$$

$$= \frac{x^2 - x - 1 + 1}{x-1} = \frac{x^2 - x}{x-1} = \frac{x(x-1)}{x-1} = x \Rightarrow$$

$$\Rightarrow \lim_{x \to 1^+} \lambda(f|x,1) = \lim_{x \to 1^+} x = 1 \qquad (2)$$

From Eq. (1) and Eq. (2):

$$f'(1) = \lim_{x \to 1} \lambda(f|x,1) = 1$$

and it follows that:

$$f'(x) = \begin{cases} 2x - 1 & , \text{if } x \in (1, +\infty) \\ 1 & , \text{if } x = 1 \\ 1 & , \text{if } x \in (-\infty, 1) \end{cases} = \begin{cases} 2x - 1 & , x \in (1, +\infty) \\ 1 & , x \in (-\infty, 1] \end{cases}$$

EXERCISES

(7) Find the derivatives of the following functions and show the implications below.

a) $f(x) = (x+1)(x^2-x+1) \Rightarrow f'(x) = 3x^2$

b) $f(x) = (x^2+1)(2x+1)(2x+3) \Rightarrow f'(x) = 2(8x^3+12x^2+7x+4)$

c) $f(x) = (2x+1)\sqrt{x} \Rightarrow f'(x) = \dfrac{6x+1}{2\sqrt{x}}$

d) $f(x) = (3x+2)^2\sqrt{x} \Rightarrow f'(x) = \dfrac{(3x+2)(15x+2)}{2\sqrt{x}}$

e) $f(x) = (x^2+5x+6)\sqrt{x} \Rightarrow f'(x) = \dfrac{5x^2+15x+6}{2\sqrt{x}}$

(8) Find the tangent line $(\ell): y = ax+b$ to the graph $(C): y = f(x)$ at $x = x_0$ for the following choices of f and x_0:

a) $f(x) = x^2+3x+1$ at $x_0 = \sqrt{2}+\sqrt{3}$

b) $f(x) = x^3+2x^2+2x+1$ at $x_0 = 1-\sqrt{3}$

c) $f(x) = x^5+4x^3$ at $x_0 = 3\sqrt{2}+2\sqrt{3}$

d) $f(x) = (x^2+3x)\sqrt{x}$ at $x_0 = 2$

(9) Show the following more generalized implications.

a) $f(x) = (ax^2+bx+c)\sqrt{x} \Rightarrow f'(x) = \dfrac{5ax^2+3bx+c}{2\sqrt{x}}$

b) $f(x) = (ax+b)^2\sqrt{x} \Rightarrow f'(x) = \dfrac{(ax+b)(5ax+b)}{2\sqrt{x}}$

(10) Consider the functions

$f(x) = x^3 + ax^2 + bx + 1, \ \forall x \in \mathbb{R}$

$g(x) = x^2 + x + 2, \ \forall x \in \mathbb{R}$

Find all $a, b \in \mathbb{R}$ such that the graphs $(C_1): y = f(x)$ and $(C_2): y = g(x)$ have the same tangent line at $x_0 = 1$

(11) Find all $a \in \mathbb{R}$ such that the line $(\ell): y = 2x + 1$ is the tangent line to $(C): y = f(x)$ with

$f(x) = x^2 + 2ax + 1, \ \forall x \in \mathbb{R}$

at $x = x_0$ and find the corresponding contact point x_0.

(12) Define the widest possible subset of $\mathbb{R}$ over which the following functions are differentiable and define the derivative $f'(x)$:

a) $f(x) = \begin{cases} 2x - 1 & , x \in [1, +\infty) \\ x^2 & , x \in (-\infty, 1) \end{cases}$

b) $f(x) = \begin{cases} x^2 + 3x & , x \in (1, +\infty) \\ (x-1)^2 + 4 & , x \in (-\infty, 1] \end{cases}$

c) $f(x) = |x^2 + 3x|, \ \forall x \in \mathbb{R}$

(Hint: rewrite as a multiple formula function)

(13) Let $f: A \to \mathbb{R}$ be a function that is differentiable at $x_0 = a$. Show that

$$\lim_{x \to a} \frac{x f(a) - a f(x)}{x - a} = f(a) - a f'(a)$$

(14) Let $f: \mathbb{R} \to \mathbb{R}$ be a function differentiable at $x_0 = 0$ with $f(0) = 0$ and $f'(0) = 0$. Show that

$$g(x) = \begin{cases} f(x)\cos(1/x) & , x \in \mathbb{R} - \{0\} \\ 0 & , x = 0 \end{cases}$$

is differentiable at $x_0 = 0$

(15) Let $f: A \to \mathbb{R}$ be a function with $x_0 = 0 \in A$. Show that

$$\left.\begin{array}{l} f \text{ differentiable at } x_0 = 0 \\ \lim\limits_{x \to 0} \dfrac{f(x)}{x} = 1 \end{array}\right\} \Rightarrow f'(0) = 1$$

(Hint: You need to show $f(0) = 0$ using the implied continuity at $x_0 = 0$ first).

