Asymptotes

Asymptotes are lines that are approached by the graph of the function under the limits $x \to x_0^{\pm}$ or $x \to +\infty$ or $x \to -\infty$. The graph may or may not make contact with an asymptote. The precise definitions are as follows: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ be a function

(1) $(l): y = ax + b$ is asymptote of $f(x)$ at $\pm\infty$ $\Leftrightarrow$

$$\begin{cases} \lim\limits_{x \to \pm\infty} \dfrac{f(x)}{x} = a \\ \lim\limits_{x \to \pm\infty} [f(x) - ax] = b \end{cases}$$

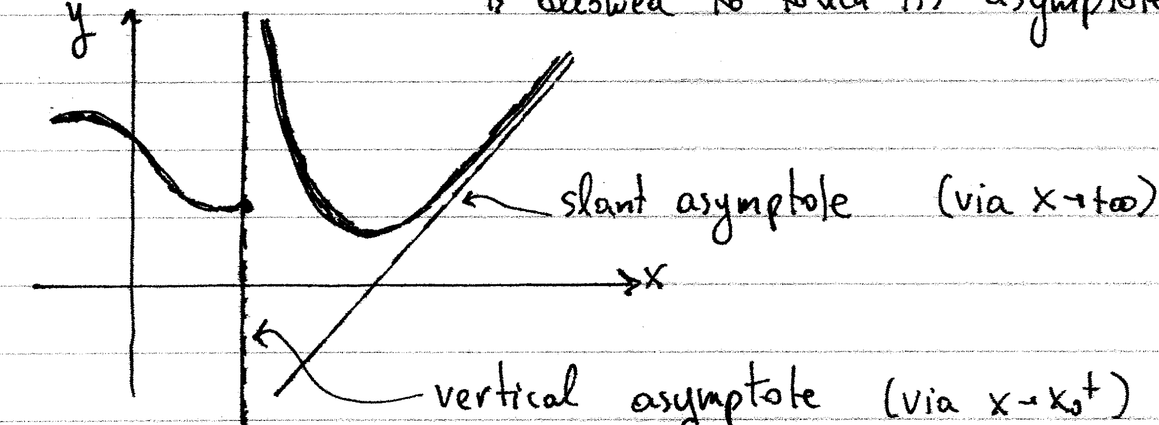
notation: $f(x) \sim ax + b$ with $x \to \pm\infty \Leftrightarrow$
$\Leftrightarrow (l): y = ax + b$ is asymptote of $f(x)$ at $\pm\infty$

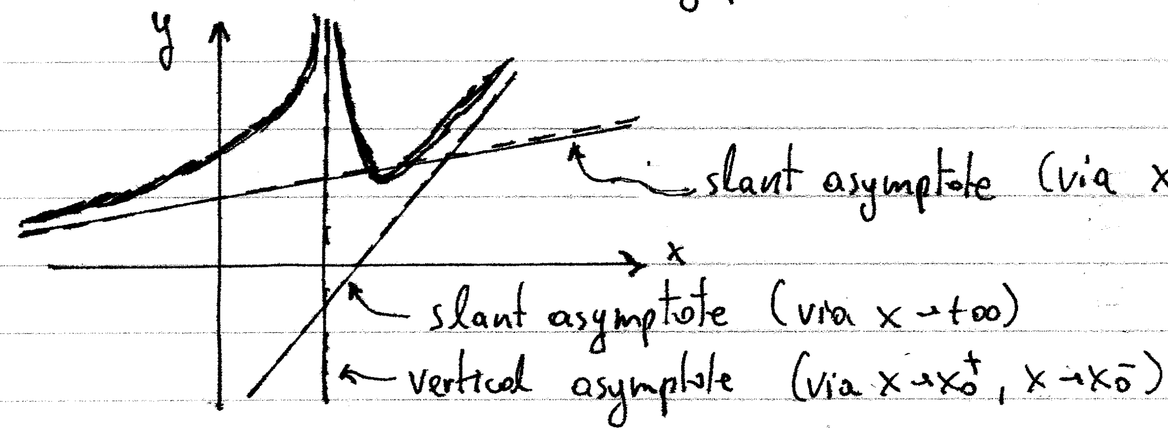
remark: If $a \neq 0$, then (l) is a slant asymptote.
If $a = 0$, then (l) is a horizontal asymptote.

