LIMITS

▼ Definition of limits

- Let $f: A_f \to \mathbb{R}$ be a function. Informally, the statement $\lim_{x\to\sigma} f(x) = L$ means that "when x approaches σ, then $f(x)$ approaches L".
- Possibilities for $x \to \sigma$:
  - $x \to x_0 \in \mathbb{R}$: approach x_0 from both sides
  - $x \to x_0^-$: approach x_0 from $x < x_0$
  - $x \to x_0^+$: approach x_0 from $x > x_0$
  - $x \to +\infty$: x becomes arbitrarily large
  - $x \to -\infty$: x becomes arbitrarily small.
- Possibilities for L:
  - $L = \ell \in \mathbb{R}$: $f(x)$ approaches a real number ℓ.
  - $L = +\infty$: $f(x)$ becomes arbitrarily large
  - $L = -\infty$: $f(x)$ becomes arbitrarily small.
- We can give 15 formal definitions for all possible combinations or combine them into 1 abstract definition. Overview of the argument:
  - a) Define neighborhoods (i.e. the concept $x \to \sigma$)
  - b) Define limit points (i.e. that σ can be approached)
  - c) Abstract limit definition
  - d) The 15 Weierstrass definitions.

Definition of neighborhoods

- Let $\delta > 0$ be given. We define the neighborhood $N(\sigma, \delta)$ to represent the set of numbers near σ, with δ controlling the meaning of "near":

$x \in N(x_0, \delta) \Leftrightarrow 0 < \lvert x - x_0 \rvert < \delta$	$x \to x_0$
$x \in N(x_0^+, \delta) \Leftrightarrow x_0 < x < x_0 + \delta$	$x \to x_0^+$
$x \in N(x_0^-, \delta) \Leftrightarrow x_0 - \delta < x < x_0$	$x \to x_0^-$
$x \in N(+\infty, \delta) \Leftrightarrow x > 1/\delta$	$x \to +\infty$
$x \in N(-\infty, \delta) \Leftrightarrow x < -1/\delta$	$x \to -\infty$

- Smaller δ makes the neighborhood tighter around σ.
- Note that $x_0 \notin N(x_0, \delta)$ and $x_0 \notin N(x_0^+, \delta)$ and $x_0 \notin N(x_0^-, \delta)$.
- Let $\varepsilon > 0$ be given. We define the interval $I(L, \varepsilon)$ to represent the set of numbers near L, including L itself if it is finite, with ε controlling the meaning of near.

$$y \in I(\ell, \varepsilon) \Leftrightarrow \lvert y - \ell \rvert < \varepsilon$$
$$y \in I(+\infty, \varepsilon) \Leftrightarrow y > 1/\varepsilon$$
$$y \in I(-\infty, \varepsilon) \Leftrightarrow y < -1/\varepsilon$$

- Note that:

$$I(+\infty, \varepsilon) = N(+\infty, \varepsilon)$$
$$I(-\infty, \varepsilon) = N(-\infty, \varepsilon)$$
$$I(\ell, \varepsilon) = N(\ell, \varepsilon) \cup \{\ell\}$$
$$\ell \in N(\ell, \varepsilon)$$

- <u>Limit point</u>

Let $f: A \to \mathbb{R}$ be a function with domain A. We say that

$$\boxed{\sigma \text{ limit point of } A \Leftrightarrow \forall \delta > 0 : N(\sigma, \delta) \cap A \neq \emptyset}$$

i.e. no matter how much we "squeeze" $N(\sigma,\delta)$ it will always overlap with A. This means that we can get as close to σ as we want using numbers from A.

EXAMPLES

a) For $A = [1,6)$

$2, 2^+, 2^-, 1, 1^+, 6, 6^-$ are all limit points of A.

$1^-, 6^+, 7, 0, +\infty, -\infty$ are NOT limit points of A.

b) For $A = (3, +\infty)$

$3, 3^+, 5, 5^+, 5^-, +\infty$ are limit points of A.

$3^-, -\infty, 1, 1^-, 1^+$ are NOT limit points of A.

- Note from the examples that endpoints of intervals can be limit points even if the point does not belong to the interval.

- For endpoints of intervals, one of the side limits (+ or −) is NOT a limit point.

c) For $A = (1,3] \cup \{5\}$

5 is NOT a limit point

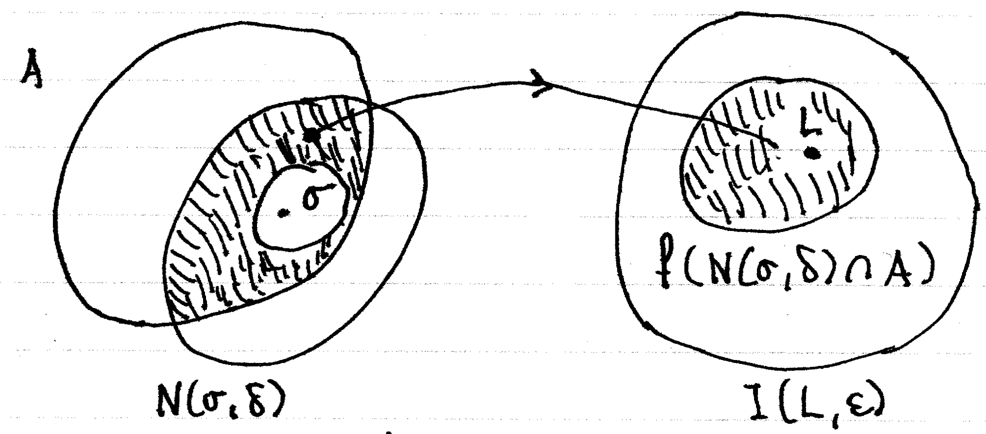
$5^+, 5^-$ are NOT limit points.

- Isolated points are not limit points either.

Abstract limit definition

- Let $f: A \to \mathbb{R}$ be a function. To define $\lim_{x\to\sigma} f(x)$ it is necessary for σ to be a limit point of A. Assume it is. Then

$$\boxed{\lim_{x\to\sigma} f(x) = L \Leftrightarrow \forall \varepsilon > 0 : \exists \delta > 0 : f(N(\sigma,\delta) \cap A) \subseteq I(L,\varepsilon)}$$



"For all $\varepsilon > 0$, there is a $\delta > 0$, such that $f(N(\sigma,\delta)\cap A)$ is contained in $I(L,\varepsilon)$"

- To interpret this statement we note that:

$f(N(\sigma,\delta)\cap A)$ = Values taken by $f(x)$ when x is NEAR σ but not equal to σ (with $x\in A$)

$I(L,\varepsilon)$ = Values near L, including L if finite

ε = How close we want $f(x)$ to be to L.

δ = How close x must be brought to σ so that $f(x)$ will be as close to L as has been required by our choice of ε.

• Thus: as we make ε smaller, it should be possible to find a smaller δ that will squeeze $f(N(\sigma,\delta)\cap A)$ back inside $I(L,\varepsilon)$.

Weierstrass definitions of limit

To construct Weierstrass definitions for the 15 possible cases, we first rewrite the general definition as follows:

$$\boxed{\lim_{x\to\sigma} f(x) = L \Leftrightarrow \\ \Leftrightarrow \forall\varepsilon>0 : \exists\delta>0 : \forall x\in A : (x\in N(\sigma,\delta)\Rightarrow f(x)\in I(L,\varepsilon))}$$

by using:

$$f(N(\sigma,\delta)\cap A)\subseteq I(L,\varepsilon) \Leftrightarrow$$
$$\Leftrightarrow \forall x\in A : (x\in N(\sigma,\delta)\Rightarrow f(x)\in I(L,\varepsilon)).$$

The definition reads:
"For all $\varepsilon>0$, there is a $\delta>0$, such that for all $x\in A$, if $x\in N(\sigma,\delta)$ then $f(x)\in I(L,\varepsilon)$"
To construct specific Weierstrass definitions we replace the two belonging conditions $x\in N(\sigma,\delta)$ and $f(x)\in I(L,\varepsilon)$ with the corresponding inequalities.

EXAMPLES

a) $\lim\limits_{x\to x_0} f(x)=\ell \Leftrightarrow \forall\varepsilon>0:\exists\delta>0:\forall x\in A:(0<|x-x_0|<\delta \Rightarrow |f(x)-\ell|<\varepsilon)$

"For all $\varepsilon>0$, there is a $\delta>0$, such that for all $x\in A$, if $0<|x-x_0|<\delta$ then $|f(x)-\ell|<\varepsilon$".

b) $\lim\limits_{x\to 3^+} f(x)=+\infty \Leftrightarrow$

$\Leftrightarrow \forall\varepsilon>0:\exists\delta>0:\forall x\in A:(3<x<3+\delta \Rightarrow f(x)>1/\varepsilon)$

c) $\lim\limits_{x\to -\infty} f(x)=4 \Leftrightarrow$

$\Leftrightarrow \forall\varepsilon>0:\exists\delta>0:\forall x\in A:(x<-1/\delta \Rightarrow |f(x)-4|<\varepsilon)$

d) $\lim\limits_{x\to 0^-} f(x)=0 \Leftrightarrow$

$\Leftrightarrow \forall\varepsilon>0:\exists\delta>0:\forall x\in A:(-\delta<x<0 \Rightarrow |f(x)|<\varepsilon)$

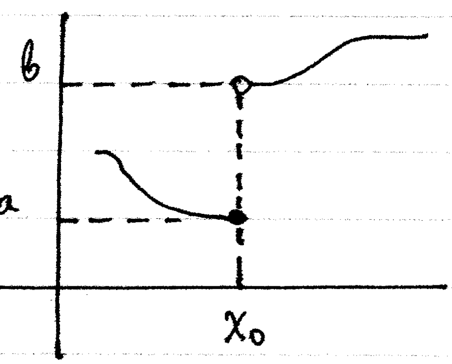
Limit does not exist

Let $f: A \to \mathbb{R}$ be a function with σ a limit point of A. We say that:

$$\lim_{x \to \sigma} f(x) \text{ does not exist} \iff \begin{cases} \forall \ell \in \mathbb{R}: \lim_{x \to \sigma} f(x) \neq \ell \\ \lim_{x \to \sigma} f(x) \neq +\infty \\ \lim_{x \to \sigma} f(x) \neq -\infty \end{cases}$$

Geometric examples

a)



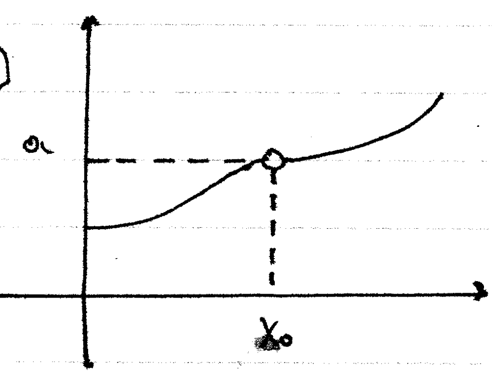
$$\lim_{x \to x_0^+} f(x) = b$$

$$\lim_{x \to x_0^-} f(x) = a$$

$$\lim_{x \to x_0} f(x) \text{ does not exist}$$

$$f(x_0) = a$$

b)

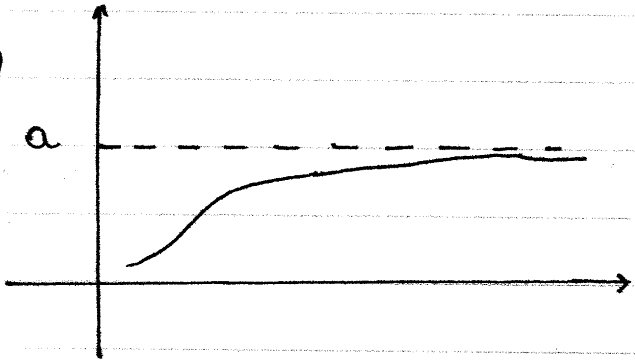


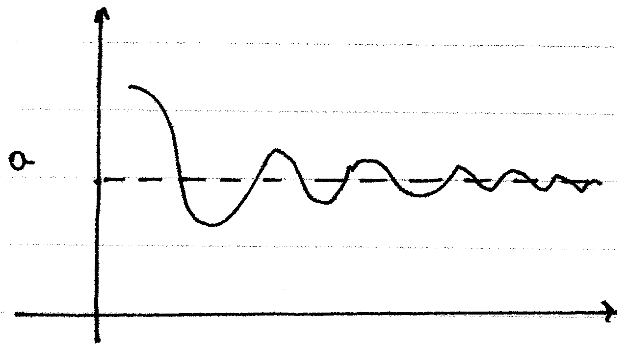
$$\lim_{x \to x_0} f(x) = a$$

$f(x_0)$ not defined!