Chain rule

- The chain rule is a superrule that is used to generate differentiation rules that are then used in problems. We seldomly use the chain rule directly.
- Recall the definition of function composition:
  For $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$, we define $f \circ g: G \to \mathbb{R}$ with

$$\begin{cases} \text{dom}(f \circ g) = \{x \in \text{dom}(g) \mid g(x) \in \text{dom}(f)\} \\ \qquad\qquad\quad = \{x \in B \mid g(x) \in A\} = G \\ \forall x \in G: (f \circ g)(x) = f(g(x)) \end{cases}$$

  Note that by definition, the belonging condition for $\text{dom}(f \circ g)$ is:

$$x \in \text{dom}(f \circ g) \Longleftrightarrow \begin{cases} x \in \text{dom}(g) \\ g(x) \in \text{dom}(f) \end{cases}$$

- The chain rule claims that:

$$\boxed{\begin{cases} g \text{ differentiable at } x_0 \\ f \text{ differentiable at } g(x_0) \end{cases} \Longrightarrow \begin{cases} f \circ g \text{ differentiable at } x_0 \\ (f \circ g)'(x_0) = f'(g(x_0))\, g'(x_0) \end{cases}}$$

We postpone the proof. Every choice of f generates a new generalized differentiation rule. For example:

1) For $f(x) = x^n$ with $n \in \mathbb{N}^*$, using

$$(x^n)' = nx^{n-1}$$

we obtain:

$$\boxed{([g(x)]^n)' = n\,[g(x)]^{n-1}\, g'(x)}$$

2) For $f(x)=\sqrt{x}$, using $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$, we obtain:

$$\boxed{\left(\sqrt{g(x)}\right)' = \frac{g'(x)}{2\sqrt{g(x)}}}$$

↳ Note that for each generalization, starting from the initial differentiation rule:

(a) All x are replaced with $g(x)$

(b) The entire result is then multiplied with $g'(x)$.

Step (a) corresponds to the $f'(g(x_0))$ factor

Step (b) corresponds to the $g'(x_0)$ factor.

We see therefore that every basic differentiation rule can give a more powerful generalized differentiation rule via the chain rule.

EXAMPLES

a) Find and factor the derivative of
$f(x) = (x^3+2x^2+x+2)^3, \ \forall x \in \mathbb{R}.$

Solution

$$
\begin{aligned}
f'(x) &= \left[(x^3+2x^2+x+2)^3\right]' = \\
&= 3(x^3+2x^2+x+2)^2 (x^3+2x^2+x+2)' \\
&= 3(x^3+2x^2+x+2)^2 (3x^2+4x+1) \\
&= 3\left[x^2(x+2)+(x+2)\right]^2 (3x^2+4x+1) \\
&= 3(x^2+1)^2(x+2)^2(3x^2+4x+1), \ \forall x \in \mathbb{R}
\end{aligned}
$$

b) Find and factor the derivative of
$f(x) = (x^2-2)^3(3x+2)^4, \ \forall x \in \mathbb{R}.$

Solution

$$
\begin{aligned}
f'(x) &= \left[(x^2-2)^3(3x+2)^4\right]' = \\
&= \left[(x^2-2)^3\right]'(3x+2)^4 + (x^2-2)^3\left[(3x+2)^4\right]' \\
&= 3(x^2-2)^2(x^2-2)'(3x+2)^4 + (x^2-2)^3 \cdot 4(3x+2)^3(3x+2)' \\
&= 3(x^2-2)^2(2x)(3x+2)^4 + (x^2-2)^3 \cdot 4(3x+2)^3 \cdot 3 \\
&= 3 \cdot 2 \cdot (x^2-2)^2(3x+2)^3\left[(x)(3x+2)+2(x^2-2)\right] \\
&= 6(x^2-2)^2(3x+2)^3(3x^2+2x+2x^2-4) \\
&= 6(x^2-2)^2(3x+2)^3(5x^2+2x-4), \ \forall x \in \mathbb{R}.
\end{aligned}
$$

