PRELIMINARIES

Sets

- A set is an unordered collection of elements. An element can be a number or another set. We denote sets with upper-case letters: A, B, C, etc.
- Notation

  $x \in A$: x is an element of the set A

  $\quad$ x belongs to A.

  $x \notin A$: x is not an element of the set A

  $\quad$ x does not belong to A.
- Fundamentally, sets are defined via belonging condition of the form:

  $$x \in A \Leftrightarrow p(x)$$

  Here, $p(x)$ is a predicate, i.e. a statement about x which is either true or false.

  "$\Leftrightarrow$" means that $x \in A$ implies $p(x)$ and $p(x)$ implies $x \in A$.

Logic Operations

Let p, q be two statements that are either true or false. We define the following logic operations:

$p \wedge q$: p true AND q true

$p \vee q$: p true OR q true

(At least one of p or q is true, or both)

$p \underline{\vee} q$: either p is true OR q is true, but not both.

$p \Rightarrow q$: if p is true, then q is true

p implies q.

$p \Leftrightarrow q$: p is true <u>if and only if</u> q is true.

p implies q and q implies p.

Quantified statements

Let A be a set and $p(x)$ be a predicate. We define the following:

$\forall x \in A : p(x) \longrightarrow$ <u>For all</u> $x \in A$, $p(x)$ is true

$\exists x \in A : p(x) \longrightarrow$ There exists at least one $x \in A$ such that $p(x)$ is true.

Set algebra

- Sets can be defined by listing elements.

  e.g.: $A = \{2, 3, 6, 8\}$

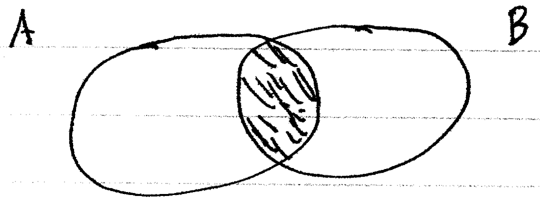
  Note that elements cannot be repeated and the order of listing is not important. The corresponding belonging condition is:

  $$x \in A \Leftrightarrow x = 2 \vee x = 3 \vee x = 6 \vee x = 8$$

- Let A, B be two sets. We define the following <u>set operations</u>; in terms of belonging conditions:

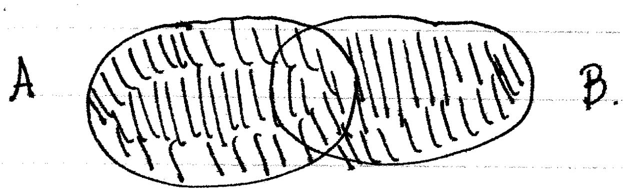
a) <u>Intersection</u> : $A \cap B$

$x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B)$



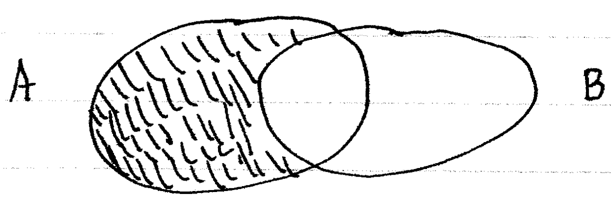
b) <u>Union</u> : $A \cup B$

$x \in A \cup B \Leftrightarrow (x \in A \vee x \in B)$.



c) <u>Difference</u> : $A - B$

$x \in A - B \Leftrightarrow (x \in A \wedge x \notin B)$.



- Let A, B be two sets. We say that
  a) $A = B$ if and only if the sets A, B have the same elements.
  b) $A \subseteq B$ if and only if all the elements of the set A also belong to the set B.

These definitions more formally are written as follows:

$$A \subseteq B \Leftrightarrow \forall x \in A : x \in B$$
$$A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$$

- We define $\emptyset = \{\}$ as the empty set, i.e. a set with no elements.