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = a$$

c)





In both cases:
$$\lim_{x\to+\infty} f(x) = a$$

Note that it is possible that $f(x) \neq a$, $\forall x \in A$ as in the top figure but $f(x)$ is allowed to satisfy $f(x) = a$ for a finite or infinite number of points x as $x \to +\infty$.

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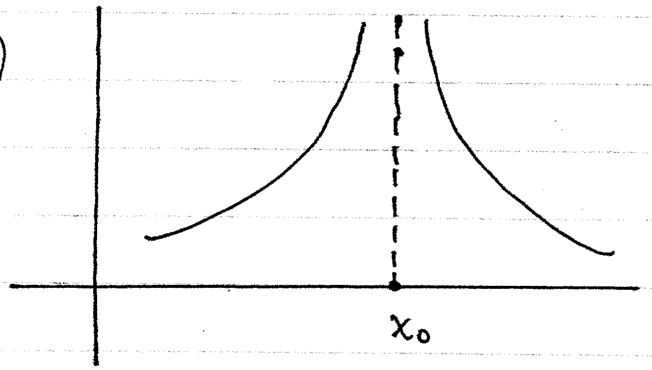
d)



$f(x_0)$ not defined

$$\lim_{x\to x_0} f(x) = +\infty$$

$$\lim_{x\to x_0^+} f(x) = +\infty$$

$$\lim_{x\to x_0^-} f(x) = +\infty$$

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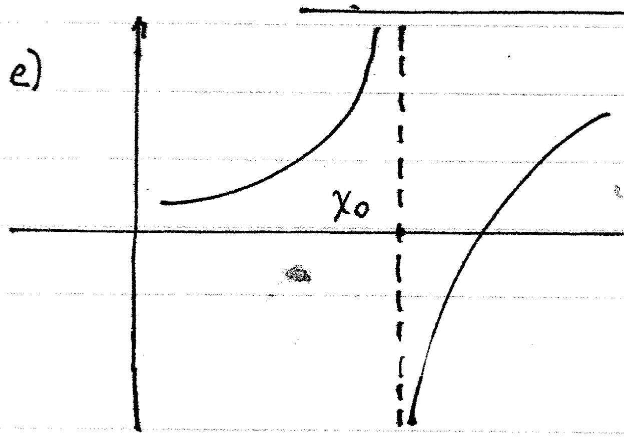
e)



$f(x_0)$ not defined

$$\lim_{x\to x_0^+} f(x) = -\infty$$

$$\lim_{x\to x_0^-} f(x) = +\infty$$

$\lim_{x\to x_0} f(x)$ does not exist.

EXERCISES

① Write the definition with quantifiers and also in English for the following statements

a) $\lim_{x \to 2} f(x) = 3$

b) $\lim_{x \to -\infty} f(x) = 9$

c) $\lim_{x \to +\infty} f(x) = -\infty$

d) $\lim_{x \to 4^+} f(x) = 3$

e) $\lim_{x \to 1^-} f(x) = +\infty$

f) $\lim_{x \to 2^-} f(x) = -\infty$

g) $\lim_{x \to 1} f(x) = +\infty$

h) $\lim_{x \to -\infty} f(x) = +\infty$

i) $\lim_{x \to +\infty} f(x) = 0$

j) $\lim_{x \to 3} f(x) = -\infty$.

Solution to 1

a) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (0 < |x-2| < \delta \Rightarrow |f(x) - 3| < \varepsilon)$

b) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (x < -1/\delta \Rightarrow |f(x) - 9| < \varepsilon)$

c) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (x > 1/\delta \Rightarrow f(x) < -1/\varepsilon)$

d) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (4 < x < 4 + \delta \Rightarrow |f(x) - 3| < \varepsilon)$

e) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (1 - \delta < x < 1 \Rightarrow f(x) > 1/\varepsilon)$

f) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (2 - \delta < x < 2 \Rightarrow f(x) < -1/\varepsilon)$

g) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (0 < |x-1| < \delta \Rightarrow f(x) > 1/\varepsilon)$

h) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (x < -1/\delta \Rightarrow f(x) > 1/\varepsilon)$

i) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (x > 1/\delta \Rightarrow |f(x)| < \varepsilon)$

j) $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (0 < |x-3| < \delta \Rightarrow f(x) < -1/\varepsilon)$

▼ Limits and operations

(1) Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ be two functions and let σ be a limit point of both A and B. We assume that: $\lim_{x \to \sigma} f(x) = l_1$ and $\lim_{x \to \sigma} g(x) = l_2$. Then:

a) $\lim_{x \to \sigma} [f(x) + g(x)] = l_1 + l_2 = \lim_{x \to \sigma} f(x) + \lim_{x \to \sigma} g(x)$

b) $\lim_{x \to \sigma} [f(x) g(x)] = l_1 l_2 = [\lim_{x \to \sigma} f(x)][\lim_{x \to \sigma} g(x)]$

c) $\forall a \in \mathbb{R} : \lim_{x \to \sigma} [a f(x)] = a l_1 = a \lim_{x \to \sigma} f(x)$

d) $\lim_{x \to \sigma} g(x) \neq 0 \Rightarrow \lim_{x \to \sigma} \left[\dfrac{f(x)}{g(x)} \right] = \dfrac{l_1}{l_2} = \dfrac{\lim_{x \to \sigma} f(x)}{\lim_{x \to \sigma} g(x)}$

e) $\lim_{x \to \sigma} |f(x)| = |l_1| = |\lim_{x \to \sigma} f(x)|$

f) $\lim_{x \to \sigma} f(x) > 0 \Rightarrow \lim_{x \to \sigma} \sqrt{f(x)} = \sqrt{l_1} = \sqrt{\lim_{x \to \sigma} f(x)}$

Trivial limits

- $P(x)$ polynomial $\Rightarrow \forall x_0 \in \mathbb{R}: \lim\limits_{x \to x_0} P(x) = P(x_0)$

- $\left.\begin{array}{l} P(x),\ Q(x) \text{ polynomials} \\ Q(x_0) \neq 0 \end{array}\right\} \Rightarrow \lim\limits_{x \to x_0} \dfrac{P(x)}{Q(x)} = \dfrac{P(x_0)}{Q(x_0)}$

- $\lim\limits_{x \to x_0} f(x) = a > 0 \Rightarrow \lim\limits_{x \to x_0} \sqrt{f(x)} = \sqrt{a}$

EXAMPLES

1) $f(x) = \sqrt{x^2+3x} \leftarrow \lim\limits_{x \to 1} f(x)$.

Rigorous proof:

$$\lim_{x \to 1} (x^2+3x) = 1^2 + 3 \cdot 1 = 4 > 0 \Rightarrow$$

$$\Rightarrow \lim_{x \to 1} f(x) = \lim_{x \to 1} \sqrt{x^2+3x} = \sqrt{4} = 2.$$

Brief version:

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \sqrt{x^2+3x} = \sqrt{1^2+3 \cdot 1} = \sqrt{4} = 2.$$

2) $f(x) = \dfrac{2x+1}{\sqrt{x^2+1}} \leftarrow \lim\limits_{x \to 2} f(x)$.

Rigorous proof.

$$\lim_{x\to 2}(x^2+1) = 2^2+1 = 4+1 = 5 > 0 \Rightarrow$$

$$\Rightarrow \lim_{x\to 2}\sqrt{x^2+1} = \sqrt{5} \quad (1)$$

$$\lim_{x\to 2}(2x+1) = 2\cdot 2+1 = 5 \quad (2)$$

From (1) and (2):

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{2x+1}{\sqrt{x^2+1}} = \frac{5}{\sqrt{5}} = \frac{5\sqrt{5}}{5} = \sqrt{5}.$$

Brief version

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{2x+1}{\sqrt{x^2+1}} = \frac{2\cdot 2+1}{\sqrt{2^2+1}} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

Indeterminent forms

1 → Form 0/0 : You need to find the cancelation that removes 0 from the numerator and denominator.

EXAMPLES

a) $f(x) = \dfrac{x^2+6x+9}{x^2+5x+6}$ $\longleftarrow$ $\lim\limits_{x\to -3} f(x)$.

Solution

▸ Note that $0/0$. Thus

$$f(x) = \frac{x^2+6x+9}{x^2+5x+6} = \frac{(x+3)^2}{(x+2)(x+3)} = \frac{x+3}{x+2} \Rightarrow$$

$$\Rightarrow \lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{x+3}{x+2} = \frac{-3+3}{-3+2} = \frac{0}{-1} = 0$$

b) $f(x) = \dfrac{x^2-3x+2}{|x-1|}$ $\longleftarrow$ $\lim\limits_{x\to 1^-} f(x), \lim\limits_{x\to 1^+} f(x)$.

Solution

▸ Note $0/0$ form.

$$f(x) = \frac{x^2-3x+2}{|x-1|} = \frac{(x-1)(x-2)}{|x-1|}.$$

It follows that:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{(x-1)(x-2)}{|x-1|} = \lim_{x\to 1^+} \frac{(x-1)(x-2)}{x-1}$$

$$= \lim_{x \to 1^+} (x-2) = 1-2 = -1$$

and

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{(x-1)(x-2)}{|x-1|} = \lim_{x \to 1^-} \frac{(x-1)(x-2)}{-(x-1)}$$
$$= \lim_{x \to 1^-} (2-x) = 2-1 = 1.$$

2 → Form 0/0 with radicals

We use the identity

$$a^2 - b^2 = (a-b)(a+b) \Rightarrow \boxed{a-b = \frac{a^2-b^2}{a+b}}$$

to eliminate the square root and expose the 0/0 cancelation.

EXAMPLES

a) $f(x) = \dfrac{\sqrt{x-1} - 2}{x-5}$ ← $\lim\limits_{x \to 5} f(x)$

Solution

$$f(x) = \frac{\sqrt{x-1} - 2}{x-5} = \frac{1}{x-5} \frac{(\sqrt{x-1})^2 - 2^2}{\sqrt{x-1} + 2} =$$

$$= \frac{1}{x-5}\,\frac{(x-1)-4}{\sqrt{x-1}+2} = \frac{1}{x-5}\,\frac{x-5}{\sqrt{x-1}+2} =$$

$$= \frac{1}{\sqrt{x-1}+2} \Rightarrow$$

$$\Rightarrow \lim_{x\to 5} f(x) = \lim_{x\to 5} \frac{1}{\sqrt{x-1}+2} = \frac{1}{\sqrt{5-1}+2} =$$

$$= \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}.$$

b) $f(x) = \dfrac{x^2-\sqrt{x}}{\sqrt{x}-1} \longleftarrow \lim\limits_{x\to 1} f(x)$

<u>Solution</u>

$$f(x) = \frac{x^2-\sqrt{x}}{\sqrt{x}-1} = \frac{\dfrac{x^4-(\sqrt{x})^2}{x^2+\sqrt{x}}}{\dfrac{(\sqrt{x})^2-1^2}{\sqrt{x}+1}} =$$

$$= \frac{(x^4-x)(\sqrt{x}+1)}{(x-1)(x^2+\sqrt{x})} = \frac{x(x^3-1)(\sqrt{x}+1)}{(x-1)(x^2+\sqrt{x})} =$$

$$= \frac{x(x-1)(x^2+x+1)(\sqrt{x}+1)}{(x-1)(x^2+\sqrt{x})} =$$

$$= \frac{x(x^2+x+1)(\sqrt{x}+1)}{x^2+\sqrt{x}} \Rightarrow$$

$$\Rightarrow \lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x(x^2+x+1)(\sqrt{x}+1)}{x^2+\sqrt{x}} =$$

$$= \frac{1\cdot(1^2+1+1)(\sqrt{1}+1)}{1^2+\sqrt{1}} = \frac{1\cdot 3\cdot 2}{2} = 3.$$

③ → Using side limits : $x \to x_0^+$, $x \to x_0^-$

From the definition we prove the following properties of side limits:

• $\lim\limits_{x \to x_0} f(x) = L \Leftrightarrow \lim\limits_{x \to x_0^+} f(x) = L \wedge \lim\limits_{x \to x_0^-} f(x) = L$
• $\left.\begin{array}{l} \lim\limits_{x \to x_0^+} f(x) = L_1 \\ \lim\limits_{x \to x_0^-} f(x) = L_2 \\ L_1 \neq L_2 \end{array}\right\} \Rightarrow \lim\limits_{x \to x_0} f(x)$ <u>does not exist</u>

• <u>Does not exist</u> means that $\lim\limits_{x \to x_0} f(x) = L$ is a false statement for all possible choices of L (i.e. $L \in \mathbb{R}$, $L = +\infty$, $L = -\infty$).