c) Find and factor the derivative of

$f(x) = (3x-5)^2\sqrt{x^2-2x}$

Solution

$$f'(x) = \left[(3x-5)^2\sqrt{x^2-2x}\,\right]' =$$

$$= \left[(3x-5)^2\right]'\sqrt{x^2-2x} + (3x-5)^2\left(\sqrt{x^2-2x}\right)' =$$

$$= 2(3x-5)(3x-5)'\sqrt{x^2-2x} + (3x-5)^2\,\frac{(x^2-2x)'}{2\sqrt{x^2-2x}} =$$

$$= 2(3x-5)\cdot 3\sqrt{x^2-2x} + (3x-5)^2\,\frac{2x-2}{2\sqrt{x^2-2x}} =$$

$$= \frac{2(3x-5)\cdot 3\sqrt{x^2-2x}\cdot 2\sqrt{x^2-2x} + (3x-5)^2(2x-2)}{2\sqrt{x^2-2x}} =$$

$$= \frac{12(3x-5)(x^2-2x) + (3x-5)^2(2x-2)}{2\sqrt{x^2-2x}} =$$

$$= \frac{(3x-5)\left[12(x^2-2x) + (3x-5)(2x-2)\right]}{2\sqrt{x^2-2x}} =$$

$$= \frac{(3x-5)(12x^2-24x+6x^2-6x-10x+10)}{2\sqrt{x^2-2x}} =$$

$$= \frac{(3x-5)((12+6)x^2+(-24-6-10)x+10)}{2\sqrt{x^2-2x}}$$

$$= \frac{(3x-5)(18x^2-40x+10)}{2\sqrt{x^2-2x}} = \frac{(3x-5)\cdot 2(9x^2-20x+5)}{2\sqrt{x^2-2x}}$$

$$= \frac{(3x-5)(9x^2-20x+5)}{\sqrt{x^2-2x}}$$

Proof of chain rule

Assume that g differentiable at x_0 and f differentiable at $g(x_0)$. It follows that $f\circ g$ can be defined on a neighborhood $N(x_0,\delta)$ for some $\delta > 0$.
We define $y_0 = g(x_0)$ and

$$F(y) = \begin{cases} \lambda(f \mid y, y_0), & \text{if } y \neq y_0 \\ f'(y_0), & \text{if } y = y_0 \end{cases}$$

We claim that $\lambda(f\circ g \mid x, x_0) = F(g(x))\,\lambda(g \mid x, x_0), \ \forall x \in N(x_0,\delta)$ (1)
To show the claim, let $x \in N(x_0,\delta)$ be given. We distinguish between the following cases:

Case 1: If $g(x) \neq g(x_0)$ then:

$$\lambda(f\circ g \mid x, x_0) = \frac{(f\circ g)(x) - (f\circ g)(x_0)}{x - x_0} = \frac{f(g(x)) - f(g(x_0))}{x - x_0} =$$

$$= \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \, \frac{g(x) - g(x_0)}{x - x_0} =$$

$$= \lambda(f \mid g(x), y_0)\, \lambda(g \mid x, x_0) = F(g(x))\, \lambda(g \mid x, x_0)$$

Case 2: If $g(x) = g(x_0)$, then:

$$\lambda(f\circ g \mid x, x_0) = \frac{(f\circ g)(x) - (f\circ g)(x_0)}{x - x_0} = \frac{f(g(x)) - f(g(x_0))}{x - x_0}$$

$$= \frac{f(g(x_0)) - f(g(x_0))}{x - x_0} = 0$$

and

$$\lambda(g|x,x_0) = \frac{g(x)-g(x_0)}{x-x_0} = \frac{g(x_0)-g(x_0)}{x-x_0} = 0$$

and therefore $\lambda(f\circ g|x,x_0) = F(g(x))\lambda(g|x,x_0)$ holds trivially since both sides are zero.

This proves the claim.