(2) $(l): x = x_0$ is a vertical asymptote of $f(x)$ $\Leftrightarrow$

$$\lim_{x \to x_0^+} f(x) \in \{+\infty, -\infty\} \vee \lim_{x \to x_0^-} f(x) \in \{+\infty, -\infty\}$$

<u>Graphic Examples</u>: Note that the graph of a function is allowed to touch its asymptotes.





Continuity can be used to rule out the existence of vertical asymptotes:

<u>Prop</u>: $\boxed{f \text{ continuous at } x_0 \in A \Rightarrow (\ell): x = x_0 \text{ NOT asymptote}}$

<u>Proof</u>

Assume that f continuous at x_0. Then:

f continuous at $x_0 \Rightarrow \lim_{x \to x_0} f(x) = f(x_0) \Rightarrow$

$$\Rightarrow \lim_{x \to x_0^+} f(x) = f(x_0) \wedge \lim_{x \to x_0^-} f(x) = f(x_0) \Rightarrow$$

$$\Rightarrow \lim_{x\to x_0^+} f(x) \notin \{+\infty, -\infty\} \wedge \lim_{x\to x_0^-} f(x) \notin \{+\infty, -\infty\}$$

$\Rightarrow (\ell): x = x_0$ is NOT vertical asymptote.

▶ Methodology

• 1 To check for slant or horizontal asymptotes, we calculate

$$a_+ = \lim_{x\to+\infty} \frac{f(x)}{x} \qquad b_+ = \lim_{x\to+\infty} [f(x) - a_+ x]$$

$$a_- = \lim_{x\to-\infty} \frac{f(x)}{x} \qquad b_- = \lim_{x\to-\infty} [f(x) - a_- x]$$

If $a_+, b_+ \in \mathbb{R}$ then $(\ell): y = a_+ x + b_+$ is slant asymptote when $x \to +\infty$.

If $a_-, b_- \in \mathbb{R}$ then $(\ell): y = a_- x + b_-$ is slant asymptote when $x \to -\infty$

If $f(x)$ is a rational function, we can do the limits $x\to+\infty$ and $x\to-\infty$ simultaneously, as shown in the examples.

• 2 To find all vertical asymptotes:

a) Use continuity to eliminate all points $x_0 \in A$ where the function has no vertical asymptote.

b) Any points $x_0 \notin A$ where x_0 is NOT a limit point can be also ruled out.

c) We check the remaining points on a case by case basis.

EXAMPLES

a) Find all asymptotes of
$f(x) = x^3 + 5x^2 + 3x + 1$.

Solution

$$a_{\pm} = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^3+5x^2+3x+1}{x} =$$

$$= \lim_{x\to\pm\infty} \frac{x^3}{x} = \lim_{x\to\pm\infty} (x^2) = +\infty \Rightarrow$$

$\Rightarrow$ no slant or horizontal asymptotes when $x \to \pm\infty$.
Also: f continuous in $\mathbb{R} \Rightarrow \forall x_0 \in \mathbb{R}$: $(\ell): x = x_0$ not asymptote
$\Rightarrow$ no vertical asymptotes.