EXAMPLE

Let $A = \{1,2,3,4,5,6\}$ and $B = \{2,3,7,8\}$. Evaluate the sets $C = (A \cup B) - (A \cap B)$. and $D = A - (A \cup B)$

Solution

$$
\begin{aligned}
C &= (A \cup B) - (A \cap B) = \\
&= (\{1,2,3,4,5,6\} \cup \{2,3,7,8\}) - (\{1,2,3,4,5,6\} \cap \{2,3,7,8\}) = \\
&= \{1,2,3,4,5,6,7,8\} - \{2,3\} = \\
&= \{1,4,5,6,7,8\}
\end{aligned}
$$

$$
\begin{aligned}
D &= A - (A \cup B) \\
&= \{1,2,3,4,5,6\} - (\{1,2,3,4,5,6\} \cup \{2,3,7,8\}) \\
&= \{1,2,3,4,5,6\} - \{1,2,3,4,5,6,7,8\} \\
&= \emptyset .
\end{aligned}
$$

Number sets

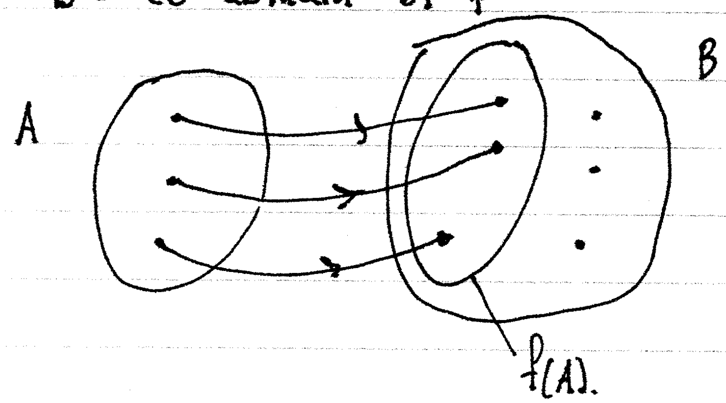
- We define the following number sets:
  1) Natural numbers : $\mathbb{N} = \{0,1,2,3,\dots\}$
  2) Integers: $\mathbb{Z} = \{0,1,-1,2,-2,3,-3,\dots\}$
  3) Rationals numbers: $\mathbb{Q}$.
     A number $x \in \mathbb{Q}$ if and only if there are two integers $a \in \mathbb{Z}$ and $b \in \mathbb{N}-\{0\}$ such that $x = a/b$.
     i.e.: $x \in \mathbb{Q} \Leftrightarrow \exists a \in \mathbb{Z} : \exists b \in \mathbb{N}-\{0\} : x = a/b$.
  4) Real numbers : $\mathbb{R}$
     The set $\mathbb{R}$ of real numbers is informally defined as the set of all numbers that can be approximated by a convergent sequence of rational numbers.
- Let A be a set. and $p(x)$ a predicate. We define
  $$S = \{x \in A \mid p(x)\}$$
  via the following belonging condition:
  $$x \in S \Leftrightarrow (x \in A \land p(x)).$$
- We use this to define intervals:

$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$	$[a,+\infty) = \{x \in \mathbb{R} \mid a \leq x\}$
$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$	$(a,+\infty) = \{x \in \mathbb{R} \mid a < x\}$
$(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$	$(-\infty,a] = \{x \in \mathbb{R} \mid x \leq a\}$
$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$	$(-\infty,a) = \{x \in \mathbb{R} \mid x < a\}$

- In general, for a subset $S \subseteq \mathbb{R}$:
  $$S \text{ interval} \Leftrightarrow \forall a,b \in S : (a < b \Rightarrow [a,b] \subseteq S)$$

Mappings and Functions

- A <u>mapping</u> $f: A \to B$ is a rule that maps every element $x \in A$ to a unique element $f(x) \in B$.
  $\mathrm{dom}(f) = A \longrightarrow$ domain of f.
  $B =$ co-domain of f



- A function $f: A \to \mathbb{R}$ is a mapping with domain $A \subseteq \mathbb{R}$ and co-domain $\mathbb{R}$.

Range of a mapping/function

- Given a mapping $f: A \to B$, the range $f(A)$ of f is the set of all elements of B for which there is at least one $x \in A$ that maps to these elements. Formally, we define $f(A)$ via the following belonging condition:

$$y \in f(A) \iff \exists x \in A : f(x) = y.$$

("$y \in f(A)$ if and only if there is at least one $x \in A$ such that $f(x) = y$").

Defining a function / implied domain

- To define a function f, it is necessary to give both

a) The algorithm for $f(x)$

b) The domain A of f.

This can be done succinctly using quantifier notation or verbally.

e.g: Let f be the function $f(x) = x^3 + x^2 + 2x + 1$ with domain $A = \mathbb{R}$.