Now, we note that

g differentiable at $x_0 \Rightarrow g$ continuous at $x_0 \Rightarrow$

$$\Rightarrow \lim_{x\to x_0} g(x) = g(x_0) = y_0 \qquad (2)$$

and

$$\begin{aligned}
\lim_{y\to y_0} F(y) &= \lim_{y\to y_0} \lambda(f|y,y_0) = && [\text{def of } F(y)] \\
&= f'(y_0) = && [f \text{ differentiable at } y_0] \\
&= F(y_0) \Rightarrow && [\text{def of } F(y)]
\end{aligned}$$

$\Rightarrow F$ continuous at y_0. (3)

Via the composition theorem, from Eq. (2) and Eq. (3):

$$\begin{aligned}
\lim_{x\to x_0} F(g(x)) &= F(\lim_{x\to x_0} g(x)) && [\text{via composition thm}] \\
&= F(y_0) && [\text{via eq. (2)}] \\
&= f'(y_0) && [\text{def of } F(y)] \\
&= f'(g(x_0)) \quad (4) && [\text{def of } y_0]
\end{aligned}$$

and it follows that

$$[f(g(x_0))]' = \lim_{x\to x_0} \lambda(f\circ g|x,x_0) = \lim_{x\to x_0} [F(g(x))\lambda(g|x,x_0)] =$$

$$= \lim_{x\to x_0} F(g(x)) \cdot \lim_{x\to x_0} \lambda(g|x,x_0) = f'(g(x_0))\, g'(x_0). \quad \square$$

EXERCISES

(16) Find and factor the derivatives of the following functions.

a) $f(x) = (x^2-3)^4$

b) $f(x) = (x^2-5x+6)^{30}$

c) $f(x) = (1+2x)^3$

d) $f(x) = (1-x^2)^2$

e) $f(x) = \sqrt{x^2+3x-1}$

f) $f(x) = \sqrt{(x^2-1)(2x+1)}$

(17) Show the following implications

a) $f(x) = (2x-1)^3(3x+2)^5 \Rightarrow f'(x) = 3(2x-1)^2(3x+2)^4(16x-1)$

b) $f(x) = (x+3)^2(5x+2)^4 \Rightarrow f'(x) = 2(x+3)(5x+2)^3(15x+32)$

c) $f(x) = (x^2-4)^3(x^2-1)^2 \Rightarrow f'(x) = 2x(x^2-1)(x^2-4)^2(5x^2-11)$

d) $f(x) = (x^2+3x+2)^3(x^2-5x+6)^5 \Rightarrow$

$\Rightarrow f'(x) = 4(x-3)^4(x-2)^4(x+1)^3(x+2)^2(4x^3-4x^2-16x+1)$

e) $f(x) = (2x+1)\sqrt{x^2+1} \Rightarrow f'(x) = \dfrac{4x^2+x+2}{\sqrt{x^2+1}}$

f) $f(x) = (x^2+3x)\sqrt{x^2-1} \Rightarrow f'(x) = \dfrac{3x^3+6x^2-2x-3}{\sqrt{x^2-1}}$

g) $f(x) = 2x\sqrt{x+1}\,\sqrt{2x+1} \Rightarrow f'(x) = \dfrac{8x^2+9x+2}{\sqrt{x+1}\,\sqrt{2x+1}}$

h) $f(x) = (5x+7)\sqrt{x^2+3x+5} \Rightarrow f'(x) = \dfrac{20x^2+59x+71}{2\sqrt{x^2+3x+5}}$

(18) Likewise show the following more generalized implications

a) $f(x) = (ax+b)\sqrt{cx+d} \Rightarrow f'(x) = \dfrac{3acx + (2ad + bc)}{2\sqrt{cx+d}}$

b) $f(x) = x^2\sqrt{ax+b} \Rightarrow f'(x) = \dfrac{x(5ax+4b)}{\sqrt{ax+b}}$

c) $f(x) = (ax+b)\sqrt{cx^2+dx+e} \Rightarrow$

$$\Rightarrow f'(x) = \frac{4acx^2 + (3ad+2bc)x + (2ae+bd)}{2\sqrt{cx^2+dx+e}}$$

▼ The quotient rule

The quotient rule is derived from the chain rule as follows.