• We use the above properties in conjunction with side limits.

EXAMPLES

a) $f(x) = \dfrac{x^2 + 2|x|}{x^2 - 2|x|} \leftarrow \lim\limits_{x \to 0} f(x)$.

Solution

• We use side limits to simplify $|x|$.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2+2|x|}{x^2-2|x|} = \lim_{x\to 0^+} \frac{x^2+2x}{x^2-2x} =$$

$$= \lim_{x\to 0^+} \frac{x(x+2)}{x(x-2)} = \lim_{x\to 0^+} \frac{x+2}{x-2} =$$

$$= \frac{0+2}{0-2} = -1 \qquad (1)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^2+2|x|}{x^2-2|x|} = \lim_{x\to 0^-} \frac{x^2-2x}{x^2+2x} =$$

$$= \lim_{x\to 0^-} \frac{x(x-2)}{x(x+2)} = \lim_{x\to 0^-} \frac{x-2}{x+2}$$

$$= \frac{0-2}{0+2} = -1 \qquad (2)$$

From (1) and (2): $\lim_{x\to 0} f(x) = -1$.

b) $f(x) = \dfrac{x^2+x}{|x|} \longleftarrow \lim_{x\to 0} f(x)$

Solution

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2+x}{|x|} = \lim_{x\to 0^+} \frac{x(x+1)}{x} =$$

$$= \lim_{x\to 0^+} (x+1) = 0+1 = 1 \qquad (1)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^2+x}{|x|} = \lim_{x\to 0^-} \frac{x(x+1)}{-x} =$$

$$= \lim_{x\to 0^-} (-x-1) = -0-1 = -1 \qquad (2)$$

From (1) and (2): $\lim_{x\to 0} f(x)$ does not exist.

EXERCISES

2 Show that

a) $\lim\limits_{x \to 3} \dfrac{x^2-9}{x^2-4x+3} = 3$

b) $\lim\limits_{x \to \sqrt{2}} \dfrac{x^4-4}{x^2-2} = 4$

c) $\lim\limits_{x \to 1} \dfrac{x^3+2x-3}{x-1}$

d) $\lim\limits_{x \to 1} \dfrac{x^2+x-2}{x^2-1} = \dfrac{3}{2}$

e) $\lim\limits_{x \to 1^-} \dfrac{x^2-2x+1}{|x|-1} = 0$

f) $\lim\limits_{x \to 1^+} \dfrac{x^2-2x+1}{|x|-1} = 0$

g) $\lim\limits_{x \to 5^-} \dfrac{|x-5|+x^2-4x-5}{x-5} = 5$

h) $\lim\limits_{x \to 1^+} \dfrac{(x^2-1)^2}{x^3-x^2-x+1} = 2$

i) $\lim\limits_{x \to -1^+} \dfrac{3x^2-3}{|x+1|} = -6$

j) $\lim\limits_{x \to 3^-} \dfrac{x^2-6x+9}{|x|-3} = 0$

3 Evaluate, if they exist, the following limits:

a) $\lim\limits_{x \to 0} \dfrac{\sqrt{9+x}-3}{x}$

b) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x^2}-1}{x}$

c) $\lim\limits_{x \to 5} \dfrac{\sqrt{x-1}-2}{x-5}$

d) $\lim\limits_{x \to 0} \dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$

e) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x^2}-1}{x}$

f) $\lim\limits_{x \to 1} \dfrac{x^2-\sqrt{x}}{\sqrt{x}-1}$

g) $\lim\limits_{x \to 2} \dfrac{\sqrt{x+3}-2}{x-1}$

h) $\lim\limits_{x \to 0} \dfrac{\sqrt{x+1}-1}{\sqrt{x+4}-2}$

④ Evaluate the following limits, if they exist:

a) $\lim\limits_{x \to 1} \dfrac{x^2-2x+1}{|x|-1}$

b) $\lim\limits_{x \to -1} \dfrac{x^2+2x+1}{|x|-1}$

c) $\lim\limits_{x \to 2} \dfrac{|x-2|+x^2-3x+2}{x-2}$

d) $\lim\limits_{x \to 3} \dfrac{x^2-5x+6+|x-3|}{x-3}$

e) $\lim\limits_{x \to 1} \dfrac{x^2+x-2}{|x|-1}$

f) $\lim\limits_{x \to 0} \dfrac{x^2+2|x|}{x^2-2|x|}$

g) $\lim\limits_{x \to 3} \dfrac{|x-3|+x^2-x-6}{x-3}$

h) $\lim\limits_{x \to 0} \dfrac{3x^2}{|x|}$

Functions with limits going to infinity

Addition and product of two functions where one of them goes to infinity can be handled by the following theorem

(2) Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ be two functions and let σ be a limit point of A and B. Let $\delta > 0$ and $a \in \mathbb{R}$. Then:

a) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) > a \\ \lim_{x \to \sigma} f(x) = +\infty \end{cases} \Rightarrow \lim_{x \to \sigma} [f(x) + g(x)] = +\infty$

b) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) < a \\ \lim_{x \to \sigma} f(x) = -\infty \end{cases} \Rightarrow \lim_{x \to \sigma} [f(x) + g(x)] = -\infty$

c) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) > a > 0 \\ \lim_{x \to \sigma} f(x) = \pm\infty \end{cases} \Rightarrow \lim_{x \to \sigma} [f(x) g(x)] = \pm\infty$

d) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) < a < 0 \\ \lim_{x \to \sigma} f(x) = \pm\infty \end{cases} \Rightarrow \lim_{x \to \sigma} [f(x) g(x)] = \mp\infty$

Note that it is not necessary for the limit $\lim_{x \to \sigma} g(x)$ to exist. It is sufficient to know how $g(x)$ is bounded. However, if $\lim_{x \to \sigma} g(x)$ does exist, we can infer the following deductions, which we summarize in the following operation tables.

$f(x) \downarrow \setminus g(x) \rightarrow$	a	$+\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	?
$-\infty$	$-\infty$	?	$-\infty$

$$\lim_{x \to \sigma} [f(x) + g(x)]$$

$f(x) \downarrow \setminus g(x) \rightarrow$	0	$p > 0$	$n < 0$	$+\infty$	$-\infty$
$+\infty$	?	$+\infty$	$-\infty$	$+\infty$	$-\infty$
$-\infty$	?	$-\infty$	$+\infty$	$-\infty$	$+\infty$

$$\lim_{x \to \sigma} [f(x) g(x)]$$

↳ The "?" correspond to indeterminate forms. It means that the limit cannot be determined without more information, and the limit may or may not exist.

Form $k/0$

- Let $f: A \to \mathbb{R}$ be a function, let $\delta > 0$, and let σ be a limit point of A. Then:

a) $\begin{cases} \forall x \in N(\sigma, \delta) \cap A : f(x) > 0 \\ \lim_{x \to \sigma} f(x) = 0 \end{cases} \Rightarrow \lim_{x \to \sigma} \frac{1}{f(x)} = +\infty$

b) $\begin{cases} \forall x \in N(\sigma, \delta) \cap A : f(x) < 0 \\ \lim_{x \to \sigma} f(x) = 0 \end{cases} \Rightarrow \lim_{x \to \sigma} \frac{1}{f(x)} = -\infty$

c) $\lim_{x \to \sigma} f(x) \in \{+\infty, -\infty\} \Rightarrow \lim_{x \to \sigma} \frac{1}{f(x)} = 0$

- From this theorem, we can show that

$$\forall a \in \mathbb{R} : \lim_{x \to a^+} \frac{1}{x-a} = +\infty$$

$$\forall a \in \mathbb{R} : \lim_{x \to a^-} \frac{1}{x-a} = -\infty$$

$$\forall k \in \mathbb{N}^* : \forall a \in \mathbb{R} : \lim_{x \to a} \frac{1}{(x-a)^{2k}} = +\infty$$

These results by themselves are sufficient for handling limits of functions that are defined as a ratio of two polynomials and yield a $k/0$ form.

EXAMPLES

a) $f(x) = \dfrac{1-3x}{(x-2)^2} \leftarrow \lim\limits_{x \to 2} f(x)$

Solution

$$f(x) = \frac{1-3x}{(x-2)^2} = (1-3x)\frac{1}{(x-2)^2} \qquad (1)$$

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = +\infty \qquad (2)$$

$$\lim_{x \to 2} (1-3x) = 1 - 3 \cdot 2 = 1 - 6 = -5 \qquad (3)$$

From Eq. (1), (2), (3): $\lim\limits_{x \to 2} f(x) = (-5)(+\infty) = -\infty$

b) $f(x) = \dfrac{2x+1}{2x-1} \leftarrow \lim\limits_{x \to 1/2^-} f(x)$.

Solution

$$f(x) = \frac{2x+1}{2x-1} = (2x+1)\frac{1}{2x-1} = \frac{2x+1}{2}\,\frac{1}{x-1/2} \qquad (1)$$

$$\lim_{x \to 1/2^-} \frac{1}{x-1/2} = -\infty \qquad (2)$$

$$\lim_{x \to 1/2^-} \frac{2x+1}{2} = \frac{2(1/2)+1}{2} = \frac{1+1}{2} = 1 \qquad (3)$$

From Eq. (1), (2), (3):

$$\lim_{x\to 1/2^-} f(x) = 1\cdot(-\infty) = -\infty$$

c) $f(x) = \dfrac{x^2+3x+2}{x^2+4x+4} \leftarrow \lim\limits_{x\to -2^+} f(x)$

Solution : Note that initially this is a 0/0 limit.

$$f(x) = \frac{x^2+3x+2}{x^2+4x+4} = \frac{(x+1)(x+2)}{(x+2)^2} = \frac{x+1}{x+2} =$$

$$= (x+1)\frac{1}{x+2} = (x+1)\frac{1}{x-(-2)}. \qquad (1)$$

$$\lim_{x\to -2^+} \frac{1}{x-(-2)} = +\infty \qquad (2)$$

$$\lim_{x\to -2^+} (x+1) = (-2)+1 = -1 \qquad (3)$$

From Eq.(1),(2),(3): $\lim\limits_{x\to -2^+} f(x) = (-1)(+\infty) = -\infty$

d) $f(x) = \dfrac{3-4x}{(x+1)^2(x^2+6x+5)} \leftarrow \lim\limits_{x\to -1} f(x).$

Solution

$$f(x) = \frac{3-4x}{(x+1)^2(x^2+6x+5)} = \frac{3-4x}{(x+1)^2(x+5)(x+1)} =$$

$$= \frac{1}{(x+1)^3}\,\frac{3-4x}{x+5} = \frac{1}{[x-(-1)]^3}\,\frac{3-4x}{x+5} \qquad (1)$$

$$\lim_{x\to -1} \frac{3-4x}{x+5} = \frac{3-4(-1)}{(-1)+5} = \frac{3+4}{4} = \frac{7}{4} > 0 \quad (2)$$

$$\lim_{x\to -1^+} \frac{1}{x-(-1)} = +\infty \Rightarrow \lim_{x\to -1^+} \frac{1}{[x-(-1)]^3} = +\infty \quad (3)$$

$$\lim_{x\to -1^-} \frac{1}{x-(-1)} = -\infty \Rightarrow \lim_{x\to -1^-} \frac{1}{[x-(-1)]^3} = -\infty \quad (4)$$

From Eq. (1), (2), (3): $\lim_{x\to -1^+} f(x) = (7/4)(+\infty) = +\infty \quad (5)$

From Eq. (1), (2), (4): $\lim_{x\to -1^-} f(x) = (7/4)(-\infty) = -\infty \quad (6)$

From Eq. (5), (6):
$\lim_{x\to -1^+} f(x) \neq \lim_{x\to -1^-} f(x) \Rightarrow \lim_{x\to -1} f(x)$ does not exist.