OR more succinctly:

Define $f(x) = x^3 + x^2 + 2x + 1, \forall x \in \mathbb{R}$.

- If the domain is not given, then we assume the default domain to be the largest possible subset of $\mathbb{R}$. This is also called, the implied domain.

EXAMPLES

a) Find the implied domain of
$f(x) = (x^2+3x+1)(x+3)$

Solution

No restrictions, therefore $A = \mathbb{R}$.

$\hookrightarrow$ The default domain of any polynomial is always $A = \mathbb{R}$.

b) Find the implied domain of
$$f(x) = \frac{3x+1}{x^3-4x}$$

Solution

Require $x^3-4x \neq 0$.

Solve $x^3-4x=0 \Leftrightarrow x(x^2-4)=0 \Leftrightarrow x(x-2)(x+2)=0$
$\Leftrightarrow x=0 \vee x-2=0 \vee x+2=0 \Leftrightarrow x=0 \vee x=2 \vee x=-2$
$\Leftrightarrow x \in \{0,-2,2\}$.

It follows that $A = \mathbb{R} - \{-2,0,2\}$.

$\hookrightarrow$ For functions of the form $f(x) = P(x)/Q(x)$ with P, Q polynomials, we require $Q(x) \neq 0$.

c) Find the implied domain of
$f(x) = \sqrt{4-x} + \sqrt{x^2+3x+2}$

Solution

Require: $\begin{cases} 4-x \geq 0 \\ x^2+3x+2 \geq 0 \end{cases}$

Solve: $4-x \geq 0 \Leftrightarrow 4 \geq x \Leftrightarrow x \in (-\infty, 4] = S_1$

For $x^2+3x+2 \geq 0$ we have:

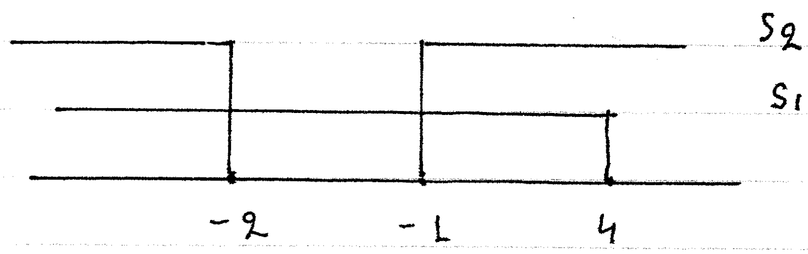
$\Delta = b^2 - 4ac = 3^2 - 4 \cdot 1 \cdot 2 = 9 - 8 = 1 \Rightarrow$

$\Rightarrow x_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a} = \dfrac{-3 \pm \sqrt{1}}{2 \cdot 1} = \dfrac{-3 \pm 1}{2} = \begin{cases} -2 \\ -1 \end{cases}$

x		-2		-1	
x^2+3x+2	$+$	0	$-$	0	$+$

It follows that

$x^2+3x+2 \geq 0 \Leftrightarrow x \in (-\infty, -2] \cup [-1, +\infty) = S_2.$



The domain is $A = S_1 \cap S_2 = (-\infty, -2] \cup [-1, 4]$

↳ For functions containing terms of the form $\sqrt{\varphi(x)}$ we require $\varphi(x) \geq 0$.

d) Find the implied domain of $f(x) = \sin x + \tan(3x)$.

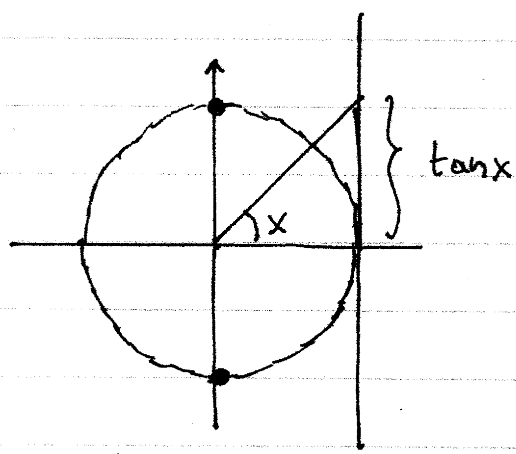
Solution

Require that $\forall k \in \mathbb{Z} : 3x \neq k\pi + \pi/2$

Solve $3x = k\pi + \pi/2 \Leftrightarrow x = k\pi/3 + \pi/6$

It follows that $A = \mathbb{R} - \left\{ \frac{k\pi}{3} + \frac{\pi}{6} \,\middle|\, k \in \mathbb{Z} \right\}$