•1 First we show that

$$\boxed{\left(\forall x \in \mathbb{R}^*: f(x) = \frac{1}{x}\right) \Rightarrow \forall x \in \mathbb{R}^*: f'(x) = \frac{-1}{x^2}}$$

Proof

Since

$$\forall x, x_0 \in \mathbb{R} - \{0\}: \lambda(f|x, x_0) = \frac{f(x) - f(x_0)}{x - x_0} = \frac{\frac{1}{x} - \frac{1}{x_0}}{x - x_0} =$$

$$= \frac{\left(\frac{x_0 - x}{x x_0}\right)}{x - x_0} = \frac{-(x - x_0)}{x x_0 (x - x_0)} = \frac{-1}{x x_0} \Rightarrow$$

$$\Rightarrow \forall x_0 \in \mathbb{R} - \{0\}: f'(x_0) = \lim_{x \to x_0} \lambda(f|x, x_0) = \lim_{x \to x_0} \left(\frac{-1}{x x_0}\right) =$$

$$= \frac{-1}{x_0 x_0} = \frac{-1}{x_0^2} \qquad \square$$

•2 Via the chain rule, this result immediately generalizes to the <u>reduced quotient rule</u>:

$$\boxed{h(x) = \frac{1}{g(x)}, \forall x \in A \Rightarrow h'(x) = \frac{-g'(x)}{[g(x)]^2}, \forall x \in A}$$

•3 Combined with the product rule, the reduced quotient rule gives the <u>quotient rule</u>:

$$\boxed{h(x) = \frac{f(x)}{g(x)}, \forall x \in A \Rightarrow h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$$

<u>Proof</u>

$$h'(x) = \left[\frac{f(x)}{g(x)}\right]' = \left[f(x) \cdot \frac{1}{g(x)}\right]' =$$

$$= f'(x)\frac{1}{g(x)} + f(x) \cdot \left[\frac{1}{g(x)}\right]' = \qquad \text{[product rule]}$$

$$= \frac{f'(x)}{g(x)} + f(x)\frac{-g'(x)}{[g(x)]^2} = \qquad \text{[reduced quotient rule]}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \qquad \square$$

EXAMPLES

a) $f(x) = \dfrac{1}{(x^2+3x)\sqrt{x}}$ $\longleftarrow$ Evaluate $f'(x)$.

Solution

$$f'(x) = \left[\frac{1}{(x^2+3x)\sqrt{x}}\right]' = \frac{-\left[(x^2+3x)\sqrt{x}\right]'}{\left[(x^2+3x)\sqrt{x}\right]^2} =$$

$$= \frac{-(x^2+3x)'\sqrt{x} - (x^2+3x)(\sqrt{x})'}{x(x^2+3x)^2} =$$

$$= \frac{-(2x+3)\sqrt{x} - (x^2+3x)\left(\frac{1}{2\sqrt{x}}\right)}{x(x^2+3x)^2} =$$

$$= \frac{-(2x+3)\sqrt{x}\,(2\sqrt{x}) - (x^2+3x)}{x(x^2+3x)^2(2\sqrt{x})} =$$

$$= \frac{-2x(2x+3) - (x^2+3x)}{x(x^2+3x)^2(2\sqrt{x})} =$$

$$= \frac{-4x^2-6x-x^2-3x}{2x^3(x+3)^2\sqrt{x}} = \frac{-5x^2-9x}{2x^3(x+3)^2\sqrt{x}} =$$

$$= \frac{-x(5x+9)}{2x^3\sqrt{x}\,(x+3)^2} = \frac{-(5x+9)}{2x^2\sqrt{x}\,(x+3)^2}$$

b) $f(x) = \dfrac{x^2+3x+2}{x^2-2x+5} \leftarrow$ Evaluate $f'(x)$.

Solution

$$f'(x) = \left[\frac{x^2+3x+2}{x^2-2x+5}\right]' =$$

$$= \frac{(x^2+3x+2)'(x^2-2x+5) - (x^2+3x+2)(x^2-2x+5)'}{(x^2-2x+5)^2}$$

$$= \frac{(2x+3)(x^2-2x+5) - (x^2+3x+2)(2x-2)}{(x^2-2x+5)^2}$$

$$= \frac{2x^3-4x^2+10x+3x^2-6x+15-2x^3+2x^2-6x^2+6x-4x+4}{(x^2-2x+5)^2}$$

$$= \frac{(2-2)x^3+(-4+3+2-6)x^2+(10-6+6-4)x+(15+4)}{(x^2-2x+5)^2}$$

$$= \frac{-5x^2+6x+19}{(x^2-2x+5)^2} = \frac{-(5x^2-6x-19)}{(x^2-2x+5)^2}.$$

EXERCISES

(19) Show the following implications

a) $f(x) = \dfrac{x^2}{x-1} \Rightarrow f'(x) = \dfrac{x(x-2)}{(x-1)^2}$

b) $f(x) = \dfrac{2x+1}{3x-2} \Rightarrow f'(x) = \dfrac{-7}{(3x-2)^2}$

c) $f(x) = \dfrac{x^2+x+1}{x^2-3x+2} \Rightarrow f'(x) = \dfrac{-(4x^2-2x-5)}{(x-2)^2(x-1)^2}$