b) Find all asymptotes of $f(x) = \dfrac{x^2(x+1)}{x^2+3x+2}$

Solution

- Slant asymptotes

$$a = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^2(x+1)}{x(x^2+3x+2)} = \lim_{x\to\pm\infty} \frac{x^2 x}{x x^2}$$

$$= \lim_{x\to\pm\infty} \frac{x^3}{x^3} = 1$$

$$b = \lim_{x\to\pm\infty} [f(x) - ax] = \lim_{x\to\pm\infty} [f(x) - x] = \lim_{x\to\pm\infty} \left[\frac{x^2(x+1)}{x^2+3x+2} - x \right]$$

$$= \lim_{x \to \pm\infty} \frac{x^2(x+1) - x(x^2+3x+2)}{x^2+3x+2} =$$

$$= \lim_{x \to \pm\infty} \frac{x^3+x^2-x^3-3x^2-2x}{x^2+3x+2} = \lim_{x \to \pm\infty} \frac{-2x^2-2x}{x^2+3x+2} =$$

$$= \lim_{x \to \pm\infty} \frac{-2x^2}{x^2} = -2$$

It follows that $(\ell): y = x - 2$ is slant asymptote as $x \to \pm\infty$.

▸ Vertical asymptotes.

• Solve $x^2+3x+2=0 \Leftrightarrow (x+1)(x+2)=0 \Leftrightarrow x+1=0 \vee x+2=0$

$$\Leftrightarrow x=-1 \vee x=-2$$

thus $\mathrm{dom}(f) = \mathbb{R} - \{-1, -2\}$.

• Since f continuous on $\mathbb{R}-\{-1,-2\} \Rightarrow$

$\Rightarrow \forall x_0 \in \mathbb{R}-\{-1,-2\}$: $(\ell): x = x_0$ is NOT a vertical asymptotes

• Note that $f(x) = \frac{x^2(x+1)}{x^2+3x+2} = \frac{x^2(x+1)}{(x+1)(x+2)} = \frac{x^2}{x+2}$

• At $x=-1$:

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2}{x+2} = \frac{(-1)^2}{-1+2} = \frac{1}{1} = 1 \Rightarrow$$

$$\Rightarrow \lim_{x \to -1^+} f(x) \notin \{+\infty, -\infty\} \wedge \lim_{x \to -1^-} f(x) \notin \{+\infty, -\infty\} \Rightarrow$$

$\Rightarrow (\ell): x = -1$ is NOT a vertical asymptote.

• At $x = -2$:

$$\left.\begin{array}{l}\lim\limits_{x\to -2^+} \dfrac{1}{x+2} = \lim\limits_{x\to -2^+} \dfrac{1}{x-(-2)} = +\infty \\ \lim\limits_{x\to -2^+} x^2 = (-2)^2 = 4\end{array}\right\} \Rightarrow$$

$$\Rightarrow \lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{x^2}{x+2} = +\infty \Rightarrow$$

$\Rightarrow (\ell): x = -2$ is vertical asymptote.

Thus: we have two asymptotes:

$(\ell_1): y = x - 2$

$(\ell_2): x = -2.$

c) $f(x) = \sqrt{3x^2 - x}$ ← Find all asymptotes

Solution

• Domain

Require $3x^2 - x \geq 0 \iff x(3x-1) \geq 0 \iff x \in (-\infty, 0] \cup [1/3, +\infty)$

x		0		1/3	
x	$-$	0	$+$		$+$
$3x-1$	$-$		$-$	0	$+$
Ineq	$+$	0	$-$	0	$+$

thus $\text{dom}(f) = (-\infty, 0] \cup [1/3, +\infty)$.

• Slant asymptotes:

$$\frac{f(x)}{x} = \frac{\sqrt{3x^2 - x}}{x} = \frac{\sqrt{x^2}\sqrt{3 - 1/x}}{x} = \frac{|x|\sqrt{3-1/x}}{x}.$$

$$= \begin{cases} \sqrt{3 - 1/x}, & x \in [1/3, +\infty) \\ -\sqrt{3-1/x}, & x \in (-\infty, 0] \end{cases} \Rightarrow$$

therefore:

$$a_+ = \lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \sqrt{3 - 1/x} = \sqrt{3 - 0} = \sqrt{3}$$

$$a_- = \lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} \left(-\sqrt{3 - 1/x}\right) = -\sqrt{3-0} = -\sqrt{3}$$