↳ For functions with terms of the form $\tan(\varphi(x))$ we require: $\forall k \in \mathbb{Z} : \varphi(x) \neq k\pi + \pi/2$.
Note that the domain of $f(x) = \tan x$ is $A = \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}$. On a trigonometric circle this means excluding the points at the top and bottom of the circle.



e) Find the implied domain of $f(x) = \cot(\pi x + \pi/4)$.

<u>Solution</u>

Require $\forall k \in \mathbb{Z} : \pi x + \pi/4 \neq k\pi$.

Solve:

$$\pi x + \pi/4 = k\pi \Leftrightarrow x + 1/4 = k \Leftrightarrow$$

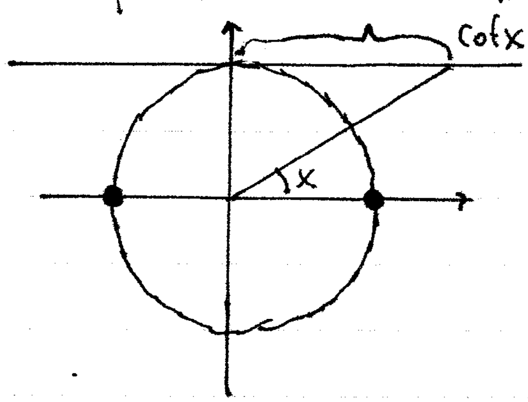
$$\Leftrightarrow x = k - \frac{1}{4} = \frac{4k-1}{4}$$

It follows that $A = \mathbb{R} - \left\{ \frac{4k-1}{4} \,\middle|\, k \in \mathbb{Z} \right\}$

↳ For functions with terms of the form $\cot(\varphi(x))$ we require: $\forall k \in \mathbb{Z} : \varphi(x) \neq k\pi$.

Note that the domain of $f(x) = \cot x$ is $A = \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}$

On a trigonometric circle this means excluding the points on the left and right of the circle.



↳ See my College Algebra notes for review of solution methods on equations, inequalities, systems of inequalites, etc.

EXERCISES

① Find the domain of the following functions.

a) $f(x) = \dfrac{x}{x^2-6x+9}$

b) $f(x) = \dfrac{x-1}{x^2+5x+6}$

c) $f(x) = \dfrac{(x+1)(x-2)}{(x-2)(x+2)}$

d) $f(x) = \dfrac{x+1}{x+2}$

e) $f(x) = \sqrt{2x-5}$

f) $f(x) = \sqrt{x^2-x-12}$

g) $f(x) = \sqrt{x^2+x+3}$

h) $f(x) = \sqrt{x^2+5x+1}$

i) $f(x) = \sqrt{x-x^3}$

j) $f(x) = -\dfrac{2}{\sqrt{3-x}}$

k) $f(x) = \sqrt{\dfrac{x+2}{x-3}}$

l) $f(x) = \dfrac{\sqrt{x+2}}{\sqrt{x-3}}$

m) $f(x) = \dfrac{1}{\sqrt{9-x^2}} + \sqrt{x^2-4}$

n) $f(x) = \sqrt{-x} + \dfrac{1}{\sqrt{5+x}}$

o) $f(x) = \sqrt{(x^3-1)(x^2-5)}$

p) $f(x) = \sqrt{x^2-4} + \sqrt{x-1}$

q) $f(x) = \dfrac{2}{|x-2|-1}$

r) $f(x) = \sqrt{3-|x|}$

s) $f(x) = \sqrt{|x-2|-1}$

t) $f(x) = \sqrt{|x|+3}$

(2) Likewise, find the domain of the following functions.

a) $f(x) = \tan(\pi x + x/3)$

b) $f(x) = \tan\left(\frac{\pi(x+1)}{2} + \frac{3\pi}{2}\right)$

c) $f(x) = \cot\left(\pi x + \frac{\pi(x+3)}{3}\right)$

d) $f(x) = \cot\left(\frac{\pi}{6} + \frac{\pi x + 2\pi(x+2)}{5}\right)$