d) $f(x) = \dfrac{x + \dfrac{x+1}{2x+1}}{x - \dfrac{x-1}{2x-1}} \Rightarrow f'(x) = \dfrac{-8x^2(2x^2-3)}{(2x+1)^2(2x^2-2x+1)^2}$

e) $f(x) = \dfrac{x^2-1}{\sqrt{x}} \Rightarrow f'(x) = \dfrac{3x^2+1}{2x\sqrt{x}}$

f) $f(x) = \dfrac{(3x-2)^3}{(2x-5)^2} \Rightarrow f'(x) = \dfrac{(3x-2)^2(6x-37)}{(2x-5)^3}$

g) $f(x) = \dfrac{(x+3)^2}{(3x-1)^4} \Rightarrow f'(x) = \dfrac{-2(x+3)(3x+19)}{(3x-1)^5}$

(20) Likewise, show that

a) $f(x) = \dfrac{\sqrt{2x+1}}{3x+2} \Rightarrow f'(x) = \dfrac{-(3x+1)}{(3x+2)^2\sqrt{2x+1}}$

b) $f(x) = \dfrac{\sqrt{2x+1}}{3x+2} \Rightarrow f'(x) = \dfrac{-(3x+1)}{(3x+2)^2\sqrt{2x+1}}$

c) $f(x) = \dfrac{\sqrt{x^2+2x}}{x^2-1} \Rightarrow f'(x) = \dfrac{-(x^3+3x^2+x+1)}{(1-x^2)^2\sqrt{x(x+2)}}$

d) $f(x) = \dfrac{x^2+1}{\sqrt{3x+2}} \Rightarrow f'(x) = \dfrac{9x^2+8x-3}{2(3x+2)\sqrt{3x+2}}$

e) $f(x) = \dfrac{(5x+6)^3}{\sqrt{x^2+3x}} \Rightarrow f'(x) = \dfrac{(5x+6)^2(20x^2+63x-18)}{2x(x+3)\sqrt{x(x+3)}}$

f) $f(x) = \dfrac{(x+1)^2}{(2x-1)^3\sqrt{2x+3}} \Rightarrow f'(x) = \dfrac{-(x+1)(6x^2+23x+23)}{(2x-1)^4(2x+3)\sqrt{2x+3}}$

(21) Given the function

$$f(x) = \frac{x\cos\omega - \sin\omega}{x\sin\omega + \cos\omega}$$

with $0 < \omega < \pi/4$, show that

$$\frac{f'(x)}{1+[f(x)]^2} = \frac{1}{x^2+1}$$

(22) Given the function $f(x) = \sqrt{x+\sqrt{1+x^2}}$, $\forall x \in \mathbb{R}$ show that:

a) $f(x) = 2\sqrt{1+x^2}\, f'(x)$

b) $4(1+x^2)f''(x) + 4xf'(x) = f(x)$

▼ Trigonometric derivatives

- The derivative of $\sin x$ can be derived via the result

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

and the trigonometric identity for factoring the sum/difference of sine functions:

$$\sin a \pm \sin b = 2\sin\left(\frac{a\pm b}{2}\right)\cos\left(\frac{a\mp b}{2}\right)$$

The main result is:

① $\boxed{\forall x\in\mathbb{R} : (\sin x)' = \cos x}$

Proof

Let $x, x_0 \in \mathbb{R}$ be given with $x \neq x_0$.

$$\lambda(\sin \mid x, x_0) = \frac{\sin x - \sin x_0}{x - x_0} = \frac{2\sin\left(\frac{x-x_0}{2}\right)\cos\left(\frac{x+x_0}{2}\right)}{x-x_0} =$$

$$= \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} \cos\left(\frac{x+x_0}{2}\right), \forall x, x_0 \in \mathbb{R} \quad (1)$$