EXERCISES

⑤ Evaluate the following limits, if they exist.

a) $\lim_{x \to 2^+} \dfrac{3x+1}{2-x}$

b) $\lim_{x \to 0^+} \dfrac{1-2x}{x}$

c) $\lim_{x \to -2} \dfrac{x^2+3x+2}{x^2+4x+4}$

d) $\lim_{x \to 3} \dfrac{2-5x}{x^2-6x+9}$

e) $\lim_{x \to 1^-} \dfrac{x^2-2x+1}{x^3-3x^2+3x-1}$

f) $\lim_{x \to 1/2^+} \dfrac{x^2+1}{2x-1}$

g) $\lim_{x \to 2^-} \dfrac{2x}{x^2-4}$

h) $\lim_{x \to 2^+} \dfrac{2x-7}{8-x^3}$

Methods for limits $x \to \pm\infty$

For limits with $x \to +\infty$ and $x \to -\infty$ we rely on the following methodology:

→ Trivial limits

① Monomials

For limits involving monomials, we use the following results:

a) $\forall k \in \mathbb{N}^*: \lim\limits_{x \to +\infty} x^k = +\infty$

b) $\forall k \in \mathbb{N}^*: \lim\limits_{x \to -\infty} x^{2k} = +\infty$

c) $\forall k \in \mathbb{N}^*: \lim\limits_{x \to -\infty} x^{2k+1} = -\infty$

d) $\forall k \in \mathbb{N}^*: \lim\limits_{x \to \pm\infty} x^{-k} = 0$

EXAMPLES

a) $\lim\limits_{x \to +\infty} (3x^5) = 3 \cdot (+\infty) = +\infty$

b) $\lim\limits_{x \to -\infty} (2x^4) = 2 \cdot (+\infty) = +\infty$

c) $\lim_{x \to -\infty} (-7x^3) = -7(-\infty) = +\infty$

d) $\lim_{x \to +\infty} \frac{3}{2x^2} = \frac{3}{2} \cdot 0 = 0$

② Polynomials

If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, then:

$$\boxed{\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n}$$

EXAMPLE

a) $\lim_{x \to -\infty} (3x - 2x^3 + 1) = \lim_{x \to -\infty} (-2x^3) = -2(-\infty) = +\infty$

b) $f(x) = (1-3x^2)^3 (x+3x^2-x^3) \leftarrow \lim_{x \to -\infty} f(x)$.

Solution

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left[(1-3x^2)^3 (x+3x^2-x^3)\right] =$$

$$= \lim_{x \to -\infty} \left[(-3x^2)^3 (-x^3)\right] = \lim_{x \to -\infty} \left[(-27x^6)(-x^3)\right]$$

$$= \lim_{x \to -\infty} (27x^9) = 27(-\infty) = -\infty.$$

(3) Rational functions

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, and
$Q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0$,

then:

$$\boxed{\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)} = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}}$$

EXAMPLES

a) $f(x) = \dfrac{x + x^3 + 1}{2x - x^2} \leftarrow \lim\limits_{x \to -\infty} f(x)$

Solution

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x + x^3 + 1}{2x - x^2} = \lim_{x \to -\infty} \frac{x^3}{-x^2} =$$

$$= \lim_{x \to -\infty} (-x) = -(-\infty) = +\infty.$$

b) $f(x) = \dfrac{3x^2 + 3x - 1}{x^2 - 2} \leftarrow \lim\limits_{x \to +\infty} f(x)$

Solution

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{3x^2 + 3x - 1}{x^2 - 2} = \lim_{x \to +\infty} \frac{3x^2}{x^2} = 3$$

c) $f(x) = \dfrac{x^2+1}{2x^4-x} \longleftarrow \lim_{x\to-\infty} f(x)$

Solution

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+1}{2x^4-x} = \lim_{x\to-\infty} \frac{x^2}{2x^4} = \lim_{x\to-\infty} \frac{1}{2x^2}$$

$$= \frac{1}{2} \cdot 0 = 0$$

d) $f(x) = \dfrac{3x^2(2x-1)^3(x^2-2)}{(3x+2)(5x^2+4x+2)^2} \longleftarrow \lim_{x\to+\infty} f(x).$

Solution

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{3x^2(2x-1)^3(x^2-2)}{(3x+2)(5x^2+4x+2)^2} =$$

$$= \lim_{x\to+\infty} \frac{(3x^2)(2x)^3 x^2}{(3x)(5x^2)^2} =$$

$$= \lim_{x\to+\infty} \frac{(3x^2)(8x^3)x^2}{(3x)(25x^4)} = \lim_{x\to+\infty} \frac{24x^7}{75x^5}$$

$$= \lim_{x\to+\infty} \frac{24x^2}{75} = \frac{24}{75} \cdot (+\infty) = +\infty$$

Indeterminate forms

1) Form ∞/∞

$$f(x) = \frac{\sqrt{g(x)}}{h(x)}, \quad f(x) = \frac{g(x)}{\sqrt{h(x)}}, \quad f(x) = \sqrt{\frac{g(x)}{h(x)}}$$

- Factor highest-order term and simplify.

CAUTION: $\boxed{\begin{array}{l} \sqrt{x^2} = |x| \\ (\sqrt{x})^2 = x \end{array}}$

EXAMPLE

$$\lim_{x\to-\infty} \frac{\sqrt{9x^2-2x+5}}{x+4} = \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{9-2/x+5/x^2}}{x(1+4/x)}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{9-2/x+5/x^2}}{1+4/x} \cdot \lim_{x\to-\infty} \frac{\sqrt{x^2}}{x} =$$

$$= \frac{\sqrt{9-0+0}}{1+0} \cdot \lim_{x\to-\infty} \frac{|x|}{x} \underset{*}{=} \sqrt{9} \cdot \lim_{x\to-\infty} \frac{-x}{x}$$

$$= 3 \lim_{x\to-\infty} (-1) = -3.$$

* Since $x \to -\infty$, assume $x<0 \Rightarrow \sqrt{x^2} = |x| = -x$.

2) Form $\infty - \infty$

A) $f(x) = Q_1(x) - Q_2(x)$
with Q_1, Q_2 rational functions.

▸ Method: Combine to one fraction, and use previous methods.

EXAMPLE

a) $f(x) = \dfrac{x^3}{x^2+1} - \dfrac{2x^2}{x-1} \longrightarrow \lim_{x\to+\infty} f(x)$.

▸ Gives $\infty - \infty$. Thus:

$$f(x) = \frac{x^3}{x^2+1} - \frac{2x^2}{x-1} = \frac{x^3(x-1) - 2x^2(x^2+1)}{(x^2+1)(x-1)}$$

$$= \frac{x^4 - x^3 - 2x^4 - 2x^2}{(x^2+1)(x-1)} = \frac{-x^4 - x^3 - 2x^2}{(x^2+1)(x-1)} \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{-x^4 - x^3 - 2x^2}{(x^2+1)(x-1)} = \lim_{x\to+\infty} \frac{-x^4}{x^3}$$

$$= \lim_{x\to+\infty} (-x) = -(+\infty) = -\infty$$

b) $f(x) = \dfrac{x^4+2x^2}{x^2-1} - \dfrac{x^5+3x+1}{x^2+1} \longrightarrow \lim_{x\to-\infty} f(x)$

▸ Gives $\infty + \infty$! Not indeterminate form so we can use a constructive argument

$$\lim_{x\to-\infty} \frac{x^4+2x^2}{x^2-1} = \lim_{x\to-\infty} \frac{x^4}{x^2} = \lim_{x\to-\infty} x^2 = +\infty \quad (1)$$

$$\lim_{x\to-\infty} \frac{x^5+3x+1}{x^2+1} = \lim_{x\to-\infty} \frac{x^5}{x^2} = \lim_{x\to-\infty} x^3 = -\infty \quad (2)$$

Thus, from (1) and (2):

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left[\frac{x^4+2x^2}{x^2-1} - \frac{x^5+3x+1}{x^2+1}\right] =$$

$$= (+\infty) - (-\infty) = (+\infty) + (+\infty) = +\infty$$

B) $f(x) = \sqrt{g(x)} - \sqrt{h(x)}$

$f(x) = g(x) - \sqrt{h(x)}$

$f(x) = \sqrt{g(x)} - h(x)$

- Use the identity $\boxed{a-b = \frac{a^2-b^2}{a+b}}$

to eliminate the square roots from the $\infty - \infty$ factor.

EXAMPLE

$f(x) = \sqrt{9x^2+x+1} - 3x \longrightarrow \lim_{x\to+\infty} f(x), \lim_{x\to-\infty} f(x)$

a) $$f(x) = \sqrt{9x^2+x+1} - 3x = \frac{(\sqrt{9x^2+x+1})^2 - (3x)^2}{\sqrt{9x^2+x+1} + 3x} =$$

$$= \frac{(9x^2+x+1) - 9x^2}{\sqrt{9x^2+x+1} + 3x} = \frac{x+1}{\sqrt{9x^2+x+1} + 3x}$$

Since

$$\sqrt{9x^2+x+1} = \sqrt{x^2}\,\sqrt{9+1/x+1/x^2} =$$
$$= |x|\sqrt{9+1/x+1/x^2} =$$
$$= x\sqrt{9+1/x+1/x^2}$$

it follows that

$$f(x) = \frac{x+1}{x\sqrt{9+1/x+1/x^2}+3x} =$$
$$= \frac{x(1+1/x)}{x\left[\sqrt{9+1/x+1/x^2}+3\right]} =$$
$$= \frac{1+1/x}{\sqrt{9+1/x+1/x^2}+3} \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} f(x) = \frac{1+0}{\sqrt{9+0+0}+3} = \frac{1}{3+3} = \frac{1}{6}$$

b) $$\lim_{x\to-\infty}(9x^2+x+1) = \lim_{x\to-\infty}(9x^2) = 9(+\infty) = +\infty$$

$$\Rightarrow \lim_{x\to-\infty}\sqrt{9x^2+x+1} = +\infty. \qquad (1)$$

$$\lim_{x\to-\infty}(-3x) = -3(-\infty) = +\infty \qquad (2)$$

From Eq. (1), (2):

$$\lim_{x\to-\infty} f(x) = (+\infty)+(+\infty) = +\infty.$$

↳ In the absense of the indeterminate form $\infty-\infty$, the limit can be evaluated directly as shown above.

EXAMPLES

⑥ Evaluate the following limits, if they exist:

a) $\lim_{x \to -\infty} (2x^4 + x^3 - 2x + 3)$

b) $\lim_{x \to +\infty} (-2x^3 + 5x^2 - 3)$

c) $\lim_{x \to -\infty} (-x^2 + 5x)$

d) $\lim_{x \to -\infty} (5x^5 + x^2 - x - 1)$

e) $\lim_{x \to +\infty} \dfrac{9x^3 - 5x + 1}{3x^3 - 2x^2}$

f) $\lim_{x \to -\infty} \dfrac{x^4 - x + 3}{x^2 + 2}$

g) $\lim_{x \to +\infty} \dfrac{2x^3 - 5x + 1}{x^5 + 3x^4}$

h) $\lim_{x \to -\infty} \dfrac{x^2 - 4}{x - 2}$

i) $\lim_{x \to -\infty} \dfrac{3}{x^2 - 9}$

j) $\lim_{x \to +\infty} \dfrac{-x^2 + x + 2}{2x - 1}$

⑦ Evaluate the following limits, if they exist:

a) $\lim_{x \to +\infty} \dfrac{\sqrt{4x^2 - 2x + 1}}{2x + 3} = 1$

b) $\lim_{x \to +\infty} \dfrac{6x^2 - x + 1}{\sqrt{x^4 + 3}} = 6$

c) $\lim_{x \to -\infty} \dfrac{\sqrt{9x^2 - 2x + 5}}{x + 4} = -3$

d) $\lim_{x \to +\infty} \dfrac{4x^3 - x + 1}{\sqrt{x^2 + x + 5}} = +\infty$

e) $\lim_{x \to +\infty} \dfrac{\sqrt{x^3 + 4} + x - 2}{x\sqrt{x} - 3\sqrt{x^3 + 1}} = -\dfrac{1}{2}$

f) $\lim_{x \to -\infty} \dfrac{\sqrt{x^2 + 2} - x + 5}{x + 4} = -2$

g) $\lim_{x \to -\infty} \sqrt{\dfrac{x - 2}{x - 1}}$

h) $\lim_{x \to +\infty} \dfrac{5x}{3x - 1 + \sqrt{9x^2 + x + 1}}$

⑧ Evaluate the following limits, if they exist:

a) $\lim\limits_{x\to-\infty} \sqrt{x^2-9}$

b) $\lim\limits_{x\to-\infty} \left(\sqrt{16x^2+x+5}-4x\right)$

c) $\lim\limits_{x\to-\infty} \left(x+4-\sqrt{x^2-x+3}\right)$

d) $\lim\limits_{x\to+\infty} \left(\sqrt{4x^2-3x}+2x\right)$

e) $\lim\limits_{x\to+\infty} \sqrt{2x^4+3x^2+1}$

f) $\lim\limits_{x\to-\infty} \sqrt{x^2+4}$

⑨ Evaluate the following limits, if they exist:

a) $\lim\limits_{x\to+\infty} \left(\dfrac{x}{x-2}-\dfrac{1}{x+2}\right)$

b) $\lim\limits_{x\to-\infty} \left(\dfrac{1-x}{2x}-\dfrac{3+x}{x^2}\right)$

c) $\lim\limits_{x\to+\infty} \left(\dfrac{x^3}{x^2+1}-\dfrac{2x^2}{x-1}\right)$

d) $\lim\limits_{x\to-\infty} \left(\dfrac{-x^4-2}{2x}+\dfrac{x^3+1}{x}\right)$

⑩ Evaluate the following limits, if they exist:

a) $\lim\limits_{x\to+\infty} \left(\sqrt{x^2+1}-x\right)$

b) $\lim\limits_{x\to-\infty} \left(\sqrt{x^2+1}-x\right)$

c) $\lim\limits_{x\to+\infty} \left(\sqrt{9x^2+x+1}-3x\right)$

d) $\lim\limits_{x\to+\infty} \left(\sqrt{4x^2+2}-\sqrt{4x^2-1}\right)$

e) $\lim\limits_{x\to+\infty} \left(x+4-\sqrt{x^2-x+3}\right)$

f) $\lim\limits_{x\to+\infty} \left(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}\right)$

g) $\lim\limits_{x\to+\infty} \left(x+1+\sqrt{x^2+x+1}\right)$

h) $\lim\limits_{x\to+\infty} \dfrac{\sqrt{x^2+1}-x}{\sqrt{x}-\sqrt{x+1}}$

Trigonometric limits

To establish the basic theory we use the following inequality:

$\forall x \in (-\pi/2, \pi/2): |\sin x| \leq |x| \leq |\tan x|$

in conjunction with the "squeeze to zero" theorem

Squeeze to zero theorem

Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ be two functions and let σ be a limit point of A. Then:

$$\left\{ \begin{array}{l} \forall x \in N(\sigma, \delta) \cap A: |f(x)| \leq g(x) \\ \lim_{x \to \sigma} g(x) = 0 \end{array} \right. \Rightarrow \lim_{x \to \sigma} f(x) = 0$$

Proof

It is sufficient to show that

$\forall \varepsilon > 0: \exists \delta > 0: \forall x \in A: (x \in N(\sigma, \delta) \Rightarrow |f(x)| < \varepsilon)$

Let $\varepsilon > 0$ be given. Since, by hypothesis,

$\lim_{x \to \sigma} g(x) = 0 \Rightarrow \exists \delta > 0: \forall x \in A: (x \in N(\sigma, \delta) \Rightarrow |g(x)| < \varepsilon)$

Choose a $\delta > 0$ such that $\forall x \in A: (x \in N(\sigma, \delta) \Rightarrow |g(x)| < \varepsilon)$ (1)

Let $x \in A$ be given and assume that $x \in N(\sigma, \delta)$. Then:

$$\begin{aligned} x \in N(\sigma, \delta) &\Rightarrow |g(x)| < \varepsilon \Rightarrow \\ &\Rightarrow |f(x)| \leq g(x) && [\text{hypothesis}] \\ &\leq |g(x)| && [\text{algebra}] \\ &< \varepsilon && [\text{Eq. } (1)] \end{aligned}$$

$\Rightarrow |f(x)| < \varepsilon$

From the above argument it follows that

$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (x \in N(\sigma, \delta) \Rightarrow |f(x)| < \varepsilon)$

$\Rightarrow \lim_{x \to \sigma} f(x) = 0$ $\square$

Limit of sinx

$$\boxed{\forall x_0 \in \mathbb{R} : \lim_{x \to x_0} \sin x = \sin x_0}$$

Proof

Let $x_0 \in \mathbb{R}$ be given and define $f(x) = \sin x - \sin x_0$. Then:

$$|f(x)| = |\sin x - \sin x_0| = \left| 2 \sin\left(\frac{x - x_0}{2}\right) \cos\left(\frac{x + x_0}{2}\right) \right| =$$

$$= 2 \left| \sin\left(\frac{x - x_0}{2}\right) \right| \cdot \left| \cos\left(\frac{x + x_0}{2}\right) \right| \leqslant$$

$$\leqslant 2 \left| \sin\left(\frac{x - x_0}{2}\right) \right| \leqslant 2 \left| \frac{x - x_0}{2} \right| =$$

$$= 2 \frac{|x - x_0|}{2} = |x - x_0| , \ \forall x \in \mathbb{R} \Rightarrow$$

$\Rightarrow \forall x \in \mathbb{R} : |f(x)| \leqslant |x - x_0|$ (1).

and

$\lim_{x \to x_0} (x - x_0) = x_0 - x_0 = 0 \Rightarrow \lim_{x \to x_0} |x - x_0| = 0$ (2)

From Eq.(1) and Eq.(2):

$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (\sin x - \sin x_0) = 0 \Rightarrow \lim_{x \to x_0} \sin x = \sin x_0$ $\square$

$$\Rightarrow \lim_{x\to x_0} \sin x = \lim_{x\to x_0}\left[(\sin x - \sin x_0) + \sin x_0\right] =$$

$$= \lim_{x\to x_0}(\sin x - \sin x_0) + \sin x_0 =$$

$$= 0 + \sin x_0 = \sin x_0$$

Composition theorem

Thm: Let $f: A\to\mathbb{R}$ and $g: B\to\mathbb{R}$ be two functions and let $h: C\to\mathbb{R}$ with $C=\{x\in B \mid g(x)\in A\}$ be defined as $\forall x\in C: h(x) = f(g(x))$. Let σ be a limit point of B and C and let $a\in A$. Then:

$$\boxed{\left.\begin{array}{l}\lim_{x\to\sigma} g(x) = a\in\mathbb{R} \\ \lim_{x\to a} f(x) = f(a)\end{array}\right\} \Rightarrow \lim_{x\to\sigma} f(g(x)) = f(a) = f\left(\lim_{x\to\sigma} g(x)\right)}$$

Proof

It is sufficient to show that

$\forall \varepsilon>0: \exists\delta>0: \forall x\in C: (x\in N(\sigma,\delta) \Rightarrow |f(g(x)) - f(a)|<\varepsilon)$

Let $\varepsilon>0$ be given. Since:

$\lim_{x\to a} f(x) = f(a) \Rightarrow$

$\Rightarrow \forall\varepsilon_0>0: \exists\delta>0: \forall x\in A: \quad (0<|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon)$

For $\varepsilon_0=\varepsilon$ we have:

$\exists\delta>0: \forall x\in A: (0<|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon)$

Choose a $\delta_0 > 0$ such that

$\forall x \in A : (0 < |x-a| < \delta_0 \Rightarrow |f(x) - f(a)| < \varepsilon)$ (1)

Since $\lim_{x \to \sigma} g(x) = a \in \mathbb{R} \Rightarrow$

$\Rightarrow \forall \varepsilon_0 > 0 : \exists \delta > 0 : \forall x \in B : (x \in N(\sigma, \delta) \Rightarrow |g(x) - a| < \varepsilon_0)$

For $\varepsilon_0 = \delta_0$, we have:

$\exists \delta > 0 : \forall x \in B : (x \in N(\sigma, \delta) \Rightarrow |g(x) - a| < \delta_0)$

Choose a $\delta_1 > 0$ such that

$\forall x \in B : (x \in N(\sigma, \delta_1) \Rightarrow |g(x) - a| < \delta_0)$. (2)

Let $x \in G$ be given and assume that $x \in N(\sigma, \delta_1)$. Then:

$x \in G \wedge x \in N(\sigma, \delta_1) \Rightarrow x \in A \wedge x \in N(\sigma, \delta_1)$ [via $G \subseteq A$]

$\Rightarrow |g(x) - a| < \delta_0$ [via Eq. (2)]

We need a stronger condition $0 < |g(x) - a| < \delta_0$, so we distinguish between the following cases:

<u>Case 1</u>: Assume that $g(x) = a$. Then

$|f(g(x)) - f(a)| = |f(a) - f(a)| = |0| = 0 < \varepsilon$.

<u>Case 2</u>: Assume that $g(x) \neq a$. Then:

$x \in G \wedge 0 < |g(x) - a| < \delta_0 \Rightarrow g(x) \in B \wedge 0 < |g(x) - a| < \delta_0$

$\Rightarrow |f(g(x)) - f(a)| < \varepsilon$ [via Eq. (1)]

In both cases we show: $|f(g(x)) - f(a)| < \varepsilon$.

and from the above argument we have shown that

$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in G : (x \in N(\sigma, \delta) \Rightarrow |f(g(x)) - f(a)| < \varepsilon)$

$\Rightarrow \lim_{x \to \sigma} f(g(x)) = f(a)$.

If we replace the statement $\lim_{x \to a} f(x) = f(a)$ with $\lim_{x \to a} f(x) = b$ we get the following corollary of the composition theorem:

<u>Corollary</u>: Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be two functions and let $h : G \to \mathbb{R}$ with $G = \{x \in B \mid g(x) \in A\}$ be defined as $\forall x \in G : h(x) = f(g(x))$. Let σ be a limit point of B and G and let $a \in A$. Then:

$$\boxed{\left.\begin{aligned} &\lim_{x \to \sigma} g(x) = a \in \mathbb{R} \\ &\lim_{x \to a} f(x) = b \in \mathbb{R} \\ &\forall x \in N(\sigma, \delta) \cap B : g(x) \neq a \end{aligned}\right\} \Rightarrow \lim_{x \to \sigma} f(g(x)) = b}$$

<u>Proof</u>

We define

$$F(x) = \begin{cases} f(x) & , \text{ if } x \in A - \{a\} \\ b & , \text{ if } x = a \end{cases}$$

Then: $\lim_{x \to a} F(x) = \lim_{x \to a} f(x) = b = F(a)$

Using the composition theorem:

$$\left\{\begin{aligned} &\lim_{x \to \sigma} g(x) = b \\ &\lim_{x \to a} F(x) = F(a) \end{aligned}\right. \Rightarrow \lim_{x \to \sigma} F(g(x)) = F(a) = b. \qquad (1)$$

Since

$$(\forall x \in N(\sigma,\delta) \cap B : g(x) \neq a) \Rightarrow (\forall x \in N(\sigma,\delta) \cap G : F(g(x)) = f(g(x)))$$

$$\Rightarrow \lim_{x \to \sigma} f(g(x)) = \lim_{x \to \sigma} F(g(x)) = b \qquad \square$$

Limits of $\cos x$, $\tan x$, $\cot x$

From the composition theorem combined with the result

$$\lim_{x \to x_0} \sin x = \sin x_0$$

it follows that:

$$\lim_{x \to \sigma} g(x) = a \Rightarrow \lim_{x \to \sigma} \sin(g(x)) = \sin a = \sin\left(\lim_{x \to \sigma} g(x)\right)$$

We use this result along with the co-factor identity:

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

to show that:

- $\boxed{\forall x_0 \in \mathbb{R} : \lim_{x \to x_0} \cos x = \cos x_0}$

Proof

Let $x_0 \in \mathbb{R}$ be given. Then:

$$\lim_{x \to x_0} \cos x = \lim_{x \to x_0} \sin\left(\frac{\pi}{2} - x\right) \quad \text{[co-factor identity]}$$

$$= \sin\left(\lim_{x \to x_0}\left(\frac{\pi}{2} - x\right)\right) \quad \text{[composition theorem]}$$

$$= \sin\left(\frac{\pi}{2} - x_0\right)$$

$$= \cos x_0 \qquad \text{[co-factor identity]} \qquad \square$$

From this result, via basic limit properties we get:

- $$\boxed{\begin{array}{l} \forall x_0 \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}: \lim\limits_{x \to x_0} \tan x = \tan x_0 \\ \forall x_0 \in \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}: \lim\limits_{x \to x_0} \cot x = \cot x_0 \end{array}}$$

Combining these results with the composition theorem, we get:

$$\boxed{\begin{array}{l} \lim\limits_{x \to \sigma} g(x) = a \Rightarrow \lim\limits_{x \to \sigma} \sin(g(x)) = \sin a = \sin\left(\lim\limits_{x \to \sigma} g(x)\right) \\ \lim\limits_{x \to \sigma} g(x) = a \Rightarrow \lim\limits_{x \to \sigma} \cos(g(x)) = \cos a = \cos\left(\lim\limits_{x \to \sigma} g(x)\right) \\ \lim\limits_{x \to \sigma} g(x) = a \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\} \Rightarrow \lim\limits_{x \to \sigma} \tan(g(x)) = \tan a \\ \lim\limits_{x \to \sigma} g(x) = a \in \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\} \Rightarrow \lim\limits_{x \to \sigma} \cot(g(x)) = \cot a \end{array}}$$

EXAMPLES

a) $f(x) = \tan\left(\dfrac{\pi x(2x+3)}{12x^2}\right) \longleftarrow \lim\limits_{x\to+\infty} f(x)$