$\forall x \in [1/3, +\infty)$: $f(x) - a_+ x = \sqrt{3x^2 - x} - (\sqrt{3})x =$

$$= \frac{(\sqrt{3x^2 - x})^2 - (x\sqrt{3})^2}{\sqrt{3x^2 - x} + x\sqrt{3}} = \frac{(3x^2 - x) - 3x^2}{|x|\sqrt{3 - 1/x} + x\sqrt{3}} =$$

$$= \frac{-x}{x\sqrt{3 - 1/x} + x\sqrt{3}} = \frac{-1}{\sqrt{3 - 1/x} + \sqrt{3}} \Rightarrow$$

$$b_+ = \lim_{x\to+\infty} [f(x) - a_+x] = \lim_{x\to+\infty} \frac{-1}{\sqrt{3-1/x} + \sqrt{3}} =$$

$$= \frac{-1}{\sqrt{3}+\sqrt{3}} = \frac{-1}{2\sqrt{3}} = \frac{-\sqrt{3}}{2\cdot 3} = \frac{-\sqrt{3}}{6}$$

$$\forall x \in (-\infty, 0]: f(x) - a_-x = \sqrt{3x^2-x} - (-\sqrt{3})x =$$

$$= \sqrt{3x^2-x} + (\sqrt{3})x = \frac{(\sqrt{3x^2-x})^2 - (x\sqrt{3})^2}{\sqrt{3x^2-x} - x\sqrt{3}} =$$

$$= \frac{(3x^2-x) - 3x^2}{|x|\sqrt{3-1/x} - x\sqrt{3}} = \frac{-x}{-x\sqrt{3-1/x} - x\sqrt{3}} =$$

$$= \frac{1}{\sqrt{3-1/x} + \sqrt{3}} \Rightarrow$$

$$\Rightarrow b_- = \lim_{x\to-\infty} [f(x) - a_-x] = \lim_{x\to-\infty} \frac{1}{\sqrt{3-1/x} + \sqrt{3}} =$$

$$= \frac{1}{\sqrt{3}+\sqrt{3}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$$

It follows that f has the following slant asymptotes:

$(l_1): y = x\sqrt{3} - \frac{\sqrt{3}}{6}$ for $x \to +\infty$

$(l_2): y = -x\sqrt{3} + \frac{\sqrt{3}}{6}$ for $x \to -\infty$

- Vertical asymptotes.

Since f continuous on $(-\infty, 0] \cup [1/3, +\infty) \Rightarrow$

$\Rightarrow \forall x_0 \in (-\infty, 0] \cup [1/3, +\infty)$: $(l): x = x_0$ is NOT vertical asymptote.

and
($\forall x_0 \in (0, 1/3)$: x_0 not a limit point of dom(f)) $\Rightarrow$
$\Rightarrow \forall x_0 \in (0, 1/3)$: $(\ell): x = x_0$ not a vertical asymptote
It follows that f does not have any vertical asymptotes.

EXERCISES

① Find all the asymptotes for the following functions

a) $f(x) = \dfrac{x-2}{x+3}$

b) $f(x) = \dfrac{x^2+5}{3x}$

c) $f(x) = \dfrac{x^2+x+1}{x-2}$

d) $f(x) = \dfrac{2}{x-1}$

e) $f(x) = \dfrac{(x+2)^3}{8x^3}$

f) $f(x) = \dfrac{x^2+1}{2x-1}$

g) $f(x) = \dfrac{x^3+2x-1}{x^2-1}$

h) $f(x) = \dfrac{x^4}{x^2+1}$

② Find all the asymptotes for the following functions

a)
$$f(x) = \begin{cases} \dfrac{1}{x-2}, & \text{if } x<2 \\ 1, & \text{if } x=2 \\ \dfrac{1}{x-2}-2, & \text{if } x>2 \end{cases}$$

b)
$$f(x) = \begin{cases} \dfrac{x+1}{x-2}, & x \neq 2 \\ 3, & x=2 \end{cases}$$

c) $f(x) = \sqrt{x^2-25}$

d) $f(x) = \sqrt{x^2+2x+5}$

e) $f(x) = \sqrt{x^2-x+1}$