Since

$$\left.\begin{array}{l} \lim_{x\to x_0} \frac{x+x_0}{2} = \frac{x_0+x_0}{2} = x_0 \\ \cos \text{ continuous on } \mathbb{R} \end{array}\right\} \Rightarrow \lim_{x\to x_0} \cos\left(\frac{x+x_0}{2}\right) = \cos x_0 \quad (2)$$

and

$$\left.\begin{aligned} &\lim_{x\to x_0} \frac{x-x_0}{2} = 0 \\ &\frac{x-x_0}{2} \neq 0, \forall x \in N(x_0,\delta) \\ &\lim_{y\to 0} \frac{\sin y}{y} = 1 \end{aligned}\right\} \Rightarrow \lim_{x\to 0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} = 1 \quad (3)$$

From Eq. (1), Eq. (2), Eq. (3):

$$(\sin x_0)' = \lim_{x\to x_0} \lambda(\sin | x, x_0) =$$

$$= \lim_{x\to x_0} \left[\frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} \cdot \cos\left(\frac{x+x_0}{2}\right) \right]$$

$$= \lim_{x\to x_0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} \lim_{x\to x_0} \cos\left(\frac{x+x_0}{2}\right) =$$

$$= 1 \cdot \cos\left(\frac{x_0+x_0}{2}\right) = \cos x_0 \qquad \square$$

↳ Note that the proof of this result depends on the continuity of cos and the limit $\lim_{x\to 0} (\sin x)/x$. Consequently continuity has to be established first before establishing differentiability.

- For the derivative of cos we use the chain rule generalization of the above result
  $$[\sin(g(x))]' = g'(x)\cos(g(x))$$
  and the cofactor identities:

$$\forall x \in \mathbb{R}: \ \sin(\pi/2 - x) = \cos x$$
$$\forall x \in \mathbb{R}: \ \cos(\pi/2 - x) = \sin x$$

as follows:

② $\boxed{(\cos x)' = -\sin x \ , \forall x \in \mathbb{R}}$

Proof

$$(\cos x)' = [\sin(\pi/2 - x)]' = (\pi/2 - x)' \cos(\pi/2 - x)$$
$$= -\cos(\pi/2 - x) = -\sin x \ , \ \forall x \in \mathbb{R}.$$

③ $\boxed{(\tan x)' = \dfrac{1}{\cos^2 x} = 1 + \tan^2 x \ , \forall x \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}}$

Proof

$$(\tan x)' = \left[\frac{\sin x}{\cos x}\right]' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} =$$
$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} \qquad (1)$$

From Eq. (1):

$$(\tan x)' = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$(\tan x)' = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} =$$
$$= 1 + \left(\frac{\sin x}{\cos x}\right)^2 = 1 + \tan^2 x \qquad \square$$

- Via the chain rule, we obtain the following generalized differentiation rules:

$(\sin x)' = \cos x$	$[\sin(g(x))]' = g'(x)\cos(g(x))$
$(\cos x)' = -\sin x$	$[\cos(g(x))]' = -g'(x)\sin(g(x))$
$(\tan x)' = \dfrac{1}{\cos^2 x}$	$[\tan(g(x))]' = \dfrac{g'(x)}{\cos^2(g(x))}$
$(\tan x)' = 1+\tan^2 x$	$[\tan(g(x))]' = [1+\tan^2(g(x))]\,g'(x)$

EXAMPLE

a) $f(x) = \sin^2(\cos^3(2x)) \longleftarrow$ Evaluate $f'(x)$.

Solution

$$f'(x) = [\sin^2(\cos^3(2x))]' =$$
$$= 2\sin(\cos^3(2x))[\sin(\cos^3(2x))]'$$
$$= 2\sin(\cos^3(2x))\cos(\cos^3(2x))[\cos^3(2x)]'$$
$$= 2\sin(\cos^3(2x))\cos(\cos^3(2x))[3\cos^2(2x)][\cos(2x)]'$$
$$= 6\sin(\cos^3(2x))\cos(\cos^3(2x))\cos^2(2x)[-\sin(2x)](2x)'$$
$$= -12\sin(\cos^3(2x))\cos(\cos^3(2x))\cos^2(2x)\sin(2x)$$

b) Show that:

$$f(x) = \cos x\sqrt{\sin(3x)} \Rightarrow f'(x) = \frac{5\cos(4x) + \cos(2x)}{4\sqrt{\sin(3x)}}$$