Solution

Since,

$$\lim_{x\to-\infty} \frac{\pi x(2x+3)}{12x^2} = \lim_{x\to+\infty} \frac{\pi x(2x)}{12x^2} = \lim_{x\to+\infty} \frac{2\pi x^2}{12x^2} =$$

$$= \frac{2\pi}{12} = \frac{\pi}{6} \notin \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\} \Rightarrow$$

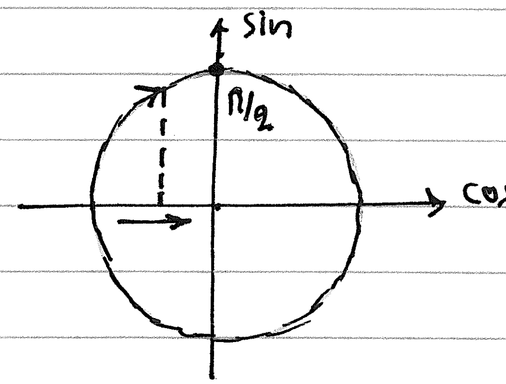
$$\Rightarrow \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \tan\left(\frac{\pi x(2x+3)}{12x^2}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

b) $f(x) = \tan x \longleftarrow \lim\limits_{x\to\pi/2^+} f(x)$

Solution

Since

$$f(x) = \tan x = \frac{\sin x}{\cos x} = (\sin x)\frac{1}{\cos x}$$



and

$$\left.\begin{array}{l} \cos x < 0, \ \forall x \in (\pi/2, \pi/2 + \pi/6) \\ \lim\limits_{x\to\pi/2^+} \cos x = \cos(\pi/2) = 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \lim_{x\to\pi/2^+} \frac{1}{\cos x} = -\infty \qquad (1)$$

and $\lim\limits_{x\to\pi/2^+} \sin x = \sin(\pi/2) = 1 \qquad (2)$

From Eq.(1) and Eq.(2)

$$\lim_{x\to\pi/2^+} f(x) = \lim_{x\to\pi/2^+}\left[(\sin x)\frac{1}{\cos x}\right] = 1(-\infty) = -\infty$$

c) $f(x) = \dfrac{x^2 - \cos x}{x\sin x} \leftarrow \lim_{x\to 0} f(x)$

Solution

Since,

$$\lim_{x\to 0}(x^2 - \cos x) = 0^2 - \cos 0 = 0 - 1 = -1 \quad (1)$$

and

$$\left.\begin{array}{l} \begin{cases} \forall x\in(0,\pi/2):(x>0 \land \sin x>0) \\ \forall x\in(-\pi/2,0):(x<0 \land \sin x<0) \end{cases} \Rightarrow \\ \Rightarrow \forall x\in(-\pi/2,0)\cup(0,\pi/2): \ x\sin x>0 \\ \quad \lim_{x\to 0} x\sin x = 0\sin 0 = 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \lim_{x\to 0}\frac{1}{x\sin x} = +\infty \quad (2)$$

it follows from Eq.(1) and Eq.(2):

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{x^2-\cos x}{x\sin x} = -1(+\infty) = -\infty$$

Zero-bounded limits

Def: Let $f: A \to \mathbb{R}$ be a function and let $S \subseteq A$. Then:

$$f \text{ bounded on } S \Leftrightarrow \exists a \in (0, +\infty): \forall x \in S: |f(x)| \leq a$$

• For example:

$\forall x \in \mathbb{R}: |\sin x| \leq 1 \Rightarrow \sin$ bounded on $\mathbb{R}$

$\forall x \in \mathbb{R}: |\cos x| \leq 1 \Rightarrow \cos$ bounded on $\mathbb{R}$

• To show that a function is bounded, it is useful to recall the following properties of absolute values:

$\forall a, b \in \mathbb{R}: |a+b| \leq |a| + |b|$

$\forall a, b \in \mathbb{R}: |a-b| \leq |a| + |b|$

$\forall a, b \in \mathbb{R}: |ab| = |a||b|$

$\forall a \in \mathbb{R}: \forall b \in \mathbb{R} - \{0\}: \left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

Thm: Let $f: A \to \mathbb{R}$ and $b: A \to \mathbb{R}$ be two functions and let σ be a limit point of A. Then:

$$\boxed{\left.\begin{array}{l} b \text{ bounded on } A \cap N(\sigma, \delta) \\ \lim\limits_{x \to \sigma} f(x) = 0 \end{array}\right\} \Rightarrow \lim_{x \to \sigma} [f(x) b(x)] = 0}$$

Proof

Since b bounded on $A \cap N(\sigma, \delta) \Rightarrow$

$$\Rightarrow \exists a \in (0, +\infty): \forall x \in A \cap N(\sigma, \delta): |b(x)| \leq a$$

and it follows that

$\forall x \in A \cap N(\sigma, \delta): \; |f(x) b(x)| = |f(x)||b(x)| \leqslant a|f(x)| \quad (1)$

We also note that

$$\lim_{x \to \sigma} f(x) = 0 \Rightarrow \lim_{x \to \sigma} |f(x)| = 0 \Rightarrow \lim_{x \to \sigma} (a|f(x)|) = 0 \quad (2).$$

From Eq. (1) and Eq. (2), via the squeeze to zero theorem we get:

$$\lim_{x \to \sigma} [f(x)\, b(x)] = 0. \qquad \square$$

EXAMPLES

a) $f(x) = \dfrac{\sin x \cos x - 3\sin(2x)\cos(2x)}{x^2+5x+3} \leftarrow \lim\limits_{x \to -\infty} f(x)$.

Solution

Note that

$$f(x) = \frac{\sin x \cos x - 3\sin(2x)\cos(2x)}{x^2+5x+3} =$$

$$= \frac{1}{x^2+5x+3} \cdot \left[\sin x \cos x - 3\sin(2x)\cos(2x)\right]$$

Define $b(x) = \sin x \cos x - 3\sin(2x)\cos(2x)$, $\forall x \in \mathbb{R}$.

$$|b(x)| = |\sin x \cos x - 3\sin(2x)\cos(2x)| \leq$$

$$\leq |\sin x \cos x| + |3\sin(2x)\cos(2x)|$$

$$= |\sin x||\cos x| + 3|\sin(2x)|\cdot|\cos(2x)|$$

$$\leq 1\cdot 1 + 3\cdot 1\cdot 1 = 1+3 = 4, \ \forall x \in \mathbb{R} \Rightarrow$$

$\Rightarrow b$ bounded on $\mathbb{R}$. (1).

and

$$\lim_{x \to -\infty} \frac{1}{x^2+5x+3} = \lim_{x \to -\infty} \frac{1}{x^2} = 0 \qquad (2)$$

From Eq. (1) and Eq. (2): $\lim\limits_{x \to -\infty} f(x) = 0$.

↳ It would have been sufficient to show that b is bounden on $(-\infty, -1/8)$ but in this case it is easy to show that b is bounded on $\mathbb{R}$.

b) $f(x) = x[1-\sin(1/x)] \longleftarrow \lim_{x\to 0} f(x)$

Solution

We note that $f(x) = x\,b(x)$, $\forall x \in \mathbb{R}-\{0\}$

where we define $\forall x \in \mathbb{R}-\{0\}: b(x) = 1-\sin(1/x)$. Since:

$$\forall x \in \mathbb{R}-\{0\}: |b(x)| = |1-\sin(1/x)| \leq |1| + |\sin(1/x)|$$

$$\leq 1+1 = 2 \Rightarrow$$

$\Rightarrow b$ bounded on $\mathbb{R}-\{0\}$ (1).

Also: $\lim_{x\to 0} x = 0$ (2)

From Eq.(1) and Eq.(2): $\lim_{x\to 0} f(x) = 0$

$\hookrightarrow$ It would have been sufficient to show that b is bounded on $(-\delta, 0)\cup(0,\delta)$. Note that using the zero-bounded theorem becomes necessary because $f(0)$ is not defined. It is possible to have zero-bounded limits that are trivial limits.

EXERCISES

(11) Evaluate the following limits, if they exist.

a) $\lim\limits_{x \to +\infty} \dfrac{\sin 4x}{x^2+5}$

b) $\lim\limits_{x \to -\infty} \dfrac{\sin 5x}{x}$

c) $\lim\limits_{x \to -\infty} \dfrac{2x \cos x}{x^2-9}$

d) $\lim\limits_{x \to +\infty} \dfrac{(2x-1)\sin 2x}{x^2+2}$

e) $\lim\limits_{x \to +\infty} \dfrac{3x^2 \sin x}{x^3+2}$

f) $\lim\limits_{x \to 0} \left(x^3 \sin(1/x)\right)$

g) $\lim\limits_{x \to +\infty} \left(\dfrac{\cos x}{x^3}\right)$

h) $\lim\limits_{x \to 1} \left[(x-1)\cos 3x\right]$

i) $\lim\limits_{x \to -\infty} \dfrac{x(\sin x + 2\cos 3x \sin 2x)}{2x^2+x+1}$

j) $\lim\limits_{x \to 0} \sin x \left[\cos(1/x^2) + \sin(1/x^2)\right]$

k) $\lim\limits_{x \to +\infty} \dfrac{x^3\left[(\sin x + \cos x)^2 + \sin x \cos x\right]}{(2x+1)^2(x^2-x+1)}$

l) $\lim\limits_{x \to +\infty} \left[\sin(1/x)\cos x + \cos(1/x)\sin x\right]$

m) $\lim\limits_{x \to 3} \dfrac{x^2-2x-3}{x+1}\left[\cos\left(\dfrac{1}{x-3}\right) + 3\sin x \sin\left(\dfrac{1}{x-3}\right)\right]$

Trigonometric 0/0 limits

Some 0/0 trigonometric limits can be handled via results that we establish via the squeeze theorem. We use the squeeze to 0 theorem to prove the squeeze theorem.

▶ Squeeze theorem

<u>Thm</u>: Let $f: A \to \mathbb{R}$, $g_1: A \to \mathbb{R}$, $g_2: A \to \mathbb{R}$ be three functions and let σ be a limit point of A. Then:

$$\left.\begin{array}{l} \forall x \in A \cap N(\sigma,\delta): g_1(x) \leq f(x) \leq g_2(x) \\ \lim_{x\to\sigma} g_1(x) = \lim_{x\to\sigma} g_2(x) = \ell \in \mathbb{R} \end{array}\right\} \Rightarrow \lim_{x\to\sigma} f(x) = \ell.$$

<u>Proof</u>

Let $x \in A \cap N(\sigma,\delta)$ be given. Then:

$$g_1(x) \leq f(x) \leq g_2(x) \Rightarrow 0 \leq f(x) - g_1(x) \leq g_2(x) - g_1(x) \Rightarrow$$
$$\Rightarrow 0 \leq |f(x) - g_1(x)| \leq g_2(x) - g_1(x)$$

and it follows that

$$\forall x \in A \cap N(\sigma,\delta): 0 \leq |f(x) - g_1(x)| \leq g_2(x) - g_1(x). \quad (1)$$

We also note that

$$\lim_{x\to\sigma} [g_2(x) - g_1(x)] = \lim_{x\to\sigma} g_2(x) - \lim_{x\to\sigma} g_1(x) = \ell - \ell = 0 \quad (2)$$

From Eq.(1) and Eq.(2), via the squeeze to zero theorem:

$$\lim_{x\to\sigma} (f(x) - g_1(x)) = 0 \Rightarrow$$
$$\Rightarrow \lim_{x\to\sigma} f(x) = \lim_{x\to\sigma} [(f(x) - g_1(x)) + g_1(x)] =$$

$$= \lim_{x\to\sigma} (f(x) - g_1(x)) + \lim_{x\to\sigma} g_1(x) =$$

$$= 0 + \ell = \ell \qquad \square$$

➤ Limits of $\sin x/x$ and $\tan x/x$

We now use the squeeze theorem combined with the inequality:

$\forall x \in (-\pi/2, 0) \cup (0, \pi/2): |\sin x| \leq |x| \leq |\tan x|$

to show the following basic results:

1) $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$

Proof

Define $f(x) = (\sin x)/x$, $\forall x \in \mathbb{R} - \{0\}$.