Solution

$$f'(x) = [\cos x\sqrt{\sin(3x)}]' =$$
$$= (\cos x)'\sqrt{\sin(3x)} + \cos x\left(\sqrt{\sin(3x)}\right)'$$
$$= -\sin x\sqrt{\sin(3x)} + \cos x\,\frac{(\sin(3x))'}{2\sqrt{\sin(3x)}} =$$
$$= -(\sin x)\sqrt{\sin(3x)} + \cos x\,\frac{(3x)'\cos(3x)}{2\sqrt{\sin(3x)}} =$$
$$= -(\sin x)\sqrt{\sin(3x)} + \cos x\,\frac{3\cos(3x)}{2\sqrt{\sin(3x)}} =$$

$$= \frac{-(\sin x)\sqrt{\sin(3x)}\left(2\sqrt{\sin(3x)}\right) + \cos x\,(3\cos(3x))}{2\sqrt{\sin(3x)}}$$

$$= \frac{-2\sin x\sin(3x) + 3\cos x\cos(3x)}{2\sqrt{\sin(3x)}} =$$

$$= \frac{-(\cos(x-3x) - \cos(x+3x)) + 3\cdot(1/2)[\cos(x-3x) + \cos(x+3x)]}{2\sqrt{\sin(3x)}}$$

$$= \frac{-2(\cos 2x - \cos 4x) + 3(\cos 2x + \cos 4x)}{4\sqrt{\sin(3x)}} =$$

$$= \frac{-2\cos 2x + 2\cos 4x + 3\cos 2x + 3\cos 4x}{4\sqrt{\sin(3x)}} =$$

$$= \frac{5\cos 4x + \cos 2x}{4\sqrt{\sin(3x)}} \qquad \square$$

↳ For the above exercise we used the following trigonometric identities:

$\cos(-x) = \cos x$

$2\cos a\cos b = \cos(a-b) + \cos(a+b)$

$2\sin a\sin b = \cos(a-b) - \cos(a+b)$.

EXERCISES

(23) Find the derivatives of the following functions

a) $f(x) = \sin(3x^2+2)$

b) $f(x) = \tan(1/x)$

c) $f(x) = \sin^3(x^2+x-5)$

d) $f(x) = \cos(\sqrt{x-1})$

e) $f(x) = \sin(\tan(x))$

f) $f(x) = \cos(\sin(x))$

g) $f(x) = \sin(\cot^2 x)$

h) $f(x) = \sin(\sqrt{2x+1})$

(24) Show that

a) $f(x) = \sin^2 x \cos x \Rightarrow f'(x) = \sin x\,(\cos(2x) + \cos^2 x)$

b) $f(x) = \tan x - \cot x \Rightarrow f'(x) = \dfrac{4}{\sin^2(2x)}$

c) $f(x) = \sqrt{x}\tan^3 x \Rightarrow f'(x) = \dfrac{\tan^2 x\,[\sin x\cos x + 6x]}{2\sqrt{x}\cos^2 x}$

d) $f(x) = \dfrac{\sin x}{\sin x + \cos x} \Rightarrow f'(x) = \dfrac{1}{1+\cos x}$

e) $f(x) = \sin(\cos^2 x)\cos(\sin^2 x) \Rightarrow f'(x) = -\sin(2x)\cos(\cos(2x))$

f) $f(x) = \dfrac{\sin x - \cos x}{\sin x + \cos x} \Rightarrow f'(x) = \dfrac{2}{\sin(2x)+1}$

g) $f(x) = (x - \sin x\cos x)^2 \Rightarrow f'(x) = 4\sin^2 x\,(x - \sin x\cos x)$

h) $f(x) = 3\sin x\sqrt{\cos(2x)} \Rightarrow f'(x) = \dfrac{3\cos(3x)}{\sqrt{\cos(2x)}}$

i) $f(x) = \dfrac{\tan(x)}{\sqrt{x}} \Rightarrow f'(x) = \dfrac{4x - \sin(2x)}{4x\sqrt{x}\cos^2 x}$

j) $f(x) = \dfrac{\tan x - \cot x}{\tan x + \cot x} \Rightarrow f'(x) = 2\sin(2x)$

k) $f(x) = \cos x\sqrt{\cos(2x)} \Rightarrow f'(x) = \dfrac{-\sin(3x)}{\sqrt{\cos(2x)}}$

l) $f(x) = \tan x\sqrt{\cot x} \Rightarrow f'(x) = \dfrac{1}{\sin(2x)\sqrt{\cot x}}$