We note that

$$\forall x \in (-\pi/2, 0) \cup (0, \pi/2): f(x) = \frac{\sin x}{x} \leq \left|\frac{\sin x}{x}\right| = \frac{|\sin x|}{|x|}$$

$$\leq \frac{|x|}{|x|} = 1 \qquad (1)$$

and

$$\begin{cases} \forall x \in (0, \pi/2): (x > 0 \wedge \sin x > 0) \\ \forall x \in (-\pi/2, 0): (x < 0 \wedge \sin x < 0) \end{cases} \Rightarrow$$

$$\Rightarrow \forall x \in (-\pi/2, 0) \cup (0, \pi/2): f(x) = \frac{\sin x}{x} > 0$$

$$\Rightarrow \forall x \in (-\pi/2, 0) \cup (0, \pi/2) : f(x) = \frac{\sin x}{x} = \left| \frac{\sin x}{x} \right| =$$

$$= \frac{|\sin x|}{|x|} \geq \frac{|\sin x|}{|\tan x|} = \left| \frac{\sin x}{\tan x} \right| =$$

$$= \left| \frac{\sin x}{\left(\frac{\sin x}{\cos x} \right)} \right| = |\cos x| \qquad (2)$$

From Eq. (1) and Eq. (2):

$$\forall x \in (-\pi/2, 0) \cup (0, \pi/2) : |\cos x| \leq f(x) \leq 1 \qquad (3)$$

and since:

$$\lim_{x \to 0} \cos x = \cos 0 = 1 \Rightarrow \lim_{x \to 0} |\cos x| = |1| = 1 \qquad (4)$$

From Eq. (3) and Eq. (4), via the squeeze theorem, we have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1.$ $\square$

2 $\boxed{\lim_{x \to 0} \frac{\tan x}{x} = 1}$

Proof

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left[\frac{\sin x}{x} \frac{1}{\cos x} \right] =$$

$$= \left[\lim_{x \to 0} \frac{\sin x}{x} \right] \left[\lim_{x \to 0} \frac{1}{\cos x} \right] =$$

$$= 1 \cdot \frac{1}{\cos 0} = 1 \cdot \frac{1}{1} = 1. \qquad \square$$

Combining these results with the composition corrolary it follows that:

Prop: Let $f: A \to \mathbb{R}$ be a function and let σ be a limit point of A. Then:

$$\begin{cases} \lim_{x \to \sigma} f(x) = 0 \\ \forall x \in A \cap N(\sigma, \delta): f(x) \neq 0 \end{cases} \Rightarrow \lim_{x \to \sigma} \frac{\sin(f(x))}{f(x)} = 1$$

$$\begin{cases} \lim_{x \to \sigma} f(x) = 0 \\ \forall x \in A \cap N(\sigma, \delta): f(x) \neq 0 \end{cases} \Rightarrow \lim_{x \to \sigma} \frac{\tan(f(x))}{f(x)} = 1$$

EXAMPLES

a) $f(x) = \dfrac{\sin(2x)}{\sin(5x)} \leftarrow \lim\limits_{x\to 0} f(x).$

Solution

$$f(x) = \frac{\sin(2x)}{\sin(5x)} = \frac{\sin(2x)}{2x}\,\frac{5x}{\sin(5x)}\,\frac{2}{5}, \quad \forall x \in \mathbb{R}-\{0\}. \quad (1)$$

$$\lim_{x\to 0} \frac{\sin(2x)}{2x} = 1 \quad (2)$$

$$\lim_{x\to 0} \frac{\sin(5x)}{5x} = 1 \Rightarrow \lim_{x\to 0} \frac{5x}{\sin(5x)} = 1 \quad (3)$$

From Eq. (1), Eq. (2), Eq. (3):

$$\lim_{x\to 0} f(x) = 1\cdot 1\cdot (2/5) = \frac{2}{5}.$$

b) $f(x) = \dfrac{1-\cos x}{x^2} \leftarrow \lim\limits_{x\to 0} f(x)$

Solution

$$f(x) = \frac{1-\cos x}{x^2} = \frac{2\sin^2(x/2)}{x^2} = \frac{2(1/2)^2\sin^2(x/2)}{(x/2)^2}$$

$$= \frac{1}{2}\left[\frac{\sin(x/2)}{x/2}\right]^2, \quad \forall x \in \mathbb{R}-\{0\}$$

Since $\lim\limits_{x\to 0} \dfrac{\sin(x/2)}{x/2} = 1 \Rightarrow$

c) $f(x) = \dfrac{\sin x - \sin(5x)}{x\cos(3x)} \longleftarrow \lim\limits_{x \to \pi/6} f(x)$.

Solution

Note that

$\cos(3(\pi/6)) = \cos(\pi/2) = 0$

$\sin(\pi/6) - \sin(5\pi/6) = \sin(\pi/6) - \sin(\pi/6) = 0$

thus this is a 0/0 limit.

$$f(x) = \frac{\sin x - \sin(5x)}{x\cos(3x)} = \frac{2\sin\left(\frac{x-5x}{2}\right)\cos\left(\frac{x+5x}{2}\right)}{x\cos(3x)}$$

$$= \frac{2\sin(-2x)\cos(3x)}{x\cos(3x)} = \frac{-2\sin(2x)}{x} \Rightarrow$$

$$\Rightarrow \lim_{x\to\pi/6} f(x) = \lim_{x\to\pi/6} \frac{-2\sin(2x)}{x} = \frac{-2\sin(\pi/3)}{\pi/6} =$$

$$= \frac{-2(\sqrt{3}/2)}{\pi/6} = \frac{-\sqrt{3}}{\pi/6} = \frac{-6\sqrt{3}}{\pi}$$

EXERCISES

(12) Evaluate the following limits, if they exist

a) $\lim\limits_{x\to 0} \dfrac{\sin 4x}{x}$

b) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$

c) $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x}$

d) $\lim\limits_{x\to 0} \dfrac{\cos x - \cos 3x}{x\sin 3x}$

e) $\lim\limits_{x\to 0} \dfrac{1+\sin x - \cos x}{1-\sin x - \cos x}$

f) $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\tan 3x}$

g) $\lim\limits_{x\to \pi/4} \dfrac{1-\cos x}{\sin x}$

h) $\lim\limits_{x\to 0} \left(x \cdot \sin(1/x)\right)$

i) $\lim\limits_{x\to 0} \dfrac{\sin 5x}{\tan 6x}$

j) $\lim\limits_{x\to \pi/4} \dfrac{\cos x - \sin x}{x-\pi/4}$

Continuity

- Let $f: A \to \mathbb{R}$ be a function. We say that

a) f continuous at $x_0 \in A \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)$

b) f continuous at $I \Leftrightarrow \forall x_0 \in I$: f continuous at x_0

- By the limit definition, it follows that:

$$\boxed{\begin{array}{l} f \text{ continuous at } x_0 \in A \Leftrightarrow \\ \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in A : (0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon) \end{array}}$$

- There are three ways a function f can fail to be continuous at x_0.
  1) $f(x_0)$ is not defined (i.e. $x_0 \notin A$)
  2) $\lim_{x \to x_0} f(x)$ does not exist.
  3) Both $f(x_0)$ and $\lim_{x \to x_0} f(x)$ exist but $\lim_{x \to x_0} f(x) \neq f(x_0)$.

Operations and continuity.

Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ with f, g continuous at $x_0 \in A$, and let $\lambda \in \mathbb{R}$. Then

a) $h_1(x) = f(x) + g(x)$ continuous at $x_0 \in A$

$h_2(x) = f(x) g(x)$ continuous at $x_0 \in A$

$h_3(x) = \lambda f(x)$ continuous at $x_0 \in A$

b) $g(x_0) \neq 0 \Rightarrow h(x) = f(x)/g(x)$ continuous at x_0
c) $\forall x \in N(x_0, \delta) : f(x) \geq 0 \Rightarrow h(x) = \sqrt{f(x)}$ continuous at x_0.

Continuity of basic functions

1) Every polynomial function
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
is continuous at $\mathbb{R}$.
2) Every rational function
$$f(x) = \frac{P(x)}{Q(x)}$$
with P, Q polynomials is continuous at each domain $A = \mathbb{R} - \{x \in \mathbb{R} \mid Q(x) = 0\}$
3) sin, cos continuous at $\mathbb{R}$
tan continuous at $\mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}$
cot continuous at $\mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}$

Continuity of function composition

$$\boxed{\left.\begin{array}{l} g \text{ continuous at } x_0 \\ f \text{ continuous at } g(x_0) \end{array}\right\} \Rightarrow \begin{array}{c} h(x) = f(g(x)) \text{ continuous} \\ \text{at } x_0 \end{array}}$$

EXAMPLE

a) Find all $a, b \in \mathbb{R}$ such that

$$f(x) = \begin{cases} x^2-3x+1 & , \text{ if } x \in (-\infty, 2) \\ ax+b & , \text{ if } x \in [2,3) \\ x^2+5x+2 & , \text{ if } x \in [3, +\infty) \end{cases}$$

is continuous on $\mathbb{R}$.

Solution

First, we note that f continuous on $\mathbb{R} - \{2,3\}$, $\forall a \in \mathbb{R}$ (1)

At $x = 2$:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 - 3x + 1) = 2^2 - 3 \cdot 2 + 1 = 4 - 6 + 1 = -1. \quad (2)$$

$$f(2) = \lim_{x \to 2^-} f(x) = -1. \quad (3)$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax+b) = 2a+b \quad (4)$$

At $x=3$:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax+b) = 3a+b \quad (5)$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^2+5x+2) = 3^2+5 \cdot 3+2 = 9+15+2 = 26 \quad (6)$$

$$f(3) = 3^2+5 \cdot 3+2 = 26 \quad (7)$$

It follows that:

f continuous on $\mathbb{R} \Leftrightarrow f$ continuous on $\{2,3\} \Leftrightarrow$

$$\Leftrightarrow \lim_{x\to 2} f(x) = f(2) \wedge \lim_{x\to 3} f(x) = f(3) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2) \\ \lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = f(3) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2a+b=-1 \\ 3a+b=26 \end{cases} \Leftrightarrow \begin{cases} 2a+b=-1 \\ a+(2a+b)=26 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2a+b=-1 \\ a+(-1)=26 \end{cases} \Leftrightarrow \begin{cases} 2\cdot 27+b=-1 \\ a=26+1=27 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a=27 \\ b=-1-2\cdot 27=-1-54=-55 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow (a,b)=(27,-55)$$

We have thus shown that

f continuous on $\mathbb{R} \Leftrightarrow (a,b)=(27,-55)$

b) Find all $a \in \mathbb{R}$ such that

$$f(x) = \begin{cases} x-2 & , \text{ if } x \in (2, +\infty) \\ x^2-(a+1)x+(a^2-1) & , \text{ if } x \in (-\infty, 2] \end{cases}$$

is continuous on $\mathbb{R}$.

Solution

We note that f continuous on $\mathbb{R} - \{2\}$, $\forall a \in \mathbb{R}$.

At $x = 2$:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x-2) = 2-2 = 0$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} [x^2-(a+1)x+(a^2-1)] = 2^2-(a+1)2+(a^2-1)$$

$$= 4-2a-2+a^2-1 = a^2-2a+1 = (a-1)^2$$

$f(2) = 2^2-(a+1)2+(a^2-1) = (a-1)^2$.

It follows that

f continuous on $\mathbb{R} \Leftrightarrow f$ continuous at $x = 2 \Leftrightarrow$

$$\Leftrightarrow \lim_{x \to 2} f(x) = f(2) \Leftrightarrow$$

$$\Leftrightarrow \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = f(2)$$

$$\Leftrightarrow (a-1)^2 = 0 \Leftrightarrow a-1 = 0 \Leftrightarrow a = 1.$$

We have thus shown that

f continuous on $\mathbb{R} \Leftrightarrow a = 1$.

c) Show that the function

$$f(x) = \begin{cases} (\sin x)[1+\cos(1/x)] & , \text{ if } x \in \mathbb{R}-\{0\} \\ 0 & , \text{ if } x=0 \end{cases}$$

is continuous on $\mathbb{R}$.

Solution

We note that f is continuous at $\mathbb{R}-\{0\}$. (1)

At $x=0$: we define $b(x) = 1+\cos(1/x)$, $\forall x \in \mathbb{R}-\{0\}$.

Then:

$|b(x)| = |1+\cos(1/x)| \leq |1| + |\cos(1/x)| \leq 1+1 = 2$, $\forall x \in \mathbb{R}-\{0\}$

$\Rightarrow$ b bounded on $\mathbb{R}-\{0\}$ (2)

and

$\lim_{x\to 0} \sin x = \sin 0 = 0$ (3)

From Eq. (2) and Eq. (3) it follows that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (\sin x)[1+\cos(1/x)] =$$

$$= 0 = f(0) \Rightarrow$$

$\Rightarrow$ f continuous at $x=0$ $\Rightarrow$

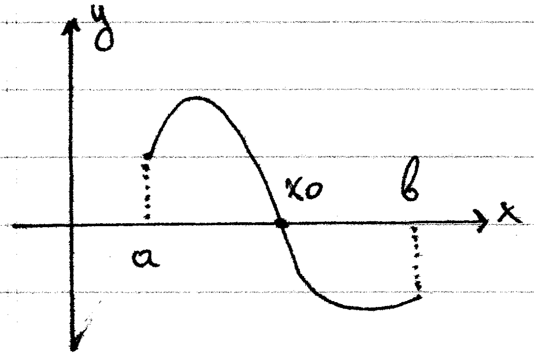
$\Rightarrow$ f continuous on $\mathbb{R}$ [via Eq. (1)].

V Consequences of continuity

(1) → Bolzano theorem

Thm: Let $f: A \to \mathbb{R}$ be a function with $[a,b] \subseteq A$. Then:

$$\boxed{\left.\begin{array}{l} f \text{ continuous on } [a,b] \\ f(a)f(b) < 0 \end{array}\right\} \Rightarrow \exists x_0 \in [a,b] : f(x_0) = 0}$$



The condition $f(a)f(b) < 0$ means that $f(a)$ and $f(b)$ have opposite signs. According to the Bolzano theorem, in order for the function to change sign, from $x=a$ to $x=b$, it has to be 0 for some $x = x_0 \in (a,b)$. A proof of the Bolzano theorem is omitted as it requires the theory of sequences.

(2) → Intermediate value theorem

Let $f: A \to \mathbb{R}$ be a function and let $S \subseteq A$. Recall that we defined $f(S)$ as

$$f(S) = \{ f(x) \mid x \in S \}$$

or via the belonging condition

$$y \in f(S) \Leftrightarrow \exists x \in S : f(x) = y$$

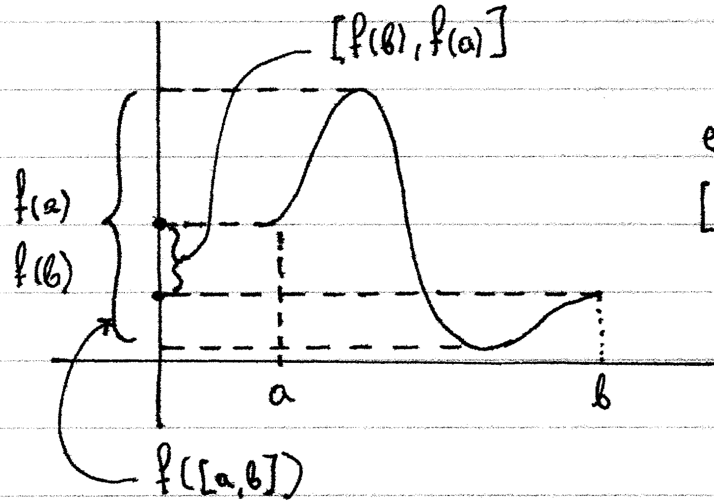
We also recall that for any two sets A and B:

$$A \subseteq B \Leftrightarrow \forall x \in A : x \in B$$

An immediate consequence of the Bolzano theorem is the intermediate value theorem:

<u>Thm</u>: Let $f: A \to \mathbb{R}$ be a function and let $[a,b] \subseteq A$. Then:

$$\boxed{\begin{array}{l} \begin{cases} f \text{ continuous on } [a,b] \\ f(a) \leq f(b) \end{cases} \Rightarrow [f(a), f(b)] \subseteq f([a,b]) \\ \begin{cases} f \text{ continuous on } [a,b] \\ f(a) \geq f(b) \end{cases} \Rightarrow [f(b), f(a)] \subseteq f([a,b]) \end{array}}$$



e.g.: $[f(b), f(a)] \subseteq f([a,b])$

▸ <u>Interpretation</u>: $f([a,b])$ represents all the values taken by the function f over the interval $[a,b]$. The interval $[f(a), f(b)]$ obviously contains all numbers y such that $f(a) \leq y \leq f(b)$. Thus the statement $[f(a), f(b)] \subseteq f([a,b])$ means that the function f takes all the values y with $f(a) \leq y \leq f(b)$.

Proof

Assume that f continuous on $[a,b]$ and $f(a) < f(b)$.

Since: $a \in [a,b] \Rightarrow f(a) \in f([a,b])$

$b \in [a,b] \Rightarrow f(b) \in f([a,b])$

we let $y \in (f(a), f(b))$ be given and we define

$g(x) = f(x) - y, \ \forall x \in [a,b]$

It follows that:

f continuous on $[a,b] \Rightarrow g$ continuous on $[a,b]$ (1)

and

$$y \in (f(a), f(b)) \Rightarrow f(a) < y < f(b) \Rightarrow \begin{cases} f(a) < y \\ y < f(b) \end{cases} \Rightarrow \begin{cases} f(a) - y < 0 \\ f(b) - y > 0 \end{cases}$$

$$\Rightarrow \begin{cases} g(a) < 0 \\ g(b) > 0 \end{cases} \Rightarrow g(a)g(b) < 0 \qquad (2)$$

From Eq.(1) and Eq.(2), via the Bolzano theorem,

$$(\exists x_0 \in (a,b) : g(x_0) = 0) \Rightarrow (\exists x_0 \in (a,b) : f(x_0) - y = 0)$$

$$\Rightarrow \exists x_0 \in (a,b) : f(x_0) = y$$

$$\Rightarrow y \in f((a,b)) \Rightarrow y \in f([a,b])$$

We have thus shown that

$$(\forall y \in [f(a), f(b)] : y \in f([a,b])) \Rightarrow \underline{[f(a), f(b)] \subseteq f([a,b])}$$

EXAMPLES

a) Show that the equation

$$\sin(\cos 3x) = 0$$

has at least one solution on $(0, \pi)$.

Solution

Define $f(x) = \sin(\cos(3x))$, $\forall x \in \mathbb{R}$.

We note that f continuous on $[0, \pi]$ (1).

and also:

$f(0) = \sin(\cos(3 \cdot 0)) = \sin(\cos 0) = \sin 1$ (2)

$f(\pi) = \sin(\cos(3\pi)) = \sin(\cos \pi) = \sin(-1) = -\sin 1$ (3)

From Eq. (2) and Eq. (3):

$f(0)f(\pi) = (\sin 1)(-\sin 1) = -\sin^2 1 < 0$ (3)

From Eq. (1) and Eq. (3):

$(\exists x_0 \in (0, \pi) : f(x_0) = 0) \Rightarrow x_0$ solves $\sin(\cos(3x)) = 0$.

b) If $a, b \in \mathbb{R}$ with $0 < a < b < \pi/2$, show that the equation

$$\frac{\sin x}{x-a} + \frac{\cos x}{x-b} = 0$$

has at least one solution $x_0 \in (a, b)$.

Solution

We note that for $x \in (a,b)$, we have $(x-a)(x-b) \neq 0$, and therefore:

$$\frac{\sin x}{x-a} + \frac{\cos x}{x-b} = 0 \Leftrightarrow (x-b)\sin x + (x-a)\cos x = 0$$

Define $f(x) = (x-b)\sin x + (x-a)\cos x$, $\forall x \in \mathbb{R}$

Then: f continuous on $[a,b]$ (1)

$f(a) = (a-b)\sin a + (a-a)\cos a = (a-b)\sin a$ (2)

$f(b) = (b-b)\sin b + (b-a)\cos b = (b-a)\cos b$ (3)

From Eq.(2) and Eq.(3):

$$f(a)f(b) = [(a-b)\sin a][(b-a)\cos b] = (a-b)(b-a)\sin a \cos b$$
$$= -(a-b)^2 \sin a \cos b.$$

We note that $a \neq b \Rightarrow (a-b)^2 > 0$

and $0 < a < \pi/2 \Rightarrow \sin a > 0$

and $0 < b < \pi/2 \Rightarrow \cos b > 0$.

It follows that

$f(a)f(b) = -(a-b)^2 \sin a \cos b < 0$ (4)

From Eq.(1) and Eq.(4), via Bolzano theorem,

$(\exists x_0 \in (a,b) : f(x_0) = 0) \Rightarrow x_0$ solves $\dfrac{\sin x}{x-a} + \dfrac{\cos x}{x-b} = 0$

③ → Continuity and Boundedness

Thm: Let $f: A \to \mathbb{R}$ be a function with $[a,b] \subseteq A$.

Then:

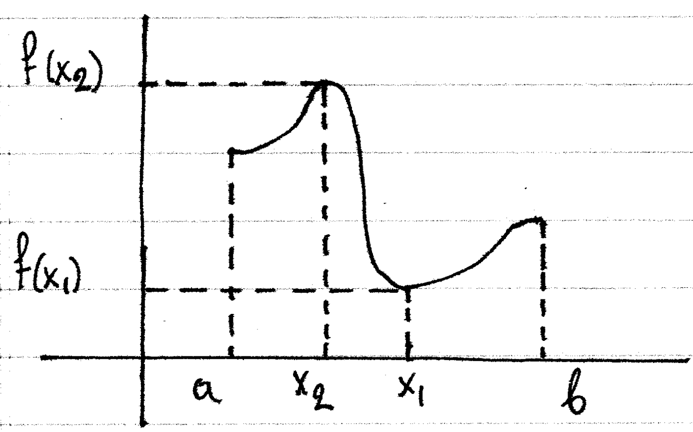
$$\boxed{f \text{ continuous on } [a,b] \Rightarrow f \text{ bounded on } [a,b]}$$

↳ Although this result is very intuitive, geometrically, a proof requires the theory of sequences and is therefore omitted.

④ → Extremum value theorem

Thm: Let $f: A \to \mathbb{R}$ be a function with $[a,b] \subseteq A$. Then:

$$\boxed{\begin{array}{l} f \text{ continuous on } [a,b] \Rightarrow \\ \Rightarrow \exists x_1, x_2 \in [a,b] : \forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2) \end{array}}$$



EXERCISES

(13) Show that the following functions are continuous in $\mathbb{R}$.

a) $f(x) = \begin{cases} x\sin(1/x) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

b) $f(x) = \begin{cases} (\sin x)/x & , x \neq 0 \\ 1 & , x = 0 \end{cases}$

(14) Find all $a \in \mathbb{R}$ for which the following functions are continuous in $\mathbb{R}$.

a) $f(x) = \begin{cases} \frac{|x|}{x} + x & , x < 0 \\ a & , x \geq 0 \end{cases}$

b) $f(x) = \begin{cases} 2x+3 & , x \leq 1 \\ ax^2 - a^2x + 7 & , x > 1 \end{cases}$

c) $f(x) = \begin{cases} 2x^2+1 & , x \leq 1 \\ \frac{x^2+ax-3}{x+1} & , x > 1 \end{cases}$

(15) Find all $a, b \in \mathbb{R}$ for which the following functions are continuous at $\mathbb{R}$:

a) $f(x) = \begin{cases} 1+2\sin x & , x < -\pi \\ a\cos x & , -\pi \leq x < 0 \\ b - 4\cos^2 x & , x \geq 0 \end{cases}$

6) $f(x)=\begin{cases} -\sin 2x & , x \leqslant -\pi/4 \\ |a\sin x + b| & , x \in (-\pi/4, \pi/4) \\ \cos 2x & , x \geqslant \pi/4 \end{cases}$

(16) Show that the equation $\sin(\cos 3x) = 0$ has at least one solution in the interval $(0, \pi)$

(17) Show that $f(x) = ax^3 + x^2 + x - 1$ with $a \neq -1$ has at least one zero in the interval $(-1, 1)$. What happens when $a = -1$?

(18) Show that $f(x) = x^3/4 + \sin(\pi x) + 3$ takes the value $5/3$ within the interval $(-2, 2)$

(19) Show that $f(x) = x^3 - \cos(\pi x) + 1$ takes the value 5 within the interval $(-2, 3)$

(20) Let a function $f: [0,1] \longrightarrow (0,1)$ be given. Show that if f is continuous at $[0,1]$ then the equation $f(x) = x$ has at least one solution at the interval $(0,1)$.

(21) Let two functions f, g be given such that they are both continuous at $[a,b]$ and $f(a) = g(b)$ and $f(b) = g(a)$. Show that there is a $c \in [a,b]$ such that $f(c) = g(c)$.

(22) Show that the equation

$$\frac{x^2+1}{x-a} + \frac{x^6+1}{x-b} = 0$$

with $a<b$ has a solution in (a,b)

(23) Let f be a function that is continuous at $[0,2\pi]$ with $f(0)=f(2\pi)$. Show that there is an $x_0 \in [0,\pi]$ such that $f(x_0+\pi) = f(x_0)$.